**Quantum Query Complexity, Confining String In An Analytically Solvable 3D Model, Hagedorn Spectrum And Equation Of State Of Yang-Mills Theories, Equation Of State Of G(2) Gauge Theory, Donald Hoffman Uses Evolutionary Game Theory To Show That Our Perceptions Of An Independent Reality Must Be Illusions, Hard Problem Of Consciousness, Evolution Has Shaped Us With Perceptions That Allow Us To Survive, Fitness Function Happens To Align With The True Structure In Reality, If You Had To Spend All That Time Figuring Blue Icon Out The Tiger Would Eat You, Everything We See Is One Big Illusion, Evolution Shapes Acceptable Solutions, Not Optimal Ones, Mathematical Model Of Consciousness, World Is Just Another Conscious Agents, When You Do A Split-Brain Operation, A Complete Transection Of The Corpus Callosum, You Get Clear Evidence Of Two Separate Consciousnesses, Physics Tells Us That There Are No Public Physical Objects, Neurons, Brains, Space … These Are Just Symbols We Use, They're Not Real And Other Executive Colloquiums Of Similar Predilections: Cacoethes Scribendi Insatiable Desire To Write Cacoēthes[1] "Bad Habit", Or Medically, "Malignant Disease" Is A Borrowing Of Greek Kakóēthes.[2] The Phrase Is Derived From A Line In The Satires Of Juvenal: Tenet Insanabile Multos Scribendi Cacoethes, Or "The Incurable Desire (Or Itch) For Writing Affects Many". See Hypergraphia. Cessante Ratione Legis Cessat Ipsa Lex When The Reason For The Law Ceases, The Law Itself Ceases A Rule Of Law Becomes Ineffective When The Reason For Its Application Has Ceased To Exist Or Does Not Correspond To The Reality Anymore. By Gratian Carpe Diem Is A Latin Aphorism, Usually Translated**

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# "Seize The Day", Taken From Book 1 Of The Roman Poet Horace's Work Odes (23 BC). Models

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# ABSTRACT

**Abstract:** **Harry Buhrman, Matthias Christandl, Jeroen Zuiddam** study nondeterministic multiparty quantum communication with (e&eb) a quantum generalization of broadcasts. Authors show that, with number-in-hand classical inputs, the communication complexity of (e) a Boolean function in this communication model equals (=) the logarithm of (e) the support rank of (e) the corresponding tensor, whereas (e) the approximation complexity in (eb) this model equals (=) the logarithm of the border support rank. This characterisation allow (eb)s us to prove a log-rank conjecture posed by (e) Villagra et al. for (e) nondeterministic multiparty quantum communication with (e&eb) message-passing. The support rank characterization of (e) the communication model connects quantum communication complexity intimately to (e&eb) the theory of asymptotic entanglement transformation and (e&eb) algebraic complexity theory. In this context, authors introduce (eb) the graph wise equality problem. For a cycle graph, the complexity of (e) this communication problem is closely related to (e&eb) the complexity of the computational problem of multiplying matrices, or more precisely, it equals (=) the logarithm of the asymptotic support rank of (e) the iterated matrix multiplication tensor. Authors employ Strassen's laser method to show that (eb) asymptotically there exist nontrivial protocols for (e) every odd-player cyclic equality problem. They exhibit an efficient protocol for (e) the 5-player problem for (e) small inputs, and authors show how Young flattenings yield (eb) nontrivial complexity lower bounds. MSC classes: 68Q17, 15A69, 16Z05, 68Q12, 81P45 ACM classes:E.4Cite as: arXiv: 1603.03757 [quant-ph] (or arXiv: 1603.03757v2 [quant-ph] for this version) **Nondeterministic quantum communication complexity: the cyclic equality game and iterated matrix multiplication Harry Buhrman, Matthias Christandl, Jeroen Zuiddam (Submitted on 11 Mar 2016 (v1), last revised 2 Apr 2016 (this version, v2)) STOP** The Quantum Query Model is a framework that allows us (eb) to express most known quantum algorithms. Algorithms represented by this model consist on (e&eb) a set of unitary operators acting over (e&eb) a finite Hilbert space and a final measurement step consisting on (e&eb) a set of projectors. In this work, **S. A. Grillo, F. L. Marquezino** prove that the application of these unitary operators before the measurement step is equivalent to (=) decomposing (e&eb) a unit vector into (e&eb) a sum of vectors and then inverting some of their relative phases. Authors also prove that the vectors of (e) that sum must fulfill a list of properties and authors call such vectors a Block Set. If we take the same measurement step on the Quantum Query Model and (e&eb) on the Block Set Formulation, then authors prove that (eb) both formulations give (eb) the same Gram matrix of output states, although the Block Set Formulation allows (eb) a much more explicit form. Therefore, the Block Set reformulation of (e) the Quantum Query Model has (e) advantages for the design of exact quantum algorithms. Finally, authors test these advantages of Block Sets by applying approach to (e&eb) the construction of quantum exact algorithms. Subjects: Quantum Physics (quant-ph); Computational Complexity (cs.CC) Cite as: arXiv: 1602.07716 [quant-ph] (or arXiv: 1602.07716v1 [quant-ph] for this version) **Quantum query without unitary operators S. A. Grillo, F. L. Marquezino (Submitted on 24 Feb 2016) STOP** Therefore, the Block Set reformulation of (e) the Quantum Query Model has (e) advantages for the design of exact quantum algorithms. Finally, authors test these advantages of Block Sets by applying approach to (e&eb) the construction of quantum exact algorithms. Subjects: Quantum Physics (quant-ph); Computational Complexity (cs.CC) Cite as: arXiv: 1602.07716 [quant-ph] (or arXiv: 1602.07716v1 [quant-ph] for this version) **Quantum query without unitary operators S. A. Grillo, F. L. Marquezino (Submitted on 24 Feb 2016) STOP** Authors present two new quantum algorithms that either find a triangle (a copy of $K_{3}$) in (eb) an undirected graph G on n nodes, or reject if (e) G is triangle free. The first algorithm uses (e) combinatorial ideas with (e&eb) Grover Search and makes (eb) $\tilde{O}(n^{10/7})$ queries. The second algorithm uses (e) $\tilde{O}(n^{13/10})$ queries and is based on (e) a design concept of Ambainis [in Proceedings of the $45$th IEEE Symposium on Foundations of Computer Science, 2004, pp. 22–31] that incorporates (e) the benefits of quantum walks into Grover Search [L. Grover, in Proceedings of the Twenty-Eighth ACM Symposium on Theory of Computing, 1996, pp. 212–219]. The first algorithm uses (e) only $O (\log n) $ qubits in its quantum subroutines, whereas (e) the second one uses (e) $O(n)$ qubits. The Triangle Problem was first treated in [H. Buhrman et al., SIAM J. Comput., 34 (2005), pp.

1324–1330], where (e) an algorithm with $O(n+\sqrt{nm})$ query complexity was presented, where (e) m is the number of edges of G.Copyright © 2007 Society for Industrial and Applied Mathematics **SIAM J. Comput., 37(2), 413–424. (12 pages) Quantum Algorithms for the Triangle Problem Published online: 16 May 2007 Keywords 05C85, 68R10, 68W99 AMS Subject Headings quantum algorithm, query complexity, triangle problem, quantum walk Publication Data ISSN (print): 0097-5397 ISSN (online): 1095-7111 Publisher: Society for Industrial and Applied Mathematics Frédéric Magniez, Miklos Santha, and Mario SzegedyDOI:10.1137/050643684 STOP Nathan Wiebe, Ashish Kapoor, Krysta M Svore** demonstrate (eb) how quantum computation can provide (eb) non-trivial improvements in the computational and statistical complexity of (e) the perceptron model. Authors develop two quantum algorithms for (e) perceptron learning. The first algorithm exploits (e&eb) quantum information processing to determine (eb) a separating hyperplane using (e) a number of steps sublinear in (eb) the number of data points N, namely O (N−−√). The second algorithm illustrates (eb) how the classical mistake bound of (e) O (1γ2) can be further improved to (e) O (1γ√) through quantum means, where γ denotes the margin. Such improvements are achieved through (e&eb) the application of quantum amplitude amplification to (e) the version space interpretation of (e) the perceptron model. Subjects: Quantum Physics (quant-ph); Learning (cs.LG); Machine Learning (stat.ML) Cite as: arXiv: 1602.04799 [quant-ph] (or arXiv: 1602.04799v1 [quant-ph] for this version) **Quantum Perceptron Models Nathan Wiebe, Ashish Kapoor, Krysta M Svore (Submitted on 15 Feb 2016) STOP Anirban Narayan Chowdhury, Rolando D. Somma** present quantum algorithms for (e) solving two problems regarding stochastic processes. The first algorithm prepares the thermal Gibbs state of a quantum system and runs in time almost linear in (e&eb) Nβ/Z−−−−−√ and polynomial in log (1/ϵ), where N is the Hilbert space dimension, β is the inverse temperature, Z is the partition function, and ϵ is the desired precision of the output state. Quantum algorithm exponentially improves (eb) the dependence on 1/ϵ and quadratically improves (eb) the dependence on β of known quantum algorithms for (e) this problem. Second algorithm estimates the hitting time of (e) a Markov chain. For a sparse stochastic matrix P, it runs in time almost linear in (eb) 1/(ϵΔ3/2), where ϵ is the absolute precision in the estimation and Δ is (=) a parameter determined by P, and whose inverse is (=) an upper bound of the hitting time. Quantum algorithm quadratically improves (eb+) the dependence on 1/ϵ and 1/Δ of the analog classical algorithm for hitting-time estimation. Both algorithms use (e) tools recently developed in the context of Hamiltonian simulation, spectral gap amplification, and solving linear systems of equations. Subjects: Quantum Physics (quant-ph) Report number: LA-UR-16-21218 Cite as: arXiv: 1603.02940 [quant-ph] (or arXiv: 1603.02940v1 [quant-ph] for this version) **Quantum algorithms for Gibbs sampling and hitting-time estimation Anirban Narayan Chowdhury, Rolando D. Somma (Submitted on 9 Mar 2016)STOP Nathan Wiebe, Ashish Kapoor, Krysta M Svore** demonstrate (eb) how quantum computation can provide (eb) non-trivial improvements in the computational and statistical complexity of (e) the perceptron model. Authors develop two quantum algorithms for (e) perceptron learning. The first algorithm exploits (e&eb) quantum information processing to determine (eb) a separating hyperplane using (e) a number of steps sublinear in (eb) the number of data points N, namely O (N−−√). The second algorithm illustrates (eb) how the classical mistake bound of (e) O (1γ2) can be further improved to (e) O (1γ√) through quantum means, where γ denotes the margin. Such improvements are achieved through (e&eb) the application of quantum amplitude amplification to (e) the version space interpretation of (e) the perceptron model. Subjects: Quantum Physics (quant-ph); Learning (cs.LG); Machine Learning (stat.ML) Cite as: arXiv: 1602.04799 [quant-ph] (or arXiv: 1602.04799v1 [quant-ph] for this version) **Quantum Perceptron Models Nathan Wiebe, Ashish Kapoor, Krysta M Svore (Submitted on 15 Feb 2016) STOP** The Quantum Query Model is a framework that allows us (eb) to express most known quantum algorithms. Algorithms represented by this model consist on (e&eb) a set of unitary operators acting over (e&eb) a finite Hilbert space and a final measurement step consisting on (e&eb) a set of projectors. In this work, **S. A. Grillo, F. L. Marquezino** prove that the application of these unitary operators before the measurement step is equivalent to (=) decomposing (e&eb) a unit vector into (e&eb) a sum of vectors and then inverting some of their relative phases. Authors also prove that the vectors of (e) that sum must fulfill a list of properties and authors call such vectors a Block Set. If we take the same measurement step on the Quantum Query

Model and (e&eb) on the Block Set Formulation, then authors prove that both formulations give (eb) the same Gram matrix of output states, although the Block Set Formulation allows (eb) a much more explicit form. Therefore, the Block Set reformulation of (e) the Quantum Query Model has (e) advantages for the design of exact quantum algorithms. Finally, authors test these advantages of Block Sets by applying approach to (e&eb) the construction of quantum exact algorithms. Subjects: Quantum Physics (quant-ph); Computational Complexity (cs.CC) Cite as: arXiv: 1602.07716 [quant-ph] (or arXiv: 1602.07716v1 [quant-ph] for this version) **Quantum query without unitary operators S. A. Grillo, F. L. Marquezino (Submitted on 24 Feb 2016) STOP** Some of the most interesting and important results concerning (e&eb) quantum finite automata are those showing that (eb) they can recognize certain languages with (much) less resources than (eb) corresponding classical finite automata \cite{Amb98,Amb09,AmYa11,Ber05,Fre09,Mer00,Mer01,Mer02,Yak10,ZhgQiu112,Zhg12}. This paper shows (eb)three results of such a type that are (=) stronger in some sense than other ones because (a) they deal with models of quantum automata with (e&eb) very little quantumness (so-called semi-quantum one- and t(e&eb) wo-way automata with one qubit memory only); (b) differences, even comparing with probabilistic classical automata, are bigger than expected; (c) a trade-off (e&eb) between the number of classical and quantum basis states needed is demonstrated in one case and (d) (e&eb)languages (or the promise roblem) used to show main results are very simple and often explored ones in automata theory or in communication complexity, with (e&eb) seemingly little structure that could be utilized. Subjects: Formal Languages and Automata Theory (cs.FL); Quantum Physics (quant-ph) Journal reference: RAIRO-Inf. Theor. Appl. 48 (2014) 187-207. Earlier versions at LATA'14 DOI: 10.1051/ita/2014003 Cite as: arXiv: 1307.2499 [cs.FL] (or arXiv: 1307.2499v2 [cs.FL] for this version) **On the state complexity of semi-quantum finite automata Shenggen Zheng, Jozef Gruska, Daowen Qiu (Submitted on 9 Jul 2013 (v1), last revised 28 Jan 2014 (this version, v2))STOP** The degree of a polynomial representing (or approximating) a function f is (=) a lower bound for the quantum query complexity of f. This observation has been a source of (e) many lower bounds on quantum algorithms. It has been an open problem whether this lower bound is (=) tight. Authors exhibit (eb) a function with polynomial degree M and quantum query complexity (M1.321...). This is the first superlinear separation between polynomial degree and (e&eb) quantum query complexity. The lower bound is shown by (e) a new, more general version of quantum adversary method. Published in: Foundations of Computer Science, 2003 Proceedings.44th Annual IEEE Symposium on Date of Conference: 11-14 Oct. 2003 Page(s): 230 – 239 ISSN: 0272-5428 Print ISBN: 0-7695-2040-5 INSPEC Accession Number: 7840089 DOI: 10.1109/SFCS.2003.1238197 Publisher: IEEE **Polynomial degree vs. quantum query complexity Author(s) A. Ambainis ; Inst. of Math. & Comput. Sci., Latvia Univ., Riga, Latvia STOP** An {\it exact} quantum algorithm requires (E) its output to be correct certainly, and this kind of algorithms has been studied mostly in (eb) the query model since the discoveries of Deutsch-Jozsa algorithm and (e,Eb) the solution of Simon's problem. In this article, authors prove the optimal exact quantum query complexity for (e) a generalized Deutsch-Jozsa problem and characterize (eb)the symmetrically partial Boolean functions in terms of (e&eb) exact quantum query complexity. More specifically, (1) we give an optimal exact quantum query algorithm with complexity k+1 for computing the symmetrically partial Boolean function DJkn defined as: DJkn(x)=1 for |x|=n/2, DJkn(x)=0 for |x| in the set {0,1,…,k,n−k,n−k+1,…,n}, and it is undefined for (e) the rest cases, where (e) n is even, |x| is the Hamming weight of x. The case of k=0 is the well-known Deutsch-Jozsa problem. (2) They prove that any symmetrically partial Boolean function of has exact quantum 1-query complexity if and only if f can be computed by the Deutsch-Jozsa algorithm. (3) They also discover the optimal exact quantum 2-query complexity for solving a variant of the Deutsch-Jozsa problem, i.e. distinguishing between inputs of Hamming weight {⌊n/2⌋,⌈n/2⌉} and Hamming weight in the set {0,n} for all odd n. Authors further discuss the optimal exact quantum algorithms for (e) distinguishing between inputs of two different Hamming weights k and l. (4). We provide an algorithm to determine the degree of (e) any symmetrically partial Boolean function. Subjects: Quantum Physics (quant-ph); Computational Complexity (cs.CC); Data Structures and Algorithms (cs.DS) Cite as: arXiv: 1603.06505 [quant-ph] (or arXiv: 1603.06505v3 [quant-ph] for this version) **Characterizations of symmetrically partial Boolean functions with exact quantum query complexity Daowen Qiu, Shenggen Zheng (Submitted on 21 Mar 2016 (v1), last**

**revised 12 Apr 2016 (this version, v3))STOP** Recently, a broadcasting multiple blind signature scheme based on (e) quantum teleportation has been proposed for (e) the first time. It is claimed to have (e) unconditional security and properties of quantum multiple signature and (e&eb) quantum blind signature. In this paper, authors analyze the security of the protocol and show that the signatory Ui can learn the signed message m (eb) by a single particle measurement and the signed message m can be modified at random by anyone according to the scheme. Furthermore, there are some participant attacks and external attacks existing in the scheme. Finally the authors' improved protocol and show that (eb) the improved scheme can resist all of the mentioned attacks. Meanwhile, the secret keys can be used again and again, which will make it more efficient and (e&eb) O practical. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1507.01351 [quant-ph] (or arXiv: 1507.01351v2 [quant-ph] for this version) **Improvement of a quantum broadcasting multiple blind signature scheme based on quantum teleportation Wei Zhang, Daowen Qiu, Xiangfu Zou (Submitted on 6 Jul 2015 (v1), last revised 28 Dec 2015 (this version, v2)) STOP** Authors explore bounds of {\em time-space tradeoffs} in language recognition on (e&erb) {\em two-way finite automata} for some special languages. They prove: (1) a time-space tradeoff upper bound for recognition of the languages LEQ(n) on {\em two-way probabilistic finite automata} (2PFA): TS=O(nlogn), whereas a time-space tradeoff lower bound on {\em two-way deterministic finite automata} is Ω(n2), (2) a time-space tradeoff (e&eb) upper bound for recognition of the languages LINT(n) on {\em two-way finite automata with (e&eb) quantum and classical states} (2QCFA): TS=O(n3/2logn), whereas a lower bound on 2PFA is TS=Ω(n2), (3) a time-space tradeoff upper bound for recognition of the languages LNE(n) on exact 2QCFA: TS=O(n1.87logn), whereas a lower bound on 2PFA is TS=Ω(n2). It has been proved (Klauck, STOC'00) that the exact one-way quantum finite automata have no advantage comparing to classical finite automata in recognizing languages. However, the result (3) shows that the exact 2QCFA do have an advantage in comparison with their classical counterparts, which has been the first example showing that the exact quantum computing have advantage in time-space tradeoff comparing to classical computing. Usually, two communicating parties, Alice and Bob, are supposed to have an access to arbitrary computational power in {\em communication complexity} model that is used. Instead of that we will consider communication complexity in such a setting that two parties are using only finite automata and we prove in this setting that quantum automata are better than classical automata and also probabilistic automata are better than deterministic automata for some well known tasks. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1507.01346 [quant-ph] (or arXiv: 1507.01346v2 [quant-ph] for this version) STO.PIn the literature, there exist several interesting hybrid models of (e) finite automata which have (e) both quantum and classical states. Authors call them semi-quantum automata. In this paper, they compare the descriptional power of these models with (e&eb) that of DFA. Specifically, authors present a uniform method that gives (eb) a lower bound on (e) the size of the three existing main models of (e) semi-quantum automata, and this bound shows that semi-quantum automata can be at most exponentially more concise than DFA. Compared with a recent work (Bianchi, Mereghetti, Palano, and Theoret. Comput. Sci., 551(2014), 102-115), our method shows the following two advantages: (i) our method is much more concise; and (ii) our method is universal, since it is applicable to the three existing main models of semi-quantum automata, instead of only a specific model. Comments: arXiv admin note: text overlap with arXiv: 1206.2131. To appear in Theoretical Computer Science Subjects: Formal Languages and Automata Theory (cs.FL); Quantum Physics (quant-ph) Cite as: arXiv: 1502.02839 [cs.FL] (or arXiv:1502.02839v2 [cs.FL] for this version)STOP The concept of promise problems was introduced and started to be systematically explored by Even, Selman, Yacobi, Goldreich, and other scholars. It has been argued that promise problems should be seen as partial decision problems and as such that they are more fundamental than decision problems and formal languages that used to be considered as the basic ones for complexity theory. The main purpose of this paper is to explore the promise problems accepted by classical, quantum and also semi-quantum finite automata. More specifically, authors first introduce two acceptance modes of promise problems, recognizability and solvability, and explore their basic properties. Afterwards, we show several results concerning descriptional complexity on promise problems. In particular, we prove: (1) there is a promise problem that can be recognized exactly by measure-once one-way quantum finite automata (MO-1QFA), but no

deterministic finite automata (DFA) can recognize it; (2) there is a promise problem that can be solved with (e&eb) error probability $\epsilon \leq 1/3$ by one-way finite automaton with (e&eb) quantum and classical states (1QCFA), but no one-way probability finite automaton (PFA) can solve it with (e&eb) error probability $\epsilon \leq 1/3$; and especially, (3) there are promise problems A(p) with prime p that can be solved {\em with any error probability} by MO-1QFA with only two quantum basis states, but they cannot be solved exactly by (e) any MO-1QFA with two quantum basis states; in contrast, the minimal PFA solving A(p) with (e&eb) any error probability (usually smaller than 1/2) has p states. Finally, authors mention a number of problems related to promise for further study. Comments: Substantial changes are made. Some new results are added Subjects: Formal Languages and Automata Theory (cs.FL); Quantum Physics (quant-ph) Cite as: arXiv: 1411.3870 [cs.FL] (or arXiv: 1411.3870v3 [cs.FL] for this version) **STOP** The potential of the exact quantum information processing is an interesting, important and intriguing issue. For examples, it has been believed that quantum tools can provide (eb) significant, that is larger than polynomial, advantages in the case of (e) exact quantum computation only, or mainly, for problems with (e&eb) very special structures. Authors will show that this is not the case. In this paper the potential of quantum finite automata producing (eb) outcomes not only with a (high) probability, but with certainty (so called exactly) is explored in the context of (e) their uses for solving promise problems and with respect to the size of automata. It is shown that for solving particular classes $\{An\}_{n=1}^{\infty}$ of promise problems, even those without (e) some very special structure, that succinctness of the exact quantum finite automata under consideration, with respect to the number of (basis) states, (e&eb) can be very small (and constant) though it grows proportional to (e&eb) n in the case deterministic finite automata (DFAs) of the same power are used. This is here demonstrated also for the case that the component languages of the promise problems solvable by DFAs are non-regular. The method used can be applied (e&eb) in finding more exact quantum finite automata or quantum algorithms for (e) other promise problems. Subjects: Formal Languages and Automata Theory (cs.FL); **Computational Complexity (cs.CC); Quantum Physics (quant-ph) Cite as: arXiv: 1404.1689 [cs.FL] (or arXiv: 1404.1689v4 [cs.FL] for this version) Bužek and M. Hillery** analyze the possibility of (e) copying (that is, cloning) arbitrary states of (e) a quantum-mechanical spin-1/2 system. Authors show that there exists (eb) a ‘‘universal quantum-copying machine’’ (i.e., transformation) which approximately copies quantum-mechanical states such that (e) the quality of its output does not (e) depend on the input. Authors also examine a machine which combines a unitary transformation and (e&eb) a selective measurement to produce good copies of (e) states in the neighborhood of a particular state. Authors discuss the problem of (e) measurement of the output states. © 1996 The American Physical Society Received 5 February 1996 DOI:http://dx.doi.org/10.1103/PhysRevA.54.1844 **Quantum copying: Beyond the no-cloning theorem V. Bužek and M. Hillery Phys. Rev. A 54, 1844 – Published 1 September 1996 STOP** The Quantum Query Model is a framework that allows us (eb) to express most known quantum algorithms. Algorithms represented by this model consist on (e&eb) a set of unitary operators acting over (e&eb) a finite Hilbert space and a final measurement step consisting on (e&eb) a set of projectors. In this work, **S. A. Grillo, F. L. Marquezino** prove that the application of these unitary operators before the measurement step is equivalent to (=) decomposing (e&eb) a unit vector into (e&eb) a sum of vectors and then inverting some of their relative phases. Authors also prove that the vectors of (e) that sum must fulfill a list of properties and authors call such vectors a Block Set. If we take the same measurement step on the Quantum Query Model and (e&eb) on the Block Set Formulation, then authors prove that both formulations give (eb) the same Gram matrix of output states, although the Block Set Formulation allows (eb) a much more explicit form. Therefore, the Block Set reformulation of (e) the Quantum Query Model has (e) advantages for the design of exact quantum algorithms. Finally, authors test these advantages of Block Sets by applying approach to (e&eb) the construction of quantum exact algorithms. Subjects: Quantum Physics (quant-ph); Computational Complexity (cs.CC) Cite as: arXiv: 1602.07716 [quant-ph] (or arXiv: 1602.07716v1 [quant-ph] for this version) **Quantum query without unitary operators S. A. Grillo, F. L. Marquezino (Submitted on 24 Feb 2016) STOP** Therefore, the Block Set reformulation of (e) the Quantum Query Model has (e) advantages for the design of exact quantum algorithms. Finally, authors test these advantages of Block Sets by applying approach to (e&eb) the construction of quantum exact algorithms. Subjects:

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The first algorithm uses (e) only $O (\log n)$ qubits in its quantum subroutines, whereas (e) the second one uses (e) $O(n)$ qubits. The Triangle Problem was first treated in [H. Buhrman et al., SIAM J. Comput., 34 (2005), pp. 1324–1330], where (e) an algorithm with $O(n+\sqrt{nm})$ query complexity was presented, where (e) m is the number of edges of G.Copyright © 2007 Society for Industrial and Applied Mathematics **SIAM J. Comput., 37(2), 413–424. (12 pages) Quantum Algorithms for the Triangle Problem Published online: 16 May 2007 Keywords 05C85, 68R10, 68W99 AMS Subject Headings quantum algorithm, query complexity, triangle problem, quantum walk Publication Data ISSN (print): 0097-5397 ISSN (online): 1095-7111 Publisher: Society for Industrial and Applied Mathematics Frédéric Magniez, Miklos Santha, and Mario SzegedyDOI:10.1137/050643684 STOP** Authors derive a necessary and sufficient condition for two pure states, each entangled in two remote systems, to be clonable by the sequential access to the two systems. The result shows (eb) to what extent the correlation to other systems can be read out from (e) a subsystem without (e) altering its marginal density operators. This extends the standard no-cloning theorem to the case of a subsystem correlated to others. Received 1 May 1998 DOI:http://dx.doi.org/10.1103/PhysRevLett.81.4264 **No-Cloning Theorem of Entangled States Masato Koashi and Nobuyuki Imoto Phys. Rev. Lett 81, 4264 – Published 9 November 1998 STOP** Authors analyze in perturbative QCD the asymptotic behaviour of (e) deep inelastic processes in the semi-hard region $x\to 0$. The study is done by (e) extending the soft gluon insertion techniques. Authors confirm and extend the analysis recently performed by (e) Ciafaloni. The main results are the following: (i) Soft gluon emission from (e) the incoming parton take place in (eb) a region where (e) the angles between incoming and outgoing partons are (=) ordered. This is due to coherent effects similar to (e) the ones in the $x\to 1$ region. (ii) Virtual corrections involving (e&eb) an internal line with energy fraction x give rise, for (e) $x\to 0$, to a new form factor of non-Sudakov type. This regularizes (e&eb) collinear singularities when (e) an emitted gluon is parallel to (e) the incoming parton. (iii) At the complete inclusive level, the new form factor plays (e&eb) the same role as (=) the virtual corrections in (eb) the Lipatov equation for (e) the Regge regime. Authors show that, in the semi-hard regime, the gluon anomalous dimension coincides with (=) the Lipatov ansatz. (iv) Authors identify the branching structure of (e) initial-state radiation including (e) the semi-hard regime. The branching is formulated as (=) a probability process given in (eb) terms of Sudakov and non-Sudakov form factors. This process, in principle, can be used to (e) extend the existing simulations of QCD cascades to (e&eb) the semi-hard regime. Research supported in part by the Italian Ministero Della Pubblica Istruzione. Copyright © 1990 Published by Elsevier B.V. **Nuclear Physics B Volume 336, Issue 1, 21 May 1990, Pages 18-85 Small-x behaviour of initial state radiation in perturbative QCD Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. S. CataniF. Fiorani, G. Marchesini STOP** Within the QCD-improved parton model and assuming the factorization theorem to (e) hold in the helicity basis and for (e) higher twist contributions, authors show how nonzero single spin asymmetries in hadron-hadron high energy and moderately large pT inclusive processes can be obtained, even in massless perturbative QCD, provided the quark intrinsic motion is taken into account. A simple model is constructed which reproduces the main features of the data on the single spin asymmetry observed in inclusive pion production in pp collisions. Copyright © 1995 Published by Elsevier B.V. **Physics Letters B Volume 362, Issues 1–4, 9 November 1995, Pages 164–172 Single spin asymmetry for $p\uparrow p \to \pi$ X in perturbative QCD M Anselminoa, M Boglionea, F Murgiab STOP.** In the {\em distributed Deutsch-Jozsa promise problem}, two

parties are to determine whether (E&EB)their respective strings x,y∈{0,1}n are at the {\em Hamming distance} H(x,y)=0 or H(x,y)=n2. Buhrman et al. (STOC' 98) proved that (eb) the exact {\em quantum communication complexity} of this problem is O(logn) while the {\em deterministic communication complexity} is (=) Ω(n). This was the first impressive (exponential) gap between quantum and (e&eb) classical communication complexity. In this paper, authors generalize the above distributed Deutsch-Jozsa promise problem to determine, for any fixed n2≤k≤n, whether H(x,y)=0 or H(x,y)=k, and show that an exponential gap between exact quantum and deterministic communication complexity still holds if k is an even such that (eb) 12n≤k<(1−λ)n, where 0<λ<12 is given. Authors also deal with a promise version of (e) the well-known {\em disjointness} problem and show also that for this promise problem there exists an exponential gap between quantum (and also probabilistic) communication complexity and (e*eb) deterministic communication complexity of (e) the promise version of (e) such a disjointness problem. Finally, some applications to (e) quantum, probabilistic and deterministic finite automata of the results obtained are (=) demonstrated. Subjects: Quantum Physics (quant-ph); Computational Complexity (cs.CC); Distributed, Parallel, and Cluster Computing (cs.DC); Formal Languages and Automata Theory (cs.FL) Cite as: arXiv: 1402.7254 [quant-ph] (or arXiv: 1402.7254v3 [quant-ph] for this version) **Generalizations of the distributed Deutsch-Jozsa promise problem Jozef Gruska, Daowen Qiu, Shenggen Zheng (Submitted on 28 Feb 2014 (v1), last revised 18 Mar 2015 (this version, v3)) STOP Authors** study scaling properties of the diffraction dissociation of virtual photons in a deep inelastic scattering. **Authors** concentrate on the total diffraction dissociation rate (e&eb) diffraction excitation mass spectrum and (e&eb) the pomeron structure function to the lowest order in perturbative QCD. We calculate the valence structure function and the strangeness and charm content of the pomeron and estimate the ocean structure function using the pomeron factorization property. We find that quarks carry ≈ 10% of pomeron's momentum. Differential cross section of the (virtual) photon-pomeron scattering is found to exhibit features typical of the hadronic two-body reactions, supporting a treatment of the Pomeron as a particle, whereas the flavor dependence of structure functions does not support the particle treatment of the pomeron. Diffraction dissociation of photons is predicted to make ≈ 15% of the total deep inelastic scattering rate at smallx and largeQ 2. Detailed predictions for the mass spectrum and angular distribution of jets produced on the valence component of the pomeron are presented. **Zeitschrift für Physik C Particles and Fields June 1992, Volume 53, Issue 2, pp 331-345 First online: Pomeron structure function and diffraction dissociation of virtual photons in perturbative QCD Nikolai N. Nikolaev, Bronislav G. Zakharov STOP Authors** present the first complete calculation of Z-boson production in association with a jet in hadronic collisions through next-to-next-to-leading order in perturbative QCD. **Authors** computation uses the recently proposed N-jettiness subtraction scheme to regulate the infrared divergences that appear in the real-emission contributions. **Authors** present phenomenological results for 13 TeV proton-proton collisions with fully realistic fiducial cuts on the final-state particles. The remaining theoretical uncertainties after the inclusion of our calculations are at the percent level, making the Z+jet channel ready for precision studies at the LHC run II. Received 11 December 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.152001 **Z-Boson Production in Association with a Jet at Next-To-Next-To-Leading Order in Perturbative QCD Radja Boughezal, John Campbell, R. Keith Ellis, Christfried Focke, Walter Giele, Xiaohui Liu, and Frank Petriello Phys. Rev. Lett. 116, 152001 – Published 14 April 2016 STOP** Inspired by the recent measurement of the ratio of Bc branching fractions to J/ψπ+ and J/ψμ+νμ final states at the LHCb detector, we study the semileptonic decays of Bc meson to the S-wave ground and radially excited 2S and 3S charmonium states with the perturbative QCD approach. After evaluating the form factors for the transitions Bc→P,V, where P and V denote pseudoscalar and vector S-wave charmonia, respectively, we calculate the branching ratios for all these semileptonic decays. The theoretical uncertainty of hadronic input parameters are reduced by utilizing the light-cone wave function for Bc meson. It is found that the predicted branching ratios range from 10−6 up to 10−2 and could be measured by the future LHCb experiment. Our prediction for the ratio of branching fractions BR (B+c→J/Ψπ+) BR(B+c→J/Ψμ+νμ) is in good agreement with the data. For Bc→Vlνl decays, the relative contributions of the longitudinal and transverse polarization are discussed in different momentum transfer squared regions. These predictions will be tested on the ongoing and forthcoming experiments.

Subjects: High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1602.08918 [hep-ph] (or arXiv: 1602.08918v1 [hep-ph] for this version) **Semileptonic decays of Bc meson to S-wave charmonium states in the perturbative QCD approach Zhou Rui, Hong Li, Guang-xin Wang, Ying Xiao STOP Authors** present a detailed phenomenological study of W-boson production in association with a jet through next-to-next-to-leading order (NNLO) in perturbative QCD. Fiducial cross sections and differential distributions for both 8 TeV and 13 TeV LHC collisions are presented, as are results for both the inclusive one-jet bin and the exclusive one-jet bin. Two different event selection criteria are considered: a general selection with standard cuts used in experimental analyses, and a boosted selection that focuses on high transverse momentum jets. **Authors** discuss the higher-order corrections in detail and identify for which observables and phase space regions the QCD perturbative expansion is under good theoretical control, and where additional work is needed. For most distributions and phase space regions the QCD perturbative expansion exhibits good convergence after the inclusion of the NNLO corrections. Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex) Cite as: arXiv: 1602.06965 [hep-ph] (or arXiv: 1602.06965v1 [hep-ph] for this version) STOP Authors present a detailed phenomenological study of Z-boson production in association with a jet through next-to-next-to-leading order (NNLO) in perturbative QCD. Fiducial cross sections and differential distributions for both 8 TeV and 13 TeV LHC collisions are presented. We study the impact of different parton distribution functions (PDFs) on predictions for the Z+jet process. Upon inclusion of the NNLO corrections, the residual scale uncertainty is reduced such that both the total rate and the transverse momentum distributions can be used to discriminate between various PDF sets. Subjects: High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1602.08140 [hep-ph] (or arXiv: 1602.08140v1 [hep-ph] for this version) **STOP Authors** compute the fully differential decay rate of the Standard Model Higgs boson into b-quarks at next-to-next-to-leading order (NNLO) accuracy in alpha_S. Authors employ a general subtraction scheme developed for the calculation of higher order perturbative corrections to QCD jet cross sections, which is based on (e)the universal infrared factorization properties of QCD squared matrix elements. **Authors** show that the subtractions render the various contributions to the NNLO correction finite. In particular, authors demonstrate analytically that the sum of integrated subtraction terms correctly reproduces (eb) the infrared poles of the two-loop double virtual contribution to this process. We present illustrative differential distributions obtained by implementing the method in a parton level Monte Carlo program. The basic ingredients of our subtraction scheme, used here for (e) the first time to compute a physical observable, are universal and can be employed for the computation of more involved processes. Subjects: High Energy Physics - Phenomenology (hep-ph) DOI: 10.1007/JHEP04(2015)036 Report number: CERN-PH-TH-2015-005, CP3-14-82 Cite as: arXiv:1501.07226 [hep-ph] (or arXiv:1501.07226v2 [hep-ph] for this version) **Higgs boson decay into b-quarks at NNLO accuracy Vittorio Del Duca, Claude Duhr, Gabor Somogyi, Francesco Tramontano, Zoltan Trocsanyi** Wiener processes (the formalization of Brownian motion random walk) provide (eb) a basis for a model of discrete, mobile elements (Dx, Dy, Dz, Dt) that 'tile' space-time. The possible migrations of these elements are shown graphically via (e&eb) computer simulation. The Wiener process is modified by (e) not taking the process to (e) infinitesimals, and by also allowing (eb) the process to be a random walk in (eb) time as well as space. While migrations of (e) a single element can easily be expressed with (e&eb) such a Wiener-like process, migrations within (e&eb) the completely tiled space-time (without introducing extra dimensions) require (e) curvature. And since the quantum vacuum energy fluctuations imply (eb) curvature fluctuations, authors use those curvature fluctuations in the model. The process of measurement (e&eb) the nature of particle spin (e&eb) and the quantum mechanical implications of the model are explored. The following results are obtained: In regions outside mass, the Schwarzschild metric is deduced (using a plausibility argument) without (e) employing the General Relativity field equations. And inside mass, the model gives (eb) the remarkable result that for spherical masses greater than (e) the Planck mass, there is (=) no wave function, so, for example, the 2-slit experiment will no (e)t (even theoretically) show interference when (e) performed with cannonballs. Comments: arXiv admin note: substantial text overlap with arXiv: 1406.5248 Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1601.07171 [quant-ph] (or arXiv: 1601.07171v1 [quant-ph] for this version) **Indeterminate Space-**

**time Quantum Mechanics: a Computer-augmented Framework using Wiener-like Processes Carlton Frederick STOP** The block universe idea, representing (eb) space-time as a fixed whole, suggests (eb) the flow of time is (=) an illusion: the entire Universe just is, with no special meaning attached to (e&eb) the present time. This paper points out that this view, in essence represented by (e) usual space-time diagrams, is based on (e) time-reversible microphysical laws, which fail to (e) capture essential features of the time-irreversible macro-physical behaviour and (e&eb) the development of emergent complex systems, including life, which exist in (eb) the real Universe. When these are taken into account, the unchanging block Universe view of space-time is (=) best replaced by (e) an evolving block Universe which extends as (=) time evolves, with (e&eb) the potential of the future continually becoming (=) the certainty of the past; space-time itself evolves, as (e) do the entities within it. However this time evolution is not related to (e&eb) any preferred surfaces in space-time; rather it is associated with (e&eb) the evolution of (e) proper time along families of world lines. Relativity and the Dimensionality of the World Relativity and the Dimensionality of the World Look **Relativity and the Dimensionality of the World Volume 153 of the series Fundamental Theories of Physics pp 49-79 Physics in the Real Universe: Time and Space-Time George F. R. Ellis STOP** In our reality, we do not have (e) perfect spaces and perfect systems. Therefore many \textit{physical laws} function approximatively. Also, the \textit{physical constants} are not universal too. Variations of their values depend from a space to another, from a system to another, from a time to another, and so on depending on many parameters. The physical laws and similarly the physical constants are t{\%} true, i{\%} indeterminate, and f{\%} false in a given space with a certain composition, and it has a different neutrosophical truth value \textless t', i', f'\textgreater in another space with another composition. That's why, instead of universal (1, 0, 0)-physical laws and universal (1, 0, 0)-physical constants, we have (t, i, f)-physical laws and respectively (t, i, f)-physical constants, meaning partially true, partially indeterminate, and partially false in each space. Therefore, one uses the \textit{neutrosophic logic}, which is a general framework for unification of many existing logics, and its components t (truth), i (indeterminacy), f (falsehood) are standard or non-standard real subsets of ]−0, 1+[ with not necessarily any connection between them. It has many applications in physics. Reference: Florentin Smarandache, \textit{Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability}, by Sitech {\&} Educational, Craiova, 140 p., 2013. To cite this abstract, use the following reference: http://meetings.aps.org/link/BAPS.2016.DAMOP.Q1.201**47th Annual Meeting of the APS Division of Atomic, Molecular and Optical Physics Monday–Friday, May 23–27, 2016; Providence, Rhode Island Session Q1: Poster Session III (4:00pm-6:00pm) 4:00 PM–4:00 PM, Thursday, May 26, 2016 Room: Exhibit Hall C Abstract ID: BAPS.2016.DAMOP.Q1.201 Abstract: Q1.00197: (t, i, f)-Physical Laws and (t, i, f)-Physical Constants Florentin Smarandache (University of New Mexico)STOP** In the view of many social scientists, the more probable a theory is, the better it is, and if we have to choose between (e&eb) two theories which are equally strong in terms of (e&eb) their explanatory power, and differ only in that one is (=) probable and the other is improbable, then we should choose (e&eb) the former. Popper rejects (e) this. Science, or to be precise, the working scientist, is interested, in Popper's view, in theories with (e&eb) a high informative content, because such theories possess (e) a high predictive power and are (=) consequently highly testable. But if this is true, Popper argues, then, paradoxical as it may sound, the more improbable a theory is (eb) the better it is scientifically, because (e) the probability and informative content of (e) a theory vary inversely—the higher the informative content of a theory (eb) the lower will be its probability, for the more information a statement contains, (eb) the greater will be the number of ways in which it may turn out to be false. Thus the statements which are of special interest to the scientist are (=) those with a high informative content and (consequentially) a low probability, which nevertheless come close to the truth. Informative content, which is in inverse proportion to probability, is (=) in direct proportion to testability. Consequently the severity of the test to which a theory can be subjected (e&eb) and by means of which it is falsified or corroborated, is (=) all-important. For Popper, all scientific criticism must be (=) piecemeal, i.e., he holds that it is not possible to (e) question every aspect of a theory at once. More precisely, while attempting to resolve (e) a particular problem a scientist of necessity accepts all kinds of (e) things as unproblematic. These things constitute (e) what Popper terms the ‘background knowledge’. However, he stresses that the background knowledge

is (=) not knowledge in the sense of (e) being conclusively established; it may be challenged (e) at any time, especially if it is suspected that its uncritical acceptance may be responsible for (e) difficulties which are subsequently encountered. Nevertheless, it is clearly not (e) possible to (e) question both the theory and the background knowledge at (eb) the same time (e.g., in conducting an experiment the scientist of necessity assumes (eb) that the apparatus used is in working order). How then can one be certain that one is questioning the right thing? The Popperian answer is that we cannot have (e) absolute certainty here, but repeated tests usually show (eb) where the trouble lies. Even observation statements, Popper maintains, are fallible, and science in his view is (=) not a quest for certain knowledge, but an evolutionary process in which hypotheses or conjectures are imaginatively proposed and tested in order to explain facts or to solve problems. Popper emphasizes both the importance of questioning the background knowledge when (e) the need arises, and the significance of the fact that observation-statements are (=) theory-laden, and hence fallible. For while falsifiability is simple as (=) a logical principle, in practice it is (=) exceedingly complicated—no single observation can ever be taken to (e) falsify a theory, for there is (=) always the possibility (a) that the observation itself is (=) mistaken, or (b) that the assumed background knowledge is (=) faulty or defective. Popper was initially uneasy with the concept of truth, and in his earliest writings he avoided asserting that a theory which is corroborated is true—for clearly if every theory is an open-ended hypothesis, as he maintains, then ipso facto it has to be at least potentially false. For this reason Popper restricted himself to the contention that a theory which is falsified is false and is known to be such, and that a theory which replaces a falsified theory (because it has a higher empirical content than the latter, and explains what has falsified it) is a 'better theory' than its predecessor. However, he came to accept Tarski's reformulation of the correspondence theory of truth, and in Conjectures and Refutations (1963) he integrated the concepts of truth and content to frame the metalogical concept of 'truthlikeness' or 'verisimilitude'. A 'good' scientific theory, Popper thus argued, has a higher level of verisimilitude than its rivals, and he explicated this concept by reference to the logical consequences of theories. A theory's content is the totality of its logical consequences, which can be divided into two classes: there is the 'truth-content' of a theory, which is the class of true propositions which may be derived from it, on the one hand, and the 'falsity-content' of a theory, on the other hand, which is the class of the theory's false consequences (this latter class may of course be empty, and in the case of a theory which is true is necessarily empty). (Stanford Encyclopedia Of Philosophy) Authors study a U (N) gauged matrix quantum mechanics which, in the large N limit, is closely related to (e&eb) the chiral WZW conformal field theory. This manifests itself in two ways. First, authors construct (eb) the left-moving Kac-Moody algebra from matrix degrees of (e) freedom. Secondly, authors compute the partition function of (e) the matrix model in terms of Schur and Kostka polynomials and show that, in the large N limit, it coincides with the partition function of the WZW model. This same matrix model was recently shown to describe non-Abelian quantum Hall states and the relationship to the WZW model can be understood in this framework. Subjects: High Energy Physics - Theory (hep-th); Strongly Correlated Electrons (cond-mat.str-el) Cite as: arXiv: 1604.05711 [hep-th] (or arXiv: 1604.05711v1 [hep-th] for this version **A Matrix Model for WZW Nick Dorey, David Tong, Carl Turner STOP** Authors present a non-perturbative study of (e) the lambda phi**4 model on (e&eb) a non-commutative plane. The lattice regularised form can be mapped onto (e&eb) a Hermitian matrix model, which enables (eb) Monte Carlo simulations. Numerical data reveal (eb) the phase diagram; at large lambda it contains a "striped phase", which is absent in (eb) the commutative case. Authors explore the question whether or not this phenomenon persists in (eb) a Double Scaling Limit (DSL), which extrapolates (e&eb) simultaneously to the continuum and to (e&eb) infinite volume, at a fixed non-commutativity parameter. To this end, authors introduce a dimensional lattice spacing based on (e) the decay of the correlation function. Results provide (eb) evidence for the existence of a striped phase even in the DSL, which implies (eb) the spontaneous breaking of translation symmetry. Due to (e) the non-locality of this model, this does not (e) contradict the Mermin-Wagner Theorem. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th) Journal reference: JHEP 1410 (2014) 56 DOI: 10.1007/JHEP10(2014)056 Report number: IFT-UAM/CSIC-14-014 Cite as: arXiv:1403.3318 [hep-lat] (or arXiv:1403.3318v2 [hep-lat] for this version) **The Continuum Phase Diagram of the 2d Non-**

**Commutative lambda phi**4 Model Héctor Mejía-Díaz, Wolfgang Bietenholz, Marco Panero STOP** A twisted version of (e) four dimensional supersymmetric gauge theory is formulated. The model, which refines (e&eb) a nonrelativistic treatment by Atiyah, appears to underlie (e&eb) many recent developments in topology of low dimensional manifolds; the Donaldson polynomial invariants of (e) four manifolds and the Floer groups of three manifolds appear naturally. The model may also be interesting from (e) a physical viewpoint; it is (=) in a sense a generally covariant quantum field theory, albeit one in which general covariance is (=) unbroken, there are (=) no gravitons, and the only excitations are (=) topological. **Article Communications in Mathematical Physics September 1988, Volume 117, Issue 3, pp 353-386 Topological quantum field theory Edward Witten STOP** The formalism of the previous paper is extended to the case of (e&eb) supersymmetric strings. The effective theory which sums up fermionic surfaces is described by (e) the supersymmetric Liouville equation. At D = 10 effective decoupling of (e) the Liouville dilaton takes place (eb) and theory coincides with (=) the old ones. At D = 3 theory is equilavent to (=) the three-dimensional Ising model, which is thus reduced to (e) the two-dimensional supersymmetric Liouville theory. **Physics Letters B Volume 103, Issue 3, 23 July 1981, Pages 211-213Quantum geometry of fermionic strings Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. A.M. Polyakov STOP** There is a persistent confusion about (e&eb) determinism and predictability. In spite of the opinions of some eminent philosophers (e.g., Popper), it is possible to understand that the two concepts are (=) completely unrelated. In few words we can say that determinism is (=) ontic and has to do with how Nature behaves, while predictability is (=) epistemic and is related to what the human beings are able to compute. An analysis of the Lyapunov exponents and the Kolmogorov-Sinai entropy shows how deterministic chaos, although with (e&eb) an epistemic character, is (=) non subjective at all. This should clarify the role and content of stochastic models in the description of the physical world **Chapter Models and Inferences in Science Volume 25 of the series Studies in Applied Philosophy, Epistemology and Rational Ethics pp 133-146 Date: 28 January 2016 Chaos and Stochastic Models in Physics: Ontic and Epistemic Aspects Sergio Caprara, Angelo Vulpiani STOP** This review is devoted to the problem of (e) thermalization in (eb) a small isolated conglomerate of (e) interacting constituents. A variety of physically important systems of (e) intensive current interest belong to (e) this category: complex atoms, molecules (including biological molecules), nuclei, small devices of condensed matter and quantum optics on nano- and micro-scale, cold atoms in optical lattices, ion traps. Physical implementations of quantum computers, where there are many interacting qubits, also fall (eb) into this group. Statistical regularities come into play through (e&eb) inter-particle interactions, which have (e) two fundamental components: mean field, which along with external conditions; forms (eb) the regular component of the dynamics, and residual interactions responsible for (e) the complex structure of (e) the actual stationary states. At sufficiently high level density, the stationary states become (=) exceedingly complicated superpositions of (e) simple quasiparticle excitations. At this stage, regularities typical of (e) quantum chaos emerge (eb) and bring in signatures of thermalization. Authors describe all the stages and the results of (e) the processes leading to (eb) thermalization, using (e) analytical and massive numerical examples for (e) realistic atomic, nuclear, and spin systems, as well as for (e) models with random parameters. The structure of stationary states, strength functions of (e) simple configurations, and concepts of entropy and (e&eb) temperature in application to (e) isolated mesoscopic systems are discussed in detail. They conclude with a schematic discussion of (e) the time evolution of such systems to (e&eb) equilibrium. Corresponding author Copyright © 2016 Elsevier B.V: All rights reserved**. Physics Reports Volume 626, 15 April 2016, Pages 1–58 Quantum chaos and thermalization in isolated systems of interacting particles Quantum chaos and thermalization in isolated systems of interacting particles F. Borgonovia, , F.M. Izrailevb, d, , L.F. Santosc, , V.G. Zelevinskyd, , STOP** Recalling Thomas Nagel's discussion concerning the difficulties associated with (e&eb)developing a scientific explanation for the nature of experience, Nagel states that current reductionist attempts fail by(e) filtering out (e) any basis for (e) consciousness and thus become (=) meaningless since they are (=) logically compatible with (e&eb) its absence. In this article we call into question the fundamental philosophy of (e) the mind-brain identity hypothesis of (e) Cognitive Theory: 'What processes in the brain give rise to (eb) awareness?' and the associated search for 'neural correlates of (e)

consciousness' (NCC). The proper scientific manner of (e) posing the query should simply be 'What processes give rise to (eb) awareness?' Authors begin to formalize the Eccles psychon and summarize one of (e) fourteen empirical protocols to test this putative model. This requires (e) a new science of Unified Field, UF Mechanics, entailing in terms of (e&eb) our current stage of development operationally completed forms of (e) quantum theory, gravitation and cosmology arising from (e) a unique derivation of the M-Theory (string theoretic) vacuum. Until now the quest for psychophysical bridging has typically been in the arena between brain and quantum geometry; and many have wondered if contemporary science is sufficient for the task. Nagel further asks 'what would be left if one removed the viewpoint of the subjective observer' and then suggests 'that the remaining properties would be the physical processes themselves or states intrinsic to the experience of awareness'. Authors examine a new theoretical framework for introducing and experimentally testing the underlying physical cosmology of these noe**tic parameters. Unified Field Mechanics & Its Applications: Elucidating the Objective Character of Experience Richard L. Amoroso, Francisco Di Biase Scientific GOD Journal STOP** An android is meant to look and acts like a human being even to the extent of being indistinguishable. Generally the simplistic distinction between a humanoid robot, a computerized machine capable of replicating a variety of complex human functions automatically, and an android is one of appearance. While one day a yottaflop (1024 bits per second) hyper-supercomputer could have a sufficient holographic database and processing power to be truly indistinguishable from a human being, the issue of the applicability of sentience (self-awareness) to an android comes to (e) the forefront. The currently dominant cognitive model of awareness closely aligned to (e&eb) the AI model, states that mind equals (=) brain and those once correct algorithms are known all of (e&eb) human intelligence could be replicated artificially. This is the so-called mechanistic view: 'The laws of physics and chemistry are sufficient to describe (eb) all living systems; no additional life principle is required'. In this work authors develop the point of view that the regime of Unified Field Mechanics (UFM) supplies (eb) an inherent action principle driving (eb) both the evolution of complex Self-Organized Living Systems (SOLS) and (e&eb) the physical processes of awareness. These UFM parameters in conjunction with (e&eb)'conscious quantum computing' (class of quantum computer modeled with (e&eb) physical parameters of mind-body interaction) putatively leads directly to(eb) the construction of sentient (or sentient-like) Androids **This is Part I of this two-part article entitled "Solving the Mind-Body Problem". Unified Field Mechanics & Its Applications: Can Sentient Androids Be Constructed? Part I: Solving the Mind-Body Problem Richard L. Amoroso STOP** Authors discuss supersymmetric quantum mechanical models with (e&eb) periodic potentials. The important new feature is that it is possible for (e) both isospectral potentials to support (eb+) zero modes, in contrast with (e&eb) the standard nonperiodic case where either one or neither (but not both) of the isospectral pair has a zero mode. Thus it is possible to have (e) supersymmetry unbroken and (e&eb) **yet also have a vanishing Witten index**. Authors present some explicit exactly soluble examples for (e) which the isospectral potentials have (e) identical band spectra, and which is (=) "self-isospectral" in the sense that the potentials have (e) identical shape, but are translated by one half-period relative to (e&eb) one another. Received 4 June 1997 DOI:http://dx.doi.org/10.1103/PhysRevD.57.1271 **Self-isospectral periodic potentials and supersymmetric quantum mechanics Gerald Dunne and Joshua Feinberg Phys. Rev. D 57, 1271 – Published 15 January 1998 STOP** Authors present the first Monte Carlo results for (e) supersymmetric matrix quantum mechanics with (e&eb) 16 supercharges at (eb) finite temperature. The recently proposed nonlattice simulation enables (eb) us to include the effects of (e&eb) fermionic matrices in a transparent and reliable manner. The internal energy nicely interpolates (e&eb) the weak coupling behavior obtained by (e) the high temperature expansion, and the strong coupling behavior predicted from (e) the dual black-hole geometry. The Polyakov line asymptotes at low temperature to (e) a characteristic behavior for (e) a deconfined theory, suggesting (eb) the absence of a phase transition. These results provide (eb) highly nontrivial evidence for (e) the gauge-gravity duality. Received 4 September 2007 DOI:http://dx.doi.org/10.1103/PhysRevLett.100.021601 **Monte Carlo Studies of Supersymmetric Matrix Quantum Mechanics with Sixteen Supercharges at Finite Temperature Konstantinos N. Anagnostopoulos, Masanori Hanada, Jun Nishimura, and Shingo Takeuchi Phys. Rev. Lett. 100, 021601 – Published 15 January 2008 STOP** There have been many claims that quantum mechanics plays a

key role in (e&eb) the origin and/or operation of biological organisms, beyond (e) merely providing (eb)the basis for (e) the shapes and sizes of biological molecules and (e&eb) their chemical affinities. These range from Schrödinger's suggestion that quantum fluctuations produce (eb) mutations, to Hameroff and Penrose's conjecture that quantum coherence in microtubules is linked to (e&eb) consciousness. Authors review some of these claims in this paper, and discuss the serious problem of decoherence. Authors advance some further conjectures about quantum information processing in (eb) bio-systems. Some possible experiments are suggested. Copyright © 2004 Elsevier Ireland Ltd. All rights reserved. **Biosystems Volume 78, Issues 1–3, December 2004, Pages 69–79Does quantum mechanics play a non-trivial role in life? P.C.W. Davies STOP** Authors study the squared width and the profile of flux tubes in (eb) compact U (1) lattice gauge theory in (eb) three spacetime dimensions. The results obtained from (e) numerical calculations in the dual formulation of (e) this confining theory are compared with (e&eb) predictions from (e) an effective bosonic-string model and from (e) the dual-superconductor model: it is found that the former fails at (eb) describing the quantitative features of the flux tube, while (e) the latter is in good agreement with (=) Monte Carlo data. The analytical interpretation of these results (in the light of the semi-classical analysis by Polyakov) is pointed out, and a comparison with (e&eb) non-Abelian gauge theories in (eb) four spacetime dimensions is discussed. Subjects:High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Journal reference: JHEP 02 (2016) 180 DOI:10.1007/JHEP02 (2016)180 Cite as: arXiv: 1601.07455 [hep-lat] (or arXiv: 1601.07455v2 [hep-lat] for this version) **Width of the flux tube in compact U(1) gauge theory in three dimensions Michele Caselle, Marco Panero, Davide Vadacchino STOP** Authors present high-precision lattice calculations of (e) the thermodynamics of (e) Yang-Mills theories with (e&eb) different gauge groups in the confining phase, Authors show that the equation of state is described remarkably well by (e) a gas of massive, non-interacting glueballs, provided that (e) an effective bosonic closed-string model is used to derive (eb) an exponentially growing Hagedorn spectrum for (e) the heavy states. In particular, this model describes (eb) very accurately the results for (e) the SU (3) theory reported by (e) Bors\'anyi et al. in JHEP 07 (2012) 056, as well as a novel set of lattice data for (e) the SU (2) theory. In addition, Authors also show (eb) that the equation of state in the deconfined phase exhibits (eb) a near perfect proportionality to the number of (e) gluon degrees of freedom, including for the Yang-Mills theory based on the exceptional, center-less gauge group G2. Subjects: High Energy Physics - Lattice (hep-lat) Cite as: arXiv: 1510.01146 [hep-lat] (or arXiv: 1510.01146v1 [hep-lat] for this version) **Universal aspects in the equation of state for Yang-Mills theories Alessandro Nada STOP** In U (1) lattice gauge theory in three spacetime dimensions, the problem of confinement can be studied analytically in a semi-classical approach, in terms of a gas of monopoles with Coulomb-like interactions. In addition, this theory can be mapped to a spin model via an exact duality transformation, which allows one to perform high-precision numerical studies of the confining potential. Taking advantage of these properties, we carried out an accurate investigation of the effective string describing the low-energy properties of flux tubes in this confining gauge theory. Authors found striking deviations from the expected Nambu-Goto-like behavior, and, for the first time, evidence for contributions that can be described by a term proportional to the extrinsic curvature of the effective string worldsheet. Such term is allowed by Lorentz invariance and its presence in the infrared regime of the U (1) model was indeed predicted by Polyakov several years ago. Results show that this term scales as expected according to Polyakov's solution, and becomes the dominant contribution to the effective string action in the continuum limit. Authors also demonstrate analytically that the corrections to the confining potential induced by the extrinsic curvature term can be related to the partition function of the massive perturbation of a c=1 bosonic conformal field theory. The implications of results for (e&eb) SU (N) Yang-Mills theories in three and in four spacetime dimensions are discussed. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Journal reference: JHEP 1501 (2015) 105 DOI: 10.1007/JHEP01 (2015)105 Report number: IFT-UAM/CSIC-14-052 Cite as: arXiv: 1406.5127 [hep-lat] (or arXiv: 1406.5127v2 [hep-lat] for this version) **A different kind of string Michele Caselle, Marco Panero, Roberto Pellegrini, Davide Vadacchino STOP Authors** present the results of a systematic, first-principles study of the spectrum and decay constants of (e) mesons for different numbers of color charges N, via lattice computations. They restrict attention to states in (eb) the non-zero isospin sector, evaluating

(e&eb) the masses associated with (e&eb) the ground-state and first excitation in (eb) the pseudoscalar, vector, scalar, and axial vector channels. results are based on (e) a new set of simulations of four dimensional SU(N) Yang-Mills theories with (e&eb) the number of colors ranging from N=2 to N=17; the spectra and the decay constants are computed in (e&eb) the quenched approximation (which becomes (=) exact in the 't Hooft limit) using Wilson fermions. After discussing the extrapolations to the chiral and large-N limits, we present a comparison of our results to some of the numerical computations and analytical predictions available in the literature - including, in particular, those from holographic computations. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th) Journal reference: JHEP 06 (2013) 071 DOI: 10.1007/JHEP06 (2013)071 Report number: HIP-2013-01/TH Cite as: arXiv: 1304.4437 [hep-lat] (or arXiv: 1304.4437v2 [hep-lat] for this version) **Mesons in large-N QCD Gunnar S. Bali, Francis Bursa, Luca Castagnini, Sara Collins, Luigi Del Debbio, Biagio Lucini, Marco Panero STOP** Recently, Berenstein et al. have proposed a duality between (e&eb)a sector of Script N = 4 super-Yang-Mills theory with (e&eb) large R-charge J, and string theory in (eb) a pp-wave background. In the limit considered, the effective't Hooft coupling has been argued to be (=) λ' = gYM2N/J2 = 1/ (μp+α') 2. Authors study Yang-Mills theory at (eb) small λ' (large μ) with (e&eb) a view to reproducing (eb) string interactions. Authors demonstrate that the effective genus counting parameter of (e) the Yang-Mills theory is (=) g22 = J4/N2 = (4πgs)2(μp+α')4, the effective two-dimensional Newton constant for (e) strings propagating on (e&eb) the pp-wave background. Authors identify g2 (λ') 1/2 as the effective coupling between a wide class of excited string states on (e&eb) the pp-wave background. Authors compute the anomalous dimensions of BMN operators at first order in (eb) g22 and λ' and interpret result as (=) the genus one mass renormalization of the corresponding string state. Authors postulate a relation between the three-string vertex function and (e&eb) the gauge theory three-point function and compare (e&eb) proposal to string field theory. They utilize this proposal, together with (e&eb) quantum mechanical perturbation theory, to (e) recompute the genus one energy shift of (e) string states, and find precise agreement with (=) gauge theory computation. **PP-wave string interactions from perturbative Yang-Mills theory Neil R. Constable2, Daniel Z. Freedman1, 2, Matthew Headrick3, Shiraz Minwalla3, Lubos Motl3, Alexander Postnikov1 and Witold Skiba2 Published 5 August 2002 •Journal of High Energy Physics, Volume 2002, JHEP07(2002 STOP John F. Donoghue, Basem Kamal El-Menoufi** discuss the derivation of the trace anomaly using (e) a non-local effective action at one loop. This provides (eb) a simple and instructive form and emphasizes (eb) infrared physics. They then use this example to explore (e&eb) several of the properties of non-local actions, including (e) displaying the action for (e) the full non-local energy-momentum tensor. As an application, authors show that the long-distance corrections at one loop lead to (eb) quantum violations of some classical consequences of the equivalence principle, for example producing a frequency dependence of the gravitational bending of light. **Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1503.06099 [hep-th] or arXiv: 1503.06099v1 [hep-th] for this version) ED trace anomaly, non-local Lagrangians and quantum Equivalence Principle violations John F. Donoghue, Basem Kamal El-Menoufi (Submitted on 20 Mar 2015) STOP John F. Donoghue, Basem Kamal El-Menoufi** discuss cosmological effects of the quantum loops of (e) massless particles, which lead to (eb) temporal non-localities in (eb) the equations of motion governing (e&eb) the scale factor a(t). For the effects discussed here, loops cause (eb) the evolution of a(t) to depend on the memory of (e) the curvature in the past with (e&eb) a weight that scales initially as 1/(t-t'). As one of primary examples authors discuss the situation with (e&eb) a large number of light particles, such that these effects occur in (eb) a region where (e) gravity may still be treated (e&eb) classically. However, they also describe the effect of (e) quantum graviton loops and the full set of Standard Model particles. Authors show that these effects decrease with (e&eb) time in an expanding phase, leading to (eb) classical behavior at late time. In a contracting phase, within our approximations the quantum results can lead to (eb) a bounce-like behavior at scales below the Planck mass, avoiding (e) the singularities required (e) classically by(e) the Hawking-Penrose theorems. For conformally invariant fields, such as the Standard Model with (e&eb) a conformally coupled Higgs, this result is (=) purely non-local and parameter independent. Subjects: General

Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th) Journal reference: Phys. Rev. D 89, 104062 (2014) DOI: 10.1103/PhysRevD.89.104062 Report number: ACFI-T14-004 Cite as: arXiv: 1402.3252 [gr-qc] (or arXiv: 1402.3252v2 [gr-qc] for this version**) Non-local quantum effects in cosmology 1: Quantum memory, non-local FLRW equations and singularity avoidance John F. Donoghue, Basem Kamal El-Menoufi STOP N.E.J Bjerrum-Bohr, John F. Donoghue, Pierre Vanhove** compute the leading post-Newtonian and quantum corrections to (e&eb) the Coulomb and Newtonian potentials using (e) the full modern arsenal of on-shell techniques; authors employ spinor-helicity variables everywhere, use (e) the Kawai-Lewellen-Tye (KLT) relations to derive (eb) gravity amplitudes from (e) gauge theory and use (e) unitarity methods to extract (e) the terms needed at one-loop order. Authors stress that results are (=) universal and thus will hold in any quantum theory of (e) gravity with (e&eb) the same low-energy degrees of (e) freedom as has been considered. Previous results for (e) the corrections to (e) the same potentials, derived historically using (e) Feynman graphs, are verified explicitly, but our approach presents (eb) a huge simplification, since starting points for (e) the computations are compact and tedious index contractions and various complicated integral reductions are eliminated from (e) the onset, streamlining the derivations. They also analyze the spin dependence of the results using (e) the KLT factorization, and show how the spinless correction in the framework is easily seen to be (=) independent of the interacting matter considered. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) DOI: 10.1007/JHEP02 (2014)111 Report number: IHES/P/13/23, IPHT-t13/019 Cite as: arXiv: 1309.0804 [hep-th] (or arXiv: 1309.0804v2 [hep-th] for this version) **On-shell Techniques and Universal Results in Quantum Gravity N.E.J Bjerrum-Bohr, John F. Donoghue, Pierre Vanhove STOP** In effective field theories it is common to identify the onset of new physics with (e&eb) the violation of tree-level unitarity. However, authors show that this is (=) parametrically incorrect in the case of chiral perturbation theory, and is (=) probably theoretically incorrect in general. In the chiral theory, authors explore (e&eb) perturbative unitarity violation as (=) a function of the number of colors and (e&eb) the number of flavors, holding the scale of the "new physics" (i.e. QCD) fixed. This demonstrates (eb) that the onset of new physics is (=) parametrically uncorrelated with (e&eb) tree-unitarity violation. When the latter scale is lower than that of new physics, (eb) the effective theory must heal its unitarity violation itself, which is expected because (e) the field theory satisfies (eb) the requirements of (e) unitarity. In the chiral theory, the self-healing results in (eb) a resonant structure with (e&eb) scalar quantum numbers. In the electroweak variant of (e) this argument, the structure must have the properties of the Higgs and must couple proportional to (e) the mass in both gauge boson and (e&eb) fermion scattering. A similar example can be seen in the case of general relativity coupled to (e&eb) multiple matter fields, where (e) iteration of the vacuum polarization diagram restores unitarity. Authors present arguments that suggest (eb) the correct identification should be connected to (e&eb) the onset of inelasticity rather than unitarity violation. Authors describe how the onset of inelasticity can occur in (eb) the effective theory, although it does not (e) appear possible to predict (eb) the onset reliably. Subjects: High Energy Physics - Phenomenology (hep-ph) DOI: 10.1103/PhysRevD.86.014025 Cite as: arXiv: 1203.5153 [hep-ph] (or arXiv: 1203.5153v2 [hep-ph] for this version) **Self-healing of unitarity in effective field theories and the onset of new physics Ufuk Aydemir, Mohamed M. Anber, John F. Donoghue STOP Recently** a short scale modified black hole metric, known as holographic metric, has been proposed in order to capture (e) the self-complete character of gravity. In this paper authors show that such a metric can reproduce (eb) some geometric features expected from (e) the quantum N-portrait beyond (e) the semi-classical limit. Authors show that for a generic N this corresponds to (e&eb) having an effective energy momentum tensor in (eb) Einstein equations or, equivalently, non-local terms in the gravity action. They also consider the higher dimensional extension of the metric and (e&eb) the case of an AdS cosmological term. Authors provide (eb) a detailed thermodynamic analysis of both cases, with (e&eb) particular reference to the repercussions on (e&eb) the Hawking-Page phase transition Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1604.03263 [gr-qc] (or arXiv: 1604.03263v1 [gr-qc] for this version) **Geometric model of black hole quantum N-portrait, extra dimensions and**

**thermodynamics Antonia M. Frassino, Sven Köppel, Piero Nicolini STOP** The general view is that all fundamental physical laws should be (=) formulated within the framework given by (e) quantum mechanics (QM). In a sense, QM therefore has (e) the character of a metaphysical theory. Consequently, if it is possible to derive QM from more basic principles (eb) these principles should be of general, philosophical nature. Here, authors derive the formalism of QM from (e) well-motivated epistemic principles. A key assumption is that in a proper physical theory, the introduction of (e) entities or distinctions that are unknowable in principle is in conflict with (e&eb) the theory. First, an epistemic formalism is developed, using (e) concepts like knowledge and potential knowledge, identifying (e&eb) a physical state S with the potential knowledge of the physical world. It is demonstrated that QM emerges from (e) this formalism. However, Hilbert spaces, wave functions and probabilities are defined in (e&eb) certain well-defined observational contexts only. This means that the epistemic formalism is (=) broader than QM. In the fundamental layer of description, the physical state S is (=) a subset of a state space S= {Z}, such that S always contains (e) many elements Z. These elements correspond to (e&eb) unattainable states of complete knowledge of the world. The evolution of S cannot be determined in terms of (e&eb) the individual evolution of the elements Z, unlike the evolution of an ensemble in classical phase space. The evolution of S is described in terms of (e&eb) sequential time n∈N, which is updated according to (e&eb) n→n+1 each time an event occurs, each time potential knowledge changes. Sequential time n can be (=) separated from (e) relational time t, which describes distances between (e&eb) events in space-time. There is an entire space-time associated with (e&eb) each n, in which t represents (eb) the knowledge at sequential time n about the temporal relations between present and (e&eb) past events. Subjects: Quantum Physics (quant-ph); History and Philosophy of Physics (physics.hist-ph) Cite as: arXiv: 1601.00680 [quant-ph] (or arXiv: 1601.00680v1 [quant-ph] for this version) **A strict epistemic approach to physics Per Östborn (Submitted on 4 Jan 2016) STOP** Edge detectors have traditionally been an essential part of (e) many computer vision systems. There are different methods that have been proposed for (e) improving edge detection in real images. This paper proposes an edge detection method based on (e) the Sobel technique and generalized type-2 fuzzy logic systems. To limit the complexity of (e) handling generalized type-2 fuzzy logic, the theory of \(\alpha \)-planes is (=) used. Simulation results are obtained with (e&eb) the Sobel operator (without fuzzy logic), then with a type-1 fuzzy logic system (T1FLS), an interval type-2 fuzzy logic system (IT2FLS) and with (e&eb) a generalized type-2 fuzzy logic system (GT2FLS). The proposed generalized type-2 fuzzy edge detection method is tested with (e&eb) synthetic images with (e&eb) promising results. To illustrate the advantages of using (e) generalized type-2 fuzzy logic in combination with (e&eb) the Sobel operator, the figure of merit of Pratt measure is applied to measure (eb) the accuracy of the edge detection process. Keywords \(\alpha \)-planes General type-2 fuzzy logic system Generalized type-2 fuzzy logic system Interval type-2 fuzzy logic system Image processing Edge detection **Methodologies and Application Soft Computing February 2016, Volume 20, Issue 2, pp 773-784 First online: 16 December 2014 An improved sobel edge detection method based on generalized type-2 fuzzy logic Claudia I. Gonzalez, Patricia Melin, Juan R. Castro, Olivia Mendoza, Oscar Castillo STOP** Spontaneous breakdown of supersymmetry and internal symmetry is studied for (e) the interaction of N chiral scalar superfields. A lemma is established which shows (eb) that the spontaneous breakdown of supersymmetry is (=) a very rare occurrence, but that it can happen (in at least the tree approximation) is shown by (e) the construction of a model. The model contains (e) three superfields (N = 3) and preserves parity. In contrast to (e) supersymmetry itself, a spontaneous breakdown of (e) internal symmetry triggered by (e) the supersymmetry is found to be (=) a frequent occurrence, and a general formalism for it is (=) set up. A mass-formula is derived and examples given. Copyright © 1975 Published by Elsevier B.V. **Nuclear Physics B Volume 96, Issue 2, 22 September 1975, Pages 331-352Spontaneous symmetry breaking for chirals scalar superfields Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. L. O'Raifeartaigh a STOP** The ratio of shear viscosity to (e&eb) volume density of entropy can be used to characterize (eb) how close a given fluid is to being perfect. Using string theory methods, authors show that this ratio is equal to (=) a universal value of (e) $\hbar/4\pi k_B$ for a large class of strongly interacting (e&eb) quantum field theories whose dual description involves (e&eb) black holes in anti–de Sitter space. Authors provide evidence that

this value may serve as (=) a lower bound for a wide class of systems, thus suggesting that black hole horizons are (=) dual to the most ideal fluids. DOI:http://dx.doi.org/10.1103/PhysRevLett.94.111601 **Viscosity in Strongly Interacting Quantum Field Theories from Black Hole Physics P. K. Kovtun, D. T. Son, and A. O. Starinets Phys. Rev. Lett. 94, 111601 – Published 22 March 2005 STOP Goro Ishiki, Sang-Woo Kim, Jun Nishimura, and Asato Tsuchiya** test the recent claim that supersymmetric matrix quantum mechanics with (e&eb) mass deformation preserving maximal supersymmetry can be used to (e) study N=4 super Yang-Mills theory on (e&eb) R×S3 in the planar limit. When the mass parameter is large, we can integrate out all of (e) the massive fluctuations around (e&eb) a particular classical solution, which corresponds to (e&eb) R×S3. The resulting effective theory for (e) the gauge field moduli at (eb) finite temperature reproduces (eb) the deconfinement phase transition in (eb) N=4 super Yang-Mills theory on (e&eb) R×S3 at weak coupling. This transition was speculated to be a continuation of the conjectured phase transition at strong coupling, which corresponds to (e&eb) the Hawking-Page transition based on (e) the gauge-gravity duality. At strong coupling, one can study the target theory by (e) the new simulation method for (e) supersymmetric matrix quantum mechanics. DOI:http://dx.doi.org/10.1103/PhysRevLett.102.111601 **Deconfinement Phase Transition in N=4 Super Yang-Mills Theory on R×S3 from Supersymmetric Matrix Quantum Mechanics Goro Ishiki, Sang-Woo Kim, Jun Nishimura, and Asato Tsuchiya Phys. Rev. Lett. 102, 111601 – Published 20 March 2009 STOP** It is shown that every one-dimensional quantum mechanical Hamiltonian H1 can have (e) a partner H2 such that H1 and H2 taken together may be viewed as (=) the components of a supersymmetric Hamiltonian. The term 'supersymmetric Hamiltonian' is taken to mean (eb) a Hamiltonian defined in terms of (e&eb)charges that obey (eb)the same algebra as that of (e) the generators of supersymmetry in field theory. The consequences of this symmetry for (er) the spectra of H1 and H2 are explored. It is shown how the supersymmetric pairing may be utilised to eliminate (e) the ground state of H1, or add a state below the ground state of H1 or maintain the spectrum of H1. It is also explicitly demonstrated that the supersymmetric pairing may be used to generate (eb) a class of anharmonic potentials with (e&eb) exactly specified spectra. The complete spectrum of an anharmonic potential so generated consists of (e) all the eigenstates of the simple harmonic oscillator and, in addition, a ground state at a specified energy E which lies arbitrarily below (eb) the E=1/2 ground state of the harmonic oscillator. **Supersymmetric quantum mechanics of one-dimensional systems C V Sukumar Journal of Physics A: Mathematical and General, Volume 18, Number 15 STOP** Authors examine the properties and consequences of (e&eb) pseudo-supersymmetry for (e) quantum systems admitting (e) a pseudo-Hermitian Hamiltonian. Authors explore the Witten index of (e) pseudo-supersymmetry and show that every pair of diagonalizable (not necessarily Hermitian) Hamiltonians with (e&eb) discrete spectra and (e&eb) real or complex-conjugate pairs of (e) eigenvalues are (=) isospectral and have (e) identical degeneracy structure except (e) perhaps for the zero eigenvalue if and only if (e) they are pseudo-supersymmetric partners. This implies (eb) that pseudo-supersymmetry is (=) the basic framework for (e) generating non-Hermitian PT-symmetric and (e&eb) non-PT-symmetric Hamiltonians with (e&eb) a real spectrum via (e&eb) a Darboux transformation, and shows (eb) that every diagonalizable Hamiltonian H with(e&eb) a discrete spectrum and (e&eb) real or complex-conjugate pairs of (e) eigenvalues may be factored as (=) H=L♯L where L is (=) a linear operator with pseudo-adjoint L♯. In particular, this factorization applies to (e&eb) PT-symmetric and Hermitian Hamiltonians. The non-degenerate two-level systems provide (eb) a class of Hamiltonians that are pseudo-Hermitian. Authors demonstrate the implications of general results for (e&eb) this class in some detail. Copyright © 2002 Elsevier Science B.V. All rights reserved. **Nuclear Physics B Volume 640, Issue 3, 30 September 2002, Pages 419–434Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians Ali Mostafazadeh STOP Michele Caselle, Alessandro Nada, Marco Panero** present a high-precision lattice calculation of (e) the equation of state in the confining phase of (e) SU(2) Yang-Mills theory. Authors show that the results are described very well by (e) a gas of massive, non-interacting glueballs, provided (e) one assumes an exponentially growing Hagedorn spectrum. The latter can be derived within (eb) an effective bosonic closed-string model, leading to (eb) a parameter-free theoretical prediction, which is in perfect agreement with (=) proposed lattice results. Furthermore, when applied to (e&eb) SU (3) Yang-Mills theory, this effective model accurately describes (eb) the lattice results reported by

Bors\'anyi et al. in JHEP 07 (2012) 056. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 07 (2015) 143 DOI: 10.1007/JHEP07 (2015)143 Cite as: arXiv: 1505.01106 [hep-lat] (or arXiv: 1505.01106v2 [hep-lat] for this version) **Hagedorn spectrum and thermodynamics of SU (2) and SU (3) Yang-Mills theories Michele Caselle, Alessandro Nada, Marco Panero STOP B. de Wit J. Hoppe, H. Nicolai** study the quantum-mechanical properties of (e) a supermembrane and examine the nature of (e) its ground state. A supersymmetric gauge theory of (e) area-preserving transformations provides (eb) a convenient framework for this study. The supermembrane can be viewed as (=) a limiting case of a class of models in supersymmetric quantum mechanics. Its mass does not depend on (=) the zero modes and vanishes only if (e) the wave function is (=) a singlet under (e&eb) supersymmetry transformations of (e) the nonzero modes. Authors exhibit (eb) the complexity of (e) the supermembrane ground state and (e&eb) examine various truncations of these models. None of these truncations has massless states. Copyright © 1988 Published by Elsevier B.V. **Nuclear Physics B Volume 305, Issue 4, 28 December 1988: Pages 545-581: On the quantum mechanics of supermembranes Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. B. de Wit J. Hoppe, H. Nicolai STOP** In this note we explain how world-volume geometries of (e) D-branes can be reconstructed within (eb) the microscopic framework where (e) D-branes are described through (e&eb) boundary conformal field theory. Authors extract the (non-commutative) world-volume algebras from (e) the operator product expansions of (e) open string vertex operators. For branes in a flat background with (e&eb) constant non-vanishing B-field, the operator products are computed perturbatively to (e) all orders in the field strength. Resulting series coincides with (=) Kontsevich's presentation of the Moyal product. After extending these considerations to (e) fermionic fields authors conclude with some remarks on the generalization of approach to (e&eb) curved backgrounds. **D-branes and deformation quantization Volker Schomerus1,2 Published 25 July 1999 Journal of High Energy Physics, Volume 1999, JHEP06(1999)STOP** Authors employ the path integral approach developed in Gangopadhyay S. and Scholtz F. E., Phys. Rev. Lett., 102 (2009) 241602 to discuss the (generalized) harmonic oscillator in (eb) a noncommutative plane. The action for (e) this system is derived in (e&eb) the coherent state basis with (e&eb) additional degrees of freedom. From this the action in the coherent state basis without (e) any additional degrees of freedom is obtained. This gives (eb) the ground-state spectrum of the system. Authors then employ the exact renormalization group approach to show (eb) that equivalence can be constructed between this (noncommutative) system and (e&eb) a commutative system **Path integral action and exact renormalization group dualities for quantum systems in noncommutative plane Sunandan Gangopadhyay Published 25 June 2015 • Copyright © EPLA, 2015 STOP Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini** present a lattice study of (e)the equation of state in Yang-Mills theory based on (e) the exceptional G(2) gauge group. As is well-known, at zero temperature this theory shares (e&eb) many qualitative features with real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e) large distances. In agreement with previous works, authors show that at finite temperature this theory features (eb) a first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation. Authors also show that the equilibrium thermodynamic observables in the deconfined phase bear striking quantitative similarities with (=) those found in SU (N) gauge theories: in particular, these quantities exhibit (eb) nearly perfect proportionality to the number of gluon degrees of freedom, and (e&eb) the trace anomaly reveals (eb) a characteristic quadratic dependence on (e&eb) the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions). Authors compare our lattice data with (e&eb) analytical predictions from (e) effective models, and discuss their implications for (e) the deconfinement mechanism and (e&eb) high-temperature properties of (e) strongly interacting, non-supersymmetric gauge theories. Results give (eb) strong evidence for (e) the conjecture that the thermal deconfining transition is governed by (e&eb) a universal mechanism, common to (e) all simple gauge groups. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03 (2015)057 Report number: DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv: 1409.8305 [hep-lat] (or arXiv: 1409.8305v2 [hep-lat] for this

version **Exceptional thermodynamics: The equation of state of G(2) gauge theory Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini STOP** In U (1) lattice gauge theory in three spacetime dimensions, the problem of confinement can be studied analytically in (e&eb) a semi-classical approach, in terms of (e&eb) a gas of monopoles with (e&eb) Coulomb-like interactions. In addition, this theory can be mapped to (e&eb) a spin model via (e&eb) an exact duality transformation, which allows (eb) one to perform high-precision numerical studies of (e) the confining potential. Taking advantage of these properties, authors carried out an accurate investigation of (e) the effective string describing (eb) the low-energy properties of flux tubes in (eb) this confining gauge theory. Authors found striking deviations from (e) the expected Nambu-Goto-like behavior, and, for the first time, evidence for (e) contributions that can be described by a term proportional to the extrinsic curvature of the effective string worldsheet. Such term is allowed by (e) Lorentz invariance and its presence in the infrared regime of the U (1) model was indeed predicted by (e) Polyakov several years ago. Results show that this term scales as expected according to (e) Polyakov's solution, and becomes (=) the dominant contribution to (e) the effective string action in the continuum limit. Authors also demonstrate analytically that the corrections to (e) the confining potential induced by (e) the extrinsic curvature term can be related to (e&eb) the partition function of (e) the massive perturbation of (e) a c=1 bosonic conformal field theory. The implications of results for SU (N) Yang-Mills theories in (e&eb) three and in four spacetime dimensions are discussed. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Journal reference: JHEP 1501 (2015) 105 DOI: 10.1007/JHEP01 (2015)105 Report number: IFT-UAM/CSIC-14-052 Cite as: arXiv: 1406.5127 [hep-lat] (or arXiv: 1406.5127v2 [hep-lat] for this version) **A different kind of string Michele Caselle, Marco Panero, Roberto Pellegrini, Davide Vadacchino STOP** In the past ten years, the ideas of supersymmetry have been profitably applied to (e&eb) many nonrelativistic quantum mechanical problems. In particular, there is now a much deeper understanding of why certain potentials are analytically solvable and (e&eb) an array of powerful new approximation methods for (e) handling potentials which are not (e) exactly solvable. In this report, authors review the theoretical formulation of (e) supersymmetric quantum mechanics and discuss many applications. Exactly solvable potentials can be understood in terms of (e&eb) a few basic ideas which include (e) supersymmetric partner potentials (e&eb) shape invariance and (e&eb) operator transformations. Familiar solvable potentials all have (e) the property of shape invariance. Authors describe new exactly solvable shape invariant potentials which include (e) the recently discovered self-similar potentials as (=) a special case. The connection between inverse scattering,(e&eb) isospectral potentials and (e&eb) supersymmetric quantum mechanics is discussed and multisoliton solutions of (e) the K dV equation are constructed. Approximation methods are also discussed within (eb) the framework of supersymmetric quantum mechanics and in particular it is shown that a supersymmetry inspired WKB approximation is (=) exact for a class of shape invariant potentials. Supersymmetry ideas give (eb) particularly nice results for the tunneling rate in (eb) a double well potential and for improving (eb+) large N expansions. Authors also discuss the problem of (e) a charged Dirac particle in (eb) an external magnetic field and other potentials in terms of (e&eb) supersymmetric quantum mechanics. Finally, authors discuss structures more general than supersymmetric quantum mechanics such as(=) parasupersymmetric quantum mechanics in which there is symmetry between a boson and (e&eb) a para-fermion of order p. Copyright © 1995 Published by Elsevier B.V. **Physics Reports Volume 251, Issues 5–6, January 1995, Pages 267–385Supersymmetry and quantum mechanics Fred Coopera, Avinash Khareb, Uday Sukhatmec STOP** The free energy of (e) the maximally supersymmetric SU (N) gauge theory at temperature T is expected to scale, in the large-N limit, as (=) N2T4 times a function of the 't Hooft coupling, f (gYM2N). In the strong coupling limit the free energy has been deduced from (e) the near-extremal 3-brane geometry, and its normalization has turned out to be Full-size image (<1 K) times that found in (eb) the weak coupling limit. In this paper authors calculate the leading correction to (e) this result in inverse powers of (e) the coupling, which originates from (e) the R4 terms in the tree-level effective action of (e) type IIB string theory. The correction to Full-size image (<1 K) is positive and of (e) order Full-size image (<1 K). Thus, f (gYM2N) increases as (e&eb) the 't Hooft coupling is decreased, in accordance with (e&eb) the expectation that is should be approaching 1 in (eb) the weak coupling limit. Authors also discuss similar corrections for (e) other conformal theories describing (eb)

coincident branes. In particular, authors suggest (eb) that the coupling independence of (e) the near extremal entropy for (e) D1-branes bound to (e) D5-branes is related to (e&eb) the vanishing of (e) the Weyl tensor of AdS3 × S3. Copyright © 1998 Published by Elsevier B.V. **Nuclear Physics B Volume 534, Issues 1–2, 23 November 1998, Pages 202–222Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory Steven S. Gubsera, Igor R. Klebanovb, 1, Arkady A. Tseytlinc, 2 STOP** A new approach to (e) N=2 supersymmetry based on (e) the concept of harmonic superspace is proposed and is used to give (eb) an unconstrained superfield geometric description of N=2 super Yang-Mills and supergravity theories as (=) well as of matter N=2 hypermultiplets. The harmonic N=2 superspace has (e) an independent coordinates, in addition to the usual ones, the isospinor harmonics ui+or- on (e&eb) the sphere SU (2)/U (1). The role of ui+or- is to relate the SU (2) group realised on the component fields to (e&eb) a U (1) group acting on (e&eb) the relevant superfields. Their introduction makes it possible to SU (2)-covariantise (eb (t), e&eb) the notion of Grassmann analyticity. Crucial for the construction is the existence of (e) an analytic subspace of the general harmonic N=2 superspace. The hypermultiplet superfields and (e&eb) the true prepotentials (pre-prepotentials) of N=2 super Yang-Mills and supergravity are (=) unconstrained superfunctions over (e&eb) this analytic subspace. The pre-prepotentials have (e) a clear geometric interpretation as gauge connections with (e&eb) respect to the internal SU (2)/U (1) directions. A radically new feature arises: the number of gauge and (e&eb) auxiliary degrees of freedom becomes (=) infinite while (e) the number of physical degrees of freedom remains (=) finite. Other new results are the massive N=2 Yang-Mills theory and (e&eb) various off-shell self-interactions of hypermultiplets. The propagators for matter and Yang-Mills superfields are given. **Unconstrained N=2 matter, Yang-Mills and supergravity theories in harmonic superspace A Galperin, E Ivanov, S Kalitzin, V Ogievetsky and E Sokatchev STOP** By assuming the existence of a novel multipronged string state for (e) D-particles interacting with (e&eb) D-brane intersections in type IIA string theory, authors are able to derive a quantum mechanical description of (e) supersymmetric Reissner-Nordstrom black holes. A supersymmetric index calculation provides (eb) evidence for (e) this conjecture. The quantum mechanical system becomes (=) two decoupled conformal quantum mechanical systems in (eb) the low energy limit. The conformal quantum mechanics has expected properties of (e) a dual description of string theory on (e&eb) AdS2×S2. Subjects: High Energy Physics - Theory (hep-th) Journal reference: JHEP 0001:006, 2000 DOI: 10.1088/1126-6708/2000/01/006 Report number: LBNL-43734 Cite as: arXiv:hep-th/9907101

(or arXiv:hep-th/9907101v5 for this version) **Supersymmetric Quantum Mechanical Description of Four Dimensional Black Holes Julie D. BlumSTOP** Using recently proposed soliton equations authors derive a basic identity for the scaling violation of (e) N=2 supersymmetric gauge theories $\Sigma ia_i \partial F/\partial a_i - 2F = 8\pi i b_1 u$. Here F is (=) the prepotential, ai's are (=) the expectation values of (e) the scalar fields in (eb) the vector multiplet, u=1/2 Tr $\langle \phi 2 \rangle$ and b1 is the coefficient of (e) the one-loop β-function. This equation holds in (eb) the Coulomb branch of all N=2 supersymmetric gauge theories coupled with (e&eb) massless matter. **TOHRU EGUCHI and SUNG-KIL YANG, Mod Phys. Lett. A 11, 131 (1996). DOI: 10.1142/S0217732396000151 PREPOTENTIALS OF N=2 SUPERSYMMETRIC GAUGE THEORIES AND SOLITON EQUATIONS STOP** The Coulomb branch of N = 2 supersymmetric gauge theories in four dimensions is described in general by (e) an integrable Hamiltonian system in (eb) the holomorphic sense. A natural construction of such systems comes from (e) two-dimensional gauge theory and (e&eb) spectral curves. Starting from this point of view, authors propose an integrable system relevant to (e&eb) the N = 2 SU (n) gauge theory with (e&eb) a hypermultiplet in the adjoint representation, and offer (eb) much evidence that it is correct. The model has (e) a Full-size image (<1 K) S-duality group (with the central element −1 of Full-size image (<1 K) acting as (=) charge conjugation); Full-size image (<1 K) permutes the Higgs, confining, and oblique confining phases in the expected fashion. We also study more exotic phases. Copyright © 1996 Published by Elsevier B.V. **Nuclear Physics B Volume 460, Issue 2, 5 February 1996, Pages 299–334Supersymmetric Yang-Mills theory and integrable systems Ron Donagia, 1, Edward Wittenb, 2 STOP A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2** present a first step towards generalizing (eb) the work of Seiberg and Witten on N = 2 supersymmetric Yang-Mills theory to (e&eb) arbitrary gauge groups. Specifically, authors propose a particular sequence of (e) hyperelliptic genus n − 1 Riemann surfaces to underly

(e&eb) the quantum moduli space of SU (n)N = 2 supersymmetric gauge theory. These curves have (e) an obvious generalization to (e) arbitrary simply laced gauge groups, which involves (e&eb) the A-D-E type simple singularities. To support proposal, authors argue that the monodromy in (e&eb) the semiclassical regime is (=) correctly reproduced. We also give some remarks on a possible relation to string theory. Copyright © 1995 Published by Elsevier B.V. **Physics Letters B Volume 344, Issues 1–4, 26 January 1995, Pages 169–175Simple singularities and N = 2 supersymmetric Yang-Mills theory A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 STOP Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2** review and extend earlier work that uses (e) the AdS/CFT correspondence to relate the black-hole–black-string transition of gravitational theories on a circle to (e&eb) a phase transition in maximally supersymmetric (1 + 1)-dimensional SU(N) gauge theories at (eb) large N, again compactified on (e&eb) a circle. Authors perform gravity calculations to determine (eb) a likely phase diagram for (e) the strongly coupled gauge theory. Authors then directly study the phase structure of (e) the same gauge theory, now at (eb) weak't Hooft coupling. In the interesting temperature regime for (e) the phase transition, the (1 + 1)-dimensional theory reduces to (e&eb) a (0 + 1)-dimensional bosonic theory, which they solve using (e) Monte Carlo methods. Authors find strong evidence that the weakly coupled gauge theory also exhibits (eb) a black hole–black string-like phase transition in (eb) the large N limit. Authors demonstrate that a simple Landau–Ginzburg-like model describes (eb) the behaviour near the phase transition remarkably well. The weak coupling transition appears to be (=) close to the cusp between a first-order and a second-order transition. **Black-hole–black-string phase transitions in thermal (1 + 1)-dimensional supersymmetric Yang–Mills theory on a circle Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 Published 29 October 2004 • 2004 IOP Publishing Ltd Classical and Quantum Gravity, Volume 21, Number 22STOP** Monitoring the orbits of stars around (e&eb) Sgr A* offers(eb)the possibility of detecting (e&eb) the precession of (e) their orbital planes due to (e) frame dragging, of (e) measuring the spin and quadrupole moment of (e) the black hole, and of (e)testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and (e&eb) tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are (=), in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for which an optical interferometer, such as GRAVITY, will detect orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to (e&eb) the timescale of precession due to (e) the quadrupole moment of (e) the black hole. As a result, this non-conservative effect is (=) a potential source of systematic uncertainty in (eb) testing the no-hair theorem with (e&eb) stellar orbits. Subjects: High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI:10.1088/0004-637X/777/1/57 Cite as:arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) **Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP** Authors perform a phase space analysis of (e) a generalized modified gravity theory with (e&eb) non-minimal coupling between geometry and (e&eb) matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that (eb) in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, Authors prove that there is (=) an attractor solution to describe (eb) the late time accelerating universe when (e) the modified gravity is chosen in (eb) a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and supernova type Ia data sets, Authors find that (eb) the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as:arXiv: 1512.09281 [gr-qc] (or arXiv: 1512.09281v1 [gr-qc] for this version) **Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang** study the P−V criticality and phase transition

in (eb) the extended phase space of (e) charged Gauss-Bonnet black holes in (eb) anti-de Sitter space, where (e) the cosmological constant appears as (=) a dynamical pressure of the system and (e&eb) its conjugate quantity is (=) the thermodynamic volume of (e) the black hole. The black holes can have (e) a Ricci flat (k=0), spherical (k=1), or hyperbolic (k=−1) horizon. Authors find that for the Ricci flat and hyperbolic Gauss-Bonnet black holes, (eb)no P−V criticality and phase transition appear, while for the black holes with a spherical horizon, even when (e) the charge of the black hole is absent, the P−V criticality and the small black hole/large black hole phase transition (eb) will appear, but it happens only in (eb) d=5 dimensions; when (e) the charge does not vanish, the P−V criticality and (e&eb) the small black hole/large phase transition always appear in (eb) d=5 dimensions; in the case of d≥6, to (e) have the P−V criticality and the small black hole/large black hole phase transition, there exists (eb) an upper bound for (e) the parameter b=α˜|Q|−2/(d−3), where (e) α~ is the Gauss-Bonnet coefficient and Q is the charge of the black hole. Authors calculate the critical exponents at the critical point and find that for all cases, they are (=) the same as those in the van der Waals liquid-gas system. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/JHEP09 (2013)005 Cite as: arXiv: 1306.6233 [gr-qc] (or arXiv: 1306.6233v4 [gr-qc] for this version) **P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang STOP** paper studies the whole process of (e) gravitational collapse and (e&eb) accretion of (e) a massless scalar field in (eb) asymptotically flat spacetime. Two kinds of initial configurations are considered. One is the initial data without (e) black hole; the other contains (e) a black hole. Under suitable initial conditions, Authors find that multi-horizon will appear which mean (eb) that the initial black hole formed by (e) gravitational collapse or existing at the beginning will instantly expand and suddenly grow (e&eb, eb), rather than grows gradually in the accretion process. A new type of critical behavior is found around (e&eb) the instant expansion. The numerical computation shows (eb) that the critical exponents are (=) universal. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Computational Physics (physics. Comp-ph) Cite as: arXiv: 1604.03363 [gr-qc] (or arXiv: 1604.03363v1 [gr-qc] for this version) **Multi-horizon and Critical Behavior in Gravitational Collapse of Massless Scalar Zhoujian Cao, Rong-Gen CAI, Run-Qiu Yang STOP** In this paper, Authors present the results on the gravitational collapse of (e) charged massless scalar field in asymptotically flat spacetime with(e&eb) a perfectly reflecting wall. Differing from previous works, Authors study the system in (eb) the double null coordinates, by which we could simulate (e&eb) the system until the black hole forms with (e&eb) higher precision but less performance time. Authors investigate the influence of charge on (e&eb) the black hole formation and the scaling behavior near the critical collapses. The gapless and gapped critical behaviors for (e) black hole mass and charge are studied numerically. Authors find that they satisfy (eb) the scaling laws for critical gravitational collapse but the gapped critical behavior is different from (e) its AdS counterpart. Subjects: General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1602.00112 [gr-qc] (or arXiv: 1602.00112v1 [gr-qc] for this version) **Multiple critical gravitational collapse of charged scalar with reflecting wall Rong-Gen CAI, Run-Qiu Yang STOP** In this paper Authors investigate the (2+1)-dimensional topological non-Fermi liquid in (eb) strongly correlated electron system, which has (e) a holographic dual description by (e) Einstein gravity in (3+1)-dimensional anti-de Sitter (AdS) space-time. In a dyonic Reissner-Nordstrom black hole background, Authors consider a Dirac fermion coupled to (e&eb) the background U (1) gauge theory and an (e&eb) intrinsic chiral gauge field bM induced by (e) chiral anomaly. UV retarded Green's function of (e) the charged fermion in (eb) the UV boundary from (e) AdS4 gravity is calculated, by imposing (e&eb) in-falling wave condition at the horizon. Authors also obtain IR correlation function of (e) the charged fermion at (eb) the IR boundary arising from (e) the near horizon geometry of (e) the topological black hole with (e&eb) index k=0,±1. By using (e) the UV retarded Green's function and IR correlation function Authors analyze the low frequency behavior of (e) the topological non-Fermi liquid at zero and (e&eb) finite temperatures, especially the relevant non-Fermi liquid behavior near (eb) the quantum critical region. In addition, Authors find that when k=±1, the effective mass of (e) bulk Dirac fermion at IR fixed point is (=) topologically quantized, consequently the Fermi momentum presents (eb) a discrete sparse distributed pattern, in contrary to (e&eb) the Ricci flat case with k=0. As demonstrated examples, Authors calculate

the spectral functions of (e) topological non-Fermi liquid due to (e) anomalous Hall effect (AHE) (|b→|≠0,b 0=0) and chiral magnetic effect (CME) (|b→|=0,b0≠0), which can be tested in topological materials such as (=) topological insulator and (e&eb) topological metal with (e&eb) strong couplings. Subjects: High Energy Physics - Theory (hep-th); Strongly Correlated Electrons (cond-mat.str-el); General Relativity and Quantum Cosmology (gr-qc) Cite as:arXiv: 1601.03865 [hep-th] (or arXiv: 1601.03865v1 [hep-th] for this version) **Topological Non-Fermi Liquid Rong-Gen Cai, Yong-Hui Qi, Yue-Liang Wu, Yun-Long Zhang STOP** The unusual magnetic materials are significant in both science and technology. However, because of (e) the strongly correlated effects, it is difficult to understand their novel properties from (e) theoretical aspects. Holographic duality offers (eb) a new approach to understanding such systems from (e) gravity side. This paper will give a brief review of Authors recent works on the applications of holographic duality in understanding (e&eb) unusual magnetic materials. Some quantitative compare between holographic results and (e&eb) experimental data will be shown and some predictions from (e) holographic duality models will be discussed. Comments: For Second LeCosPA Symposium at Taiwan National University, Taipai, Dec. 14-18, 2015 Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1601.02936 [hep-th] (or arXiv:1601.02936v1 [hep-th] for this version) **Understanding strongly coupling magnetism from holographic duality Rong-Gen Cai, Run-Qiu Yang STOP** It was shown recently that, without jeopardizing the success of the Λ cold dark matter model on cosmic scales, the modified Newtonian dynamics (MOND) can be derived as (=) an emergent phenomenon when (e) axionlike dark matter particles condense into (e&eb) superfluid on the galactic scales. Authors propose in this paper a Dirac-Born-Infeld (DBI) scalar field conformally coupled to (e&eb) the matter components. To maintain the success of MOND phenomenon of (e) dark matter superfluid on (eb) the galactic scales, the fifth force introduced by (e) the DBI scalar should be screened on (e&eb) the galactic scales. It turns out that the screening effect naturally leads to (eb) a simple explanation for (e) a longstanding puzzle that the MOND critical acceleration coincides with (=) present Hubble scale. This galactic coincidence problem is solved, provided that (e) the screened DBI scalar also plays the role of (e&eb) dark energy on the cosmic scales. Subjects:General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 93, 023515 (2016) DOI: 10.1103/PhysRevD.93.023515 Cite as:arXiv:1511.00627 [gr-qc] (or arXiv:1511.00627v3 [gr-qc] for this version) **Dark matter superfluid and DBI dark energy Rong-Gen Cai, Shao-Jiang Wang STOP** The tensor-vector-scalar (TeVeS) model is (=) considered a viable theory of (e) gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here Authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (e&eb) the TeVeS model. Furthermore, Authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to Planck data, and find that (eb) it is ruled out by cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 92, 083505 (2015) DOI: 10.1103/PhysRevD.92.083505 Cite as: arXiv: 1412.4073 [astro-ph.CO] (or arXiv: 1412.4073v3 [astro-ph.CO] for this version) **Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang STOP** The group of unimodular transformations on the roots of polynomials over GF(2) is considered, and those polynomials with (e&eb) symmetries in (eb) the unimodular group are identified. The cross-correlation function between (e&eb) two maximum-length linear shift register sequences of (e) the same degree is shown to be (=) computable as an explicit linear transformation, in (eb) matrix form, on (e&eb) either one of the sequences, regarded as (=) a vector. The underlying vector space is (=) the "cyclotomic algebra," generated by (e) the cyclotomic cosets, or by what Gauss termed the **"periods" of the cyclotomic equation**. A variety of numerical examples are worked in detail. **Information Sciences Volume 1, Issue 1, December 1968, Pages 87-109 Theory of transformation groups of**

**polynomials over GF(2) with applications to linear shift register sequences Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more link. Solomon W. Golomb** stop Authors study the P−V criticality and phase transition in (eb) the extended phase space of charged Gauss-Bonnet black holes in (eb) anti-de Sitter space, where (e) the cosmological constant appears as (=) a dynamical pressure of (e) the system and its conjugate quantity is (=) the thermodynamic volume of the black hole. The black holes can have (e) a Ricci flat (k=0), spherical (k=1), or hyperbolic (k=−1) horizon. Authors find that for the Ricci flat and hyperbolic Gauss-Bonnet black holes, (eb) no P−V criticality and phase transition appear, while (e) for the black holes with a spherical horizon, even when the charge of the black hole is absent, t(eb) he P−V criticality and the small black hole/large black hole phase transition will appear, but it happens only in d=5 dimensions; when the charge does not vanish, the P−V criticality and (e&eb) the small black hole/large phase transition always appear in (eb) d=5 dimensions; in the case of d≥6, to have (e) the P−V criticality and (e&eb)the small black hole/large black hole phase transition, there exists (ebb) an upper bound for the parameter b=α˜|Q|−2/(d−3), where (e) α~ is the **Gauss-Bonnet coefficient** and Q is (=) the charge of the black hole. Authors calculate the critical exponents at the critical point and find that (eb) for all cases, (eb) they are the same as those in the van der Waals liquid-gas system. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/JHEP09(2013)005 Cite as: arXiv:1306.6233 [gr-qc] (or arXiv:1306.6233v4 [gr-qc] for this version) **P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang STOP** Recently Hořava proposed a renormalizable gravity theory in (eb) four dimensions which reduces to (e&eb) Einstein gravity with (e&eb) a non-vanishing cosmological constant in IR but with (e&eb) improved UV behaviors. Here, authors study an IR modification which breaks ``softly" (e) the detailed balance condition in (eb) Hořava model and allows (eb) the asymptotically flat limit as well. Authors obtain the black hole and cosmological solutions for **``arbitrary" cosmological constant** that represent (eb) the analogs of the standard Schwarzschild-(A)dS solutions which can be asymptotically (A)dS as well as flat and authors discuss their thermodynamical properties. They also obtain solutions for (e) FRW metric with (e&eb) an arbitrary cosmological constant. Authors study its implication to (e&eb) the dark energy and find that it seems to be consistent with (e&eb) current observational data. **The black hole and cosmological solutions in IR modified Hořava gravity Mu-In Park Published 29 September 2009 • Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP.** The Einstein equations with (e&eb) a negative cosmological constant admit (e) black hole solutions which are asymptotic to (e) anti-de Sitter space. Like black holes in asymptotically flat space, these solutions have (e) thermodynamic properties including (e) a characteristic temperature and intrinsic entropy equal to (=) one quarter of the area of the event horizon in Planck units. There are however some important differences from (e) the asymptotically flat case. A black hole in anti-de Sitter space has (e) a minimum temperature which occurs when (e) its size is of the order of the characteristic radius of (e) the anti-de Sitter space. For larger black holes the red-shifted temperature measured at (eb) infinity is greater. This means that such black holes have (e) positive specific heat and can be in stable equilibrium with (e&eb) thermal radiation at a fixed temperature. It also implies (eb) that the canonical ensemble exists for asymptotically anti-de Sitter space, unlike the case for asymptotically flat space. One can also consider the microcanonical ensemble. One can avoid (e) the problem that arises in asymptotically flat space of (e) having to put the system in (eb) a box with unphysical perfectly reflecting walls because (e) the gravitational potential of anti-de Sitter space acts as a box of finite volume. Communicated by A. Jaffe **Communications in Mathematical Physics December 1983, Volume 87, Issue 4, pp 577-588 Thermodynamics of black holes in anti-de Sitter space S. W. Hawking, Don N. Page STOP** In previous work authors introduced the concept of black-hole entropy, which they identified with (e&eb) the surface area of the black hole in question expressed in units of (e) the Planck length squared. Authors suggested that the appropriate generalization of the second law for a region containing a black hole is (=) that the black-hole entropy plus (+) the common entropy in the black-hole exterior never decreases. Here authors establish the validity of this law for the infall of an entropy-bearing system into (e&eb) a much larger and more massive generic stationary black hole. To do this authors determine (eb) a general lower bound for the increase in black-hole entropy, and an upper bound for the entropy of

the system, while allowing for quantum effects at each stage. In passing authors show that the generalized second law is a statistical law which becomes (=) over-whelmingly probable in the limit of a macroscopic system. Authors also consider briefly more general situations. Finally, we give two simple examples of predictions made by the generalized second law for (e) black-hole formation processes. Received 17 September 1973 DOI:http://dx.doi.org/10.1103/PhysRevD.9.3292 **Generalized second law of thermodynamics in black-hole physics Jacob D. Bekenstein Phys. Rev. D 9, 3292 – Published 15 June 1974 STOP** The Lovelock gravity consists of (e) the dimensionally extended Euler densities. The geometry and horizon structure of (e) black hole solutions could be quite complicated in (eb) this gravity; however, authors find that some thermodynamic quantities of the black holes like the mass, Hawking temperature and entropy have (e) simple forms expressed in terms of (e) horizon radius. The case with black hole horizon being a Ricci flat hypersurface is (=) particularly simple. In that case the black holes are (=) always thermodynamically stable with (e&eb) a positive heat capacity and their entropy still obeys (e&eb) the area formula, which is no longer valid for (e) black holes with positive or negative constant curvature horizon hypersurface. In addition, for black holes in (eb) the gravity theory of Ricci scalar plus a 2n-dimensional Euler density with (e&eb) a positive coefficient, thermodynamically stable small black holes always exist in (eb) D=2n+1 dimensions, which are absent in (eb) the case without (e) the Euler density term, while (e) the thermodynamic properties of the black hole solutions with (e&eb) the Euler density term are qualitatively similar to those of black holes without (e) the Euler density term as D>2n+1. Copyright © 2004 Published by Elsevier B.V. **Physics Letters B Volume 582, Issues 3–4, 4 March 2004, Pages 237–242 A note on thermodynamics of black holes in Lovelock gravity Rong-Gen CAI STOP** There are a number of similarities between black-hole physics and (e, eb, =) thermodynamics. Most striking is the similarity in the behaviors of black-hole area and (e&eb) of entropy: Both quantities tend to increase (eb+) irreversibly. In this paper we make this similarity the basis of (e) a thermodynamic approach to black-hole physics. After a brief review of the elements of the theory of information, authors discuss black-hole physics from the point of view of information theory. Authors show that it is natural to introduce (eb) the concept of black-hole entropy as (=) the measure of information about (e&eb) a black-hole interior which is (=) inaccessible to an exterior observer. Considerations of simplicity and consistency, and dimensional arguments indicate (eb) that the black-hole entropy is equal to (=) the ratio of the black-hole area to the square of the Planck length times a dimensionless constant of (e) order unity. A different approach making use of the specific properties of Kerr black holes and of concepts from information theory leads to (eb) the same conclusion, and suggests (eb) a definite value for the constant. The physical content of (e) the concept of black-hole entropy derives from (e) the following generalized version of the second law: When common entropy goes down a black hole, (eb) the common entropy in (eb) the black-hole exterior plus the black-hole entropy never decreases. The validity of this version of the second law is supported by (e) an argument from information theory as well as by several examples. Received 2 November 1972 DOI:http://dx.doi.org/10.1103/PhysRevD.7.2333 **Black Holes and Entropy Jacob D. Bekenstein Phys. Rev. D 7, 2333 – Published 15 April 1973 STOP** The effects of power-law Maxwell field on the Van der Waals like phase transition of higher-dimensional dilaton black holes are probed in detail. It is shown that the Smarr relation gains corrections due to (e) the effects of (e) both the power-law Maxwell field and (e&eb) the dilaton field while (e) thermodynamic volume is exactly the same as (=) that of Einstein-Maxwell-dilaton black holes. Authors successfully derive the analytic solutions of (e) critical point and carry out (e&eb) some check to ensure that these critical quantities are (=) positive. It is shown that the constraint on the parameters turns out to be (=) 0<\alpha^2<1, which is tighter than that in (eb) the non-extended phase space. It is also shown (eb) that these critical quantities and (e&eb) the ratio P_cv_c/T_c are affected by (e&eb) the power-law Maxwell field. Moreover, critical exponents are found to coincide with those of other AdS black holes, showing (eb) the powerful influence of mean field theory. Comments: references added Subjects: General Relativity and Quantum Cosmology (gr-qc) Cite as:arXiv:1601.05500 [gr-qc] (or arXiv:1601.05500v2 [gr-qc] for this version **Effects of power-law Maxwell field on the Van der Waals like phase transition of higher dimensional dilaton black holes Jie-Xiong Mo, Gu-Qiang Li, Xiao-Bao Xu STOP** A consistent and unified picture for (e) critical phenomena of charged AdS black holes in f(R) gravity is drawn in this paper. Firstly, we investigate the

phase transition in canonical ensemble. Authors derive the explicit solutions corresponding to (e&eb) the divergence of CQ. The two solutions merge into (e&eb) one when (e) the condition Qc=−13R0−−−√ is satisfied. The curve of specific heat for Q<Qc has two divergent points and can be divided into (e&eb) three regions. Both the large radius region and the small radius region are (=) thermodynamically stable with (e&eb) positive specific heat while (e) the medium radius region is unstable with (e&eb) negative specific heat. However, when Q>Qc, the specific heat is always positive, implying (eb) the black holes are locally stable and no phase transition will take place. Secondly, both the T−r+ curve and T−S curve f(R) AdS black holes are investigated and they exhibit Van der Vaals like behavior as the P−v curve in the former research. Critical physical quantities are obtained and they are consistent with (=) those derived from the specific heat analysis. Authors carry out numerical check of Maxwell equal area law for the cases Q=0.2Qc, 0.4Qc, 0.6Qc, 0.8Qc. The relative errors are (=) amazingly small and can be negligible. So the Maxwell equal area law holds for (e) T−S curve of f(R) black holes. Thirdly, authors establish geometrothermodynamics for (e) f(R) AdS black hole to examine the phase structure. It is shown that the Legendre invariant scalar curvature R would diverge exactly where (e) the specific heat diverges. To summarize, the above three perspectives are consistent with (e&eb) each other, thus providing (eb) a unified picture which deepens the understanding of (e) critical phenomena of (e) f(R) AdS black holes. Subjects: General Relativity and Quantum Cosmology (gr-qc) Cite as:arXiv:1602.01251 [gr-qc] (or arXiv:1602.01251v1 [gr-qc] for this version) **A consistent and unified picture for critical phenomena of f(R) AdS black holes Jie-Xiong Mo, Gu-Qiang Li, Yu-Cheng Wu STOP** The state equation of the charged AdS black hole is reviewed in (e&eb) the T−r plane. Thinking of the phase transition, the T−S, P−V, P−v graphs are plotted and then the equal area law is used (e) in the three cases to get (eb) the phase transition point (P, T). The analytical phase transition point relations for (e) P-T of charged AdS black hole has been obtained successfully. By comparing the three results, authors find that the equal area law possibly cannot (e) be used directly for P−v plane. According to the T−S, P−V results, they plot (e&eb) the P−T−Q graph and find that for a highly charged black hole a very low temperature condition is required for (e) the phase transition. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI:10.1140/epjc/s10052-015-3641-0 Cite as: arXiv: 1503.07658 [gr-qc] (or arXiv:1503.07658v1 [gr-qc] for this version) **A note on Maxwell's equal area law for black hole phase transitionShan-Quan Lan, Jie-Xiong Mo, Wen-Biao Liu STOP** Recently, extended phase space thermodynamics of Lovelock AdS black holes has been of great interest. To provide insight from (e) a different perspective and gain (e) a unified phase transition picture, non-extended phase space thermodynamics of (e) (n+1)-dimensional charged topological Lovelock AdS black holes is (=) investigated detailedly in (eb) the grand canonical ensemble. Specifically, the specific heat at (eb) constant electric potential is calculated and phase transition in the grand canonical ensemble is discussed. To probe the impact of the various parameters, they utilize (e) the control variate method and solve (e) the phase transition condition equation numerically for the case k=1, −1. There are two critical points for (e) the case n=6, k=1 while there is only one for other cases. For k=0, there exists no phase transition point. To figure out the nature of phase transition in the grand canonical ensemble, authors carry out an analytic check of the analog form of (e) Ehrenfest equations proposed by Banerjee et al. It is shown that Lovelock AdS black holes in (eb) the grand canonical ensemble undergo (e&eb) a second order phase transition. To examine the phase structure in (eb) the grand canonical ensemble, authors utilize the thermodynamic geometry method and calculate both the Weinhold metric and Ruppeiner metric. It is shown that for both analytic and graphical results that the divergence structure of (e) the Ruppeiner scalar curvature coincides with (=) that of the specific heat. Research provides (eb) one more example that Ruppeiner metric serves as (=) a wonderful tool to (e) probe the phase structures of (e) black holes. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: Eur. Phys. J. C75 (2015) 5, 211 DOI:10.1140/epjc/s10052-015-3434-5 Cite as: arXiv: 1503.01956 [gr-qc] (or arXiv: 1503.01956v1 [gr-qc] for this version **Non-extended phase space thermodynamics of Lovelock AdS black holes in grand canonical ensemble Jie-Xiong Mo, Wen-Biao Liu STOP** To provide an analytic verification of the nature of phase transition at the critical point of P−V criticality, the original expressions of Ehrenfest equations have been introduced directly. By treating the cosmological constant and (e&eb) its conjugate quantity as (=) thermodynamic pressure and volume respectively, authors carry out

analytical check of (e) classical Ehrenfest equations. To show that approach is universal, authors investigate not only higher-dimensional charged AdS black holes (e&eb), but also rotating AdS black holes. Not only are the examples of Einstein gravity shown, but also the example of modified gravity is presented for (e) Gauss-Bonnet AdS black holes. The specific heat at constant pressure CP, the volume expansion coefficient α and the isothermal compressibility coefficient κT are found to diverge exactly at the critical point. It has been verified that both Ehrenfest equations hold at (eb) the critical point of P−V criticality in (eb) the extended phase spaces of (e) AdS black holes. So the nature of the critical point of P−V criticality of AdS black holes has been demonstrated analytically to be (=) a second-order phase transition. These results are consistent with (e&eb) the nature of liquid-gas phase transition at the critical point. In this sense, extant research would deepen the understanding of (e) the relations of AdS black holes and (e&eb) liquid-gas systems. Moreover, successful approaches to introduce (eb) the original expressions of Erhenfest equations directly into (e&eb) black hole phase transition research demonstrate again (eb) that black hole thermodynamics is closely related (e&eb) to classical thermodynamics, which allows (eb) us to borrow techniques from (e) classical thermodynamics to investigate the thermodynamics of black holes. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 89, 084057 (2014) DOI: 10.1103/PhysRevD.89.084057 Cite as:arXiv: 1404.3872 [gr-qc] (or arXiv:1404.3872v1 [gr-qc] for this version) **Ehrenfest scheme for P−V criticality of higher dimensional charged black holes, rotating black holes and Gauss-Bonnet AdS black holes Jie-Xiong Mo, Wen-Biao Liu STOP** Aiming at a unified phase transition picture of (e) the charged topological black hole in (eb) Ho\v{r}ava-Lifshitz gravity, authors investigate this issue not only in (eb) canonical ensemble with (e&eb) the fixed charge case but also in grand-canonical ensemble with (e&eb) the fixed potential case. Authors firstly perform the standard analysis of the specific heat,(e&eb) the free energy and the Gibbs potential, and (e&eb) then study its geometrothermodynamics. It is shown that the local phase transition points (eb) not only witness the divergence of (e) the specific heat, but also witness the minimum temperature and the maximum free energy or Gibbs potential. They also witness the divergence of (e) the corresponding thermodynamic scalar curvature. No matter which ensemble is chosen, the metric constructed can successfully produce (eb) the behavior of the thermodynamic interaction and (e&eb) phase transition structure while other metrics failed to (e) predict the phase transition point of (e) the charged topological black hole in former literature. In grand-canonical ensemble, authors have discovered the phase transition which has not been reported before. It is similar to (e) the canonical ensemble in which the phase transition only takes place when k=−1. But it also has its unique characteristics that the location of the phase transition point depends on (eb) the value of potential, which is different from (e) the canonical ensemble where (e) the phase transition point is (=) independent of the parameters. After an analytical check of Ehrenfest scheme, authors find that the new phase transition is (=) a second order one. It is also found that the thermodynamics of (e) the black hole in Horava-Lifshitz gravity is quite different from that in (eb) Einstein gravity. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: JHEP1310(2013)056 DOI:10.1007/JHEP10(2013)056 Cite as: arXiv:1404.2497 [gr-qc] (or arXiv:1404.2497v1 [gr-qc] for this version) **A unified phase transition picture of the charged topological black hole in Horava-Lifshitz gravity Jie-Xiong Mo, Xiao-Xiong Zeng, Gu-Qiang Li, Xin Jiang, Wen-Biao Liu STOP** To understand the effect of third order Lovelock gravity, P−V criticality of (e) topological AdS black holes in Lovelock-Born-Infeld gravity is investigated. The thermodynamics is further explored with some more extensions and details than the former literature. A detailed analysis of the limit case $\beta\to\infty$ is performed for (e) the seven-dimensional black holes. It is shown that for the spherical topology, P−V criticality exists for (e) both the uncharged and charged cases. Results demonstrate again that the charge is (=) not the indispensable condition of P−V criticality. It may be attributed to (e) the effect of higher derivative terms of curvature because (e) similar phenomenon was also found for (e) Gauss-Bonnet black holes. For k=0, there would be (=) no P−V criticality. Interesting findings occur in (eb) the case k=−1, in which positive solutions of critical points are found for (e) both the uncharged and (e&eb) charged cases. However, the P−v diagram is (=) quite strange. To check whether these findings are physical, authors give the analysis on the non-negative definiteness condition of (e) entropy. It is shown that for any nontrivial value of α, the entropy is (=) always positive for any specific volume v. Since no P−V

criticality exists for (e) k=−1 in Einstein gravity and Gauss-Bonnet gravity, findings can be related with (e&eb) the peculiar property of (e) third order Lovelock gravity. The entropy in third order Lovelock gravity consists of (e) extra terms which is absent in the Gauss-Bonnet black holes, which makes (eb) the critical points satisfy (eb)the constraint of non-negative definiteness condition of (e) entropy. Authors also check the Gibbs free energy graph and (e&eb) the "swallow tail" behavior can be observed. Moreover, the effect of nonlinear electrodynamics is (=) also included in research. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: Eur. Phys. J. C (2014) 74:2836 DOI: 10.1140/epjc/s10052-014-2836-0 Cite as: arXiv: 1401.0785 [gr-qc] (or arXiv: 1401.0785v1 [gr-qc] for this version) **P-V Criticality of Topological Black Holes in Lovelock-Born-Infeld Gravity Jie-Xiong Mo, Wen-Biao Liu STOP** Authors investigate the phase transitions of (e) black holes with (e&eb) conformal anomaly in canonical ensemble from (e) different perspectives. Some interesting and novel phase transition phenomena have been discovered. Firstly, authors discuss the behavior of the specific heat and (e&eb) the inverse of the isothermal compressibility. It is shown that there are (=)striking differences in Hawking temperature and (e&eb) phase structure between black holes with (e&eb)conformal anomaly and those without (e) it. In the case with conformal anomaly, there exists (eb) local minimum temperature corresponding to (e&eb) the phase transition point. Phase transitions take place not only from (e) an unstable large black hole to (e) a locally stable medium black hole but also from (e) an unstable medium black hole to (e) a locally stable small black hole. Secondly, authors probe in detail the dependence of phase transitions on (=) the choice of parameters. The results show (eb) that black holes with (e&eb) conformal anomaly have much richer phase structure than (e) those without it. There would be two, only one or no phase transition points depending on the parameters we have chosen. The corresponding parameter region is derived both numerically and graphically. Thirdly, geometrothermodynamics are built up to (e) examine the phase structure authors have discovered. It is shown that Legendre invariant thermodynamic scalar curvature diverges (e&eb) exactly where the specific heat diverges. Furthermore, critical behaviors are investigated by (e) calculating the relevant critical exponents. It is proved that these critical exponents satisfy (eb) the thermodynamic scaling laws, leading to the conclusion that critical exponents and the scaling laws can reserve even when we consider conformal anomaly. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc)Journal reference: Advances in High Energy Physics (2014), Article ID 739454DOI:10.1155/2014/739454 Cite as: arXiv:1312.0679 [hep-th] (or arXiv:1312.0679v3 [hep-th] for this version) **Phase transitions, geometrothermodynamics and critical exponents of black holes with conformal anomaly Jie-Xiong Mo, Wen-Biao Liu STOP** Adopting the strong field limit approach, we investigated the strong gravitational lensing in a Brane-World black hole, which means (eb) that the strong field limit coefficients and (e&eb) the deflection angle in this gravitational field are obtained. With this result, it can be said with certainly that the strong gravitational lensing is related to (e&eb) the metric of gravitational fields closely, the cosmology parameter {\alpha} and the dark matter parameter \b{eta} come from (e) the Brane-World black hole exerts a great influence on it. Comparing with (e&eb) the Schwarzschild-AdS spacetime and (e&eb) the Schwarzschild-XCMD spacetime, the parameters {\alpha}, \b{eta} of (e) black holes have (e) the similar effects on (e&eb) the gravitational lensing. In some way, authors infer that the real gravitational fields in our universe can be described by (e) this metric, so the results of (e) the strong gravitational lensing in this spacetime will be more reasonable for (e) us to observe. Finally, it has to be noticed that the influence which the parameters {\alpha}, \b{eta} exerted on (e&eb) the main observable quantities of (e) this gravitational field is (=) discussed. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/s10773-015-2545-y Cite as: arXiv:1506.08410 [gr-qc] (or arXiv:1506.08410v1 [gr-qc] for this version) **Strong Gravitational Lensing in a Brane-World Black Hole GuoPing Li, Biao Cao, Zhongwen Feng, Xiaotao Zu STOP** In this Letter, authors investigate the effects of dark energy on P−V criticality of charged AdS black holes by considering the case of the RN-AdS black holes surrounded by quintessence. By treating the cosmological constant as (=) thermodynamic pressure, authors study its thermodynamics in (eb) the extended phase space. It is shown that quintessence dark energy does not (e) affect the existence of small/large black hole phase transition. For the case $\omega_q=-2/3$ authors derive analytic expressions of (e) critical physical quantities, while (e) for cases $\omega_q\neq-2/3$ appeal to (e) numerical method for help. It is shown that

quintessence dark energy affects (e&eb) the critical physical quantities near (eb) the critical point. Critical exponents are also calculated. They are exactly the same as (=) those obtained before for (e) arbitrary other AdS black holes, which implies (eb) that quintessence dark energy does not (e) change the critical exponents. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO) Journal reference: Physics Letters B (2014), pp.256-260 DOI: 10.1016/j.physletb.2014.06.047 Cite as:arXiv:1407.0011 [gr-qc] (or arXiv:1407.0011v1 [gr-qc] for this version) **Effects of dark energy on P-V criticality of charged AdS black holes Gu-Qiang Li STOP** Monitoring the orbits of stars around Sgr A* offers the possibility of detecting the precession of (e) their orbital planes due to frame dragging, of measuring the spin and quadrupole moment of the black hole, and of testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are, in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for (e) which an optical interferometer, such as GRAVITY, will detect (e&eb) orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to the timescale of precession due to (e) the quadrupole moment of the black hole. As a result, this non-conservative effect is a potential source of systematic uncertainty in testing the no-hair theorem with stellar orbits.Subjects:High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1088/0004-637X/777/1/57 Cite as: arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) **Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP** Authors perform a phase space analysis of a generalized modified gravity theory with non-minimal coupling between geometry and matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, authors prove that there is an attractor solution to describe (eb)the late time accelerating universe when (e) the modified gravity is(=) chosen in a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and (e&eb) supernova type Ia data sets; authors find that the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1512.09281 [gr-qc] (or arXiv:1512.09281v1 [gr-qc] for this version) **Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong STOP** The tensor-vector-scalar (TeVeS) model is considered a viable theory of gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (eb) the TeVeS model. Furthermore, authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to (e) Planck data, and find that (eb) it is ruled out by (e) cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference: Phys. Rev. D 92, 083505 (2015) DOI:10.1103/PhysRevD.92.083505 Cite as: arXiv:1412.4073 [astro-ph.CO] (or arXiv:1412.4073v3 [astro-ph.CO] for this version) **Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang STOP** Using the anti–de Sitter/conformal field theory correspondence, authors relate the shear viscosity η of the finite-temperature N=4 supersymmetric Yang-Mills theory in the large N, strong-coupling regime with (e&eb) the absorption cross section of (e) low-energy gravitons by (e) a near-extremal black three-brane. **G. Policastro, D. T. Son, and A. O. Starinets** show that in the limit of zero frequency this cross section coincides with

(=) the area of the horizon. From this result they find η=π8N2T3. They conjecture that for finite't Hooft coupling g2YMN the shear viscosity is (=) η=f(g2YMN)N2T3, where f(x) is a monotonic function that decreases from O(x−2ln−1(1/x)) at small x to π/8 when x→∞. Received 14 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevLett.87.081601 **Shear Viscosity of Strongly Coupled N=4 Supersymmetric Yang-Mills Plasma G. Policastro, D. T. Son, and A. O. Starinets Phys. Rev. Lett. 87, 081601 – Published 2 August 2001 STOP** Authors use the AdS/CFT correspondence to determine the rate of energy loss of a heavy quark moving through (e&eb) Script N = 4 SU (Nc) supersymmetric Yang-Mills plasma at large't Hooft coupling. Using the dual description of the quark as a classical string ending on a D7-brane, **Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1** use a complementary combination of analytic and numerical techniques to determine (eb) the friction coefficient as a function of quark mass. Provided strongly coupled Script N = 4 Yang-Mills plasma is (=) a good model for hot, strongly coupled QCD, results may be relevant for (e) charm and bottom physics at RHIC. **Energy loss of a heavy quark moving through Script N = 4 supersymmetric Yang-Mills plasma Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1 Published 11 July 2006 Journal of High Energy Physics, Volume 2006, JHEP07(2006) STOP Benjamin Svetitsky** calculates the classical drag and diffusion coefficients for (e) a charmed quark propagating in the quark-gluon plasma. Both coefficients turn out rather large, so that (1) a charmed quark created when (e) the plasma is hot will be (=) stopped before propagating 1 fm and (2) subsequent diffusion will be fast. The first effect should serve to (e) increase the yield of J/ψ mesons in (eb) relativistic heavy-ion collisions, while the second should work in the opposite direction. In any case, the two effects should dominate (e) the dynamics of a $c\bar{c}$ pair. Received 16 November 1987 DOI: http://dx.doi.org/10.1103/PhysRevD.37.2484 **Diffusion of charmed quarks in the quark-gluon plasma Benjamin Svetitsky Phys. Rev. D 37, 2484 – Published 1 May 1988 STOP** Using holographic duality, authors present results for (e) both head-on and off-center collisions of Gaussian shock waves in (eb) strongly coupled N=4 supersymmetric Yang-Mills theory. The shock waves superficially resemble (e&eb) Lorentz contracted colliding protons. The collisions results in (eb) the formation of a plasma whose evolution is well described by (e) viscous hydrodynamics. The size of the produced droplet is R∼1/Teff where (e) Teff is the effective temperature, which is (=) the characteristic microscopic scale in strongly coupled plasma. These results demonstrate (eb) the applicability of hydrodynamics to microscopically small systems and bolster (eb) the notion that hydrodynamics can be applied to (e&eb) heavy-light ion collisions as well as some proton-proton collisions. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1601.01583 [hep-th] (or arXiv: 1601.01583v2 [hep-th] for this version) **How big are the smallest drops of quark-gluon plasma? Paul M. Chesler STOP Paul M. Chesler, Krishna Rajagopal** calculate how the energy and (e&eb) the opening angle of jets in N=4 SYM theory evolve as they propagate through (e&eb) the strongly coupled plasma of that theory. **Paul M. Chesler, Krishna Rajagopal** define the rate of energy loss dEjet/dx and (e&eb) the jet opening angle in a straightforward fashion directly in the gauge theory before (e) calculating both holographically, in (eb) the dual gravitational description. In this way, authors rederive the previously known result for (e) dEjet/dx without (e) the need to introduce (e&eb) a finite slab of plasma. They obtain a striking relationship between the initial opening angle of the jet, which is to say the opening angle that it would have had if it had found itself in vacuum instead of in plasma, and (e&eb) the thermalization distance of the jet. Via this relationship, they show that N=4 SYM jets with (e&eb) any initial energy that have the same initial opening angle and the same trajectory through (e&eb) the plasma experience the same fractional energy loss. They also provide an expansion that describes how the opening angle of the N=4 SYM jets increases slowly as they lose (e) energy, over (e&eb) the fraction of their lifetime when (e) their fractional energy loss is (=) not yet large. Authors close by looking ahead toward potential qualitative lessons from (e) results for QCD jets produced in (eb) heavy collisions and propagating through (e&eb) quark-gluon plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1511.07567 [hep-th] (or arXiv:1511.07567v2 [hep-th] for this version) **On the Evolution of Jet Energy and**

**Opening Angle in Strongly Coupled Plasma Paul M. Chesler, Krishna Rajagopal STOP Paul M. Chesler, Niki Kilbertus, Wilke van der Schee study** the collision of planar shock waves in (eb) AdS5 as a function of (e) shock profile. In the dual field theory the shock waves describe (eb) planar sheets of energy whose collision results in (eb) the formation of a plasma which behaves (e&eb) hydrodynamically at late times. They find that the post-collision stress tensor near (eb) the light cone exhibits (eb) transient non-universal behavior which depends on (=) both the shock width and (e&eb) the precise functional form of (e) the shock profile. However, over a large range of shock widths, including those which yield (eb) qualitative different behavior near (eb) the future light cone, and for (e) different shock profiles, authors find universal behavior in (eb) the subsequent hydrodynamic evolution. Additionally, authors compute the rapidity distribution of (e) produced particles and find it to be well described by (e) a Gaussian. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) DOI: 10.1007/JHEP11 (2015)135 Cite as: arXiv: 1507.02548 [hep-th] (or arXiv: 1507.02548v2 [hep-th] for this version) **Universal hydrodynamic flow in holographic planar shock collisions Paul M. Chesler, Niki Kilbertus, Wilke van der Schee STOP** Using numerical holography, authors study the collision, at non-zero impact parameter, of bounded, localized distributions of energy density chosen to (e) mimic relativistic heavy ion collisions, in (eb) strongly coupled N = 4 supersymmetric Yang-Mills theory. Both longitudinal and transverse dynamics in the dual field theory are (=) properly described. Using the gravitational description we solve 5D Einstein equations, without dimensionality reducing symmetry restrictions, to find the asymptotically anti-de Sitter spacetime geometry. Implications of **Paul M. Chesler, Laurence G. Yaffe** results on the understanding of early stages of heavy ion collisions, including the development of transverse radial flow, are discussed. Comments: Enlarged JHEP version, 14 pages Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1501.04644 [hep-th] (or arXiv:1501.04644v2 [hep-th] for this version) **Holography and off-center collisions of localized shock waves Paul M. Chesler, Laurence G. Yaffe STOP Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** study the dynamic after a smooth quench across a continuous transition from the disordered phase to the ordered phase. Based on scaling ideas, linear response and (e&eb) the spectrum of unstable modes, authors develop a theoretical framework, valid for (e) any second order phase transition, for the early-time evolution of the condensate in the broken phase. Their analysis unveils (eb) a novel period of non-adiabatic evolution after the system passes through (e&eb) the phase transition, where (e) a parametrically large amount of coarsening occurs before (e) a well-defined condensate forms. Formalism predicts (eb) a rate of defect formation parametrically smaller than (eb) the Kibble-Zurek prediction and yields (eb) a criterion for (e) the break-down of Kibble-Zurek scaling for sufficiently fast quenches. **Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** numerically test formalism for (e) a thermal quench in (eb) a 2 + 1 dimensional holographic superfluid. These findings, of (e) direct relevance in a broad range of fields including cold atom, condensed matter, statistical mechanism and cosmology, ar (=)e an important step towards a more quantitative understanding of dynamical phase transitions. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); Statistical Mechanics (cond-mat.stat-mech) Journal reference: Phys. Rev. X 5, 021015 (2015) DOI: 10.1103/PhysRevX.5.021015 Cite as: arXiv:1407.1862 [hep-th] (or arXiv:1407.1862v2 [hep-th] for this version) **Defect formation beyond Kibble-Zurek mechanism and holography Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu STOP Paul M. Chesler, Krishna Rajagopal** present calculations in which an energetic light quark shoots through (e&eb) a finite slab of strongly coupled N=4 supersymmetric Yang-Mills (SYM) plasma, with thickness L, focussing on what comes out on (eb) the other side. Authors find that even when the "jets" that emerge from (e) the plasma have lost a substantial fraction of (e) their energy they look in almost all respects like "jets" in (eb) vacuum with (e&eb) the same reduced energy. The one possible exception is (=) that the opening angle of the "jet" is (=) larger after passage through (e&eb) the slab of plasma than before. Along the way, authors obtain (eb) a fully geometric characterization of energy loss in (eb) the strongly coupled plasma and show that (eb) dE out/dL∝L2/x2stop−L2−−−−−−−−√, where Eout is the energy of the "jet" that emerges from (e) the slab

of plasma and x stop is (=) the (previously known) stopping distance for (e) the light quark in an infinite volume of (e) plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Journal reference: Phys. Rev. D 90, 025033 (2014) DOI: 10.1103/PhysRevD.90.025033 Cite as: arXiv: 1402.6756 [hep-th] (or arXiv: 1402.6756v1 [hep-th] for this version) **Jet quenching in strongly coupled plasma Paul M. Chesler, Krishna Rajagopal STOP** Quasinormal modes are (=) eigenmodes of dissipative systems. Perturbations of classical gravitational backgrounds involving (e&eb) black holes or branes naturally lead to (eb) quasinormal modes. The analysis and classification of (e) the quasinormal spectra require solving (e) non-Hermitian eigenvalue problems for (e) the associated linear differential equations. Within the recently developed gauge-gravity duality, these modes serve as (=) an important tool for (e) determining the near-equilibrium properties of (e) strongly coupled quantum field theories, in particular their transport coefficients, such as (=) viscosity, conductivity and diffusion constants. In astrophysics, the detection (e&eb) of quasinormal modes in gravitational wave experiments would allow (eb) precise measurements of (e) the mass and spin of black holes as well as new tests of general relativity. This review is meant as an introduction to the subject, with a focus on the recent developments in the field. **Quasinormal modes of black holes and black branes Emanuele Berti1,2, Vitor Cardoso1,3 and Andrei O Starinets4 Published 24 July 2009 • 2009 IOP Publishing Ltd Classical and Quantum Gravity, Volume 26, Number 16 STOP Gary T. Horowitz and Veronika E. Hubeny** investigate the decay of a scalar field outside (e) a Schwarzschild anti–de Sitter black hole. This is determined by computing (e&eb) the complex frequencies associated with (e&eb) quasinormal modes. There are qualitative differences from (e) the asymptotically flat case, even in (eb) the limit of small black holes. In particular, for a given angular dependence, the decay is (=) always exponential—there are (=) no power law tails at late times. In terms of the AdS-CFT correspondence, a large black hole corresponds to (e&eb) an approximately thermal state in the field theory, and the decay of the scalar field corresponds to (e&eb) the decay of a perturbation of this state. Thus one obtains the time scale for the approach to (e) thermal equilibrium. **Gary T. Horowitz and Veronika E. Hubeny** compute (e&eb) these time scales for the strongly coupled field theories in three, four, and six dimensions, which are (=)dual to string theory in asymptotically AdS spacetimes. Received 27 September 1999 DOI: http://dx.doi.org/10.1103/PhysRevD.62.024027 **Quasinormal modes of AdS black holes and the approach to thermal equilibrium Gary T. Horowitz and Veronika E. Hubeny Phys. Rev. D 62, 024027 – Published 27 June 2000 STOP** Gravitational waves emitted by (e) perturbed black holes or relativistic stars are dominated by (e) `quasinormal ringing', damped oscillations at (eb) single frequencies which are (=) characteristic of the underlying system. These quasinormal modes have been studied for (e) a long time, often with the intent of describing the time evolution of (e) a perturbation in terms of (e&eb) these modes in a way very similar to a normal-mode analysis. In this review, authors summarize how quasinormal modes are defined and computed. **Hans-Peter Nollert** finds why they have been regarded as closely analogous to normal modes, and discovers why they are actually quite different. Authors also discuss how quasinormal modes can be used in (eb) the analysis of a gravitational wave signal, such as will hopefully be detected in (e&eb) the near future. **Quasinormal modes: the characteristic `sound' of black holes and neutron stars Hans-Peter Nollert STOP** Perturbations of black holes, initially considered in the context of possible observations of astrophysical effects, have been studied for the past 10 years in string theory, brane-world models, and quantum gravity. Through the famous gauge/gravity duality, proper oscillations of (e) perturbed black holes, called quasinormal modes, allow (eb) for the description of (e) the hydrodynamic regime in (eb) the dual finite temperature field theory at (eb) strong coupling, which can be used to (e) predict the behavior of quark-gluon plasmas in (eb) the nonperturbative regime. On the other hand, the brane-world scenarios assume (eb) the existence of extra dimensions in nature, so that (e) multidimensional black holes can be formed in (eb) a laboratory experiment. All this stimulated active research in the field of perturbations of (e) higher-dimensional black holes and branes during recent years. In this review recent achievements on various aspects of black hole perturbations are discussed such as decoupling of variables in the perturbation equations, quasinormal modes (with special emphasis on various numerical and analytical methods of calculations), late-time tails, (e&eb) gravitational stability, (e&eb) anti–de Sitter/conformal field theory interpretation of (e) quasinormal modes, and holographic superconductors.

Authors also touch on state-of-the-art observational possibilities for (e) detecting quasinormal modes of black holes. DOI: http://dx.doi.org/10.1103/RevModPhys.83.793 © 2011 American Physical Society **Quasinormal modes of black holes: From astrophysics to string theory R. A. Konoplya and Alexander Zhidenko Rev. Mod. Phys. 83, 793 – Published 11 July 2011 STOP Vitor Cardoso and José P. S. Lemos** study the quasinormal modes (QNM) of electromagnetic and gravitational perturbations of a Schwarzschild black hole in (eb) an asymptotically anti–de Sitter (AdS) spacetime. Some of the electromagnetic modes do not (e) oscillate; they only decay, since they have (e) pure imaginary frequencies. The gravitational modes show (eb) peculiar features: the odd and even gravitational perturbations no longer have (e) the same characteristic quasinormal frequencies. There is a special mode for odd perturbations whose behavior differs completely from (e) the usual one in scalar and electromagnetic perturbations in AdS spacetime, but has a similar behavior to the Schwarzschild black hole in an asymptotically flat spacetime: the imaginary part of the frequency goes as 1/r+, where r+ is the horizon radius. **Vitor Cardoso and José P. S. Lemos** also investigate the small black-hole limit showing that the imaginary part of the frequency goes as r2+. These results are important to the AdS/CFT conjecture since, according to it, the QNM's describe the approach to equilibrium in the conformal field theory. Received 12 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevD.64.084017 **STOP Gianluca Calcagni** studies the ultraviolet complete non-relativistic theory recently proposed by Hořava. After introducing a Lifshitz scalar for a general background, authors analyze the cosmology of (e) the model in Lorentzian and Euclidean signature. Vacuum solutions are found and it is argued the existence of (e) non-singular bouncing profiles. They find a general qualitative agreement with both the picture of Causal Dynamical Triangulations and (e&eb) Quantum Einstein Gravity. However, inflation driven by (e) a Lifshitz scalar field on a classical background might not (e) produce a scale-invariant spectrum when (e) the principle of detailed balance is assumed. **Cosmology of the Lifshitz universe Gianluca Calcagni Published 25 September 2009 • Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Petr Hořavaa, , Edward Wittenb** propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as (=) the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. **Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Petr Hořava1** proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2. This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point" with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. **Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009)STOP Petr Hořava** presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the **Cotton tensor**. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's

general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short distances. **Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009STOP** Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of (e) quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from (e) rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. **Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova,c** address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable" version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. **On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** identify signatures of chaos in the dynamics of discord in a multiqubit system collectively modeled as (=) a quantum kicked top. Evolution of discord between any two qubits is quasiperiodic in regular regions, while (e) in chaotic regions the quasiperiodicity is lost. As the initial wave function is varied from (e) the regular regions to the chaotic sea, a contour plot of the time-averaged discord remarkably reproduces (eb) the structures of the classical stroboscopic map. **Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** also find surprisingly **opposite behavior** of two-qubit discord versus (e&eb) entanglement of the two qubits as measured by the concurrence. Results provide evidence of signatures of (e) chaos in dynamically generated discord. http://dx.doi.org/10.1103/PhysRevE.91.032906©2015 American Physical Society **Signatures of chaos in the dynamics of quantum discord Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose Phys. Rev. E 91, 032906 – Published 10 March 2015 STOP.** How do closed quantum many-body systems driven out of (e) equilibrium eventually achieve equilibration? And how do these systems thermalize, given that they comprise so many degrees of freedom? Progress in answering these—and related—questions has accelerated in recent years—a trend that can be partially attributed to success with experiments performing quantum simulations using ultracold atoms and trapped ions. **J. Eisert, M. Friesdorf & C. Gogolin** provide (eb) an overview of this progress, specifically in studies probing dynamical equilibration and (e&eb) thermalization of systems driven out of (e) equilibrium by quenches, ramps and periodic driving. In doing so, they also address (e&eb) topics such as the eigenstate thermalization hypothesis, typicality, transport, many-body localization and universality near phase transitions, as well as future prospects for quantum simulation. **Quantum many-body systems out of equilibrium J. Eisert, M. Friesdorf& C. Gogolin Nature Physics 11, 124–130 (2015) doi: 10.1038/nphys3215 Received 11**

**August 2014 Accepted 03 December 2014 published online 03 February 2015 STOP.** Quantum teleportation is (=) one of the most important protocols in quantum information. By exploiting the physical resource of entanglement, quantum teleportation serves as a key primitive in a variety of quantum information tasks and represents an important building block for quantum technologies, with a pivotal role in the continuing progress of quantum communication, quantum computing and quantum networks. **Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein** review the basic theoretical ideas behind quantum teleportation and its variant protocols. They focus on the main experiments, together with (e&eb) the technical advantages and disadvantages associated with (e&eb) the use of the various technologies, from photonic qubits and (e&eb) optical modes to atomic ensembles, trapped atoms, and (e&eb) solid-state systems. Analysing the current state-of-the-art, authors finish by discussing open issues, challenges and potential future implementations. Cite as: arXiv: 1505.07831 [quant-ph] (or arXiv: 1505.07831v1 [quant-ph] for this version) **Advances in Quantum Teleportation Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein (Submitted on 28 May 2015) STOP. Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are channels for which the coherent information is zero, but which nonetheless have capacity. Transmitting data reliably over (e&eb) noisy communication channels is one of the most important applications of (e) information theory, and is well understood for (e) channels modeled by classical physics. However, when quantum effects are involved, (eb) one do not know how to compute channel capacities. This is because the formula for (e) the quantum capacity involves (e&eb) maximizing the coherent information over (e&eb) an unbounded number of (e) channel uses. In fact, entanglement across channel uses can even increase the coherent information from zero to non-zero. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** study the number of channel uses necessary to detect positive coherent information. In all previous known examples, two channel uses already sufficed. It might be that only a finite number of channel uses are always sufficient. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are (=) channels for which the coherent information is (=) zero, but which nonetheless have (e) capacity. **Unbounded number of channel uses may be required to detect quantum capacity Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk Nature Communications 6, Article number: 6739 doi:10.1038/ncomms7739 Published 31 March 2015 STOP.** Characterizing the behaviour of strongly coupled quantum systems out of (e) equilibrium is a cardinal challenge for (e) both theory and experiment. With diverse applications ranging from the dynamics of the quark–gluon plasma to transport in novel states of quantum matter, establishing universal results and organizing principles out of equilibrium is crucial. **M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm** present a universal description of energy transport between quantum critical heat baths in arbitrary dimension. Current-carrying non-equilibrium steady state (NESS) is a Lorentz-boosted thermal state. In the context of gauge/gravity duality this reveals an intimate correspondence between far-from-equilibrium transport and black hole uniqueness theorems. Authors provide analytical expressions for (e) the energy current and the generating function of (e) energy current fluctuations, together with (e&eb) predictions for experiment. **Energy flow in (eb) quantum critical systems far from (e) equilibrium M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm Nature Physics 11, 509–514 (2015) doi: 10.1038/nphys3320: Published online 04 May 2015 STOP.** Properties of systems near quantum critical points (QCPs) have been studied extensively. A QCP is (=) a point across which the symmetry of (e) the ground state of a quantum system changes in (eb) a fundamental way; such a point can be accessed by (e) changing some parameter, say $\lambda$, in the Hamiltonian governing the system. The change in the ground state across (e&eb) a QCP is mediated by (e) quantum fluctuations. Unlike conventional thermal critical points, thermal fluctuations do not (e) play a crucial role in such transitions. Similar to its thermal counterparts, the low-energy physics near (e&eb) a QCP is associated with a number of critical exponents which characterize the universality class of such a transition. Among these exponents, the dynamical critical exponent z provides (eb) the signature of the relative scaling of space and time at (eb) the

transition and has (e) no counterpart in thermal phase transitions. The other exponent which is going to be important for (e) the purpose of this review is (=) the well-known correlation length exponent ν. These exponents are formally defined as follows. As we approach the critical point at λ=λc, the correlation length diverges as ξ∼|λ−λc|−ν, while (e) the gap between the ground state and first excited state vanishes as (e) ΔE∼ξ−z∼|λ−λc|zν. Exactly at the critical point λ=λc, the energy of the low-lying excitations vanishes at (eb) some wave number k 0 as ω∼|k−k0|z. The critical exponents are independent of (e) the details of the microscopic Hamiltonian; they depend only on a few parameters such as the dimensionality of the system and the symmetry of the order parameter. These features render (eb) the low-energy equilibrium physics of a quantum system near a QCP truly universal. **Quantum Quenching, Annealing and Computation Lecture Notes in Physics Volume 802, 2010, pp 21-56 Date: 08 Mar 2010 Non-equilibrium Dynamics of Quantum Systems: Order Parameter Evolution, Defect Generation, and Qubit Transfer S. Mondal, D. Sen, K. Sengupta STOP Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl** propose a class of qubit networks that admit (e) the perfect state transfer of (e) any quantum state in a fixed period of time. Unlike many other schemes for (e) quantum computation and communication, these networks do not (e) require qubit couplings to be switched on and off. When restricted to (e) N-qubit spin networks of identical qubit couplings, they show that (eb) 2log3N is the maximal perfect communication distance for (e) hypercube geometries. Moreover, if one allows fixed but different couplings between the qubits, then (eb) perfect state transfer can be achieved over (e&eb) arbitrarily long distances in a linear chain. **Perfect State Transfer in Quantum Spin Networks Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl Phys. Rev. Lett. 92, 187902 – Published 4 May 2004 STOP.** As with classical information processing, a quantum information processor requires (e) bits (qubits) that can be independently addressed (e&eb) and read out, long-term memory elements to store (e) arbitrary quantum states1, 2, and the ability to transfer (e&eb) quantum information through (e&eb) a coherent communication bus accessible to (e) a large number of qubits3, 4. Superconducting qubits made with (e&eb) scalable microfabrication techniques are (=) a promising candidate for the realization of (e) a large-scale quantum information processor5, 6, 7, 8, and 9. (For references please see the paper). Although these systems have successfully passed tests of (e) coherent coupling for (e) up to four qubits10, 11, 12, 13, communication of individual quantum states between (e&eb) superconducting qubits via (e&eb) a quantum bus has not yet been realized. Here, authors perform an experiment demonstrating the ability to coherently transfer quantum states between (e&eb) two superconducting Josephson phase qubits through a quantum bus. This quantum bus is (=) a resonant cavity formed by an open-ended superconducting transmission line of length 7 mm. After preparing an initial quantum state with (e&eb) the first qubit, this quantum information is transferred and stored (e&eb) as a nonclassical photon state of (e) the resonant cavity, then retrieved later by (e) the second qubit connected to (e&eb) the opposite end of the cavity. Beyond simple state transfer, these results suggest (eb) that a high-quality-factor superconducting cavity could also function as (=) a useful short-term memory element. The basic architecture presented here can be expanded, offering (eb) the possibility for the coherent interaction of (e&eb) a large number of superconducting qubits. **Nature 449, 438-442 (27 September 2007) | doi:10.1038/nature06124; Received 18 April 2007; Accepted 25 July 2007 Coherent quantum state storage and transfer between two phase qubits via a resonant cavity Mika A. Sillanpää1, Jae I. Park1 & Raymond W. Simmonds1 STOP. Chui-Ping Yang, Shih-I Chu, and Siyuan** present a scheme to achieve maximally entangled states, controlled (e&eb) phase-shift gate, and SWAP gate for (e) two superconducting-quantum-interference-devices (SQUID) qubits, by (e) placing SQUIDs in a microwave cavity. **Chui-Ping Yang, Shih-I Chu, and Siyuan** also show how to transfer quantum information from one SQUID qubit to (e&eb) another. In this scheme, no transfer of quantum information between (e&eb) the SQUIDs and the cavity is required, the cavity field is (=) only virtually excited and thus the requirement on the quality factor of the cavity is (=) greatly relaxed. http://dx.doi.org/10.1103/PhysRevA.67.042311 **Possible realization of entanglement, logical gates, and quantum-information transfer with (e&eb) superconducting-quantum-interference-device qubits in cavity QED Chui-Ping Yang, Shih-I Chu, and Siyuan Han Phys. Rev. A 67, 042311 – Published 17 April 2003 STOP. L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** explore the capability of spin-1/2 chains to act (e&eb) as quantum channels for (e) both teleportation and transfer of

(e) qubits. Exploiting (e) the emergence of (e) long-distance entanglement in (eb) low-dimensional systems [Phys. Rev. Lett. 96, 247206 (2006)], here **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** show (eb) how to obtain high communication fidelities between (e&eb) distant parties. An investigation of protocols of teleportation and (e&eb) state transfer is presented, in (eb) the realistic situation where (e) temperature is included. Basing setup on (eb) antiferromagnetic rotationally invariant systems, both protocols are represented by (e) pure depolarizing channels. **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** propose a scheme where channel fidelity close to 1 can be achieved on (e&eb) very long chains at moderately small temperature. **Qubit Teleportation and (e&eb) Transfer across Antiferromagnetic Spin Chains L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia Phys. Rev. Lett. 99, 060401 – Published 6 August 2007 STOP. David Petrosyan, Guy Bensky, Gershon Kurizki, Igor Mazets, Johannes Majer, and Jörg Schmiedmayer P** examine the possibility of coherent reversible information transfer between solid-state superconducting qubits and (e&eb) ensembles of ultracold atoms. Strong coupling between these systems is mediated by (e&eb) a microwave transmission line resonator that interacts near resonantly with (e&eb) the atoms via (e&eb) their optically excited Rydberg states. Solid-state qubits can then be used to (e) implement rapid quantum logic gates, while (e) collective metastable states of (e) the atoms can be employed for (e) long-term storage and (e&eb) optical readout of quantum information. **Reversible state transfer between superconducting qubits and (e&eb) atomic ensembles David Petrosyan, Guy Bensky, Gershon Kurizki, Igor Mazets, Johannes Majer, and Jörg Schmiedmayer Phys. Rev. A 79, 040304(R) – Published 14 April 2009 STOP.** Quantum computers have (e) the capability of out-performing their classical counterparts for (e) certain computational problems1. Several scalable quantum-computing architectures have been proposed. An attractive architecture is a large set of (e) physically independent qubits arranged in three spatial regions where (1) the initialized qubits are (=) stored in (eb) a register, (2) two qubits are brought together to (e) realize a gate and (3) the readout of (e) the qubits is carried out2, 3. For a neutral-atom-based architecture, a natural way to connect these regions is to use (e) optical tweezers to move qubits within (e&eb) the system. **Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier** demonstrate the coherent transport of (e) a qubit, encoded on (e&eb) an atom trapped in (eb) a submicrometre tweezer, over (e&eb) a distance typical of the separation between atoms in (eb) an array of optical traps4, 5, 6. Furthermore, authors transfer a qubit between (e&eb) two tweezers, and show that this manipulation also preserves (e&eb) the coherence of the qubit. **Nature Physics 3, 696 - 699 (2007) Published online: 12 August 2007 | doi: 10.1038/nphys698 Subject Categories: Atomic and molecular physics | Optical physics | Quantum physics | Information theory and (e&eb) computation Two-dimensional transport and transfer of (e) a single atomic qubit in (eb) optical tweezers Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier STOP.** In optically controlled quantum computers it may be favorable to address (e&eb) different qubits using (e) light with different frequencies, since (e) the optical diffraction does not then limit the distance between (e&eb) qubits. Using qubits that are close to each other enable (erb) qubit-qubit interactions and (e&eb) gate operations that are strong and fast in comparison to (e&eb) qubit-environment interactions and decoherence rates. However, as qubits are addressed in frequency space, great care has to be taken when designing the laser pulses, so that they perform (e&eb) the desired operation on (eb) one qubit, without (e) affecting other qubits. Complex hyperbolic secant pulses have theoretically been shown to be excellent for (e) such frequency-addressed quantum computing [I. Roos and K. Molmer, Phys. Rev. A 69, 022321 (2004)]—e.g., for use in quantum computers based on optical interactions in rare-earth-metal-ion-doped crystals. The optical transition lines of (e) the rare-earth-metal-ions are (=) inhomogeneously broadened and therefore the frequency of (e) the excitation pulses can be used to (e) selectively address qubit ions that are (=) spatially separated by (e&eb) a distance much less than a wavelength. , **Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter demonstrate experimentally** frequency-selective transfer of (e) qubit ions between (e&eb) qubit states using (e) complex hyperbolic secant pulses. Transfer efficiencies better than 90% were obtained. Using (e) the complex

hyperbolic secant pulses it was also possible to create (eb) two groups of ions, absorbing (e) at specific frequencies, where (e) 85% of the ions at one of the frequencies were shifted out of (e) resonance with (e&eb) the field when (e) ions in the other frequency group were excited. This procedure of selecting (e&eb) interacting ions, called qubit distillation, was carried out (e&eb) in preparation for two-qubit gate operations in (eb) the rare-earth-metal-ion-doped crystals. The techniques for frequency-selective state-to-state transfer developed here may be also useful also for (e) other quantum optics and quantum information experiments in (eb) these long-coherence-time solid-state systems. **Experimental demonstration of efficient and selective population transfer and qubit distillation in a rare-earth-metal-ion-doped crystal Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter Phys. Rev. A 71, 062328 – Published 23 June 2005 STOP. Martin Leijnse and Karsten Flensberg** propose a method to coherently transfer (e&eb) quantum information, and to create (eb) entanglement, between topological qubits and (e&eb) conventional spin qubits. Suggestion uses (e) gated control to transfer an electron (spin qubit) between (e&eb) a quantum dot and edge Majorana modes in (eb) adjacent topological superconductors. Because of the spin polarization of (e) the Majorana modes, the electron transfer translates spin superposition states into (e&eb) superposition states of the Majorana system, and vice versa. It is also shown how a topological superconductor can be used to (e) facilitate long-distance quantum information transfer (e&eb) and entanglement between (e&eb) spatially separated spin qubits. **Quantum Information Transfer between Topological and (e&eb) Spin Qubit Systems Martin Leijnse and Karsten Flensberg Phys. Rev. Lett. 107, 210502 – Published 18 November 2011 http://dx.doi.org/10.1103/PhysRevLett.107.210502 STOP. Gogyan** proposes a method that enables efficient conversion of the quantum information frequency between (e&eb) different regions of a spectrum of light based on (e) recently demonstrated (eb) strong parametric coupling between (e&eb) two narrow-band single-photon pulses propagating in (eb) a slow-light atomic medium [N. Sisakyan and Yu. Malakyan, Phys. Rev. A, 75, 063831 (2007)]. It is shown that an input qubit at (eb) telecom wavelength is transformed into (e&eb) another at a visible domain in (eb) a lossless and shape-conserving manner while (e) keeping the initial quantum coherence and (e&eb) entanglement. These transformations can be realized with quantum efficiency close to (e) its maximum value. http://dx.doi.org/10.1103/PhysRevA.81.024304 **Qubit transfer between (e&eb) photons at telecom and visible wavelengths in (eb) a slow-light atomic medium A. Gogyan Phys. Rev. A 81, 024304 – Published 25 February 2010 STOP. N. Cleland and M. R. Geller** propose a quantum computing architecture based on (e) the integration of nanomechanical resonators with (e&eb) Josephson-junction phase qubits. Resonators are GHz-frequency, dilatational disk resonators, which couple to (e&eb) the junctions through (e&eb) a piezoelectric interaction. System is analogous to (e) a collection of tunable few-level atoms (the Josephson junctions) coupled to (e&eb) one or more electromagnetic cavities (the resonators). Architecture combines desirable features of solid-state and optical approaches and may make (eb) quantum computing possible in a scalable, solid-state environment. **Superconducting Qubit Storage and Entanglement with (e&eb) Nanomechanical Resonators A. N. Cleland and M. R. Geller Phys. Rev. Lett 93, 070501 – Published 10 August 2004 STOP. D. N. Matsukevich, A. Kuzmich** report on the coherent quantum state transfer from (e) a two-level atomic system to a single photon Entanglement between (e&eb) a single photon (signal) and a two-component ensemble of (e) cold rubidium atoms is used to (e) project the quantum memory element (the atomic ensemble) onto (e&eb) any desired state by measuring (eb) the signal in a suitable basis. Atomic qubit is read out by (e) stimulating directional emission of (e) a single photon (idler) from (e) the (entangled) collective state of (e) the ensemble. Faithful atomic memory preparation and readout are verified by (e) the observed correlations between (e&eb) the signal and (e&eb) the idler photons. These results enable (eb) implementation of (e) distributed quantum networking. **Science 22 October 2004: Vol. 306 no. 5696 pp. 663-666 DOI: 10.1126/science.1103346 Quantum State Transfer Between Matter and (e&eb) Light D. N. Matsukevich, A. Kuzmich STOP. Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi** investigate the most general conditions under (e&eb) which a finite ferromagnetic long-range interacting (e&eb) spin chain achieves (eb) unitary fidelity and the shortest transfer time in transmitting (e&eb) an unknown input qubit. A deeper insight into system dynamics, allows (eb) to identify an ideal system involving (e&eb) sender and receiver only. However, this two-spin ideal chain is unpractical due to (e) the rapid decrease of (e) the coupling

strength with (e&eb) the distance. Therefore authors propose an optimization scheme for (e) approaching the ideal behavior, while keeping the interaction strength still reasonably high. The procedure is (=) scalable with the size of the system and straightforward to implement. **Perfect state transfer in (eb) long-range interacting spin chains Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi Phys. Rev. A 78, 022325 – Published 18 August 2008 STOP. L Fedichkin, M Yanchenko and K. A Valiev** investigated the use of (e) quantum bits (qubits) - semiconductor quantum dots containing (e) one electron and each consisting of (e) two tunnel-connected parts - as basic elements of the quantum computer. Numerical solution of a Schrödinger equation taking account of (e) the Coulomb field of (e) adjacent electrons shows (eb) that in such structures (e&eb)the realization of (e) a full set of basic logic operations, which are necessary for (e) fulfillment of quantum computations, is (=) possible. Durations of one- and two-qubit operations versus (e&eb) qubit geometry are obtained. Decoherence rates due to (e) spontaneous emission of (e) phonons and acoustic phonons (both piezoelectric and deformation) are evaluated. Analysis of these rates shows (eb) the proposed qubit to be (eb) coherent enough to work for (e) an unlimited time. **L Fedichkin et al 2000 Nanotechnology 11 387 doi:10.1088/0957-4484/11/4/339 Coherent charge qubits based on GaAs quantum dots with a built-in barrier L Fedichkin, M Yanchenko and K A Valiev ©2015 American Physical Society STOP.** In recent years **S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P have** investigated the fidelity of (e) the quantum state transfer (QST) of two qubits by means of (e) an arbitrary spin-12 network, on (e&eb) a lattice of any dimensionality. Under the assumptions that the network Hamiltonian preserves (eb) the magnetization and that a fully polarized initial state is taken for (e) the lattice, they obtain (eb) a general formula for the average fidelity of (e) the two qubits QST, linking it to (e&eb) the one- and two-particle transfer amplitudes of (e) the spin excitations among the sites of (e&eb) the lattice. **S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P** apply this formalism to (e&eb) a 1D spin chain with (e&eb) XX-Heisenberg type nearest-neighbour interactions adopting (e&eb) a protocol that is a generalization of the single qubit one proposed in Paganelli et al. [Phys. Rev. A 87, 062309 (2013)]. They report that a high-quality two qubit QST can be achieved provided one can control (e&eb) the local fields at sites near the sender and receiver. Under such conditions, they obtain (eb) an almost perfect transfer in a time that scales (e&eb) either linearly or, depending on the spin number, quadratically with (e&eb) the length of the chain. **Transfer of arbitrary two-qubit states via (e&eb) a spin chain S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina Phys. Rev. A 91, 042321 – Published 16 April 2015 http://dx.doi.org/10.1103/PhysRevA.91.042321 ©2015 American Physical Society STOP.** Combining techniques of cavity quantum electrodynamics (e&eb) quantum measurement, and (e&eb) quantum feedback, Norbert **Kalb, Andreas Reiserer, Stephan Ritter, and Gerhard Rempe** realize the heralded transfer of (e) a polarization qubit from a photon onto (e&eb) a single atom with 39% efficiency and 86% fidelity. The reverse process, namely, qubit transfer from (e) the atom onto a given photon, is demonstrated with 88% fidelity and an estimated efficiency of up to 69%. In contrast to previous work based on (e) two-photon interference, scheme is (=) robust against photon arrival-time jitter and achieves (eb) much higher efficiencies. Thus, it constitutes (eb) a key step toward the implementation of (e) a long-distance quantum network. Journal reference: Phys. Rev. Lett. 114, 220501 (2015) DOI:10.1103/PhysRevLett.114.**220501 Cite as:arXiv:1503.06709 [quant-ph] (or arXiv:1503.06709v2** [quant-ph] for this version) **Heralded Storage of a Photonic Quantum Bit in a Single Atom Norbert Kalb, Andreas Reiserer, Stephan Ritter, Gerhard Rempe 2 Jun 2015 STOP.** Current high-throughput DNA sequencing technologies enable (eb) acquisition of billions of data points through (e&eb) which myriad biological processes can be interrogated, including (e) genetic variation (e&eb) chromatin structure (e&eb) gene expression patterns, small RNAs and protein–DNA interactions. **Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker** describe the MethylC-sequencing (MethylC-seq) library preparation method, a 2-d protocol that enables (eb) the genome-wide identification of cytosine DNA methylation states at (eb) single-base resolution. The technique involves (e&eb) fragmentation of genomic DNA followed by (e) adapter ligation, bisulfite conversion and (e&eb) limited amplification using (e) adapter-specific PCR primers in preparation for (e) sequencing. To date, this protocol has been successfully applied to (e&eb) genomic DNA isolated from (e) primary cell culture, (e&eb) sorted cells and (e&eb) fresh tissue from over (e&eb) a thousand plant and animal

samples. **NATURE PROTOCOLS | PROTOCOL MethylC-seq library preparation for base-resolution whole-genome bisulfite sequencing Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker Nature Protocols 10, 475–483 (2015) doi:10.1038/nprot.2014.114 Published online 18 February 2015 STOP.** Starting from a product initial state, equal-time correlations in (eb) nonrelativistic quantum lattice models propagate (e&eb) within (eb) a lightcone-like causal region. The presence of entanglement in (eb) the initial state can modify (e&eb) this behavior, enhancing and accelerating (eb, eb+) the growth of correlations. **Michael Kastner** gives a quantitative description, in the form of Lieb-Robinson-type bounds on (e&eb) equal-time correlation functions, of (e) the interplay of dynamics vs.(e&eb) initial entanglement in quantum lattice models out of (e) equilibrium. **Michael Kastner** tests the bounds against (e&eb) model calculations, and also Cite as: arXiv: 1507.00529 [quant-ph] (or arXiv: 1507.00529v1 [quant-ph] for this version) **Entanglement-enhanced spreading of (e) correlations Michael Kastner (Submitted on 2 Jul 2015) STOP.** A scheme based on (e) coherent tunneling by (e) adiabatic passage (CTAP) of exchange-only spin qubit quantum states in a linearly arranged double quantum dot chain is demonstrated. Logical states for the qubit are defined by adopting the spin state of three electrons confined in a double quantum dot. The possibility to obtain gate operations entirely with electrical manipulations makes this qubit a valuable architecture in the field of quantum computing for the implementation of quantum algorithms. Effect of the external control parameters as well as the effect of the dephasing on the coherent tunneling in the chain is studied. During adiabatic transport, within a constant energy degenerate eigenspace, the states in the double quantum dots internal to the chain are not populated, while transient populations of the mixed states in the external ones are predicted. **Coherent tunneling by adiabatic passage of an exchange-only spin qubit in a double quantum dot chain E. Ferraro, M. De Michielis, M. Fanciulli, and E. Prati Phys. Rev. B 91, 075435 – Published 26 February 2015 STOP.** Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. **Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova, c** address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is Manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. **On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Petr Hořava** presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-

counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the Cotton tensor. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short distances. **Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009 STOP Petr Hořava1** proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2.This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point" with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. **Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009) STOP Petr Hořavaa, , Edward Wittenb** propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. **Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP** Hořava-Lifshitz theory of gravity with detailed balance is plagued by (e) the presence of a negative bare (or geometrical) cosmological constant which makes (eb) its cosmology clash with (e&eb) observations. Authors argue that adding the effects of (e) the large vacuum energy of quantum matter fields, this bare cosmological constant can be approximately compensated to account for (e) the small observed (total) cosmological constant ΛOBS, thus resulting in (eb) a self-contained model of gravity and particle physics. Even though authors donot cannot address the fine-tuning problem in this way, they establish a relation between the smallness of ΛOBS and (e&eb) the scale ℓUV at which dimension 4 corrections to (e) the Einstein gravity become (eb) significant for cosmology. This scale turns out to be (eb) ℓUV sime 5 ℓP for ΛOBS sime 0 and authors therefore argue that the smallness of ΛOBS guarantees that (eb) Lorentz invariance is broken only at (eb) very small scales. They provide a first rough estimation for the values of (e) the parameters of the theory μ and ΛW.Four-dimensional Quantum Einstein Gravity (QEG) is likely to be (=) an asymptotically safe theory which is applicable at (e&eb) arbitrarily small distance scales. On sub-planckian distances it predicts (eb) that spacetime is a fractal with (e&eb) an effective dimensionality of 2. The original argument leading to (eb) this result was based upon (e) the anomalous dimension of Newton's constant. In the present paper authors demonstrate that also the spectral dimension equals 2 microscopically, while it is equal to 4 on macroscopic scales. This result is an exact consequence of asymptotic safety and does not (e) rely on any truncation. Contact is made with recent Monte Carlo simulations. Authors calculate the renormalized vacuum energy of (e) a massless scalar field confined between (e&eb) two nearby parallel plates formed by (e) ideal uncharged conductors, placed tangentially to (e) the surface of a sphere with (e&eb) mass M and radius R. This study will take into account the static and spherically symmetric solution of (e) Hořava-Lifshitz gravity found by (e) Kehagias-Sfetsos (KS), in (eb) both weak field and infrared limits. A slight amplification of (e) the Casimir force between (e&eb) the conducting plates is found. Thermal corrections to (e&eb) the Casimir energy are analyzed. Based on (e) current Casimir effect measurements, a constraint on (e) the ω

parameter of KS metric is (=) also obtained. Received 1 October 2013 DOI: http://dx.doi.org/10.1103/PhysRevD.88.104035 **Hořava-Lifshitz gravity effects on Casimir energy in weak field approximation and infrared regime C. R. Muniz, V. B. Bezerra, and M. S. Cunha Phys. Rev. D 88, 104035 – Published 26 November 2013 STOP** Recently Hořava proposed a renormalizable gravity theory in (eb) four dimensions which reduces to (e&eb) Einstein gravity with (e&eb) a non-vanishing cosmological constant in (eb) IR but with improved UV behaviors. Here, I study an IR modification which breaks (e) ``softly'' the detailed balance condition in (eb) Hořava model and allows (eb) the asymptotically flat limit as well. Author obtains the black hole and cosmological solutions for (e) ``arbitrary'' cosmological constant that represent (eb) the analogs of the standard Schwarzschild-(A) dS solutions which can be (=) asymptotically (A)dS as well as flat and he discusses their thermodynamical properties Author also obtains solutions for FRW metric with (e&eb) an arbitrary cosmological constant. He studies its implication to (e&eb) the dark energy and finds (eb) that it seems to be consistent with current observational data. **The black hole and cosmological solutions in IR modified Hořava gravity Mu-In Park Published 29 September 2009 Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP** Let X be a compact complex manifold and E be a holomorphic vector bundle on X. Given a deformation (X,E) of the pair (X,E) over a small polydisk B centered at the origin, authors study the jumping phenomenon of the cohomology groups dimCHq(Xt,Et) near t=0. Authors show that there are precisely two cohomological obstructions to the stability of dimCHq (Xt, Et), which can be expressed explicitly in terms of the Maurer-Cartan element associated to the deformation. This generalizes the results of X. Ye. Subjects: Differential Geometry (math.DG); Algebraic Geometry (math.AG); Complex Variables (math.CV) Cite as: arXiv: 1601.06472 [math.DG] (or arXiv: 1601.06472v1 [math.DG] for this version) This article gives an exposition of the deformation theory for (e) pairs (X, E), where (e) X is a compact complex manifold and E is (=) a holomorphic vector bundle over X, adapting (e&eb) an analytic viewpoint \`{a} la Kodaira-Spencer. By introducing and exploiting an auxiliary differential operator, we derive the Maurer--Cartan equation and differential graded Lie algebra (DGLA) governing the deformation problem, and express them in terms of differential-geometric notions such as the connection and curvature of E, obtaining a chain level refinement of the classical results that the tangent space and obstruction space of the moduli problem are respectively given by the first and second cohomology groups of the Atiyah extension of E over X. As an application, we give examples where deformations of pairs are unobstructed Subjects: Differential Geometry (math.DG); Complex Variables (math.CV) Journal reference: Complex Manifolds 3 (2016), 16-40 DOI:10.1515/coma-2016-0002 Cite as: arXiv:1406.6753 [math.DG] (or arXiv:1406.6753v4 [math.DG] for this version) **A differential-geometric approach to deformations of pairs (X, E) Kwokwai Chan, Yat-Hin Suen and On the jumping phenomenon of dimCHq (Xt, Et) Kwokwai Chan, Yat-Hin Suen STOP** Authors study the dynamic service migration problem in mobile edge-clouds that host cloud-based services at the network edge. This offers the benefits of reduction in network overhead and latency but requires service migrations as user locations change over time. It is challenging to make these decisions in an optimal manner because of the uncertainty in node mobility as well as possible non-linearity of the migration and transmission costs. In this paper, we formulate a sequential decision making problem for service migration using the framework of Markov Decision Process (MDP). Our formulation captures general cost models and provides a mathematical framework to design optimal service migration policies. In order to overcome (e) the complexity associated with (e&eb) computing the optimal policy, we approximate the underlying state space by the distance between the user and service locations. We show that the resulting MDP is exact for uniform one-dimensional mobility while it provides a close approximation for uniform two-dimensional mobility with a constant additive error term. We also propose a new algorithm and a numerical technique for computing the optimal solution which is significantly faster in computation than traditional methods based on value or policy iteration. We illustrate the effectiveness of our approach by simulation using real-world mobility traces of taxis in San Francisco. Comments: in Proc. of IFIP Networking 2015Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1506.05261 [cs.DC] (or arXiv: 1506.05261v1 [cs.DC] for this version) **Dynamic Service Migration in Mobile Edge-Clouds Shiqiang Wang, Rahul Urgaonkar, Murtaza Zafer, Ting**

**He, Kevin Chan, Kin K. Leung STOP** Let kE denote the group algebra of an elementary abelian p-group of rank r over an algebraically closed field of characteristic p. We investigate the functors Fi from kE-modules of constant Jordan type to vector bundles on Pr−1(k), constructed by Benson and Pevtsova. For a kE-module M of constant Jordan type, we show that restricting the sheaf Fi(M) to a dimension s−1 linear subvariety of Pr−1(k) is equivalent to restricting M along a corresponding rank s shifted subgroup of kE and then applying Fi. In the case r=2, we examine the generic kernel filtration of M in order to show that Fi(M) may be computed on certain sub quotients of M whose Loewy lengths are bounded in terms of i. More precise information is obtained by applying similar techniques to the nth power generic kernel filtration of M. The latter approach also allows us to generalise our results to higher ranks r. Subjects: Representation Theory (math.RT) MSC classes: 16G10, 20C20 Cite as: arXiv: 1504.01994 [math.RT] (or arXiv: 1504.01994v1 [math.RT] for this version) **Modules of constant Jordan type, pullbacks of bundles and generic kernel filtrations Shawn Baland, Kenneth Chan STOP** Mobile micro-cloud is an emerging technology in distributed computing, which is aimed at providing seamless computing/data access to the edge of the network when a centralized service may suffer from poor connectivity and long latency. Different from the traditional cloud, a mobile micro-cloud is smaller and deployed closer to users, typically attached to a cellular basestation or wireless network access point. Due to (e) the relatively small coverage area of each basestation or access point, when a user moves across areas covered by different base stations or access points which are attached to (e&eb) different micro-clouds, issues of service performance and service migration become (=) important. In this paper, authors consider such migration issues. They model the general problem as (=) a Markov decision process (MDP), and show that, in the special case where (e) the mobile user follows(eb) a one-dimensional asymmetric random walk mobility model, the optimal policy for (e) service migration is (=) a threshold policy. Authors obtain the analytical solution for (e) the cost resulting from arbitrary thresholds, and then propose an algorithm for finding the optimal thresholds. The proposed algorithm is more efficient than standard mechanisms for solving MDPs. Comments: in Proc. of IEEE MILCOM 2014, Oct. 2014 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) DOI: 10.1109/MILCOM.2014.145 Cite as: arXiv: 1503.05141 [cs.DC] (or arXiv: 1503.05141v1 [cs.DC] for this version) **Mobility-Induced Service Migration in Mobile Micro-Clouds Shiqiang Wang, Rahul Urgaonkar, Ting He, Murtaza Zafer, Kevin Chan, Kin K. Leung STOP** Seamless computing and data access is enabled by the emerging technology of mobile micro-clouds (MMCs). Different from traditional centralized clouds, an MMC is typically connected directly to a wireless base-station and provides services to a small group of users, which allows users to have instantaneous access to cloud services. Due to the limited coverage area of base-stations and the dynamic nature of mobile users, network background traffic, etc., the question of where to place the services to cope with these dynamics arises. In this paper, we focus on dynamic service placement for MMCs. We consider the case where there is an underlying mechanism to predict the future costs of service hosting and migration, and the prediction error is assumed to be bounded. Our goal is to find the optimal service placement sequence which minimizes the average cost over a given time. To solve this problem, we first propose a method which solves for the optimal placement sequence for a specific look-ahead time-window, based on the predicted costs in this time-window. We show that this problem is equivalent to a shortest-path problem and propose an algorithm with polynomial time-complexity to find its solution. Then, we propose a method to find the optimal look-ahead window size, which minimizes an upper bound of the average cost. Finally, we evaluate the effectiveness of the proposed approach by simulations with real-world user-mobility traces. Comments: in Proc. of IEEE ICC 2015 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1503.02735 [cs.DC] (or arXiv: 1503.02735v1 [cs.DC] for this version) **Dynamic Service Placement for Mobile Micro-Clouds with Predicted Future Costs Shiqiang Wang, Rahul Urgaonkar, Kevin Chan, Ting He, Murtaza Zafer, Kin K. Leung STOP** Authors give a mathematically rigorous construction of Rozansky-Witten's 3-dimensional σ-model as a perturbative quantum field theory (QFT) by applying Costello's approach using the Batalin-Vilkovisky (BV) formalism. The quantization of our model is obtained via the technique of configuration

spaces. We also investigate the observable theory following the work of Costello-Gwilliam. In particular, we show that the cohomology of local quantum observables on a genus g handle body is given by H∗(X,(Λ∗TX)⊗g), where X is the target hyperk\"ahler manifold. We further give a mathematical definition of the partition function and prove that it coincides with the Rozansky-Witten invariants. Subjects: Quantum Algebra (math.QA); High Energy Physics - Theory (hep-th); Algebraic Geometry (math.AG); Differential Geometry (math.DG); Geometric Topology (math.GT) Cite as: arXiv:1502.03510 [math.QA] (or arXiv:1502.03510v1 [math.QA] for this version) **A mathematical foundation of Rozansky-Witten theory Kwokwai Chan, Naichung Conan Leung, Qin Li STOP Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2** study the perturbative dynamics of (e) noncommutative field theories on (eb) Script Rd, and find (eb) an intriguing mixing of the UV and the IR. High energies of virtual particles in loops produce (eb) non-analyticity at low momentum. Consequently, the low energy effective action is (=) singular at zero momentum even when (e) the original noncommutative field theory is (=) massive. Some of the nonplanar diagrams of (e) these theories are (=) divergent, but they interpret these divergences as (=) IR divergences and deal with (e&eb) them accordingly. Authors explain how this UV/IR mixing arises from (e) the underlying noncommutativity. This phenomenon is reminiscent of (e) the channel duality of (e) the double twist diagram in (eb) open string theory. **Noncommutative perturbative dynamics Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 Published 15 March 2000 • Journal of High Energy Physics, Volume 2000, JHEP02 (2000 STOP** Authors give a derivation of the gravitational Hamiltonian starting from (e) the Einstein - Hilbert action, keeping track of (e) all surface terms. This derivation can be applied to (e&eb) any spacetime that asymptotically approaches (eb) a static background solution. The surface term that arises in (eb) the Hamiltonian can be taken as (=) the definition of the `total energy', even for (e) spacetimes that are not (e) asymptotically flat. (In the asymptotically flat case, it agrees with (eb) the usual ADM energy.) They also discuss the relation between the Euclidean action and (e&eb) the Hamiltonian when (e) there are horizons of infinite area (e.g. acceleration horizons) as well as (e&eb) the usual finite area black hole horizons. Acceleration horizons seem to be (=) more analogous to (e) extreme than nonextreme black holes, since they find evidence that their horizon area is not (e) related to the total entropy **The gravitational Hamiltonian, action, entropy and surface terms S W Hawking and Gary T Horowitz Classical and Quantum Gravity, Volume 13, Number 6 http://dx.doi.org/10.1088/0264-9381/13/6/017 STOP** Recent observations of Type 1a supernovae indicating (eb) an accelerating universe have (e) once more drawn attention to the possible existence, at the present epoch, of (e) a small positive Λ-term (cosmological constant). In this paper authors review both observational and theoretical aspects of (e) a small cosmological Λ-term. Authors discuss the current observational situation focusing on cosmological tests of Λ including (e) the age of the universe, high redshift supernovae (e&eb) gravitational lensing (e&eb) galaxy clustering and (e&eb) the cosmic microwave background. They also review the theoretical debate surrounding Λ: the generation of Λ in models with (e&eb) spontaneous symmetry breaking and through(e&eb) quantum vacuum polarization effects — mechanisms which are known to give rise to (eb) a large value of Λ hence leading to (eb) the "cosmological constant problem." More recent attempts to generate (eb) a small cosmological constant at the present epoch using (e) either field theoretic techniques or by (e) modelling a dynamical Λ-term by scalar fields are also extensively discussed. Anthropic arguments favouring a small Λ-term are briefly reviewed. A comprehensive bibliography of recent work on Λ is provided. **ARUN SAHNI and ALEXEI STAROBINSKY, Int. J. Mod Phys. D 09, 373 (2000) DOI: 10.1142/S0218271800000542 THE CASE FOR A POSITIVE COSMOLOGICAL Λ-TERM STOP** Authors explore the observational adequacy of (e) a class of unified dark energy and dark matter (UDE/M) models with (e&eb) a fast transition. Constraints are set using (e) a combination of geometric probes— some low redshift ones and (e&eb) some high redshift ones (CMB-related included). The transition is phenomenologically modeled by (e) two different transition functions corresponding to (e&eb) a fast and to an ultrafast transition, respectively. They find that the key parameters governing (e&eb) the transition can be (=) well constrained, and from (e) the statistical point of view it follows (eb) that the models cannot be discarded when compared to (e&eb) LCDM. Authors find the intriguing result that standard or input parameters such as Ωm and Ωb are (=) far better constrained than in (eb) LCDM, and this is the case for (e) the derived or output parameter

measuring (eb) the deceleration value at present, q0. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043537 © 2016 American Physical Society **Cosmological constraints on fast transition unified dark energy and dark matter models Ruth Lazkoz, Iker Leanizbarrutia, and Vincenzo Salzano Phys. Rev. D 93, 043537 – Published 22 February 2016 STOP** Authors report on constraints to (e) the cosmological jerk parameter (j) and to possible variability of (e) the fine-structure constant (Δα/α) based on (e) stochastic quintessence models of (e) dark energy, discussed by Chongchitnan and Efstathiou [Phys. Rev. D 76, 043508 (2007)]. Authors confirm the results by (e) these authors in the sense that many viable solutions can be obtained, obeying (e&eb) current observational constraints in low redshifts. They add the observables j and Δα/α to (e&eb) this conclusion. However, they find peculiarities that may produce (eb) in the nearby universe, potential observational imprints in (eb) future cosmological data. Authors conclude, for redshifts z≲3, that (i) j(z) fluctuates due to (e&eb) the stochasticity of the models, reaching (eb) an amplitude of up to 5% relatively to (e&eb) the Λ cold dark matter model value (jΛCDM=1); and (ii) by contrasting two distinct ("extreme") types of solutions, variabilities in α(z), linked to (e&eb) a linear coupling (ζ) between the dark energy and (e&eb) electromagnetic sectors, are (eb) weakly dependent on redshift, for (e) couplings of the order |ζ|∼10−4, even for (e) large variations in the equation of state parameter at (eb) relatively low redshifts. Nonlinear couplings produce (eb) an earlier and steeper onset of (e) the evolution in Δα/α (z), but can still accommodate (e) the data for weak enough couplings. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043509 © 2016 American Physical Society **The Void Galaxy Survey: Star Formation Properties B. Beygu1,6,★, K. Kreckel2, J. M. van der Hulst1, T. H Jarrett3, R. Peletier1, R. van de Weygaert1, J. H. van Gorkom4 and M. A. Aragon-Calvo5 STOP Stochastic quintessence models: Jerk and fine-structure variability constraints Christine C. Dantas and André L. B. Ribeiro Phys. Rev. D 93, 043509 – Published 8 February 2016 STOP** Authors study the star formation properties of 59 void galaxies as part of the Void Galaxy Survey (VGS). Current star formation rates are derived from (e) $\rm {H\alpha} $ and recent star formation rates from (e) near-UV imaging. In addition, infrared 3.4 $\rm {\mu m}$, 4.6 $\rm {\mu m}$, 12 $\rm {\mu m}$ and 22 $\rm {\mu m}$ WISE emission is used as (=) star formation and mass indicator. Infrared and optical colours show that (eb) the VGS sample displays (eb) a wide range of dust and metallicity properties. They combine these measurements with (e&eb) stellar and $\rm {H\small {I}} $ masses to measure (eb) the specific SFRs ($\rm {SFR/M_ {*}} $) and (e&eb) star formation efficiencies ($\rm {SFR/M_ {H\small {I}}}$). Authors compare the star formation properties of their sample with (e&eb) galaxies in (eb) the more moderate density regions of (e) the cosmic web, 'the field'. They find that specific SFRs of (e) the VGS galaxies as (=) a function of stellar and $\rm {H\small {I}}$ mass are similar to (e&eb, =) those of the galaxies in these field regions. Their $\rm {SFR\alpha} $ is slightly elevated than (e) the galaxies in the field for (e) a given total $\rm {H\small {I}} $ mass. In the global star formation picture presented by (e) Kennicutt-Schmidt, VGS galaxies fall into (eb)the regime of low average star formation and correspondingly (e&eb) low $\rm {H\small {I}} $ surface density. Their mean $\rm {SFR\alpha /M_{H\small {I}}}$ and (e&eb) $\rm {SFR\alpha /M_{*}}$ are of (e) the order of $\rm {10^{-9.9}}$ $\rm {yr^{-1}}$. Authors conclude that while the large scale underdense environment must play (e&eb) some role in galaxy formation and growth through accretion, they find that even with respect to other galaxies in the more mildly underdense regions, the increase in star formation rate is (=) only marginal. Key words galaxies: star formation galaxies: formation —galaxies: structure large-scale structure of universe© 2016 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Authors study the formation and evolution of (e) filamentary configurations of dark matter haloes in (eb) voids. Extant investigation uses (e) the high-resolution Λ cold dark matter simulation Cosmo Grid to (e) look for (e) void systems resembling the VGS_31 elongated system of (e) three interacting galaxies that was recently discovered by (e) the Void Galaxy Survey inside (eb) a large void in the Sloan Digital Sky Survey galaxy redshift survey. H I data revealed these galaxies to be embedded in (eb) a common elongated envelope, possibly embedded in (eb) intravoid filament. In the Cosmo Grid simulation authors look for systems similar to (e) VGS_31 in mass, size and environment. They find (eb) a total of eight such systems. For these systems, authors study the distribution of (e) neighbour haloes, the assembly and evolution of (e) the main haloes and (e&eb) the dynamical evolution of the haloes, as well as the evolution of (e) the large-scale

structure in which the systems are (=) embedded. The spatial distribution of the haloes follows that of (e&eb) the dark matter environment. Authors find that VGS_31-like systems have (e) a large variation in formation time, having (e) formed between (e&eb) 10 Gyr ago and the present epoch. However, the environments in which the systems are embedded evolved to (e) resemble each other substantially. Each of the VGS_31-like systems is embedded in (eb) an intravoid wall, that no later than z = 0.5 became the only prominent feature in its environment. While parts of the void walls retain (eb) a rather featureless character, authors find that around half of them are marked by (e) a pronounced and rapidly evolving substructure. Five haloes find themselves in a tenuous filament of a few h−1 Mpc long inside (eb) the intravoid wall. Finally, authors compare the results to (e&eb) observed data from VGS_31. Study implies that (eb) the VGS_31 galaxies formed in (eb) the same (proto) filament, and did not (e) meet just recently. The diversity amongst the simulated halo systems indicates (eb) that VGS_31 may not (e) be typical for (e) groups of galaxies in voids. Key words galaxies: formation galaxies: interactions cosmology (e&eb) theory dark matter large-scale structure of Universe © 2013 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society **Assembly of filamentary void galaxy configurations Steven Rieder1, 2, Rien van de Weygaert3, Marius Cautun3, Burcu Beygu3 and Simon Portegies Zwart1First published online August 20, 2013 STOP** Authors measure the star formation properties of (e) two large samples of galaxies from (e) the SDSS in large-scale cosmic voids on (e&eb) time scales of 10 Myr and 100 Myr, using (e) Hα emission line strengths and GALEX FUV fluxes, respectively. The first sample consists of 109,818 optically selected galaxies. They find that void galaxies in this sample have (e) higher specific star formation rates (SSFRs; star formation rates per unit stellar mass) than (e&eb) similar stellar mass galaxies in (eb) denser regions. The second sample is (=) a subset of (e) the optically selected sample containing (e) 8070 galaxies with (e&eb) reliable HI detections from ALFALFA. For the full HI detected sample, SSFRs do not (e) vary systematically with (e&eb) large-scale environment. However, investigating only the HI detected dwarf galaxies reveals (eb) a trend towards higher SSFRs in (eb) voids. Furthermore, authors estimate the **star formation** rate per unit HI mass (known as the star formation efficiency; SFE) of a galaxy, as (=) a **function of environment**. For the overall HI detected population, authors notice no (e) environmental dependence. (AC=DC=0)Limiting the sample to dwarf galaxies again reveals (eb) a trend towards higher SFEs in (eb) voids. These results suggest (eb) that void environments provide (eb) a nurturing environment for (e) dwarf galaxy evolution allowing (eb) for higher specific star formation rates and efficiencies. Comments: submitted to ApJ 12-2015 Subjects: Astrophysics of Galaxies (astro-ph.GA) Cite as: arXiv: 1601.04092 [astro-ph.GA]**.** Can Grover's algorithm speed up (eb+) search of a physical region—for example a 2-D grid of size √ n × √ n? The problem is that √n time seems to be needed for (e) each query, just to move amplitude across the grid. Here authors show that this problem can be surmounted (e) refuting a claim to (e) the contrary by Benioff. In particular, authors show how to search a d-dimensional hypercube in (eb) time O (√n) for d ≥ 3, or O (√nlog5/2n) for d = 2. More generally, authors introduce a model of (e) quantum query complexity on graphs, motivated by (e) fundamental physical limits on (e&eb) information storage, particularly the holographic principle from (e) black hole thermodynamics. Results in this model include (e) almost-tight upper and lower bounds for (e) many search tasks; a generalized algorithm that works for (e) any graph with good expansion properties, not just hypercubes; and relationships among several notions of 'locality' for (e) unitary matrices acting on graphs. As an application of results, authors give an O ( √ n)-qubit communication protocol for (e) the disjointness problem, which improves (eb+) an upper bound of Høyer and de Wolf and matches (e&eb) a lower bound of Razborov. ACM Classification: F.1.2, F.1.3 AMS Classification: 81P68, 68Q10 Key words and phrases: Quantum computing, (e&eb) Grover search, amplitude amplification, (e&eb) quantum communication complexity, disjointness, lower bounds ∗This work was mostly done while the author was a **PhD student at UC Berkeley, supported by an NSF Graduate Fellowship and by ARO grant DAAD19-03-1-0082.†Supported by an IQC University Professorship and by CIAR**. STOP Grover's quantum search algorithm provides (eb) a way to speed up (eb) combinatorial search, but is not (e) directly applicable to searching (e&eb) a physical database. Nevertheless, Aaronson and Ambainis showed (eb) that a database of N items laid out in (eb) d spatial dimensions can be searched in time of (e) order N−−√ for d>2, and in time of order N−−√poly (logN) for d=2. Authors consider an alternative

search algorithm based on (e) a continuous-time quantum walk on (e&eb) a graph. The case of the complete graph gives (eb) the continuous-time search algorithm of (e) Farhi and Gutmann, and other previously known results can be used to show that (eb) N−−√ speedup can also be achieved on (e&eb) the hypercube. Authors show that full N−−√ speedup can be achieved on (eb) a d-dimensional periodic lattice for d>4. In d=4, the quantum walk search algorithm takes time of (e) order N−−√poly (logN), and (e&eb) in d<4, the algorithm does not (e) provide substantial speedup. Received 16 June 2003 DOI:http://dx.doi.org/10.1103/PhysRevA.70.022314 **Spatial search by quantum walk Andrew M. Childs and Jeffrey Goldstone Phys. Rev. A 70, 022314 – Published 23 August 2004 STOP** Quantum walks are considered in (eb) a one-dimensional random medium characterized by (e) static or dynamic disorder. Quantum interference for (e) static disorder can lead to (eb) Anderson localization which completely hinders (e) the quantum walk and it is contrasted with (e&eb) the decoherence effect of (e) dynamic disorder having (e) strength W, where (e) a quantum to (e&eb) classical crossover at time tc∝W−2 transforms (e&eb) the quantum walk into (e&eb) an ordinary random walk with (e&eb) diffusive spreading. Authors demonstrate this localization and decoherence phenomena in (eb) quantum carpets of the observed time evolution, authors relate results to (e&eb) previously studied models of (e) decoherence for quantum walks, and examine (e&eb) in detail a dimer lattice which corresponds to(e&eb) a single qubit subject to randomness. Received 11 August 2007 DOI:http://dx.doi.org/10.1103/PhysRevA.77.022302 **Quantum walks on a random environment Yue Yin, D. E. Katsanos, and S. N. Evangelou Phys. Rev. A 77, 022302 – Published 5 February 2008 STOP** Feedback quantum control, where (e) the sample `teaches' a computer-controlled arbitrary light form generator to find (e&eb) the optimal light field, is experimentally demonstrated for (e) a molecular system. Femtosecond pulses tailored by a computer-controlled acousto-optic pulse shaper excite (eb) fluorescence from laser dye molecules in (eb) solution. Fluorescence and laser power are (=) monitored, and the computer uses (e) the experimental data and a genetic algorithm to (e) optimize population transfer from ground to (e&eb) first excited state. Both efficiency (the ratio of excited state population to (e&eb) laser energy) and (e&eb) effectiveness (total excited state population) are optimized. Potential use as an `automated theory tester' is discussed. Copyright © 1997 Elsevier Science B.V. All rights reserved. **Chemical Physics Letters Volume 280, Issues 1–2, 28 November 1997, Pages 151–158 Feedback quantum control of molecular electronic population transfer Christopher J Bardeena, Vladislav V Yakovleva, Kent R Wilsona, Scott D Carpenterb, Peter M Weberc, Warren S Warrend STOP** Quantum robots are described as (=) mobile quantum computers and ancillary systems that move in and interact with (e&eb) arbitrary environments. Their dynamics is given as (=) tasks which consist of (e) sequences of (e) alternating computation and (e&eb) action phases. A task example is considered in which a quantum robot searches (e&eb) a space region to find the location of a system. The possibility that the search can be (=) more efficient than a classical search is examined by (e) considering use of (e) Grover's Algorithm to process the search results. For reversible searches this is (=) problematic for two reasons. One is (=) the removal of entanglements generated by (e) the search process. The other is that even if the entanglement can be avoided, (eb) the search process in 2-D space regions is (=) no more efficient than a classical search. However, quantum searches of space regions with (e&eb) 3 or more dimensions are (=) more efficient than classical searches. Reasons why quantum robots are interesting independent of these results are briefly summarized. Subjects: Quantum Physics (quant-ph) Journal reference: AMS Contemporary Math Series, Vol 305, (2002) Cite as: arXiv: quant-ph/0003006 (or arXiv: quant-ph/0003006v2 for this version) **Space Searches with a Quantum Robot Paul Benioff (Physics Division, Argonne National Lab, Argonne, IL) (Submitted on 1 Mar 2000 (v1), last revised 26 Jun 2001 (this version, v2))STOP** The relationship between the foundations of mathematics and (e&eb) physics is a topic of of much interest. This paper continues this exploration by (e) examination of the effect of space and time dependent number scaling on (e&eb) theoretical descriptions of (e) some physical and geometric quantities. Fiber bundles provide (eb) a good framework to introduce (eb) a space and time or space time dependent number scaling field. The effect of the scaling field on (e&eb) a few nonlocal physical and geometric quantities are described. The effect on gauge theories is to introduce (eb) a new complex scalar field into (e&eb) the derivatives appearing in Lagrangians. U (1) invariance of Lagrangian terms does not (e) affect the real part of the scaling field. For this field, any mass is possible. The scaling

field is also shown to affect (e&eb) quantum wave packets and (e&eb) path lengths, and (e&eb) geodesic equations even on (e&eb) flat space. Scalar fields described so far in physics, are (=) possible candidates for (e) the scaling field. The lack of direct evidence for (e) the field in physics restricts the scaling field in that the gradient of (e) the field must be close to (e) zero in (eb) a local region of (e) cosmological space and time. There are no restrictions outside (e) the region. It is also seen that the scaling field does not (e) affect comparisons of computation or measurements outputs with (e&eb) one another. However it does affect (e&eb) the assignment of numerical values to the outputs of computations or measurements. These are needed because (e) theory predictions are in terms of (e&eb) numerical values. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum Information Processing, 15(3), 1081-1102, (2016) DOI: 10.1007/s11128-015-1084-z Cite as: arXiv: 1508.01732 [math-ph] (or arXiv: 1508.01732v1 [math-ph] for this version) **Space and time dependent scaling of numbers in mathematical structures: Effects on physical and geometric quantities Paul Benioff STOP** The purpose of this paper is to put the description of number scaling and its effects on (e&eb) physics and geometry on (e&eb) a firmer foundation, and to make it more understandable. A main point is that two different concepts, number and (e&eb) number value are combined in the usual representations of (e) number structures. This is valid as (e) long as just one structure of each number type is being considered. It is not valid when (e) different structures of each number type are (=) being considered. Elements of base sets of (e) number structures, considered by (e) themselves, have no meaning. They acquire meaning or value as (e) elements of a number structure. Fiber bundles over (e&eb) a space or space time manifold, M, are described. The fiber consists of (e) a collection of many real or complex number structures and (e&eb) vector space structures. The structures are parameterized by (e) a real or complex scaling factor, s. A vector space at a fiber level, s has (e) as scalars, real or complex number structures at (eb) the same level. Connections are described that relate scalar and (e&eb) vector space structures at both neighbor M locations and (e&eb) at neighbor scaling levels. Scalar and vector structure valued fields are (=) described and covariant derivatives of these fields are (=) obtained. Two complex vector fields, each with (e&eb) one real and one imaginary field, appear, with one complex field associated with (e&eb) positions in M and the other with (e&eb) position dependent scaling factors. A derivation of (e) the covariant derivative for (e) scalar and vector valued fields gives (eb) the same vector fields. The derivation shows (eb) that the complex vector field associated with (e&eb) scaling fiber levels is (=) the gradient of a complex scalar field. Use of these results in gauge theory shows (eb) that the imaginary part of the vector field associated with (e&eb) M positions acts (e&eb) like the electromagnetic field. The physical relevance of the other three fields, if any, is (=) not known. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Proceedings SPIE STA conference, Vol. 9500, 9500 0X, 2015 DOI: 10.1117/12.2176080 Cite as: arXiv: 1503.05600 [math-ph] (or arXiv: 1503.05600v1 [math-ph] for this version) **Principal fiber bundle description of number scaling for scalars and vectors: Application to gauge theory Paul Benioff STOP** This work uses (e) fiber bundles as a framework to describe some effects of number scaling on (e&eb) gauge theory and some geometric quantities. A description of number scaling and fiber bundles over (e&eb) a flat space time manifold, M, is followed by (e&eb) a description of gauge theory. A fiber at point x of M contains (e) a pair of scaled complex number and (e&eb) vector space structures, Ccx×Vcx for each c in GL(1,C). A space time dependent scalar field, g, determines, (eb) for each x, the scaling value of (e) the vector space structures that contain (e) the values of a vector valued matter field at (eb) x. The vertical components of (e) connections between neighboring fibers are taken to be (=) the gradient field A(x)+iB(x), of g. Abelian gauge theory for (e) these fields gives (eb) the result that B is (=) massless and no mass restrictions for (e) A. Addition of an electromagnetic field dies not (e) change these results. In the Mexican hat Higgs mechanism B combines with (e&eb) a Goldstone boson to create massive vector bosons, the photon field, and the Higgs field. For geometric quantities the fiber bundle is (=) a tangent bundle with (e&eb) a fiber at point x containing (e) scaled pairs, Rrx×Trx of real number and (e&eb) tangent space structures for (e) each x and and nonnegative real r. B is zero everywhere. The A field affects (e&eb) path lengths and the proper times of (e) clocks along paths. It also appears in (eb) the geodesic equation. The lack of physical evidence for (e) the

gradient field, A(x)+iB(x) means (eb) that it either couples very weakly to (e&eb) matter fields or that it is close to zero for all x in (eb) a local region of cosmological space and (e&eb) time. It says nothing about (e&eb) the values outside the local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Quantum Studies: Mathematics and Foundations, 2, 289-313, 2015 DOI: 10.1007/s40509-015-0049-3 Cite as: arXiv: 1412.1493 [math-ph] (or arXiv: 1412.1493v2 [math-ph] for this version) **Fiber bundle description of number scaling in gauge theory and geometry Paul Benioff STOP** Stronger foundations for (e) earlier work on the effects of (e) number scaling, and local mathematics is described. Emphasis is placed on the effects of scaling on (e&eb) coordinate systems. Effects of scaling are represented by (e) a scalar field, θ, which appears (eb) in gauge theories as (=) a spin zero boson. Gauge theory considerations led to (eb) the concept of local mathematics, as expressed through (e&eb) the use of universes, as collections of (e) local mathematical systems at (eb) each point, x, of a space time manifold, M. Both local and global coordinate charts are (=) described. These map M into (e&eb) either local or global coordinate systems within (eb) a universe or between (e&eb) universes, respectively. The lifting of global expressions of (e) nonlocal physical quantities, expressed by (e) space and or time integrals or derivatives on M, to (e&eb) integrals or derivatives on coordinate systems, is (=) described. The assumption of local mathematics and (e&eb) universes makes integrals and derivatives, on (e&eb) M or on global charts, meaningless. They acquire (e) meaning only when mapped into (e&eb) a local universe. The effect of scaling, by including (e) the effect of θ into (e&eb) the local maps, is described. The lack of experimental evidence for (e) θ so far shows that (eb) the coupling constant of θ to (e&eb) matter fields must be very small compared to **the fine structure constant**. Also the gradient of θ must be (=) very small in the local region of (e) cosmological space and time occupied by (e) us as observers. So far, there are no known restrictions on (e&eb) θ or its gradient in regions of space and/or time that are far away from (e) our local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum information and computation XII, proceedings of SPIE conference, 8-9 May, 2014, Vol. 9123, 91230Q-1 DOI: 10.1117/12.2049705 Cite as:arXiv: 1405.3217 [math-ph] (or arXiv: 1405.3217v1 [math-ph] for this version) **Effects of mathematical locality and number scaling on coordinate chart use Paul Benioff STOP** Effects of local availability of (e) mathematics (LAM) and (e&eb) space time dependent number scaling on (e&eb) physics and, especially, geometry are described. LAM assumes (eb) separate mathematical systems as (=) structures at each space time point. Extension of gauge theories to include freedom of (e) choice of scaling for number structures, and other structures based on (e) numbers, results (eb) in a space time dependent scaling factor based on (e) a scalar boson field. Scaling has no effect on (e&eb) comparison of experimental results with (e&eb) one another or with theory computations. With LAM all theory expressions are elements of (e) mathematics at some reference point. Changing the reference point introduces (eb) (external) scaling. Theory expressions with integrals or derivatives over (e&eb) space or time include scaling factors (internal scaling) that cannot be removed by (e) reference point change. Line elements and path lengths, as integrals over (e&eb) space and/or time show (eb) the effect of scaling on geometry. In one example, the scaling factor goes to 0 as the (e&eb) time goes to 0, the big bang time. All path lengths, and values of physical quantities, are crushed to (eb) 0 as t goes to 0. Other examples have (e) spherically symmetric scaling factors about some point, x. In one type, a black scaling hole, the scaling factor goes to (e&eb) infinity as the distance, d, between any point y and x goes to 0. For scaling white holes, the scaling factor goes to (e&eb) 0 as d (e&eb) goes to 0. For black scaling holes, path lengths from (e) a reference point, z, to y become (=) infinite as (e) y approaches x. For white holes, path lengths approach (e&eb) a value much less than the unscaled distance from z to (e) x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.2014360 Cite as: arXiv: 1306.4613 [quant-ph] (or arXiv: 1306.4613v1 [quant-ph] for this version) **Effects of gauge theory based number scaling on geometry Paul Benioff STOP** In gauge theories, separate vector spaces, Vx, are assigned to (e) each space time point x with (e&eb) unitary operators as maps between basis vectors in neighboring Vx. Here gauge theories are extended by replacing (e&eb) the single underlying set of complex scalars, C, with (e&eb) separate sets, Cx, at each x, and including (e) scaling between the

Cx. In gauge theory Lagrangians, number scaling shows (eb) as a scalar boson field, B, with (e&eb) small coupling to (e&eb) matter fields. Freedom of number scaling is extended to (e&eb) a model with (e&eb) separate number structures assigned to (e) each point x. Separate collections, Ux, of all mathematical systems based on (e) numbers, are assigned to (e) each x. Mathematics available to (e) an observer, Ox, at x is (=) that in Ux. The B field induces (eb) scaling between structures in the different Ux. Effects of B scaling on (e&eb) some aspects of (e) physics and geometry are described. The lack of experimentally observed scaling means (eb) that B (z) is essentially constant for all points, z, in a region, Z, that can be occupied by (e) us as observers. This restriction does not (e) apply outside Z. The effects of B scaling on (e&eb) line elements, curve lengths, and distances between points are examined. Oz's description, using (e) Uz in Z, of elements at points, x, outside Z, includes (e) scaling from x to (e&eb) z. Integrals over curves include (e) scaling factors inside the integrals. Two examples are discussed. One shows that B (t) can be such that mathematical, physical, and geometric quantities approach (eb) zero as the time t approaches zero. This mimics the big bang in that distances (e) approach zero. Examples of black and white scaling holes are described in (eb) which B(x) is plus or minus infinity at a point x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1211.3381 [quant-ph] (or arXiv: 1211.3381v2 [quant-ph] for this version) **Gauge theory extension to include number scaling by boson field: Effects on some aspects of physics and geometry Paul Benioff STOP** Local availability of (e) mathematics and number scaling provide (eb) (an approach to (e) a coherent theory of physics and mathematics. Local availability of mathematics assigns (e&eb) separate mathematical universes, U_{x}, to (e) each space time point, x. The mathematics available to (e) an observer, O_{x}, at x is contained in (eb) U_{x}. Number scaling is based on (e) extending the choice freedom of vector space bases in gauge theories to (e) choice freedom of underlying (e&eb) number systems. Scaling arises in (eb) the description, in U_{x}, of (e) mathematical systems in U_{y}. If a_{y} or \psi_{y} is (=) a number or a quantum state in U_{y}, then (eb)the corresponding number or state in (eb) U_{x} is r_{y,x}a_{x} or r_{y,x}\psi_{x}. Here a_{x} and \psi_{x} are (=) the same number and state in U_{x} as (=) a_{y} and \psi_{y} are in (eb) U_{y}. If y=x+\hat{\mu}dx is a neighbor point of x, then (eb) the scaling factor is r_{y,x}=\exp(\vec{A}(x)\cdot\hat{\mu}dx) where \vec{A} is a vector field, assumed here to be (=) the gradient of a scalar field. The effects of scaling and local availability of mathematics on (e&eb) quantum theory show that scaling has (e) two components, external and internal. External scaling is shown above for (e) a_{y} and \psi_{y}. Internal scaling occurs in expressions with integrals or derivatives over (e&eb) space or space time. An example is (=) the replacement of the position expectation value, \int\psi^{*}(y)y\psi(y)dy, by \int_{x}r_{y,x}\psi^{*}_{x}(y_{x})y_{x}\psi_{x}(y_{x})dy_{x}. This is an integral in U_{x}. The good agreement between quantum theory and (=) experiment shows (eb) that scaling is negligible in (eb) a space region, L, in which experiments and calculations can be done, and results compared. L includes the solar system, but the speed of light limits the size of L to a few light years. Outside of L, at cosmological distances, the limits on (e&eb) scaling are not present. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.917731 Cite as: arXiv: 1205.0200 [quant-ph] (or arXiv: 1205.0200v1 [quant-ph] for this version) **Local Availability of mathematics and number scaling: Effects on quantum physics Paul Benioff STOP** The work is based on two premises: local availability of mathematics to an observer at any space time location, and (e&eb) the observation that number systems, as (=) structures satisfying axioms for (e) the number type being considered, can be scaled by arbitrary, positive real numbers. Local availability leads to (eb) the assignment of mathematical universes, Vx, to each point, x, of space time. Vx contains (e) all the mathematics that an observer, Ox, at x, can know. Each Vx contains (e) many types of mathematical systems. These include (e) the different types of numbers (natural numbers, integers, rationals, and real and complex numbers), Hilbert spaces, algebras, and many other types of systems. Space time dependent scaling of number systems is used to (e) define representations, in Vx, of real and complex number systems in (eb) Vy. The representations are scaled by (e) a factor ry, x relative to (e&eb) the systems in Vx. For y a neighbor point of x, ry, x is (=) the exponential of the scalar product of (e) a gauge field, A→(x), and the vector from x to (e&eb) y. For y distant from x, ry, x is a path integral from x to (e&eb) y. Some consequences of the two premises will be examined.

Number scaling has no effect on (e&eb) general comparisons of numbers obtained as (=) computations or as experimental outputs. The effect is limited to (e) mathematical expressions that include (e) space or space time integrals or derivatives. The effect of A→ on (e&eb) wave packets and canonical momenta in quantum theory, and some properties of A→ in gauge theories, is described. Subjects: Quantum Physics (quant-ph); Information Theory (cs.IT); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) Journal reference: Chapter 2, in Quantum Theory, I. Cotaescu, Editor, Intech open access publisher, 2012 Cite as:arXiv:1110.1388 [quant-ph] (or arXiv:1110.1388v2 [quant-ph] for this version) **Effects on quantum physics of the local availability of mathematics and space time dependent scaling factors for number systems Paul Benioff STOP** Gauge theories can be described by (e) assigning a vector space V(x) to (e&eb) each space time point x. A common set of complex numbers, C, is usually assumed to be (=) the set of scalars for all the V{x}. This is expanded here to assign (e&eb) a separate set of scalars, C{x}, to V{x} for each x. The freedom of choice of bases, expressed by (e) the action of a gauge group operator on (e&eb) the V{x}, is expanded here to include (e) the freedom of choice of (e) complex scale factors, c_{y, x}, as (=) elements of GL (1, C) that relate C{y} to (e&eb) C{x}. A gauge field representation of c_{y, x} gives (eb) two gauge fields, A (x) and iB(x). Inclusion of these fields in (eb) the covariant derivatives of Lagrangians results in (eb) A(x) appearing as (=) a gauge boson for which mass is (=) optional and B(x) as (=) a massless gauge boson. B(x) appears to be (=) the photon field. The nature of A(x) is not known at present. One does know that the coupling constant of A(x) to (e) matter fields is very small compared to (e&eb) the fine structure constant. Subjects: Mathematical Physics (math-ph); Quantum Physics (quant-ph) Journal reference:Proc.SPIE Int.Soc.Opt.Eng. 8057:80570X, 2011 DOI: 10.1117/12.895454 Cite as: arXiv: 1105.0633 [math-ph] (or arXiv: 1105.0633v1 [math-ph] for this version) **New Gauge Fields from Extension of Parallel Transport of Vector Spaces to Underlying Scalar Fields Paul Benioff STOP** This work is based on (e) the field of reference frames based on (e) quantum representations of (e) real and complex numbers described in other work. Here frame domains are expanded to include (e) space and time lattices. Strings of qukits are described as (=) hybrid systems as they are both mathematical and (e&eb) physical systems. As mathematical systems they represent numbers. As physical systems in each frame the strings have (e) a discrete Schrodinger dynamics on the lattices. The frame field has an iterative structure such that the contents of (e) a stage j frame have images in (eb) a stage j-1 (parent) frame. A discussion of parent frame images includes (e) the proposal that points of stage j frame lattices have (e) images as (=) hybrid systems in parent frames. The resulting association of energy with (e&eb) images of lattice point locations, as (=) hybrid systems states, is discussed. Representations and images of other physical systems in (eb) the different frames are also described. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Advances in Mathematical Physics, Vol. 2009, ID 452738 DOI: 10.1155/2009/452738 Cite as: arXiv: 0809.4499 [quant-ph] (or arXiv: 0809.4499v2 [quant-ph] for this version) **A Possible Approach to Inclusion of Space and Time in Frame Fields of Quantum Representations of Real and Complex Numbers Paul Benioff STOP** A quantum theory representations of real (R) and complex (C) numbers are given that is based on (e) states of single, finite strings of (e) qukits for (e) any base k > 1. Both unary representations and (e&eb) the possibility that qukits with (e&eb) k a prime number are (=) elementary and the rest composite are discussed. Cauchy sequences of (e) qukit string states are defined from (e) the arithmetic properties. The representations of (e) R and C, as equivalence classes of these sequences, differ from (e&eb) classical kit string state representations in two ways: the freedom of choice of basis states, and the fact that each quantum theory representation is (=) part of a mathematical structure that is itself based on (e) the real and complex numbers. These aspects enable (eb) the description of 3 dimensional frame fields labeled by (e) different k values, different basis or gauge choices, and different iteration stages. The reference frames in the field are based on (e) each R and C representation where (e) each frame contains (e) representations of *(e) all physical theories as mathematical structures based on (e) the R and C representation. Approaches to integrating this with physics are described. It is observed that R and C values of physical quantities, matrix elements, etc. which are viewed in (eb) a frame as elementary and featureless, are seen in a parent frame as equivalence classes of (e) Cauchy sequences of qukit string states. Subjects:Quantum Physics (quant-ph); High

Energy Physics - Theory (hep-th); General Mathematics (math.GM) Journal reference:Advances in Quantum Computation, K. Mahdavi and D. Koslover, Eds., Contemporary Mathematics Vol. 482, pp125-163, (2009) Cite as: arXiv:0709.2664 [quant-ph] (or arXiv:0709.2664v2 [quant-ph] for this version) **Reference Frame Fields based on Quantum Theory Representations of Real and Complex Numbers Paul Benioff STOP** This paper extends earlier work on quantum theory representations of natural numbers N, integers I, and rational numbers Ra to describe (eb) a space of these representations and transformations on (e&eb) the space. The space is parameterized by (e) 4-tuple points in a parameter set. Each point, (k,m,h,g), labels (e&eb) a specific representation of X = N, I, Ra as a Fock space F^{X}_{k,m,h} of (e) states of (e) finite length strings of qukits q and a string state basis B^{X}_{k,m,h,g}. The pair (m,h) locates the q string in (eb) a square integer lattice I \times I, k is the q base, and the function g fixes the gauge or basis states for (e) each q. Maps on the parameter set induce (eb) transformations on on the representation space. There are two shifts, a base change operator W_{k',k}, and (e&eb) a basis or gauge transformation function U_{k}. The invariance of (e) the axioms and theorems for N, I, and Ra under (e&eb) any transformation is discussed along with (e&eb) the dependence of the properties of W_{k',k} on (e&eb) the prime factors of k' and k. This suggests (eb) that one consider prime number q's, q_ {2}, q_{3}, q_{5}, etc. as (=) elementary and the base k q's as composites of (e) the prime number q's. Subjects: Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); Mathematical Physics (math-ph) Cite as: arXiv: 0704.3574 [quant-ph] (or arXiv: 0704.3574v1 [quant-ph] for this version) **Space of Quantum Theory Representations of Natural Numbers, Integers, and Rational Numbers Paul Benioff STOP** In this paper fields of quantum reference frames based on (e) gauge transformations of (e) rational string states are described in a way that, hopefully, makes (eb) them (eb) more understandable than their description in an earlier paper. The approach taken here is based on three main points: (1) There are (=) a large number of different quantum theory representations of (e) natural numbers, integers, and rational numbers as states of qubit strings. (2) For each representation, Cauchy sequences of rational string states give (eb) a representation of the real (and complex) numbers. A reference frame is associated to (e&eb) each representation. (3) Each frame contains (e) a representation of (e) all mathematical and physical theories that have (e) the representations of the real and complex numbers as (=) a scalar base for the theories. These points and other aspects of the resulting fields are then discussed and justified in some detail. Also two different methods of relating the frame field to (e&eb) physics are discussed. Comments: 24 pages, 5 figures; submitted to proceedings, 3rd Feynman Festival, University of Maryland, 2006 Journal reference: Journal of Physics: Conference Series 70 **Fields of quantum reference frames based on different representations of rational numbers as states of qubit strings Paul Benioff STOP System studied is as follows:** Quantum Query Complexity, Confining String In An Analytically Solvable 3D Model, Hagedorn Spectrum And Equation Of State Of Yang-Mills Theories, Equation Of State Of G(2) Gauge Theory, Donald Hoffman Uses Evolutionary Game Theory To Show That Our Perceptions Of An Independent Reality Must Be Illusions, Hard Problem Of Consciousness, Evolution Has Shaped Us With Perceptions That Allow Us To Survive, Fitness Function Happens To Align With The True Structure In Reality, If You Had To Spend All That Time Figuring Blue Icon Out The Tiger Would Eat You, Everything We See Is One Big Illusion, Evolution Shapes Acceptable Solutions, Not Optimal Ones, Mathematical Model Of Consciousness, World Is Just Another Conscious Agents, When You Do A Split-Brain Operation, A Complete Transection Of The Corpus Callosum, You Get Clear Evidence Of Two Separate Consciousnesses, Physics Tells Us That There Are No Public Physical Objects, Neurons, Brains, Space

**Key words:** Quantum Query, Quantum Complexity, Confining String Analytically Solvable 3D Model, Hagedorn Spectrum,Equation Of State Of Yang-Mills Theories, Equation Of State Of G(2) Gauge Theory, Donald Hoffman, Evolutionary Game Theory, Perceptions Of An Independent Reality, Must Be Illusions, Hard Problem Of Consciousness, Evolution Has Shaped Us With Perceptions That Allow Us To Survive, Fitness Function Happens To Align With The True Structure In Reality, If You Had To

Spend All That Time Figuring Blue Icon Out The Tiger Would Eat You, Everything We See Is One Big Illusion, Evolution Shapes, Acceptable Solutions, Not Optimal Ones, Mathematical Model Of Consciousness, World Is Just Another Conscious Agents, When You Do A Split-Brain Operation, A Complete Transection Of The Corpus Callosum, You Get Clear Evidence Of Two Separate Consciousnesses, Physics Tells Us That There Are Public Physical Objects, Neurons, Brains, Space,

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# PREFACE

**Note:** Some of the following concepts are modelled; some are not. For those which are not modelled kindly see one of the papers in the series. This note has the axiomatic predication to give a holistic view of the study. Here we talk of the characterstics of systems which satisfy the condition of cosmological constant or any other for that matter or variable or axiomatic predication. . There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential blackholes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and

characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. "All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ...." (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To "deconstruct" is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between "speech" and "writing." This binary logic functions in an illicit way to establish speech as the means of giving "presence" to the world, while writing is deemed derivative and inferior In Derrida's sense of "grammatology," however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of "differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. **THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html** Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive,

disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE. Towards the end of circumvention of typing hundreds of superscripts and subscripts which would be a sardine squash and the concomitant operational difficulties, model is presented in piece meal of nine modules. Logarithms are to be taken in respect of those which incorporate more than one variable in bra-ket. Values of log (ab) and log (a+b) are readily available. Anti logarithm shall be taken at the end to predict, project, prognosticate the value of the variable. This is true for tensors, vectors and other variables too. Affirmational assertion, and explanation justification, statement of vindication, annotational commentary, and explicational glossary for each and every system changes and physical interpretation of results is one thing that is to be with earnest endeavour and feverish and febrile expediency. Editions never ever mean the same and identicality of parameters and this has been explicitly and unmistakably stated in the model apriori itself. Akin and analogous, cognate and concurrent, correspondent and congruent, comparable and complementary, synonymous and duple, tantamount and agnate, commensurate and correlative representation is only to highlight the importance and subterfuge the replication of work. It is my fervent solicitation to kindly bear with me for any lapses, notwithstanding the orchestrated efforts for a paper without any minor errors. Postulation predication, conclusive presumption, differential presuppositions, underscored decidedly axiomatic statement of the statement, equation form the bye word or the watch word in the aggrandizement-amplification, caricature and crock, understatement- unembellishment, elocution-emphasis, enunciation-inflection, announcement-argument, articulation, assertion, asseveration, choice of words, commentary and communication, declaration and

definition, delivery-diction, elucidation-emphasis, enunciation-execution, explanation, exposition-formulation, idiom, interpretation and intonation tone and tenor of stratification. Now, how do find the reaction of systems to these singularities. You do the same thing a boss does for you. "Problematize" the events and see how you behave. I will resort to "pressure tactics". "intimidation of deriding report", or "cut in the increment" to make you undergo trials, travails and tribulations. I am happy to see if you improve your work; but may or may not be sad if you succumb to it and hang yourself! We do the same thing with systems. systems show conducive response, felicitous reciprocation or behave erratically with inner roil, eponymous radicality without and with blitzy conviction say like a solipsist nature of bellicose and blustering particles, or for that matter coruscation, trepidiational motion in fluid flows, or seemingly perfidious incendiaries in gormandizing fellow elementary particles, abnormal ebullitions, surcharges calumniations and unwarranted(you think so but the system idoes not!) unrighteous fulminations. Amount of emotion or affect that is appropriate or expected in an given form of interaction. Particular individuals and diffuse obligations (see c and d) are associated with affectivity, whereas contacts with many individuals (universalistic) in a bureaucracy may be devoid of emotion and characterized by affective neutrality. Affective neutrality may refer to self discipline and the deferment of gratification (eg Weber's spirit of capitalism). In contrast, affectivity may be associated with expressing emotions. Adams and Sydie also refer to affective neutrality being associated with ego control (p. 15). (Parsons: Wikipedia). So the point that is made here is "like we problematize the "events" to understand the human behaviour we have to "problematize" the events of systems to understand their behaviour. This statement is made in connection to the fact that there shall be creation or destruction of particles or complete obliteration of the system (blackhole evaporation) or obfuscation of results. Some systems are like "inside traders" they will not put signature atoll! How do you find they did it! Anyway, there are possibilities of a CIA finding out as they recently did! So we can do the same thing witn systems to. This is accentuation, corroboration, fortification, .fomentatory note to explain the various coefficients we have used in the model as also the dissipations called door. In the bank example we have clarified that various systems are individually conservative, and their conservativeness extends holisticallytoo.that one law is universal does not mean there is complete adjudication of nonexistence of totality or global or holistic figure. Total always exists and "individual" systems always exists, if we donot bring Kant in to picture! For the time being let us not! Equations would become more ensorcelled and frenzied..... philosophy merges with ontology; ontology merges with univocity of being; analogy has always a theological vision; not a philosophical vision; one becomes adapted to the forms of god; self and world; the univocity of being does not mean that there is one and the same being; on the contrary, beings are multiple and different they are always produced by disjunctive synthesis; and they themselves are disintegrated and disjoint and divergent; membra disjuncta.the univocity of being signifies that that being is a voice that is said and it is said in one and the same "consciousness". In consciousness research, two rival sets of theories can be recognized: (A) Scientific material interpretations of consciousness are based on axioms that view consciousness in the context of highly advanced intentional processing of information in which subject-object relations evolve, and (B) humanistic interpretations of consciousness are based on axioms that view consciousness in the context of, say, "centered pulsations" that enable a conscious agent to act from his or her center of awareness. In this paper I will argue for the selection of axioms that favor humanistic interpretations of consciousness. The Pursuit of Autonomy Interdisciplinary Observations to Human Consciousness: Wautischer, Helmut (1993) the Pursuit of Autonomy. Interdisciplinary Observations to Human Consciousness: Social Neuroscience Bulletin, 6 (4). pp. 52-56.Everything about which consciousness is spoken about being is the same for everything for which it is said like gravity; it occurs therefore as an unique event for everything; for everything for which it happens; eventum tantum; it is the ultimate form for all of the forms; and all these forms are disjointed; it brings about resonance and ramification of its disjunction; the univocity of being merges with the positive use of the disjunctive synthesis, and this is the highest affirmation of its univocity like gravity; it is the eternal resurrection or a return itself, the affirmation of all chance in a single moment, the unique cast for all throws; a simple rejoinder for Einstein's god does not play dice; one being, one consciousness, for all forms and all times. a single instance for all that exists, a single phantom for all the living; a single voice for very hum of voices; or a single silence for all the silences; a

single vacuum for all the vacuums; consciousness should not be said without occurring; if consciousness is one unique event in which all the events communicate with each other; univocity refers both to what occurs to what it is said. This is attributable to all states of bodies and states of affairs and the expressible of every proposition. So univocity of consciousness means the identity of the noematic attribute and that which is expressed linguistically and sense fully. Univocity means that it does not allow consciousness to be subsisting in a quasi state and but expresses in all pervading reality; **CONSCIOUSNESS AND ITS UNIQUENESS June 12, 2012 at 8:40pm (See Deleuze Logic of sense, Wikipedia and Stanford encyclopedia for more details)** Atrocious contrivance and device, gratificational primogeniture, calamitous dodge and expedient, depraved, destructive, disastrous, execrable gambit and gimmick, great solace and succor, iniquitous, injurious, loathsome, low, maleficent, malevolent machination and maneuver, promethaleon of candor, frankness, honesty, honor, ingenuousness, innocence, openness, reality, sincerity, truthfulness, spiteful, stinking, ugly, unpleasant, unpropitious play and ploy, progenitor of jurisprudence and circumspection, racket and ruse, savvy, scam, stratagem, subterfuge, tactic and wile, forthrightness, honesty, truthfulness embodied and personified is the structural predisposition and dispensation of the complex. Nobody need have to take the Brahman-Anti Brahman agency to be a recalcitrant, repugnant, and refractory proposition. God gives and also takes away. That is the point made. And of course lesser gods, and God doth follow them and they follow God. That is why they are lesser gods. And wield the power on lesser mortals like us. When we mean gratification and deprivation we mean celestial Brahman Anti Brahman and Terrestial Brahman and Anti Brahman agencies producing such happiness or sadness by any means such as (blending- amalgamation, adulterant-adulteration, amalgam-amalgamation, blend- combination, composite- compound, debasement-denaturant, fusion-hybrid, intermixture-reduction, amalgam- mixture, composite- compound, fusion-mishmash, aggregate- alloy, amalgamation-blend, commixture-composite, composition-compost, conglomerate-fusion, goulash-medley, mishmash-stew, blend-coalition, commixture-compound, federation-heating, integration-junction, melting-merger, synthesis-unification, assortment- combination, adulteration-alloy, assimilation-association, batter-blend, brewed combine, composite compound, concoction confection, conglomeration cross of money, orientation(dis), penury creation, murder, mayhem plunder, pillage, apocalypse, Armageddon in various forms and permutation and combination of Manichaean artifice, agathokakological malevolence, agley chicane fourberie, fraud, furtiveness, gambit, awry and bad ,flagitious mechanizations, cacodemonic gambit, hanky-panky, intrigue, machination, deprecatory and diabolic underhandedness and wiles, energumenical skullduggery, goetic chicanery, lenocinant stratagem, malefic and maleficent strategy, malominous sell, sellout, sham subterfuge, peccable, humbug, imposture, misrepresentation peccant fraud, fraudulence, hoax, qued , scelerate pretense, ruse, sell, sellout, sham stratagems' of maladroit manouevures , sinistral ,tortious and venal sophistry, contingency enterprise, endangerment emprise, jeopardizing adventure, jejune jeremiad, avoidable inertia, latent passiveness: with each action accounted and showing balance in the nature's general ledger as also actionlessness:e). All these are recorded in Nature's general ledger. Neuron DNA encapsulates all the actions passions, actions, interactions, transactions or the lack of it at various levels of individual, collective and cosmic levels. In celestial Brahman Anti Brahman ledger there is cessation of dualities. One more point to be made is if 5=3+2, then we can say 5 gormandizes 3 or 5 gobbles up 2. Quintessentially such a statement arises out of the difference of LHS and RHS term wise. It must be remembered despite all theoretical abstractions and generalizations be it proton mass or Higgs Boson, or topology of a space, it is notwithstanding the fundamentality of nature is just a number. Just everything in Physics and mathematics is just a number. Let us assume I buy rice in Bangalore (District) worth 5k.g. I can always divide 5k.g.s of rice in categories of Bangalore (South), Bangalore (west), Bangalore (East) and Bangalore (South). On similar basis, there are billions of neutrons, protons and concomitant interactions which would satisfy the given numerical value. Example of gravity above makes things clear. Everything atleast is a function of time and someplace or some other coordinates, subordinates or superordinates. Mass of proton is 2.47 does not mean total mass of all protons is 2.47. It is this argument that is used in the classification scheme. By the very nature, self the unmonitoring but the witness consciousness agency identifies itself with something, be it a profession, an actor on the screen, when it does not have a identity. Such examples are legion, a Charlie Chaplin in "The Kid" or Raj kapoor

in “Jagte Raho”. These are people who just wanted to live and living without being put under surveillance every moment is the greatest part of their life. This is “Absolute Subjectivity” of Shiva. Such a state is always craving for identification for in pure consciousness all identities are lost. It is this “identitylessness” or having very minimal identity is what pervades most in India and this is due to expansion of individual consciousness. When that happens, be it education, or wealth, the “identity’ becomes so well entrenched, they become the exact opposite of “pure consciousness”,” impure consciousness” (Shakti). This is a state of “Absolute or relativistic objectivity”. There was this case of a lady who went to such an extent of planning, she was sure that whatever she does must work out. When it did not, she lapsed in to “identity crisis”. Such thing as this might happen in well developed countries where the competition is extremal and to be achieved at any cost. Realisation that one is not the doer but the witness consciousness is just like watching a movie in which one has acted. It is in this sense “advaita”, “non duality’ seems to be of cardinal and preeminent importance for the analytical treat ment of “identity’ and “identity lessness”, both happen in space time. “Multiverses” in this sense is also like a different film of an actor released at different theaters and it is in this sense we state that that every object in space time is a multiverse. You must have totally immersed in doing a “string theory” problem or watching a “film”, oblivious of surroundings. Essentially creation (space time) also is what is happening in mind and honed in to you billion times. Identification of anything with a designation or signification is one that leads to association within spacetime, which would only have pathological consequences and detrimental ramification as stated above or positive implications of “happiness” because of the identity itself. Search or ‘show” has to go on! It gives a sense of consummation. a watching pot never boils; donot go to check nature; you are disturbing it; it will never show its true colours bhakti and Virakthi, gratification and deprivation, raga and dwesha is essential to live in space. It is the quantum interest thereof that produces energy to live .gratification and deprivation which form the bastion, pillar, post and stylobate and sentinel of all human actions depends on tams, rajas, and sattva of individual or insentient object. Verily tamas, rajas and satva are the characterstics of space itself, nay the behavioural pattern and attitudinal orientation. There are peaceful places like Vienna or Sringeri and there are violent places like others. Space is warped and woofed by the Akshara, the quantum information of primordial syllable Om. cosmic general ledger is the one vibration of which is brahman; quantum information possessions thereof form the sentinel of all transactions (Gargi to Yajnavalkya: my interpretation)adhocism, nonseqientiality of events, anti sequentiality of events, simultaneity of phenomenon, concurrent happenings , improbability of phenomenon happening itself are characterstics of brahman-anti brahman (Terrestial and celestial) are the cause of non predictability of happenings like in black hole, or in the areas of operation of brahman-anti brahman (Terrestial and celestial). brahman AntiBrahman actions, interactions, decisions, circumspection, adhocism, satisfies sensitive to initial conditions; topologically mixing; dense periodic orbits, the axiomatic predications and postulation alcovishness of chaos theory. Like gravity is a characteristic of space, so is tamas, rajas, sattva, with spaces knotted together by quantum information (for models see below) .accidents are preplanned actions of Terrestial and celestial Brahman AntiBrahman. All actions are perpetrated by them, be it train on fire or death. Belief systems play an important role in the determination of tamas, rajas, and satva state of human beings. There is neither cause nor effect. All actions are performed by Terrestial and celestial Brahman anti Brahman and it is off the record. psychic energy is negative equivalent to –mc^2, so that second law of thermodynamics is satisfied on a holistic basis vis-à-vis material energy ParaBrahaman is infinite individual, collective and cosmic consciousnesses are infinite (concomitant general ledgers) for they are in the evolution mode with respect to each other in that order. Later emanates from the former as transactions emanate from the conservative state of assets and liabilities. Is not zero true for infinite conservative systems inside space and time? zero is the only consummation, consolidation and conservation. if fifty out of seventy members come out of the room at whatever time you come out, talk about the grand limbo of oblivion and hibernation of lifeless world you are going to and if you are not a celebrity, then it is not coincidence, it is coordination. Nature is playing the same trick on you. That a generalised law is there and you can measure does not make things less good albeit recalcitrant, refractory and rogue particles occur time and again, by and large. Entering in to all living beings through his constant part on the new moon night, he is born therefrom next morning. Absolute ParaBrahaman (zero) projects

himself again and again on the screen of individual consciousness which appears of depairing despondent world. Some where the balancing act is done on karma. cosmic general ledger reflects upon itself to keep all transactional ties conservative (Brihadarnkayaka Upanishad: 1.5 14) for models see next paper through its divine power ( i interpret man creates god; it is attribution and ascription of various Shaktis would the formation of god takes place like space and time) the self assumes (eb) the form of a deity man can contemplate and venerate, even though siva, the pure subject, can never in fact be an object of meditation (adhyeya). Until we realise our true identity with Sankara, he is worshipped and conceived to be a reality alien to ourselves. While we are in the realm of creation, he too is a creation or mode or appearing of the absolute, manifest to us in meditation, through his freedom as an eternal, omniscient being. There is no gulf between the created and the uncreated creator: nothing in reality, although an object of knowledge, ceases to be Siva: this is the reason why meditation [on this or that aspect] of reality bestows its fruit. The world of the senses and mind appears to the well awakened (suprabuddha) as a theophany, an eternal revealing of god in his creation. The doctrine of vibration declares that "there is no state in word, meaning or thought, either at the beginning, middle or end, which is not Siva." 59 to utter any word is, in reality, to intone a sacred formula.every act is a part of Siva's eternal cosmic liturgy, every movement of the body a ritual gesture (mudras), and every thought, god's thought. By what path are you not attainable? What words do not speak of you? in which meditation are you not an object of contemplation? What indeed are you not, o lord? 61 spandashakti, which accounts for the appearing of all things, is also the means by which deity in its many varied forms appears to man. Ksemaraja concludes: the ultimate object of worship of any theistic school differs not from the Spanda principle. The diversity of meditation is due solely to the absolute freedom of Spanda. 62 Sankara is not only the supreme object of devotion; as the static polarity of the absolute, he is the inner reality which holds together its Siva and Sakti outer manifestations. Phenomena are patterns of cognitions projected onto the surface of self-luminous siva-consciousness. There they become apparent, directly revealed to consciousness according to their manifest form. Siva is accordingly symbolized as the ground or surface of awareness, smooth and even like a screen (samabhittitalopama). 64 inscribed on this screen (kutfya) are the countless manifest forms which appear within it rendering it as diverse and beautiful as a fossil ammonite (falagrama). 65 Siva is the sacred ground upon which the cosmic mandalas are drawn, the absolute surface of inscription which bears the mark (cihna) of the universe. Abhinava writes: the variety of this world can only be manifest if the highest lord, who is essentially the pure light of consciousness, exists; just as a surface is necessary for a picture. if external objects were perceived in isolation then, because 'blue' and 'yellow', etc., are self-confined and the perceptions [we have of them] refer to their objects alone and so are insentient, mute and dumb in relation to one another . . . how would it be possible to be aware that an object is variegated? but just as depths and elevations can be represented by lines on a smooth wall, and we perceive [a female figure and think], similarly it is possible to be aware of differences in the variegated (contents of experience) only if all the diverse perceptions are connected together on the one wall of the universal light of consciousness. 66 Siva is the perfect artist who, without need of canvas or brush, paints the world pictures. the instant he imagines it, it appears spontaneously, perfect in every respect. The colours he uses are the varying shades and gradations of his own Spanda energy and the medium his own consciousness. Intractable disaffect, Mutinous obdurate, Rebellious recalcitrant, Seditious refractory, Insubordinate insurgent, antagonistically antipodean, discrepant incompatible, Contumacious coadjutant, Conflicting converse, Unyielding obdurate, the antagonistic and anachronistic dispositional ties of Anti Brahman are in fact the sine qua non of Brahman only (See Vishwaroopa of Mahavishnu, which contains the polarised positives on RHS and negatives on LHS). The universe is coloured with the dye of its own nature (svabhava) by the power of Siva's consciousness (a//). Rajanaka rama says: homage to him who paints the picture of the three worlds, thereby displaying in full evidence his amazing genius {pratibha)\ to Sambhu who is beautiful with the hundreds of appearances laid out by the brush of his own unique, subtle and pure energy. 68 analogously, at the microcosmic level, all the cognitions and emotions, etc., which make up the individual personality form the outward flow of essentially introverted consciousness. They are specific pulsations (vise$aspanda) or aspects of the universal pulsation (samanyaspanda) of pure t consciousness. at the lower level, within the domain of maya, they represent the play in the fettered soul of the three primary qualities (gunas) or

feeling-tones' which permeate to varying degrees his daily experience. These are: 1) sattva — the quality of goodness and luminosity which accompanies blissful experience both aesthetic and spiritual. 2) Rajas — the passion or agitation which oscillates between the extremes of 'light' and 'darkness' and characterises inherently painful experiences. 3) Tamas — the torpor and delusion which accompany states of inertia and ignorance. 69 the liberated soul recognises that these three are the natural and uncreated powers of pure consciousness. for him they are manifest respectively as: 1) Sankara 's power of knowledge (jnana) — the light of consciousness (prakasa); 2) the power of action (kriya) — the reflective awareness of consciousness (vimarsa)\ 3) the power of maya — which does not mean here the world of diversity, but the initial subtle distinction which appears between subject and object in pure consciousness. 70(doctrine of vibration: mark S.G. Dyczkowski) .to "deconstruct" is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. at the heart of western metaphysics, for example, derrida finds the opposition between "speech" and "writing." this binary logic functions in an illicit way to establish speech as the means of giving "presence" to the world, while writing is deemed derivative and inferior in derrida's sense of "grammatology," however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of "differance" (language organized around difference and deferred, or mediated, understandings), derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. The postmodern turn in philosophy: theoretical provocations and normative deficits by Steven best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html. you cannot see beyond consciousness for that is the limit of your storage of information or the basis for understanding further information. Turiya is empty bucket; it is from which prajna the measure of arises; turiya measures the number of quantum information stores in other three states of waking, dreams, and dreamless deep sleep (Manduka Upanishad; my interpretation for models next paper).let there be path where angels tread: it is a force never born never dies the brahman an anti brahman verily there are parallel universes in (eb) in the ambit of space and time; the society of sinners and the sinned with the grammar of evil; parallel universes contain(e) multiverses in them; all physics, philosophy, break down(e) in these universes; astronomical gratification ,deprivation, glorification, mortification, projections, axes, dimensions, rotations, foldings, concurrences, motions, shocks, are the axiomatic predications of multiverses and parallel universes; death is when self (the witness consciousness; register of individual general ledger) leaves the body; ostensibly parallel seem to have (e) an objective; parallel universes donot (e) have objectives; their paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, affection- appreciation, awareness- discernment, emotion-feeling, gut reaction, heart insight, intuition, judgment, keenness, perceptiveness, rationale, sensation, sense, sensitiveness, sensitivity, sentiment, susceptibility, taste, vibes, admiration, aesthetic sense, affection, appraisal, assessment, attraction, awareness, cognizance, commendation, comprehension, enjoyment, esteem, estimation, grasp, high regard, knowledge, liking, love, perceptional realization, recognitional regard, relish, respect, responsiveness, sensibility, sensitiveness, sensitivity, sympathy, understanding, valuation, acknowledgment, airing, baring, , confessional defenselessness, denudation- denunciation, disclosure, display, divulgence-exhibition, hazard, introduction, jeopardy, laying open, liability, manifestation, nakedness, openness, peril, presentation, publicity, revelation, risk, showing, susceptibility , susceptiveness,susceptivity, unfolding, unmasking, unveiling, vulnerability, vulnerableness, action, affection, appreciation and ardor, behavior, capacity, compassion, concern, cultivation and culture, delicacy, discernment and discrimination, emotion, empathy, faculty, fervor, fondness, heat,

imagination, impression and intelligence, intensity and intuition, judgment, keenness- palpability, passion, pathos, pity, reaction, refinement, sensibility, sensitivity, sentiment, sentimentality , sharpness, spirit, sympathy, tangibility, taste, tenderness, understanding, warmth, fond memories, hearts and flowers, homesickness, longing, pining, reminiscence, ,schmaltz, sentimentality wistfulness, yearning satiation is only ostensible; their job is to get the job done; their business is to mind other's business ;all laws of physics and finance breakdown; it is antilaw,piscatorial Piratish and bubonic bucaneerishness; there is free transactions between in multiverses; when reflected on the screen of consciousness they appear as extant existential universe. Transactions are astronomical and gratification deprivation, glorification and mortification incalculable; they have no barriers of region, religion, language which they create for others. theirs objectives are: corruption criminality, debasement debauchery, degradation evil immoral lewdness licentiousness, perversion sinfulness, vice viciousness, vileness wickedness, improbity corruption, atrocity decadence, degeneration degradation, depravity evil, immorality impurity, infamy iniquity, looseness lubricity, perversion profligacy, sinfulness turpitude,vice viciousness, vulgarity wickedness, disgrace bad reputation, abasement abuse, baseness black eye, blemish blur, brand comedown, contempt contumely, corruption culpability, debasement debasing, defamation degradation, derision disbarment, discredit disesteem, disfavor dishonor, disrepute disrespect, humbling humiliation ignominy ill repute, infamy ingloriousness, meanness obloquy,odium opprobrium, pollution ,put-down reproach, scandal scorn, slander slight, slur spot, stain stigma, taint tarnish, turpitude venality, enormity Horribleness , abomination atrociousness, atrocity crime and the positive aspectionalities thereof .let there not be any incursions where evil mercenaries in the garb of mendicants thrive with adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatious gainsaying or conformingly pugnacious Manichaean artifice, agathokakological malevolence, agley chicane fourberie, fraud, furtiveness, gambit, awry and bad ,flagitious mechanizations, cacodemonic gambit, hanky-panky, intrigue, machination, deprecatory and diabolic underhandedness and wiles, energumenical skullduggery, execrable and goetic chicanery, lenocinant stratagem, malefic and maleficent strategy, malominous sell, sellout, sham subterfuge, peccable, humbug, imposture, misrepresentation peccant fraud, fraudulence, hoax, qued , scelerate pretense, ruse, sell, sellout, sham stratagems' of maladroit manouevures , sinistral ,tortious and venal sophistry, contingency enterprise, endangerment emprise, jeopardizing adventure, jejune jeremiad, avoidable inertia, latent passiveness . statement for Mephistophelean mercenaries philosophy merges with ontology; ontology merges with univocity of being; analogy has always a theological vision; not a philosophical vision; one becomes adapted to the forms of god; self and world; the univocity of being does not mean that there is one and the same being; on the contrary, beings are multiple and different they are always produced by disjunctive synthesis; and they themselves are disintegrated and disjoint and divergent; membra disjuncta.the univocity of being signifies that that being is a voice that is said and it is said in one and the same "consciousness". everything about which consciousness is spoken about being is the same for everything for which it is said like gravity; it occurs therefore as an unique event for everything; for everything for which it happens; eventum tantum; it is the ultimate form for all of the forms; and all these forms are disjointed; it brings about resonance and ramification of its disjunction; the univocity of being merges with the positive use of the disjunctive synthesis, and this is the highest affirmation of its univocity like gravity; it is the eternal resurrection or a return itself, the affirmation of all chance in a single moment, the unique cast for all throws; a simple rejoinder for Einstein's god does not play dice; one being, one consciousness, for all forms and all times. a single instance for all that exists, a single phantom for all the living; a single voice for very hum of voices; or a single silence for all the silences; a single vacuum for all the vacuums; consciousness should not be said without occurring; if consciousness is one unique event in which all the events communicate with each other; univocity refers both to what occurs to what it is said. This is attributable to all states of bodies and states of affairs and the expressible of every proposition. So univocity of consciousness means the identity of the noematic attribute and that which is expressed linguistically and sense fully. Univocity means that it does not allow consciousness to be subsisting in a quasi state and but

expresses in all pervading reality. Negative energy stabilising wormholes are considered candidates for (er) darkmatter. An analysis of negative energy might be a useful ingredient of the thoughts of travel at speeds more than the velocity of light. Peirce explains that an "imperfect continuum" possesses "topical singularities, or places of lower dimensionality" which break continuity, while "the parts of a perfect continuum have the same dimensionality as the whole." (CP 4.642) Circular continua without singularities are time and space. (CP 6.210-212) By using math as his lens, Peirce envisioned singularities or lower dimensional holes in the imperfect cosmologic continuum, while maintaining that a probabilistic law of chance exists that entropy decreases **(Wikipedia).** On the side of philosophy**,** quite explicitly, Deleuze provides a critique of Husserl's "sleight-of-hand" change from a static categorical phenomenology to one supposedly more dynamically constitutive of reality. This a clear cut **chagrined circumvention**, **contraventional curbing**, **defeated disgruntlement**, **dissatisfactional downer**, **dragged failure**, **fizzled foil**, **grievance hindrance**, **impedimental irritation**, letdown, nonfulfillment, nonsuccess, obstruction, old one-two, **resent mental setback**, unfulfillment of the earlier theories makes Sartre contend that human existence is a conundrum whereby each of us exists, for as long as we live, within an overall condition of nothingness (nothing-ness)—that ultimately allows for free consciousness. But simultaneously, within our being (in the physical world), we are constrained to make continuous, conscious choices. It is this dichotomy that causes anguish, because choice (subjectivity) represents a limit on freedom within an otherwise unbridled range of thoughts. Subsequently, humans seek to flee our anguish through action-oriented constructs such as escapes, visualizations, or visions (such as dreams) designed to lead us toward some meaningful end, such as necessity, destiny, determinism (God), etc. Thus, in living our lives, we often become unconscious actors—Bourgeois, Feminist, Worker, Party Member, Frenchman, Canadian or American—each doing as we must to fulfill our chosen characters' destinies. However, Sartre contends our conscious choices (leading to often unconscious actions) run counter to our intellectual freedom. Yet we are bound to the conditioned and physical world—in which some form of action is always required. This leads to failed dreams of completion, as Sartre described them, because inevitably we are unable to bridge the void between the purity and spontaneity of thought and all-too constraining action; between the being and the nothingness that inherently coincide in our self. Deleuze insists on the absolute non-resemblance between what conditions (metastable, pre-individual plane of singularities) and what is conditioned (actualized or the individuated 'thing'). Only then do "the conditions of the true genesis become apparent," Deleuze writes (Logic of Sense 105). Without addressing this issue, Deleuze's borrowings from Simondon, Lautman and structuralism remain untethered. Because of Bowden's decision to completely sidestep Deleuze's overt negotiations with phenomenology, his otherwise sophisticated commentary suggests a pedagogical narrowness. If, the omission of phenomenology strengthens Bowden's focus on the ontological priority of the event over substantial ontologies via the genealogy of figures and movements Deleuze outlines, it also incompletely articulates the problem of the sense-event as Deleuze formulates it. We have lost any idea of whom Deleuze takes to be his primary adversary, against whom he stakes out his position. This worry is most clearly manifested in Bowden's description of Deleuze's ontological prioritizing of events as offering a "transcendental ontology." For Bowden Deleuze's transcendental ontology defines events that "are ontologically prior to substances, 'all the way down'" (82). But Bowden provides a second definition: it is that the priority of events over worldly individuals means that the world of events "is not something external to the conditions of knowledge" (69). **SEAN BOWDEN the Priority of Events: Deleuze's Logic of Sense Sean Bowden, the Priority of Events: Deleuze's Logic of Sense, Edinburgh University Press, 2011, 296pp., $40.00 (hbk), ISBN 9780748643646. Reviewed by David Scott, Coppin State University Weltschmerz, agony, apprehension, blues, depression, dread, mid-life crisis, misgiving, nervousness, uneasiness, angst were expressed with such effulgent words…… aggrandizement-amplification, caricature and crock, understatement- unembellishment, elocution-emphasis, enunciation-inflection, announcement, argument, articulational assertion, asseverational choice of words, commentary and communication, declaration and definition, delivery, diction, elucidation, emphasis, enunciation-execution, explanation, exposition-formulation, idiom, interpretation and intonation about the human predicament is brought out in Being and Nothingness. Being and Nothingness offers a critique of Sigmund Freud's theory of**

**the unconscious, based on the claim that consciousness is essentially self-conscious. Sartre also argues that Freud's theory of repression is internally flawed. According to Sartre, in his clinical work, Freud encountered patients who seemed to embody a particular kind of paradox—they appeared to both know and not know the same thing. In response, Freud postulated the existence of the unconscious, which contains the "truth" of the traumas underlying the patients' behavior. This "truth" is actively repressed, which is made evident by the patients' resistance to its revelation during analysis. Yet what does the resisting if the patients are unaware of what they are repressing? Sartre finds the answer in what Freud calls the "censor". "The only level on which we can locate the refusal of the subject," Sartre writes, "is that of the censor.** The Holographic Principle, as conceived in current physics, applies to fields, and perhaps, even to more elementary entities, called strings and branes. These more elementary entities remain hypothetical at the time of this writing. There are many layers between the level of fields and that of neurons. We know that neurons are surrounded by a surface phospholipid membrane, which supports an electrochemical process that is fundamentally necessary to human experience. As the information supported by the membrane surface according to the Holographic Principle is proportional to its area, we should expect to find that the neural correlates of consciousness most clearly associated with those parts of the neuron that have the have a high proportion of neuronal surface area, and, in particular, receptive surface area. Dendrites account for an average of about 90% of the neuron is receptive surface area (Wong, 2002), and so would be expected to be the most important neural structures in information processes. We find this to be true **The Holographic Principle Theory of Mind MARK GERMINE Institute for Psycho science (Wikipedia).** Discordant-discrepant, dissonant-hard-line, implacable, incompatible, incongruous- inconsistent, inexorable-inflexible, inharmonious-intransigent, opposed, reluctant, unappeasable, uncompromising, unfriendly some of these expatiations and enumerations might sound, a more generalised interpretations is given by a Vedantin. Look at this arresting passage: The teacher continued to be silent. When addressed a second and third time he said: "I am teaching, but you do not follow. The Self is silence." The undetermined and unthinkable character of the Brahman is a consequence of the absolute's eternal and immutable nature. To concede the existence of a real universe is, from the Vedantin's point of view, to posit the existence of a reality apart from the Brahman. Nor can we simply identify a real universe with the absolute unless we are prepared to compromise its unchanging, absolute status. The criterion of authenticity is immutability. Reality never changes; only that which is less than real can appear to do so. Reality is constant in the midst of change. What this means essentially is that there is change although nothing changes. This impossible situation is reflected in the ultimate impossibility of change itself. That which does not exist prior to its changing and at the end, after it has changed, must be equally non-existent between these two moments. Although the world of change appears to be real, it cannot be so. Change, according to the Vedantin, presupposes a loss of identity. Reality cannot suffer transformation; if it were to do so, it would become something else and the real would be deprived of its reality. The immortal can never become mortal, nor can the mortal become immortal. The ultimate nature of anything cannot change. Change of any sort is merely apparent (vivarta); the world of change and becoming is a false super- imposition (adhyaropa, adhyasa) on the absolute. In cosmic terms, the mistake (bhranti) consists of the supposition that the real Brahman is the unreal universe and the unreal universe is the real Brahman. In microcosmic terms, it is the mistake of falsely conceiving the body, mind or even one's personality to be the Self. In the same way as the image of a snake is falsely superimposed on a rope, similarly the universe is falsely projected onto the real substratum, the Brahman. Ignorance is not merely a personal lack of knowledge, but a cosmic principle. As such it is called "Maya," the indefinable factor (anirvacaniya) that brings this mistake in identity about. The reality status of this cosmic illusion is also indefinable: on the one hand it is not Brahman, the sole reality; on the other hand it is not absolutely non-existent like a hare's horn or the son of a barren woman. Brahman is the source of world appearances only in the sense of being their unconditioned ground or essential nature. The universe is false not because it has no nature of its own but because it does have one. Just as the illusion of a snake disappears when one sees that it is nothing but a rope, similarly cancellation (badha) of the empirically real occurs when the absolute reality of the Brahman is realised. Thus, according to Vedanta, appearance implies the real, while the real need not imply appearance. To appear is essentially to appear in place of the real, but to be real is not

necessarily to appear. All things -exist because the absolute exists. It is their Being. Thus the very existence of phenomena implies their non-existence as independent realities. When they are known to be as they are, in the fullest sense of their existence, their phenomenal nature disappears leaving the ground of Being naked and accessible. This approach was validated by a critique of experience. The Vedanta established that space, time and the other primary categories of our daily experience can have no absolute existence. It was therefore necessary to make a distinction between relative truths — that accepted by the precritical common man — and an absolute truth discovered at a higher level of consciousness. The Saiva absolutist 17 rejects any theory that maintains that the universe is less than real. From his point of view a doctrine of two truths, one absolute and the other relative, endangers the very foundation of monism. The Kashmiri Saiva approach is integral: everything is given a place in the economy of the whole. It is equally wrong to say that reality is either one or diverse. Those who do so fail to grasp the true nature of things which is neither as well as both. "We do not" says Abhinavagupta, "base our contention that [reality] is one because of the contradictions inherent in saying that it is dual. It is your approach {paksa) that accepts this [method] While, if [duality and oneness] were in fact [to contradict each other], they would clearly be two [distinct realities]." The Vedantin, who maintains that non-duality is the true nature of the absolute by rejecting duality as only provisionally real, is ultimately landed in a dualism between the real and illusory by the foolishness of his own excessive sophistry (vacafadurvidya). Oneness is better understood as the coextensive unity (ekarasa) of both duality and unity. They are equally expressions of the absolute. Gopinath Kaviraj says: According to Sarikara, Brahman is truth and Maya is inexplicable (anirvacaniya). Hence the [Advaitin's] endeavour to demonstrate the superiority of Advaita philosophy is turned against his own system. It tarnishes the picture of its philosophical perfection and profundity. He cannot accept Maya to be a reality, therefore his non-dualism is exclusive. The whole system is based on renunciation and elimination and thus is not all-embracing.... By accepting Maya to be Brahman (brahmamayi), eternal (nitya) and real (satyarupa), Brahman and Maya [in the Tantra] become one and coextensive. **The Doctrine of Vibration: An Analysis of the Doctrines and Practices of Kashmir Shaivism Mark S. G. Dyczkowski. Concept of being from Shaivite point of view:** Solutional behaviour and stability analysis of such systems is of paramount importance, notwithstanding the assumptions made which are not farfetched, in that it throws light on the fact whether they could be candidates for yet unsolved problems, providing Autheticative determination, unimpeachable validation, incontrovertible establishment, apodictic evidence, reliable roll out, unfailing cinquecento **quattrocento trecento,** incontrovertible indication for the wide amplitudinal ramificatory usage of the thesis propounded. The world is now the other person's world, a foreign world that no longer comes from the self, but from the other. The other person is a "threat to the order and arrangement of your whole world...Your world is suddenly haunted by the Other's values, over which you have no control"(Satre: being and Nothingness) Sartre states that many relationships are created by people's attraction not to another person, but rather how that person makes them feel about themselves by how they look at them. This is a state of emotional alienation whereby a person avoids experiencing **their subjectivity by identifying themselves with "the look" of the other**. The consequence is **conflict**. In order to maintain the person's own being, the person must control the other, but must also control the **freedom of the other "as freedom**". These relationships are a profound manifestation of "bad faith" as for-itself is replaced with the other's freedom. The purpose of either participant is not to exist, but to maintain the other participant's looking at them. This system is often mistakenly called "love", but it is, in fact, nothing more than emotional alienation and denial of freedom. In essence we are always simulating and selling ourselves at the cost of our freedom. Both as a script writer and participant consciousness, human beings experienceâ complete subjective feelings and sometimes absolute subjective feelings, which if remains unchecked might lead to pathological ramifications. (Italics mine) Perspicuous forbearance, Sophisticated seasoning, Partcipational observation, Background actuality, Existential worldliness, Existential strife, Predicational anteriorities, Character consonance, Ontological consonance, Primordial exactitude, Phenomenological correlates, Accolytish representation, Atrophied asseveration. Anamensial alienisms, Anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orinetatilities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state form the bastion, pillar,

post, stylobate and sentinel of the classification scheme and doxa. appropriate and diagnostic, differentiating-discriminating, distinctive-distinguishing, emblematic- especial, essentially exclusive, fixed, idiosyncratic-inborn, inbred, indicative and individual, individualistic-inherent, individualizing and ingrained, inherent, innate, local, marked, native, normal, original, particularly- peculiar, personal, private, proper, regular, representative-singular, special, specific, symbolic, symptomatic- unique these form the classification and stratification measure that is suitable notwithstanding the constancy in many a case, as in gravity, which depends on the mass and distance for two given objects**.** Like Kant and Bergson, Deleuze considers traditional notions of space and time as unifying forms imposed by the subject. He therefore concludes that pure difference is non-spatio-temporal; it is an idea, what Deleuze calls "the virtual". (The coinage refers to Proust's definition of what is constant in both the past and the present: "real without being actual, ideal without being abstract.") Proust, Le Temps Retrouvé, ch. III: see the fourth line from the bottom of this page, or, in English translation, the thirteenth paragraph here: "I began to discover the cause by comparing those varying happy impressions which had the common quality of being felt simultaneously at the actual moment and at a distance in time, because of which common quality the noise of the spoon upon the plate, the unevenness of the paving-stones, the taste of the madeleine, imposed the past upon the present and made me hesitate as to which time I was existing in. Of a truth, the being within me which sensed this impression, sensed what it had in common in former days and now, sensed its extra-temporal character, a being which only appeared when through the medium of the identity of present and past, it found itself in the only setting in which it could exist and enjoy the essence of things, that is, outside Time. [...] **Nothing but a moment of the past Much more perhaps; something which being common to the past and the present, is more essential than both. [...] a marvellous expedient of nature had caused a sensation to flash to me—sound of a spoon and of a hammer, uneven paving-stones**—simultaneously in the past which permitted my imagination to grasp it and in the present in which the shock to my senses caused by the noise had effected a contact between the dreams of the imagination and that of which they are habitually deprived, namely, the idea of existence—and thanks to that stratagem had permitted that being within me to secure, to isolate and to render static for the duration of a lightning flash that which it can never wholly grasp, a fraction of Time in its pure essence. When, with such a shudder of happiness, I heard the sound common, at once, to the spoon touching the plate, to the hammer striking the wheel, to the unevenness of the paving-stones in the courtyard of the Guermantes' mansion and the Baptistry of St. Mark's, it was because that being within me can only be nourished on the essence of things and finds in them alone its subsistence and its delight. It languishes in the observation by the senses of the present sterilised by the intelligence awaiting a future constructed by the will out of fragments of the past and the present from which it removes still more reality, keeping that only which serves the narrow human aim of utilitarian purposes. But let a sound, a scent already heard and breathed in the past be heard and breathed anew, simultaneously in the present and in the past, real without being actual, ideal without being abstract, then instantly the permanent and characteristic essence hidden in things is freed and our true being which has for long seemed dead but was not so in other ways awakes and revives, thanks to this celestial nourishment"While Deleuze's virtual ideas superficially resemble Plato's forms and Kant's ideas of pure reason, they are not originals or models, nor do they transcend possible experience; instead they are the conditions of actual experience, the internal difference in itself. "The concept they [the conditions] form is identical to its object."[Desert Islands, p. 36.] A Deleuzean idea or concept of difference is therefore not a wraith-like abstraction of an experienced thing, it is a real system of differential relations that creates actual spaces, times, and sensations. [See "The Method of Dramatization" in Desert Islands, and "Actual and Virtual" in Dialogues. In fact it not as if the style and substance of science and philosophers have been entirely different. look at the conformality and consonance in the Gaussian Theory and Deleuze's statements on identity and the loss of it in the following passage…….That is, identity is a continuous process that finds its limit in exhaustion. As Deleuze will write, "…the logical relation of causality is inseparable from a physical process of signalling, without which it would not be translated into action. By 'signal' we mean a system with orders of disparate size, endowed with elements of dissymmetry; by 'sign' we mean what happens within such a system, what flashes across the intervals when a communication takes place between disparates. The sign is indeed an effect, but an effect with two aspects: in one of

these it expresses, qua sign, the productive dissymmetry; in the other it tends to cancel it" (dr, 20). An actualized entity just is such a signal-sign system, remaining in perpetual communication with the virtual multiplicity it actualizes, constituting its own elements and identity. Such is Deleuze's account of what Badiou refers to as the operation of the "count-for-one"… an operation that is immanent to being itself and energetic in character. yet how are we to discern the local nature of ontological situations, the manner in which there is no global, overarching global situation in which all local situations are embedded as if parts of a whole? in part the above passage already responds to this question. if series must be brought into relation to communicate in order for causality to occur, and if the resonance of series is always a matter of chance, of a throw of the dice, like Lacan's objet a where no series enjoys primacy of model over copy, then all we have are local situations without a total situation. Unlike Heidegger's analysis of Dasein where Dasein is always being-in-the-world, Deleuze's simulacra or actualizations are always divergent, enjoying only local relations. Heidegger argues that in order to understand the being of Dasein we must understand the manner in which Dasein is a being-in-the-world. By contrast, Deleuze's concept of multiplicity provides us with the principle of a new structuralism, of a new local ontology, that allows us to understand the immanent organization of a multiplicity without referring it to an embedding global space. as DeLanda so beautifully puts it in relation to the mathematician gauss, "…when gauss began to tap into these differential resources, a curved two-dimension surface was studied using the old Cartesian method: the surface was embedded in three-dimensional space complete with its own fixed set of axes; then, using those axes, coordinates would be assigned to every point of the surface; finally, the geometric links between points determining the form of the surface would be expressed as algebraic relations between the numbers. but gauss realized that the calculus, focusing as it does on infinitesimal points on the surface itself (that is, operating entirely with local information), allowed the study of the surface without any reference to a global embedding space. basically, gauss developed a method to implant the coordinate axes on the surface itself (that is, a method to implant the coordinate axes on the surface itself (that is, a method of 'coordinatizing' the surface) and, once points had been so translated into numbers, to use differential (not algebraic) equations to characterize their relations. as the mathematician and historian Morris Kline observes, by getting rid of the global embedding space and dealing with the surface through its own local properties, 'gauss advanced the totally new concept that a surface is a space in itself'" (intensive science and virtual philosophy, 11-12).according to the old Cartesian method, we can only outline the properties of a space by relativizing it to a global space in terms of which it is then mapped. by contrast, gauss is able to explore a space in terms of its intrinsic metric and organization as a local space, without referencing it to a whole of which it is conceptualized as a part. **The Image of Thought** "Image of thought "permeates both popular and philosophical discourse. According to this image, thinking naturally gravitates towards truth. Thought is divided easily into categories of **truth and error**. The model for thought comes from the educational institution, in which a master sets a problem and the pupil produces a solution which is either true or false. This image of the subject supposes that there are different faculties, each of which ideally grasps the particular domain of reality to which it is most suited. In philosophy, this conception results in discourses predicated on the argument that "**Everybody knows..."** the truth of some basic idea. Descartes, for example, appeals to the idea that everyone can at least think and therefore exists. Deleuze points out that philosophy of this type attempts to eliminate all objective presuppositions while maintaining subjective ones. Deleuze maintains, with Artaud, that real thinking is one of the most difficult challenges there is. Thinking requires a confrontation with stupidity, the state of being formlessly human without engaging any real problems. One discovers that the real path to truth is through the production of sense: the creation of a texture for thought that relates it to its object. Sense is the membrane that relates thought to its other. Accordingly, learning is not the memorization of facts but the coordination of thought with a reality. "**As a result, 'learning' always takes place in and through the unconscious, thereby establishing the bond of a profound complicity between nature and mind".** Deleuze's alternate image of thought is based on difference, which creates a dynamism that traverses individual faculties and conceptions. This thought is fundamentally energetic and signifying: if it produces propositions, these are wholly secondary to its development. Philosophy merges with ontology; ontology merges with univocity of being; analogy has always a theological vision; not a philosophical vision; one becomes adapted to

the forms of god; self and world; the univocity of being does not mean that there is one and the same being; on the contrary, beings are multiple and different they are always produced by disjunctive synthesis; and they themselves are disintegrated and disjoint and divergent; membra disjuncta.the univocity of being signifies that that being is a voice that is said and it is said in one and the same "consciousness". Everything about which consciousness is spoken about being is the same for everything for which it is said like gravity; it occurs therefore as an unique event for everything; for everything for which it happens; even tum tan tum;it is the ultimate form for all of the forms; and all these forms are disjointed; it brings about resonance and ramification of its disjunction; the vorticity of being merges with the positive use of the disjunctive synthesis, and this is the highest affirmation of its univocity like gravity; it is the eternal resurrection or a return itself, the affirmation of all chance in a single moment, the unique cast for all throws; a simple rejoin der for Einstein's god does not play dice; one being, one consciousness, for all forms and all times. a single instance for all that exists, a single phantom for all the living; a single voice for every hum of voices or a single silence for all the silences; a single vacuum for all the vacuousness consciousness should not be said without conjuring if consciousness is one unique event in which all the events communicate with each other; univocity refers both to what occurs to what it is said. The attributable to all states of bodies and states of affairs and the expressible of every proposition So univocity of consciousness means the identity of thé noematic attribute and that which is expressed linguistically and sense-fully. Univovity.We mean that it does not allow consciousness to be subsisting in aquasi state and but expresses in all pervading reality. (Reference: LOS: Deleuze) Deleuze sometimes spoke of getting up behind an author and "creating a monster", as if his reading methodology somehow consisted of a distortion of the philosopher's or author's thought. Certainly there is a case to be made for this with regard to the process of deterritorialization, where something is wrested from a territory and deterritorialized upon a new territory, like the animal paw that is deterritorialized from the earth and reterritorialize on the branch. however, it seems to me that the more interesting aspect of Deleuze's approach to other philosophers and art is not so much his "monstrous creations" (i seldom find them particularly monstrous), but rather their Gaussian or Riemannian style, where he explores them in terms of their own internal organization and metric, without reducing them to something alien such as history, society, biology (reductivism), or the signifier. What we find in Deleuze's approach to phenomena is a Gaussian technique. for instance, take Deleuze's books on cinema, his book on Francis bacon, or his study of sacher-masoch. In the first instance, Deleuze's carefully separates cinema from narrative and the signifier, studying it in terms of its specific organization pertaining to the production of images. in the case of bacon, Deleuze doesn't look for an underlying narrative or "meaning", but instead studies the manner in which bacon composes and organizes his images and lines, both in terms of their production and actuality, so as to liberate a logic of sensation. finally, in approaching sacher-masoch, Deleuze allows Sacher-Masoch's novels to speak for themselves in terms of their desire and relation to pain, stalwartly refusing to reduce masochism to the complement of sadism. in each case, we have a local exploration of a "space" of multiplicities that is extremely precise. Sean Bowden the priority of events: **Deleuze's logic of sense sean Bowden, the priority of events: Deleuze's logic of sense, Edinburgh university press, 2011, 296pp., $40.00 (hbk), isbn 9780748643646 Reviewed by David Scott, Coppin state university and Gaussian spaces and multiplicities a note on Gaussian spaces and multiplicities posted by larval subjects under uncategorized.** What you "see" is not always what you "get." Many people mistakenly take their own visions literally without "seeing through" the various possibilities that are outside their belief system, knowledge or skill base. Deeper reality is not remote in the physical sense but in a psychological sense. The archetypes of the collective unconscious are arrayed behind the scenes of current worldwide conditions, of crisis and confusion. They mirror our own states back at us, whether we perceive them as such or interpret them plausibly or not. The noise of ordinary consciousness and beliefs drowns out the signal. Unconsciousness is the background of our ordinary awareness. Our organism is very much at the center of such effects. The organismic source is our human bodies and the focus of human consciousness. The fantasy principle dethrones (e) reality, but can be dissociative or compensatory. The human mind is (=) a meme-Scape. Pre-conceived concepts vie with (e&eb) structures, concepts with images. Like scientists who ignore (e) assumed truths, we leapfrog over (e&eb) our beliefs and personality deficits, claiming (eb) idiosyncratic imagination is (=) literal

reality. It couldn't be further from the truth and symbolism is utterly lost. The metaphor that might heal us enslaves (e) us. Perhaps images like the holographic universe have an implicate order. Can we have a sense of the cosmos in the world without projecting myriad fantasies on it that we embrace literally? Has the world become so horrible it is unreasonable to be realistic? We may need to look at our drives and wishes, rather than the fantasy content. Psyche constructs reality. Our experience of so-called reality is always mediated by our image of it. Even if all the contents of the psyche are real, that doesn't mean they are realistic. That psyche is real is still a radical proposition, but psychic politics certainly color the self-image and ideas of everyone. We observe and participate with images. It is not a question of nature or nurture (genes alone or experience alone). Rather, everything is (=) both. We inherit (e&eb) the structures that make our experience (=) what it is. But the structure itself is "empty," and each human culture "fills" it with its own specific adaptations. It is difficult to define an archetype and set boundaries that distinguish it from others. In a hologram each part contains all the information but in lower resolution. Archetypes have this holographic quality. There are patterns within patterns within patterns. Some overlap with others, and some are nested inside others. Archetypal realities, passed on through DNA, are expressed in distinctive neuronal tracts in the brain. They include customs and laws regarding property, incest, marriage, kinship, and social status or roles; myths and legends; beliefs about the supernatural; gambling, adultery, homicide, schizophrenia, and the therapies to deal with them. A mythic and visionary language of immediate experience encompasses themes of deepest, highest, and ultimate concern. Most fantasy-based individuals are at a complete loss to coherently explain their own conventional behavior much less anomalous events and their deep meaning, much less the cultural unconscious or mythological unconscious matrix. But they try, and become utterly entrenched in their belief that they are right about the nature of the world and reality. We have pseudo-memories about (e&eb) our personal lives. Why not more so for our collective life? The subject matter often revolves around catastrophe, creation and the mythopoeic forces of mankind. Ignorant of such dynamics, interpretive mistakes and displaced psychic contents proliferate into errors of fact. Propaganda, media distortions, memes, and disinformation compound the social problem of misapprehension further. Shameless self-promotion by personalities of such ideas leads to cults. They make up myths about the myths of by-gone eras. Roiling unconscious images can be fatally confusing. Thought illusions culminate in projections and projections of mythology. Jung suggested symbols live only as long as they are pregnant with meaning. Philosophy arose from criticism of myth, from discussing and challenging it. In science, we criticize, reject and eliminate theories. At the edge of the abyss of the unknown, new signs and symbols emerge. Credible theories and paradigms must include biology, physics, and neurophysiology. One of the reasons people "see God", or a guru, or anomalies may be because our brains are constructed to see reality through the eyes of others. There are heaps of mirror neurons which are there to make us feel the 'other'. Mirror neurons do for psychology what DNA did for biology. They provide a unifying framework and help explain a host of mental abilities. As in the psychochemical processes of empathy or falling in love, a complex feedback loop sustains a state of mind. But when we empathically transpose ourselves into someone else's position, we expose ourselves to that reality -- cognitively and emotionally. The unconscious complicates empathy, both ways. Mirror neurons might well play a role in bonding, language and self-awareness. Naively, we take too much as self-evident. But 'seeing' does not always 'believe', though many make this error or leap in logic and formulate their choices and future accordingly. Yet, there is only one way to learn what consciousness is. Experience. But we have no satisfactory explanatory edifice for consciousness. Would such a theory release in each of us our own inner knowledge of the creativity of our own consciousness, and its infinite possibilities? The problem is trying to define a verb, a dynamic, as if it were a noun. But we do recognize the effect of consciousness. It functions to mediate states of consciousness, high and low psychobiological arousal. Consciousness is the subconscious lifted up by the physical body. When the body fails, the consciousness collapses back into the subconscious. All our thoughts come from the subconscious which can see our intentions but not our world. This relates somehow to intention being imaginary and not of the physical frictionized world (King). Gerald Edelman postulates that the flows of information in the brain are mediated through're-entrant' feedback loops. As evolution provides new cognitive functions, new re-entrant loops are established. Even language itself is an archetype -- a chaotic field of dynamic associations. A subtle net of

tropes, grammar, symbols, and meaning, the program language begins in limbic resonance. Some phenomena generate their own language patterns, nomenclature, and internal coherence of meaning and representation. n a holographic universe, even time and space could no longer be viewed as fundamentals. Because concepts such as location break down in a universe in which nothing is truly separate from anything else, time and three-dimensional space would also have to be viewed as projections of this deeper order. At its deeper level reality is a sort of super hologram in which the past, present, and future all exist simultaneously. This suggests that given the proper tools it might even be possible to someday reach into the super holographic level of reality and pluck out scenes from the long-forgotten past. Or not. A fantasy of such penetration or phenomenon inside the head is not the same as that penetration. **Jung in the 21st Century: Synchronicity and science By John Ryan Haule Mind Control Countermeasures.** Adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrograde, inimically inverse, violatively unsimilar, diversely dissimilar some statements in Vedanta, it is not superfluous or redundant. It is evolutionary. Autheticative determination, unimpeachable validation, incontrovertible establishment, apodictic evidence, reliable roll out, unfailing cinquecento **quattrocento trecento,** incontrovertible indication of the evolutionary process towards understanding is ever present in all, its thematic and discursive form. it is the relentless holding on to a theory this is despicable albeit the theory is not. Look at this paragraph. It is the 'I am the body/mind' belief that gives rise to the 'I am not (e) the world' belief. These two beliefs are (=) co-created."Consciousness projects (e&eb) the appearance of the mind, body and world by taking the shape of thinking, sensing and perceiving." "Attention is Consciousness with an object. When the object vanishes, attention simply remains what it always is, Consciousness.""There is (=) no purpose to (e) meditation. The purpose is already accomplished." "Everything that is experienced is experienced by, through, in and as Consciousness." "The seen cannot be separated from seeing and seeing cannot be separated from (e) Consciousness." "The Reality of any experience is not hidden in the appearance; it is expressed by the appearance." "Once we see that everything is Consciousness... Maya still dances, but it is a dance of love not seduction." There are some truly excellent sections such as a long one taking, as a starting point, the following quotation from the painter Cézanne: "Everything vanishes, falls apart, doesn't it? Nature is always the same but nothing in her that appears to us lasts. Our art must render the thrill of her performance, along with her elements, the appearance of all her changes. It must give us a taste of her Eternity." Rupert begins: "That statement must be one of the clearest and most profound expressions of the nature and purpose of art in our era." And he goes on for some 14 pages to elaborate on this claim, examining the nature of the 'elements' Existence, appearance and Consciousness and their relationship. "This Reality is the support or ground of the appearance. The appearance may be an illusion, but the illusion itself is real. There is an illusion. It has Reality." He refers to the rope-snake metaphor and says that: "We do not see anything new. We see in a new way." "So that we know that nature is real, that there is something present, that there is a reality to it, even if everything that appears to us is insubstantial and fleeting." **The Transparency of Things' by Rupert Spira Book Review by Dennis Waite The following is a review of Rupert's book: 'The Transparency of Things: Contemplating the Nature of Experience'. There is an essay from the book here and the essay on Cézanne may be read at the awakened eye website For models see Mitigation Of Beam-Induced Backgrounds, Multi-Particle Azimuthal Correlations, Search For Extra Dimensions, Multi-Higgs-Boson Cascade, Top Quark Pair (Tt) Production Charge Asymmetry, $\Delta Y \equiv Yt - Yt^{-}$, Reflects The Asymmetry In $Tt^{-}$ Production, Ultra-Relativistic Electrons And Other Letters Of Interest: A Synecdochal Syncretism: Et Lux In Tenebris Lucet: Light Shines In The Darkness Models.** What then is the relation of Higgs boson and consciousness? Functionalist approaches generally assume that conscious experience appears as a novel property at a critical level of computational complexity. On the surface this would seem to deal with issues 1 and 3, however a conscious threshold has neither been identified nor predicted, and there are no apparent differences in electrophysiological activities between non-conscious and conscious activity. Regarding the nature of experience (why we are not unfeeling "zombies") functionalism offers no testable predictions. Problem 2) of 'binding' in vision and self is often attributed by functionalists to temporal correlation (e.g. coherent 40 Hz), but it is unclear why temporal correlation per se should bind experience without an explanation of experience. As

functionalism is based on deterministic computation, it is also unable to account for Penrose's proposed non-computability (4), or free will (5). Something may be missing. To address these issues, various proposals have been suggested in which macroscopic quantum phenomena are connected to the brain's known neural activity. For the problem of unitary binding, Marshall (1989) suggested that coherent quantum states known as Bose-Einstein condensation occurred among neural proteins (c.f. Penrose, 1987; Bohm and Hiley, 1993; Jibu and Yasue, 1995). Pre-conscious to conscious transitions was identified by Stapp (1992) with collapse of a quantum wave function in pre-synaptic axon terminals (c.f. Beck and Eccles, 1992). In another proposal, protein assemblies called microtubules within the brain's neurons are viewed as self-organizing quantum computers ("orchestrated objective reduction - Orch OR" e.g**. Penrose and Hameroff, 1995; Hameroff and Penrose 1996a; 1996b; c.f. Hameroff 1997; 1998a; 1998b; 1998c; 1998d). Stability analysis of such theories would certainly provide a gateway for further bringing in the "reality" in space time.** String theory suggests that the graviton particle would not be detected from a particle collision because it would have "jumped" into another dimension. So, if the graviton exists and the theory is correct, there should be a "gap" in the energy profile of all the particles ejected from the collision. To understand this, think of breaking the pack of balls in a game of pool. The total energy of all the balls after the collision will depend on the force with which the cue ball was hit. So, after the collision, the energy of all the balls is measured and added-up to make sure it is the same as the energy of the cue ball. BUT - imagine that there is some energy missing, because one of the pool balls has "gone missing" - into a higher dimension. (POSTED BY THEREALJEFFHAL**) Talbot says** "If the universe is a hologram, in some sense it suggests that there may be two very drastically different levels of reality: the concrete reality that we see when we look at [things]... and at some deep level there's a level of reality where everything dissolves into an ocean of energy that is holographically interconnected.". A theory of everything must encompass all the variations however contradictory it may be. Nor need any theory be sent in to grand limbo of oblivion and hibernation or a differentiated groundless less ness because there are some irregularities which are like audit discrepancies need to be rectified. Although the language of religion and science seem different, both are talking about the central order of nature relative to DNA and information, as is Peirce, Petrus, Bacon, and pharaonic Egypt. This path-ordering, active information suggests that the central pattern or law of our holographic cosmos may be modeled on the biophysical battleground of gene regulation in bacteriophage Lambda, one of nature's most efficient and highly evolved mechanisms identified by experimental praxis. Egyptian texts support that the Lambda lifestyles of lysogeny and lysis can be understood as two paths to two living systems, photosynthesis and its reversal chemiluminescence, the cosmic key for evolution of mind and continuity. The texts support that the Lambda genome is the world-heart of two ways. The viral lifestyle of lysogeny controlled by cI protein results in matter, classical spacetime, our world of projected shadow, photosynthesis, sexual genesis or vertical gene transfer, mind in the human body, and the imperfect continuum. In contrast, the viral lifestyle of lysis controlled by cro protein and lactose metabolism results in transformation to energy, the quantum, chemiluminescence, asexual genesis or HGT, mind in the cosmos, and the perfect continuum. Some scientists believe our cosmos is entangled with another (Chown 2007). On the holographic quantum level, perhaps this is just the competitive entanglement of cI and cro proteins cycling in perpetual motion—an inherent law of nature elucidating the assembly of things. In light of modern experiments, physicist John Wheeler (1988) speculated that the **act of conscious observation functions with quantum mechanics, catalyzing a probability or outcome**. So, if an observing consciousness has knowledge of the classical path and its magnetic field orientation, an act of measurement or choice might entangle space to reify the holographic quantum paths to a perfect continuum. Peirce guesses that to be drawn into a new system, a particle must have the right mass, the right velocity in the right direction, the right attraction, and it must present itself at the right point (EP 1. 270). Similarly, a finite probability exists for any particle that approaches the black hole event horizon to bounce back, dependent on the incoming particle's energy, its charge, and its projection of the orbital momentum on the axis of rotation of the black hole (Kuchiev 2003; 2004; 2004a), the same conditions in Egyptian texts (King 2004; 2006).Only an approximate conclusion remains: chance may be knowledge of the cosmos' holographic mode of operation and magnetic fields; habit breaking may result in the transformation of energy by a violation of the second kind; and evolution of mind or

energy might be possible through HGT via a cosmic viral code of Lambda-genesis. “Order is simply thought embodied in arrangement;” (CP 6.490). Accordingly, the original meaning of the signs may relate to a viral biophysics of crystallized mind for human evolution via a natural chemical pathway to a perfect continuum. Again, this semiotics of evolvability and order recurs in time with the chaotic dream of reason that projects a pervasive falsification of our perceptual world. As Wallace Stevens envisions in “Connoisseur of Chaos”, perhaps “the pensive man may see.” **Peircean Process Metaphysics Origin of Science Quantum and Classical Laws Ancient Egyptian Evolutionary Biophysics** Adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrograde, inimically inverse, violatively unsimilar, diversely dissimilar are the views of epistemologists in the analysis of raison d’etre of knowledge. Epistemology is the study of knowledge and those things closely related to it: justification, what it takes for you to be justified, the relation between knowledge and justification, whether you can have any justified beliefs at all, and if so, how you come to know (or justifiably believe) things, how you can use what you know (or justifiably believe) to come to know (or justifiably believe) other things, and whether and why it's valuable to know instead of merely having justified and true beliefs. The literature on epistemology is vast. Here's a very brief summary of some epistemological discussions. Concerning knowledge, many epistemologists think knowledge is justified true belief, where the justification you have is linked to the truth of the matter in the right kind of way, though what this way is a matter of debate; some epistemologists think knowledge can't be analyzed this way. Concerning justification's relation to knowledge, some epistemologists think we don't need to be justified to know, and some think we do need to be. Concerning what it takes to be justified, some epistemologists think that what it takes for you to be justified are only factors internal to the believer (Internalists). Others think it takes an external factor, like reliable or well-functioning cognitive faculties (Externalists). Skeptics argue that we can't have any justified beliefs at all, and many epistemologists reply to the skeptic's arguments. Concerning how we use what we know (justifiably believe) to come to know (justifiably believe) other things, some epistemologists (Foundationalists) argue that there are bedrock propositions that we know (justifiably believe), and we build our knowledge (justified beliefs) on those. Others (Coherentists) argue that there aren't bedrock propositions; rather, a set of beliefs is justified as a whole, and several beliefs can be mutually supporting. Concerning the value of knowledge, some argue that knowledge is intrinsically valuable. Others have argued that knowledge is valuable only because of the role it plays in practical reasoning, and others argue that knowledge isn't more valuable than justified and true belief, but there are other epistemic states such as understanding, that do have value above their proper subparts. **Edited y Matthew McGrath (University of Missouri, Columbia). It is not as if there is an anathema or misnomer in the philosophy or physics. Infact most of the pnysicists have used philosophical ideas towards the end of consummation of their their and their concomitant justification thereof.** The eternal return produces becoming-active. It is sufficient to relate the will to nothingness to the eternal return in order to realize that reactive forces do not return. However far they go, however deep the becoming-reactive of forces, reactive forces will not return. The small, petty, reactive man will not return. Affirmation alone returns, this that can be affirmed alone returns, joy alone returns. Everything that can be denied, everything that is negation, is expelled due to the very movement of the eternal return. We were entitled to dread that the combinations of nihilism and reactivity would eternally return too. The eternal return must be compared to a wheel; yet, the movement of the wheel is endowed with centrifugal powers that drive away the entire negative. Because Being imposes itself on becoming, it expels from itself everything that contradicts affirmation, all forms of nihilism and reactivity: bad conscience, ressentiment..., we shall witness them only once. [...] The eternal return is the Repetition, but the Repetition that selects, the Repetition that saves. Here is the marvelous secret of a selective and liberating repetition. There is no need to remind the reader that neither the image of a centrifugal movement nor the concept of a negativity-rejecting repetition appears anywhere in Nietzsche's writings, and indeed Deleuze does not refer to any text in support of this interpretation. Further, one could highlight that Nietzsche never formulates the opposition between active and reactive forces, which constitutes the broader framework of Deleuze's interpretation. For some years, Marco Brusotti has called attention to the fact that Deleuze introduced a dualism that does not exist in Nietzsche's writings. To be sure, the German philosopher describes a certain number of **"reactive" phenomena**

**(for example, in the second essay of the Genealogy of Morality, § 11, he talks about "reactive affects" [reaktive Affekte], "reactive feelings" [reaktive Gefühlen], reactive men [reaktive Menschen]); but these are nonetheless the result of complex ensembles of configurations of centers of forces that remain in themselves active.** Neither the word nor the concept of "reactive forces" ever appears in Nietzsche's philosophy. **After the discovery of the two principles of thermodynamics began a debate about the dissipation of energy and the thermal death of the universe which framed the modern renewal of the debate between the linear and circular conceptions of time**. Scientists such as Thomson, Helmholtz, Clausius, and Boltzmann and-by way of Kant, Hegel and Schopenhauer-philosophers such as Dühring, Hartmann, Engels, Wundt and Nietzsche have tried to address this problem by using the force of scientific argumentation and of philosophical discussion. Whoever believed in an origin and a final end to the motion of the universe (be it in the physical form of the gradual loss of heat, or in the metaphysical form of a final state of the "world process"), relied on the second principle of thermodynamics or on the demonstration of the thesis of Kant's first cosmological antinomy. On the contrary, those who refused to admit a final state to the universe used Schopenhauer's argument of infinity a parte ante-according to which if a final state were possible, it should already have established itself in the infinity of time past-to propose henceforth a number of alternative solutions. Scientists would propose the hypothesis that energy could have re-concentrated after a cosmic conflagration, thus reversing the tendency towards dissipation. Those belonging to the monistic and materialistic tradition relied on the first principle of thermodynamics and on the infinity of matter, space and time, and regarded the universe as an eternal succession of new forms. A certain critical agnosticism was widespread among scientists and philosophers, oftentimes through a reaffirmation of the validity of Kant's antonymic conflict, this movement avoided to take a stand on specifically speculative issues. Other German philosophers, like Otto Caspari, or Johann Carl Friedrich Zöllner, had reintroduced an organicist and pan-psychical conception of the universe, investing atoms with the ability to escape any state of balance. Indeed, it is probably one of Otto Caspari's works, The Correlation of Things (Der Zusammenhang der Dinge. Gesammelte philosophische Aufsätze (Bleslau: Trewendt, 1881)), which awakened Nietzsche's interest for all things cosmological, in that summer of 1881, in Sils-Maria. addressing Schopenhauer and Eduard von Hartmann's mystical pessimism according to which the world is the creation of a stupid and blind essence (which, after having created the world by mistake, comes to the realization that it had made a mistake and strives to return it to nothingness) Caspari stresses that it is nothing short of mystical to imagine that the world may have been borne out of a an originary and undifferentiated state. Where would it have drawn the first impulse? But, continues Caspari, even if the world had received this first impulse from some deus ex machina, there is no doubt that, in the temporal infinity of past time thus far, it would have either attained the end of the process (but this is impossible because the world would then have ended), or it would be necessarily bound to repeat indefinitely this original mistake, and the entire process that accompanies it. But then, what is the process of the world? We must now take one more step back and understand further the process of the world according to von Hartmann. Hartmann objects that the regressive movement postulated by Schopenhauer is possible only in thought: it remains nothing more than an "ideal postulate" with no real object and which "does not teach us anything about the real process of the world that unfolds in a movement contrary to this backwards movement of thought" (Hartmann, Philosophie des Unbewussten, third edition (1871): 772). Hartmann affirms that if unlike Schopenhauer one admits the reality of time and of the world process, one must also admit that the process must be limited in the past and therefore that there must be an absolute beginning. In Hartmann's mind, failure to do so would result in positing the contradictory **concept of an accomplished infinity**: "The infinity that from the point of view of regressive thinking, remains an ideal postulate, which no reality may correspond to, must, for the world, whose process is, on the contrary, a progressive movement, open up to a determinate result; and here the contradiction comes to light" (Hartmann (1871): 772). What really "comes to light" in this passage is the fact that Hartmann does not provide a demonstration but a petitio principii. Indeed, the concept of the world process analytically contains the concept of a beginning of the world. In all rigor, it is therefore impossible to demonstrate these concepts with reference to each other. Secondly, Hartmann's view that one is bound to accept the reality of the world process even if one rejects the ideality of Schopenhauer's time is mistaken. Hartmann believes that if time is

real there must be a world process with both an **absolute beginning and an absolute end. Without any justification, Hartmann jumps from Schopenhauer's negated time to oriented time.** Hartmann's view is that the world process leads into a final state absolutely identical to the initial state. However, it follows from this that even as the cosmic adventures of the unconscious come to a close, we are still haunted by the specter of a new will and of another beginning of the world process. This exposes a serious internal flaw of Hartmann's system insofar as it jeopardizes the possibility of a final liberation from existence and suffering. This is why in the last pages of his work, "On the Last Principles," he painstakingly calculates the degree of probability of a reawakening of the volitional faculty of the unconscious. Insofar as the will is entirely free, unconditioned and a-temporal, the possibility of a new volition is left to pure mathematical chance and is therefore ½. Hartmann further stresses that if the will were embedded in time, the probability of the repetition would amount to 1 and the process of the world would be bound to begin again, in an eternal return which would completely preclude the possibility of a final liberation. Fortunately, this is not the case since-according to Hartmann's remarkable logic-the world-process develops through time, but the original will is outside of time. In fact, one may even affirm, along the lines of Hartmann's peculiar theory of probability, that every new beginning gradually reduces the probability of the next beginning: let n be the number of times that the will is realized, the probability of any new realization is ½ n. "But it is clear that the probability ½ n diminishes as n increases, in a way that suffices to reassure us in practice" (Hartmann (1869): 663). In his famous work entitled **On the Conservation of Force (1847), Hermann von Helmholtz** had divided the totality of the energy in the universe between potential energy and kinetic energy and affirmed the reciprocal convertibility of the two. In 1852, William Thomson pointed out that there exists a sub-ensemble within kinetic energy, heat, which, once it has been generated, is no longer entirely convertible into potential energy-or into any other form of kinetic energy. Considering that the (partial) reconversion of heat into labor is possible only in situations that present a disparity in temperature, and that heat tends to pass from warmer to cooler bodies by spreading on an even temperature level through space, Thomson concluded that the universe tends towards a final state where any energetic transformations, every movement and every form of life will cease. Caspari's atoms (which bring to mind those in Leibniz's Monadology) resemble some sort of biological monads, endowed with internal states. For Caspari, every atom obeys the ethical imperative to participate in the conservation of the general organism and its movement does not only follow the simple physical kind of interaction but also an a priori law ensuring. If a balance of forces had been attained at any moment, this moment would still be going on: therefore, it never happened. The present state contradicts this proposition. Supposing that a certain state rigorously identical with the present state had, one day, existed, this supposition is not refuted by the present state. As one of the infinite possibilities, it is necessary that the present state had been given anyway, since until now an infinite period of time has already unfolded. **If equilibrium were possible, it must have occurred**; and if the present state has already taken place, then so too the one that preceded it as well as the one preceding that one. Therefore it has already taken place a second time, a third time and so on. And likewise it shall take place again a second time, a third time. Innumerable times forwards and backwards. This amounts to saying that all becoming occurs within a repetition of an innumerable number of absolutely identical states. [.] The immovability of forces, their equilibrium is a conceivable case, but it has not occurred. As a result the number of possibilities is greater than the number of realities. -The fact that nothing identical recurs may be explained not thanks to chance, but only thanks to an intention infiltrated within the essence of force. Indeed, supposing an enormous amount of cases, the random occurrence of the same combination is more probable than the same combination never recurring. The quantum of force in the universe is determinate and not "infinite": let us beware [hüten wir uns] from such conceptual extravaganza! Therefore the number of situations, modifications, combinations and developments of this force is doubtless enormous and practically "immeasurable," but in any case this number is determinate and not infinite. On the other hand, the time in which the universe exerts its force is infinite. That is to say, that force is eternally identical and eternally active: -until the present instant an infinity has already taken place, that is to say that all possible developments must have already taken place. Consequently, the present development must be a repetition and therefore both this that was born from it and this that shall be born from it and so on both forwards and

backwards. Everything has taken place an innumerable number of times because the overall situation of all forces always recurs (FP 11 [202] of 1881). Nietzsche regards the mechanistic vision as more plausible and less anthropomorphic than organicism. However, faced with the two major **cosmological models of his time, the mechanistic model and the organic model, Nietzsche wishes to return its polymorphous, proteiform, unstructured** and chaotic character to nature of which the perfectly non-theological and non-teleological theory of eternal return is the strongest seal. This is the first of the "new battles" which come to whoever is aware of the consequences of the death of God: take any antropomorphism away from nature. In the preparatory papers, the third book of the Gay Science is entitled "Gedanke eines Gottlosen / Thoughts of a Godless One." Aphorism 109 of this book, which immediately follows the famous aphorism against the shadows of God, **Boltzmann's theory belongs to the third phase of the debate on thermodynamics and cosmology. The publication of Thomson's brief paper "On a Universal Tendency in Nature to the Dissipation of Mechanical Energy" of 1852 signals the beginning of the scientific controversies on the problem of the dissipation of energy and opened the first phase of the debate, which was announced in the Reflexions on the Motor Powers of Fire by Sadi Carnot and whose conclusion is represented by Clausius's recapitulative article on the concept of entropy in 1865.[44] The second phase started in 1867 when, at the forty-first congress of German scientists and doctors, Clausius gave a lecture on "The Second Principle of the Mechanistic Theory of Heat," where he applied the results of his research on thermodynamics to the universe**. It is true that in his famous lecture of 1854, Helmholtz had already presented the cosmic consequences of the second principle, but Clausius' contribution had a strong impact on German culture. This is because in this lecture he robustly rejected the possibility to consider the universe as an eternal and self-renewing circle, an ewiger Kreislauf in which force and matter are in constant transformation, as was heretofore affirmed by the materialism of the scientists and philosophers, and he did so in the name of the second principle of thermodynamics. In this way, the debate on the principles of thermodynamics gained great importance in European Culture starting in 1867. In the two first phases, it is Thomson's mechanism that predicts the thermal death of the universe. In the third phase, on the contrary, the meaning of the term mechanism changes radically.[46] In accordance with the apocalyptic climate of this period dominated by the "rebirth of idealism," the "overcoming of scientific materialism" and the "bankruptcy of science", the mechanistic paradigm which had accompanied the birth of modern science became challenged on **account of the second principle of thermodynamics**. According to the theorem of the quasi-periodicity of the motions of mechanical systems demonstrated by Poincaré as part of the problem of the three bodies (1890), a mechanical system must evolve according to a quasi-periodical movement and consequently it must always return-sooner or later-to the initial state. **An easily established theorem informs** us that a limited world obeying solely the laws of mechanics shall always pass through a state closely similar to its initial state. On the contrary, according to established experimental laws (supposing we grant them absolute value and wish to push their consequences to the end), the universe is directed towards a final state, which once it is attained, it shall not be able to escape. In this final state, which shall be like a sort of death, all material bodies shall be at rest at the same temperature. **Poincaré's theorem seems** therefore incompatible with the second principle of thermodynamics, which predicts a unidirectional movement of all natural phenomena until the whole universe is brought to a total standstill. Wilhelm Ostwald and the entire energeticist school of thought contended that the principles of thermodynamics were fundamentally new, and could not be re-incorporated to traditional physics and that they should serve as a basis for a new science that regards the qualitative diversity of energy and its tendency to degradation as its axioms. Against energeticism and in an effort to bring entropic phenomena back into the theoretical framework of mechanism, Ludwig Boltzmann introduced the concept of probability in physics, not as an instrument of calculation, but as an explicative principle. In Boltzmann's statistical thermodynamics, the increase of entropy assumed by Clausius is re-interpreted as an increase in molecular chaos. As a result, it becomes possible to explain mechanistically the evolution of closed systems endowed with increasing entropic value, without it committing us to granting absolute value to the second principle of thermodynamics. **Moreover, one no longer needs to fear the thermal death of the universe insofar as the state of equilibrium will in principle** never be complete, but rather will be attained only statistically, leaving open the

possibility of fluctuations towards less probable states. Boltzmann's critics remarked that this hypothesis involved two paradoxes called the objection of reversibility (Umkehreinwand), and that of repetition (Wiederkehreinwand). I shall only address here the second one since it coincides with the theory of the eternal return. Based on Poincaré's theorem quoted above, Ernst Zermelo objected to Boltzmann that his model of the universe suggested that after a finite (if admittedly very long) time the system would return to its initial position. In his first response to Zermelo, Boltzmann avoids committing himself directly to cosmological questions and he only observes that, in the case of concrete thermodynamic systems, the time of recurrence may be extremely long. For example, in normal conditions of pressure and temperature, one-centimeter cube of gas requires 1010 years to reach a molecular configuration identical to the original one! However, following a response by Zermelo, Boltzmann wrote a new article where he outlines a cosmological picture that he will re-use later in his conclusion to his famous Lectures on Gas Theory. In this cosmological picture, Boltzmann considers the universe as a closed system with constant entropy, within which some fluctuations occur, creating islands of negative entropy. Our solar system originates in one of these fluctuations. As Clausius correctly pointed out, the entropy of our solar system increases constantly as the solar system gets closer to the state of chaos and of the thermal death of the rest of the cosmos. However, in other zones of the universe, some new fluctuations and new islands appear, so that thermal death is never generalized. Here we are given a grand cosmic image, in which the solar system and the sparkle of life that was lit on planet earth are only a fluctuation of order from within a dominant entropic tendency. Life, and the order on which it is based are exceptions, transitory forms taking place in the realm of the shapeless, they are islands of the cosmos that will soon be re-absorbed into chaos. According to Poincaré's theorem, our island will have to be reborn, to develop and die innumerable times in a strictly identical fashion. This happened an infinite number of times during the past eternity and it shall take place again an infinite number of times in the eternity to come. **Boltzmann accepts the "paradox" of recurrence-that is the eternal return of the same-as a legitimate consequence of the probabilistic conception of thermodynamics**. It may be rejected for ethical reasons, it may be stored away as an abstract speculation or dismissed along with other cosmic fantasies, but it cannot be rejected on the basis of any rigorously scientific viewpoint. **The Eternal Return: Genesis and Interpretation BY PAUL D'IORIO(excerpts). Nietzsche found the template of a materialistic cosmology based upon the first principle of thermodynamics. During the same year, he could have found a model of an organistic solution to the problem of thermal death of the universe as well as a discussion on the conformation of space in Zöllner's book On the Nature of Comets** (Johann Carl Friedrich Zöllner, Über die Natur der Kometen. Beiträge zur Geschichte und Theorie der Erkenntnis (Leipzig, Engelmann, 1872), 299 f. and 313 f.); Assumed Association In Space Time Of Particles Or Men Leads To Detrimental Ramification Be In Terms Of Progress Of “Subject Matter” Or In The Pathological Disequilibrium Of Gratification Deprivation Complex Which Again Has Mental Turbulence In Case Collective Consciousness Approve It (Raaga) Or Pernicious Implications In Case The Theory Is Not Accepted; After All What Is A Theory: Set Of Correlations And Causations Which Are “Tested” Against A Collective Consciousness Background, Which Yields “Results’ Which Are Themselves In “Question” Due To The Very Perception And The Simulation Prospective. The Definition Of Quantum Theorists' Terms, Such As Wavefunctions And Matrix Mechanics, Progressed Through Many Stages. For Instance, Erwin Schrödinger Originally Viewed The Electron's Wavefunction As Its Charge Density Smeared Across The Field, Whereas Max Born Reinterpreted It As The Electron's Probability Density Distributed Across The Field. Although The Copenhagen Interpretation Was Originally Most Popular, Quantum Decoherence Has Gained Popularity. Thus The Many-Worlds Interpretation Has Been Gaining Acceptance Moreover, The Strictly Formalist Position, Shunning Interpretation, Has Been Challenged By Proposals For Falsifiable Experiments That Might One Day Distinguish Among Interpretations, As By Measuring An AI Consciousness Or Via Quantum Computing. More Or Less, All Interpretations Of Quantum Mechanics Share Two Qualities: They Interpret Formalism—A Set Of Equations And Principles To Generate Predictions Via Input Of Initial Conditions They Interpret A Phenomenology—A Set Of Observations, Including Those Obtained By Empirical Research And Those Obtained Informally, Such As Humans' Experience Of An Unequivocal World Two Qualities Vary Among Interpretations: Ontology—Claims About What Things, Such As Categories And Entities,

Exist In The World Epistemology—Claims About The Possibility, Scope, And Means Toward Relevant Knowledge Of The World. Now How Does A Person In A Crowd Behave? They He Would Due To Herd Mentality; Ok; That Is One Theory; Simulation By Terrestial Brahman Anti Brahman Will Make Them Behave In A Highly Organised And Coordinated Manner; With Association Of Things And Individuals Already Established Resultant Orientation Of The Actions Hall Be Only Oriented Towards The Figment Of Imagination And Product Of Puerile Prognostication Formally Established By Terrestial Brahman Anti Brahman Before Such A Coordinated And Organised Execution Of Exercise Is Conducted; Analogically, If A Simulation Is Going On In A Bunch Of Particles Or Electrons, And If Some Agency Say Celestial Brahman Anti Brahman Agency Is Simulating It, There Shall Be Coordination And Organisation So That You See "Reality" While The Truth Shall Remain Hidden. Result Is The Suppression Of Information, Delineation Of Disinformation, And The Concomitant Development Of Mushrooms Of Concocted And Fabricated Theories To That "Reality"; In The Case Of Terrestial Brahman Anti Brahman Agency Intention May Be To Bulldoze And Bowdlerize You, Baffle And Befuddle You In To Submission; In The Case Of Celestial Brahman Anti Brahman Agency Only Reason Could Be To Shield The Truth Being Disseminated; But Why?;May Be You Should Never Become God Which You Are Trying To; May Be God Is Really Playing Dice; What Does That Result In To ; You Become A Scezophreniac Associating Again And Again What You Had Been Until Now Of Particles And Other Concomitants Again And Again Leading To Flawed Inferences. Confusing The Epistemic With The Ontic, Like If One Were To Presume That A General Law Actually "Governs" Outcomes—And That The Statement Of A Regularity Has The Role Of A Causal Mechanism—Is A Category Mistake. In A Broad Sense, Scientific Theory Can Be Viewed As (=) Offering Scientific Realism—Approximately True Description Or Explanation Of The Natural World—Or Might Be Perceived With Antirealism. A Realist Stance Seeks The Epistemic And The Ontic, Whereas An Antirealist Stance Seeks Epistemic But Not The Ontic. In The 20th Century's First Half, Antirealism Was Mainly Logical Positivism, Which Sought To Exclude Unobservable Aspects Of Reality From Scientific Theory. Since The 1950s, Antirealism Is More Modest, Usually Instrumentalism, Permitting Talk Of Unobservable Aspects, But Ultimately Discarding The Very Question Of Realism And Posing Scientific Theory As A Tool To Help Humans Make Predictions, Not To Attain Metaphysical Understanding Of The World. The Instrumentalist View Is Carried By The Famous Quote Of David Mermin, "Shut Up And Calculate", Often Misattributed To Richard Feynman. Quantum Mechanical Wave Function (Absolutely Squared) Describes The Completed Interference Pattern, It Must Describe An Ensemble Interpretations Of Quantum Mechanics From Wikipedia. Philosophy Of Science, The Distinction Of Knowledge Versus Reality Is Termed Epistemic Versus Ontic. A General Law Is A Regularity Of Outcomes (Epistemic), Whereas A Causal Mechanism May Regulate The Outcomes (Ontic). A Phenomenon Can Receive Interpretation Either Ontic Or Epistemic. For Instance, Indeterminism May Be Attributed To Limitations Of Human Observation And Perception (Epistemic), Or May Be Explained As A Real Existing Maybe Encoded In The Universe (Ontic). Confusing The Epistemic With The Ontic, Like If One Were To Presume That A General Law Actually "Governs" Outcomes—And That The Statement Of A Regularity Has The Role Of A Causal Mechanism—Is A Category Mistake. Interpretations Of Quantum Mechanics From Wikipedia. Reality Is Constant In The Midst Of Change. Note The Conservation Of Individual, Collective And Cosmic General Ledgers Are What Are Being Spoken Of. Not The Changing Transactions, But The Immutable Refractory Zero. (Italics Mine) What This Means Essentially Is That There Is Change Although Nothing Changes. Like There Are Changing Transactions And Still The Assets And Liabilities Will Be Equal. This Impossible Situation Is Reflected (E&Eb) In The Ultimate Impossibility Of Change Itself. That Which Does Not Exist Prior To Its Changing And At The End, After It Has Changed, Must Be Equally Non-Existent Between These Two Moments. Although The World Of Change Appears To Be Real, It Cannot Be So. (Real)Change, According To The Vedantin, Presupposes A Loss Of Identity. Reality Cannot Suffer Transformation If It Were To Do So, It Would Become Something Else And The Real Would Be Deprived Of Its Reality. The Immortal Can Never Become Mortal, Nor Can The Mortal Become Immortal The Doctrine Of Vibration: An Analysis Of The Doctrines And Practices Of Kashmir Shaivism Mark S. G. Dyczkowski. Concept Of Being From Shaivite Point Of View: In His Famous Work Entitled On The Conservation Of Force (1847), Hermann Von

Helmholtz Had Divided The Totality Of The Energy In The Universe Between Potential Energy And Kinetic Energy And Affirmed The Reciprocal Convertibility Of The Two. In 1852, William Thomson Pointed Out That There Exists A Sub-Ensemble Within Kinetic Energy, Heat, Which, Once It Has Been Generated, Is No Longer Entirely Convertible Into Potential Energy-Or Into Any Other Form Of Kinetic Energy. Considering That The (Partial) Reconversion Of Heat Into Labor Is Possible Only In Situations That Present A Disparity In Temperature, And That Heat Tends To Pass From Warmer To Cooler Bodies By Spreading On An Even Temperature Level Through Space, Thomson Concluded That The Universe Tends Towards A Final State Where Any Energetic Transformations, Every Movement And Every Form Of Life Will Cease: The Eternal Return: Genesis And Interpretation BY PAUL D'IORIO: The Ultimate Nature Of Anything Cannot Change. Change Of Any Sort Is Merely Apparent (Vivarta). The World Of Change And Becoming Is A False Super- Imposition (Adhyaropa, Adhyasa) On The Absolute. It Appears So On The Screen Of Individual, Collective Consciousness, Albeit One Cannot Be So Sure Of Cosmic, For There Are Many Reasons That Cosmic Transactions Also Had To Be Subjected To Change By ParaBrahaman. In Cosmic Terms, The Mistake (Bhranti) Consists Of The Supposition That The Real Brahman Is The Unreal Universe And The Unreal Universe Is The Real Brahman. In Microcosmic Terms, It Is The Mistake Of Falsely Conceiving The Body, Mind Or Even One's Personality To Be The Self. Note We Have Taken Self As Witness Consciousness. Notwithstanding There Is Nothing Wrong In The Conception Of Self As A Dynamical Presupposition, In That Individual Consciousness And Collective Consciousness And Even Cosmic Consciousness Evolutes In To ParaBrahaman. In The Same Way As The Image Of A Snake Is Falsely Superimposed On A Rope, Similarly The Universe Is Falsely Projected Onto The Real Substratum, The Brahman. Ignorance Is Not Merely A Personal Lack Of Knowledge, But A Cosmic Principle. As Such It Is Called "Maya," The Indefinable Factor (Anirvacaniya) That Brings This Mistake In Identity About. The Reality Status Of This Cosmic Illusion Is Also Indefinable: On The One Hand It Is Not Brahman, The Sole Reality; On The Other Hand It Is Not Absolutely Non-Existent Like A Hare's Horn Or The Son Of A Barren Woman. Note Here The Vedantin Avoids Talking About Zero Which We Have Avidly And Assiduously Stressed About. Nothing Is The Factor That Exists. Also See the Remarks about Satre's Being And Nothingness. Brahman Is The Source Of World Appearances Only In The Sense Of Being Their Unconditioned Ground Or Essential Nature. The Universe Is False Not Because It Has No Nature Of Its Own But Because It Does Have One. The Doctrine Of Vibration: An Analysis Of The Doctrines And Practices Of Kashmir Shaivism Mark S. G. Dyczkowski. Concept Of Being From Shaivite Point Of View: It Is The 'I Am The Body/Mind' Belief That Gives Rise To The 'I Am Not The World' Belief. These Two Beliefs Are (=) Co-Created."Consciousness Projects The Appearance Of The Mind, Body And World By Taking (Eb) The Shape Of Thinking, Sensing And Perceiving.""Attention Is Consciousness With An Object. When The Object Vanishes, Attention Simply Remains What It Always Is Consciousness.""There Is No Purpose To Meditation. The Purpose Is Already Accomplished.""Everything That Is Experienced Is Experienced By, Through, In And As Consciousness.""The Seen Cannot Be Separated From Seeing And Seeing Cannot Be Separated From Consciousness." "**The Reality Of Any Experience Is Not Hidden In The Appearance; It Is Expressed By The Appearance The Transparency Of Things**' By Rupert Spira Book Review By Dennis Waite The Following Is A Review Of Rupert's Book: 'The Transparency Of Things: Contemplating The Nature Of Experience'. There Is An Essay From The Book Here And The Essay On Cézanne May Be Read At The Awakened Eye Website Nature Rejects The Naiveté That Seeks Absolute Truth. We Are Beginning To Realize, Individually And Culturally, That "Realities" Are All Human Constructions. The Task Becomes One Of "Catching Ourselves In The Act" Of Creating Our Own "Reality" From The Flow Of Events. Human Truth Is Always An Engagement Of Mind With Experience. The Challenge Of The Therapist In These Times Of Chaotic Change Is To Validate The Concept That We Don't Have To Fear The Collapse Of What We Think We Are. Strange Attractors: Transference, Holography, And An **Archetype Burke, J. (2003). Strange Attractors: Transference, Holography, And An Archetype (Doctoral Dissertation, Pacifica Graduate Institute, 2003).** When we get to the topic and understanding of the Transcendental Logic, the brilliant and baffling Transcendental Deduction (TD) Recall the two movements just discussed, the one from experience to its conditions and the one from the forms of valid inference to the concepts

that we must use in all judging (the Categories). This duality led Kant to his famous question of right (quid juris) (A84=B116): with what right do we apply the Categories, which are not acquired from experience, to the contents of experience? (A85=B117). Kant's problem here is not as arcane as it might seem. It reflects an important question: How is it that the world as we experience it conforms to our logic? In briefest form, Kant thought that the trick to showing how it is possible for the Categories to apply to experience is to show that it is necessary that they apply (A97).TD has two sides, though Kant never treats them separately John Marlado, accounts, attributes, and ascribes the philosophy of Nishida KitarÅ is lucid, insightful, and deeply informative despite the highly questionable argument that gives the book its structure. In comparison with other, technically more accurate accounts of Nishida's philosophy, it is broader in scope than most but more single-minded in its approach. The analysis is for the most part limited to four books that have been translated into English, out of approximately 16 volumes of philosophical essays in Japanese, but it aims to capture the contours and most important details of Nishida's entire philosophy. To appreciate the achievement of this account, one would need only to read, in English translation or in the Japanese original, any of the volumes with their streams of repetitive, puzzling expressions and circuitous, often dead- end argumentation. Wilkinson's elucidation of details in the passages he samples makes it easier to understand why Nishida is considered Japan's greatest academic philosopher. A critical look at the frame of the book and some of its own claims reveals where, to my mind, it has gone astray. The argument that frames this book is repeated throughout: Nishida's philosophy is an attempt to articulate, in a characteristically Western philosophical manner, the Zen experience he had as a young man. Nishida first made the attempt by using the conceptual frameworks of Western philosophers, and after they failed him he developed his own, largely by recasting Buddhist insights. In the end, a philosophical articulation of the Zen worldview in Western terms or even in Eastern terms recast by conceptual, Western philosophical methods approved impossible. What Nishida attempted is incommensurable with Western philosophy. While the author first offers these statements as hypotheses (p. 2), one premise of his argument that there is such a thing as "the Zen world-view" or "the Zen experience ‘‘remains unquestioned; and another that Nishida's pivotal experience was the Zen experienceâis asserted as "beyond question" (p. 151; see also pp. 28, 48, 10, 114).Wilkinson's presentation of Zen in the first chapter is drawn largely from D. T. Suzuki, with help from Suzuki's portage Abe Masao; some comments on time, birth, and death in; and a few other sources. The result is a caricature that imagines Zen as an invariable, unitary experience. In fact, Dagan’s teachings on practice as the manifestation of enlightenment do not sit well with D. T. Suzuki's emphasis on a satori experience, and Suzuki does not enlighten us about what Zen in its various practices has historically been. The author would have benefited from acquaintance with other presentations by Zen teachers, such as that of ShunryÅ Suzuki'sZen Mind Beginner's Mind, with historical accounts of Zen traditions, and with contrasts between DÅgen's view of time and Sam Â· sÄraand that of other Buddhist philosophers. A little research in contemporary Zen scholarship would have shown how implausible is the talk of "nirvana or awareness of mu" (p. 24), "the basic Zen position [that] never varies" (p. 3), "the Zen vision" (pp. 5, 7), "Zen epistemology" , "Zen experience" (p. 59), "the conception of time involved in Zen" (p. 15), and "the fundamental Zen assertion that there is no absolute time" (p. 16). Even if DÅgen's notion that time and being are nondual (p. 16) epitomized most of Zen, Nishida's early notion of "a transcendent, unchanging reality apart from time" (p. 51) would contrast sharply with it. Equally questionable is Wilkinson's contention that Nishida's "pure experience" is equivalent to Zen satori. The notion of pure experience Nishida developed in his first major work may be pivotal for his entire philosophy, but he never claims or implies that any part of his philosophy presupposes an exceptional experience like satori that grounds his convictions and renders them unverifiable to those who have not had the experience, as Wilkinson intimates (pp. 55, 159, and 161). Nishida presents his notions of pure experience and intellectual intuition, the grasp of its unity, and later... Perennialism and Constructivism is studied by Randolph T Dible : experiencer or the observer and the transcendental subjectivity itself Pure objectivity is essentially a deconstruction, apophatic and negativa The encountering of the experiencer or observer—transcendental subjectivity itself—at the foundation of the world leads inevitably to the recognition of pure objectivity as ultimate reality (which can be taken as its ultimate deconstruction, analogous to the apophatic or via negativa), from which objects derive their value, weight, significance, meaning or objectivity. In this way, pure

objectivity can be seen as the supra-self-evident Axiological Axiom, so to speak, even Unconditional Love, in romantic terms. This axiology (value theory) has a structure inverse to the relationship between transcendental subjectivity as the radical unity of pure self-reference and on the other hand, the world of forms, as mere traces (representations, indications) of the unique, original "first distinction" Spencer-Brown speaks of at the foundation of his calculus. That is, all forms (i.e., distinctions, differences) would reduce to being the first distinction, also known as the marked state, which can be called penultimate reality (pure self-reference or transcendental subjectivity: the Spirit which animates us), except that forms are complimentary to their content, which is their objectivity or value, which would reduce to the unmarked state or ultimate reality. It is the incongruity of form (thoughts; Whitehead's "negative prehensions") and value (feelings; Whitehead's "positive prehensions," or my notion of objectivity, meaning and qualia; in short, the non-formal aspects of experience) that holds forms open and keeps them from absolute reduction. This accounts for the brute, concrete persistence of the "functional illusion"-- to use a term from Dzogchen Buddhism-- of the world. Thus this system has an axiology of metaphysical objectivity grounded on the ideal of pure objectivity as the source of all value, meaning and significance, itself the very fecundity of profundity, which is the motive of drawing the distinction in the first place **(Wikipedia, Kant's writings and Logic Of Sense)** Nano rings were created by accident while intending to make quantum dots. They have interesting optical properties associated with excitons and the Aharonov–Bohm effect. Application of these rings used as light capacitors or a buffer includes photonic computing and communications technology. Analysis and measurement of geometric phases in mesoscopic rings is ongoing. Several experiments, including some reported in 2012, show Aharonov-Bohm oscillations in charge density wave (CDW) current versus magnetic flux, of dominant period h/2e through CDW rings up to 85 **μm** in circumference above 77 K. This behavior is similar to that of the superconducting quantum interference devices (seeSQUID). **Aharonov–Bohm nano rings (Wikipedia) Stability analysis of nano rings would help progressively accentuate the research about the materials division that uses nano rings.** From the profound contemplation and wisdom of the Buddha and Mahabodhisattvas, we know there is no such reality. Instead, egolessness (non-self) is the only path to understand the reality of the deluded life. All existences are subject to the law of causes and conditions. These include the smallest particles, the relationship between the particles, the planets, and the relationship between them, up to and including the whole universe! From the smallest particles to the biggest matter, there exists no absolute independent identity. Egolessness (non-self) implies the void characteristics of all existence. Egolessness (non-self) signifies the non-existence of permanent identity for self and existence (dharma). Sunyata stresses the voidness characteristic of self and existence (dharma). Sunyata and egolessness possess similar attributes. As we have discussed before, we can observe the profound significance of sunyata from the perspective of inter-dependent relationships. Considering dharma-nature and the condition of nirvana, all existences are immaterial and of a void-nature. Then we see each existence as independent of each other. But then we cannot find any material that does exist independent of everything else. So egolessness also implies void-nature! From the observation of all existences, we can infer the theory of nirvana and the complete cessation of all phenomena. From the viewpoint of phenomena, all existences are so different from each other, that they may contradict each other. They are so chaotic. In reality, their existence is illusionary and arises from conditional causation. They seem to exist on one hand, and yet do not exist on the other. They seem to be united, but yet they are so different to one another. They seem to exist and yet they do cease! Ultimately everything will return to harmony and complete calmness. This is the nature of all existence. It is the final resting place for all. If we can understand this reality and remove our illusions, we can find this state of harmony and complete calmness. **Teachings in Chinese Buddhism (6) sunyata (emptiness) in the Mahayana context (Wikipedia) Many models have been given concatenating Buddhist doctrines and Vaccum energy. These models essentially throw light on the essence of self(witness consciousness) and the cosmic general ledger (cosmic consciousness) while the imbalance in the gratification deprivation complex always remain unbalanced when it comes to individual general ledger , advocating the evolutionary theory of individual consciousness to cosmic consciousness. It is to be noted that it is not the question of knowing all the transactions but the wisdom of what lies behind the debit- Credit, Debit-Debit, or Credit-Credit principle that is most important for assimilation.** Look at this

simple thesis to prove the point stated in the foregoing. Thesis: 'The World Has A Beginning In Time, And In Space It Is Also Enclosed In Boundaries.' Proof: 'For If One Assumes That The World Has No Beginning In Time, Then Up To Every Given Moment In Time An Eternity Is Elapsed, And Hence An Infinite Series Of States Of Things In The World, Each Following Another, Has Passed Away. But Now The Infinity Of A Series Consists Precisely In The Fact That It Can Never Be Completed Through A Successive Synthesis. Therefore An Infinitely Elapsed World-Series Is Impossible, So A Beginning Of The World Is A Necessary Condition Of Its Existence, Which Was The First Point To Be Proved (Kant, Critique Of Pure Reason, Tr. By R. Guyer And A. W. Wood (Cambridge: Cambridge University Press, 1998): B 454, P. 470). As Quoted In The Eternal Return: Genesis And Interpretation BY PAUL D'IORIO"The Cyclical Hypothesis, So Heavily Criticized By Nietzsche (VP II 325 And 334), Arises In This Way." In Fact, Nietzsche Was Not Criticizing The Cyclical Hypothesis But Only The Particular Form Of That Hypothesis Presented In Vogt's Work. All Of Nietzsche's Texts Without Exception Speak Of The Eternal Return As The Repetition Of The Same Events Within A Cycle Which Repeats Itself Eternally. If Deleuze's Interpretation Holds That The Eternal Return Is Not A Circle, Then What Is It? A Wheel Moving Centrifugally, Operating A "Creative Selection," "Nietzsche's Secret Is That The Eternal Return Is Selective" Says Deleuze: The Eternal Return Produces Becoming-Active. It Is Sufficient To Relate The Will To Nothingness To The Eternal Return In Order To Realize That Reactive Forces Do Not Return. However Far They Go, However Deep The Becoming-Reactive Of Forces, Reactive Forces Will Not Return. The Small, Petty, Reactive Man Will Not Return. Affirmation Alone Returns, This That Can Be Affirmed Alone Returns, Joy Alone Returns. Everything That Can Be Denied, Everything That Is Negation, Is Expelled Due To The Very Movement Of The Eternal Return. We Were Entitled To Dread That The Combinations Of Nihilism And Reactivity Would Eternally Return Too. The Eternal Return Must Be Compared To A Wheel; Yet, The Movement Of The Wheel Is Endowed With Centrifugal Powers That Drive Away The Entire Negative. Because Being Imposes Itself On Becoming, It Expels From Itself Everything That Contradicts Affirmation, All Forms Of Nihilism And Reactivity: Bad Conscience, Ressentiment..., We Shall Witness Them Only Once. [...] The Eternal Return Is The Repetition, But The Repetition That Selects, The Repetition That Saves. Here Is The Marvelous Secret Of A Selective And Liberating Repetition. There Is No Need To Remind The Reader That Neither The Image Of A Centrifugal Movement Nor The Concept Of A Negativity-Rejecting Repetition Appears Anywhere In Nietzsche's Writings, And Indeed Deleuze Does Not Refer To Any Text In Support Of This Interpretation. Further, One Could Highlight That Nietzsche Never Formulates The Opposition Between Active And Reactive Forces, Which Constitutes The Broader Framework Of Deleuze's Interpretation. For Some Years, Marco Brusotti Has Called Attention To The Fact That Deleuze Introduced A Dualism That Does Not Exist In Nietzsche's Writings. To Be Sure, The German Philosopher Describes A Certain Number Of "Reactive" Phenomena (For Example, In The Second Essay Of The Genealogy Of Morality, § 11, He Talks About "Reactive Affects" [Reaktive Affekte], "Reactive Feelings" [Reaktive Gefühlen], Reactive Men [Reaktive Menschen]); But These Are Nonetheless The Result Of Complex Ensembles Of Configurations Of Centers Of Forces That Remain In Themselves Active. Neither The Word Nor The Concept Of "Reactive Forces" Ever Appears In Nietzsche's Philosophy. The Eternal Return: Genesis And Interpretation BY PAUL D'IORIO Some Parts Are Deleted Due To Spatial Constraints. In the midst of thoughts about the eternal return we find at least two other thematic axes On the one hand, the view of the **world as a constant flux of forces without any goal, law, or rules of becoming. A chaos sive natura de-divinized and de-anthropomorphized which constitutes the "ontological substratum**" of the whole of Nietzsche's reflexions On the other hand, an ensemble of fragments of an anthropologico-sociological character, designing a path of liberation leading to the creation of superior individuals by way of a profound transformation of their instinctual structure. This transformation must be achieved by a practice of solitude and internal struggle towards the liberation from the ancient representations of the world and from the incorporated herd values. For an analysis of these thematic perspectives, see Paolo D'Iorio, La linea e il circolo. Cosmologia e filosofia dell'eterno ritorno in Nietzsche (Genova: Pantograf, 1995): 233-322. With most of the parameters quantified, with assumptions there must be picturesque view of the self and non self, space and time, science and philosophy with the concatenational consummations. After The Chapter "On Redemption," Where Zarathustra Dares Not Expose His

Doctrine, The Eternal Return Begins To Be Enunciated In Part Three Of The Work. In The First Place, It Is The Dwarf Who Formulates It In The Chapter "On The Vision And The Riddle." Facing The "Gate Of The Instant" Which Symbolizes The Two Infinities That Stretch Towards The Past And The Future, The Dwarf Whispers: "All Truth Is Crooked, Time Itself Is A Circle" The Dwarf Represents The Spirit Of Gravity, And He Embodies The Herd Morality, "The Belittling Virtue" Which Is The Title Of Another Chapter From Part III. The Dwarf Can Endure The Eternal Return Without Great Difficulties Because He Has No Aspirations; Unlike Zarathustra He Does Not Wish To Climb The Mountains That Symbolize Elevation And Solitude. In Two Unpublished Notes, From The Summer And The Fall Of 1883, Nietzsche Writes: The Doctrine Is At First Favored By The Rabble, Before It Gets To The Superior Men. The Doctrine Of Recurrence Will First Smile To The Rabble, Which Is Cold And Without Any Strong Internal Need. It Is The Most Ordinary Of Life Instincts, Which Gives Its Agreement First. Hence, The Content Of The Doctrine Is The Same, But Whereas The Dwarf Can Endure It (Because He Interprets It According To The Pessimistic Tradition For Which "Nothing Is New Under The Sun"), Zarathustra, Who Is The "Advocate Of Life" Regards The Eternal Return As The Strongest Objection To Existence, And As The Rest Of The Dream Suggests, He Does Not Yet Succeed In Accepting It After The Vision At The Gate Of The Instant, The Chapter Is Brought To An End By The Enigma Of The Shepherd. The Eternal Return: Genesis And Interpretation BY PAUL D'IORIO .How does the memory affect the GD complex or for that matter the gratification deprivation of an individual or a collective identity? Non-Markovian local-in-time master equations give a relatively simple way to describe the dynamics of open quantum systems with memory effects. Despite their simple form, there are still many misunderstandings related to the physical applicability and interpretation of these equations. Here, we clarify these issues both in the cases of quantum and classical master equations. Further introduction of the concept of a classical non-Markov chain signified through negative jump rates in the chain configuration. **E-M Laine et al 2012 J. Phys. B: At. Mol. Opt. Phys. 45 154004 doi:10.1088/0953-4075/45/15/154004 Local-in-time master equations with memory effects: applicability and interpretation.** Supposing That There Were Indeed An "Energy Of Contraction" Constant In All Centers Of Force Of The Universe, It Remains To Be Explained Where Any Difference Would Ever Originate. It Would Be Necessary For The Whole To Dissolve Into An Infinite Number Of Perfectly Identical Existential Rings And Spheres, And We Would Therefore Behold Innumerable And Perfectly Identical Worlds COEXISTING [Nietzsche Underlines This Word Twice] Alongside Each Other. Is It Necessary For Me To Admit This? Is It Necessary To Posit An Eternal Coexistence On Top Of The Eternal Succession Of Identical Worlds? The Eternal Return: Genesis And Interpretation BY PAUL D'IORIO. Science is usually portrayed as dehumanizing. Brave New World epitomizes this fear. "The more we understand the world, the more it seems completely pointless" (Steven Weinberg). Certainly science can seem chilling when conceived in the abstract as a metaphysical world-picture. We may seem to find ourselves living in a universe with all the human meaning stripped out: Participants in a soulless dance of molecules, or harmonics of pointlessly waggling superstrings and their braneworld cousins. Nature seems loveless and indifferent to our lives. What rights have we to be happy? **(Brave New World: Aldous Huxley).** Religion can afford to be dogmatic but science cannot. Look at this passage which counters both scientists and philosophers alike: However, Hartmann Objects That The Regressive Movement Postulated By Schopenhauer Is Possible Only In Thought: It Remains Nothing More Than An "Ideal Postulate" With No Real Object And Which "Does Not Teach Us Anything About The Real Process Of The World That Unfolds In A Movement Contrary To This Backwards Movement Of Thought" (Hartmann, Philosophie Des Unbewussten, Third Edition (1871): 772). Hartmann Affirms That If Unlike Schopenhauer One Admits The Reality Of Time And Of The World Process, One Must Also Admit That The Process Must Be Limited In The Past And Therefore That There Must Be An Absolute Beginning. In Hartmann's Mind, Failure To Do So Would Result In Positing The Contradictory Concept Of An Accomplished Infinity: "The Infinity That From The Point Of View Of Regressive Thinking, Remains An Ideal Postulate, Which No Reality May Correspond To, Must, For The World, Whose Process Is, On The Contrary, A Progressive Movement, Open Up To A Determinate Result; And Here The Contradiction Comes To Light" (Hartmann (1871): 772). What Really "Comes To Light" In This Passage Is The Fact That Hartmann Does Not Provide A Demonstration But A Petitio Principii. Indeed, The Concept Of The World

Process Analytically Contains The Concept Of A Beginning Of The World. In All Rigors, It Is Therefore Impossible To Demonstrate These Concepts With Reference To Each Other. Secondly, Hartmann's View That One Is Bound To Accept The Reality Of The World Process Even If One Rejects The Ideality Of Schopenhauer's Time Is Mistaken. Hartmann Believes That If Time Is Real There Must Be A World Process With Both An Absolute Beginning And An Absolute End. Without Any Justification, Hartmann Jumps From Schopenhauer's Negated Time To Oriented Time. The Eternal Return: Genesis And Interpretation BY PAUL D'IORIO. In the last years several theoretical papers discussed if time can be an emergent property deriving from ) quantum correlations. Here, to provide an insight into how this phenomenon can occur, authors present an experiment that illustrates Page and Wootters' mechanism of "static" time, and Gambini et al. subsequent refinements. A static, entangled state between a clock system and the rest of the universe is perceived as evolving by internal observers that test the correlations between the two subsystems. They implement this mechanism using an entangled state of the polarization of two photons, one of which is used as a clock to gauge the evolution of the second: an "internal" observer that becomes correlated with the clock photon sees the other system evolve, while an "external" observer that only observes global properties of the two photons can prove it is static.Phys. Rev. A 89, 052122 (2014)DOI:10.1103/PhysRevA.89.052122 arXiv: 1310.4691 [quant-ph] **Time from quantum entanglement: an experimental illustration Ekaterina Moreva.** Time is an entanglement .When the new ideas of quantum mechanics spread through science like wildfire in the first half of the 20th century, one of the first things physicists did was to apply them to gravity and general relativity. The results were not pretty. It immediately became clear that these two foundations of modern physics were entirely incompatible. When physicists attempted to meld the approaches, the resulting equations were bedeviled with infinities making it impossible to make sense of the results. Then in the mid-1960s, there was a breakthrough. The physicists John Wheeler and Bryce DeWitt successfully combined the previously incompatible ideas in a key result that has since become known as the Wheeler-DeWitt equation. This is important because it avoids the troublesome infinites—a huge advance. But it didn't take physicists long to realise that while the Wheeler-DeWitt equation solved one significant problem, it introduced another. The new problem was that time played no role in this equation. In effect, it says that nothing ever happens in the universe, a prediction that is clearly at odds with the observational evidence. This conundrum, which physicists call 'the problem of time', has proved to be thorn in flesh of modern physicists, who have tried to ignore it but with little success. Then in 1983, the theorists Don Page and William Wootters came up with a novel solution based on the quantum phenomenon of entanglement. This is the exotic property in which two quantum particles share the same existence, even though they are physically separated. Entanglement is a deep and powerful link and Page and Wootters showed how it can be used (e) to measure time. Their idea was that the way pair of entangled particles evolves is a kind of clock that can be used to measure change. But the results depend on how the observation is made. One way to do this is to compare the change in the entangled particles with an external clock that is entirely independent of the universe. This is equivalent to god-like observer outside the universe measuring the evolution of the particles using an external clock. In this case, Page and Wootters showed that the particles would appear entirely unchanging—that time would not exist in this scenario. But there is another way to do it that gives a different result. This is for an observer inside the universe to compare the evolution of the particles with the rest of the universe. In this case, the internal observer would see a change and this difference in the evolution of entangled particles compared with everything else is an important a measure of time. This is an elegant and powerful idea. It suggests that time is an emergent phenomenon that comes about because of the nature of entanglement. And it exists only for observers inside the universe. **Time is an entanglement** (Wikipedia). Danish religious philosopher Soren Kierkegaard carried out a systematic critique of the pretensions of reason and an abstract rationalism which he believed that the modern age was nurturing. Condemning reflection as a "danger" that ensnares people in logical delays and machinations, Kierkegaard compared it to a prison. Reflection is for him a form of captivity, a bondage which "can only be broken by [passionate] religious inwardness" (1978: 81). **Reflection seduces individuals into thinking its possibilities are "much more magnificent than a paltry decision"** (1978: 82). It leads them to act **"on principle,"** to dwell on the deliberation of the context of their actions and the calculation of their worth or outcome. Kierkegaard argues that this drives away feeling, inspiration, and

spontaneity, all of which are crucial for true inner being and a vital relation to God. For Kierkegaard, as Nietzsche would later agree, genuine inner being (and culture) is characterized by the tautness and tension of the soul which characterizes passionate existence. But the "coiled springs of life relationships ... lose their resilience" in reflection (1978: 78) and "everything becomes meaningless externality, devoid of [internal] character" (1978: 62). Kierkegaard thus contributes to the development of an irrationalist tradition that has echoes in some later postmodern thought. Kierkegaard might well have agreed with his contemporary Fyodor Dostoyevsky, who wrote: "An intelligent [reflective] man cannot seriously become anything ... excessive consciousness is a disease" (1974: 3, 5). In an age overtaken by rules and regulations, genuine action -- which Kierkegaard assumes to be subjective and spontaneous -- is frustrated at every turn. Complaining that we are too "sober and serious" (1978: 71) even at banquets, Kierkegaard bemoans the fact that even suicides are premeditated (1978: 68)! "That a person stands or falls on his actions is becoming obsolete; instead, everybody sits around and does a brilliant job of bungling through with the aid of some reflection and also by declaring that they all know very well what has to be done" (1978: 73). Thus, it is passion, not reflection, that guarantees "a decent modesty between man and man [and] prevents crude aggressiveness" (1978: 62). "Take away the passion and the propriety also disappears" (1978: 64). The ambiguity in the word "passion" may cause some confusion here. To say that the age and its individuals are "passionless" is not to say there are no emotions whatsoever, but rather that there is no true spiritual inwardness and depth, no intensively motivated action and commitment. It suggests that passion exists only in a simulated, pseudo-form, "the rebirth of passion" through "talkativeness" (1978: 64). "Chattering" for Kierkegaard gets in the way of "essential speaking" and merely "reflects" inconsequential events (1978: 89-99). Hence, in “the present age,” emotions -- which in fact are all too pronounced -- have been transformed into negative forces. Anticipating Nietzsche's genealogy of the "slave revolt" in morality, Kierkegaard claims that the "enthusiasm" of the prior age of Revolution, a "positively unifying principle,” has become a vicious "envy," a "negatively unifying principle" (1978: 81), a leveling force in its own right insofar as those lacking in talent and resources want to tear down those who have them. **THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS By Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html.** In November 2011, the United States government argued before the US Supreme Court that it wants to continue utilizing GPS tracking of individuals without first seeking a warrant. In response, Justice Stephen Breyer questioned what this means for a democratic society by referencing Nineteen Eighty-Four. Justice Breyer asked, "If you win this case, then there is nothing to prevent the police or the government from monitoring 24 hours a day the public movement of every citizen of the United States. So if you win, you suddenly produce what sounds like 1984... "[65]. What is to be said in unmistakable terms is that when it comes to national security and innocent citizens there is question of compromise even it be an infringement and encroachment of so called privacy which has been made a word to subterfuge the activities that are far said to be fair. Quantum theory has provoked intense discussions about its interpretation since its pioneer days. One of the few scientists who have been continuously engaged in this development from both physical and philosophical perspectives is Carl Friedrich von Weizsäcker. The questions he posed were and are inspiring for many, including the authors of this contribution. Weizsäcker developed Bohr’s view of quantum theory as a theory of knowledge. **Harald Atmanspacher, Hans Primas** show that such an epistemic perspective can be consistently complemented by Einstein’s ontically oriented position. **Time, Quantum and Information 2003, pp 301-321 Epistemic and Ontic Quantum Realities Harald Atmanspacher, Hans Primas Robert W. Spekkens** present a toy theory that is based on a simple principle: the number of questions about the physical state of a system that are answered must always be equal to the number that are unanswered in a state of maximal knowledge. Many quantum phenomena are found to have analogues within this toy theory. These include the noncommutativity of measurements, interference, the multiplicity of convex decompositions of a mixed state, the impossibility of discriminating nonorthogonal states, the impossibility of a universal state inverter, the distinction between bipartite and tripartite entanglement, the monogamy of pure entanglement, no cloning, no broadcasting, remote steering, teleportation, entanglement swapping, dense coding, mutually unbiased bases, and many others. The diversity and quality of these analogies is taken as evidence for the view that quantum states are

states of **incomplete knowledge** rather than states of reality. A consideration of the phenomena that the toy theory **fails to reproduce, notably, violations of Bell inequalities** and the existence of a Kochen-Specker theorem, provides clues for how to proceed with this research program.DOI: http://dx.doi.org/10.1103/PhysRevA.75.032110 **Evidence for the epistemic view of quantum states A toy theory Phys Rev A 75, 032110 – Published 19 March 2007 Robert W. Spekkens In such specialized and autonomous areas, it is to the eminent and erudite authors to interpret the contradistinctions and contradictions' in the dove tailing explaining. This I feel would be yeoman service in the creation of Nature's general ledger. M. S. Leifer and O. J. E. Maroney** examine the relationship between quantum contextuality (in both the standard Kochen-Specker sense and in the generalized sense proposed by Spekkens) and models of quantum theory in which the quantum state is maximally epistemic. We find that preparation noncontextual models must be maximally epistemic, and these in turn must be Kochen-Specker noncontextual. This implies that the Kochen-Specker theorem is sufficient to establish both the impossibility of maximally epistemic models and the impossibility of preparation noncontextual models. The implication from preparation noncontextual to maximally epistemic then also yields a proof of Bell's theorem from an Einstein-Podolsky-Rosen-like argument. DOI: http://dx.doi.org/10.1103/PhysRevLett.110.120401 Received 25 August 2012Published 20 March 2013 © 2013 American Physical Society **Maximally Epistemic Interpretations of the Quantum State and Contextuality Phys. Rev. Lett 110, 120401 – Published 20 March 2013 M. S. Leifer and O. J. E. Maroney.** Epistemological, onitic and ontological aspects particularly of quantum mechanics have invited profound thinkers to contribute their mite to the growing literature on the subject, which includes the contributions from the subterranean realm and ceratoid dualism of philosophy like that Chaos theory of Deleuze, Derrida , Kierkegaard and others. **Andrei Khrennikov** shows that the Dirac-von Neumann formalism for quantum mechanics can be obtained as an approximation of classical statistical field theory. This approximation is based on the Taylor expansion (up to terms of the second order) of classical physical variables – maps f: $\Omega \rightarrow R$, where $\Omega$ is the infinite-dimensional Hilbert space. The space of classical statistical states consists of Gaussian measures $\rho$ on $\Omega$ having zero mean value and dispersion $\sigma 2\ (\rho) \approx h$. This viewpoint to the conventional quantum formalism gives the possibility to create generalized quantum formalisms based on expansions of classical physical variables in the Taylor series up to terms of nth order and considering statistical states $\rho$ having dispersion $\sigma 2(\rho) = h\ n$ (for $n = 2$ we obtain the conventional quantum formalism). **December 2005, Volume 18, Issue 7, pp 637-650 Date: 28 Nov 2005 Generalizations of Quantum Mechanics Induced by Classical Statistical Field Theory Andrei Khrennikov** It is widely accepted that consciousness or, in other words, mental activity is in some way correlated to the behavior of the brain or, in other words, material brain activity. Since quantum theory is the most fundamental theory of matter that is currently available, it is a legitimate question to ask whether quantum theory can help us to understand consciousness. Several approaches answering this question affirmatively, proposed in recent decades, will be surveyed. It will be pointed out that they make different epistemological assumptions, refer to different neurophysiological levels of description, and adopt quantum theory in different ways. For each of the approaches discussed, these imply both problematic and promising features which will be indicated. **Discrete Dynamics in Nature and Society Volume 2004 (2004), Issue 1, Pages 51-73 http://dx.doi.org/10.1155/S102602260440106X Quantum theory and consciousness: an overview with selected examples Harald Atmanspacher Copyright © 2004 Hindawi Publishing Corporation. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.** Another thought provoking and investigatory article is by Peter G. Lewis et al. Perhaps the quantum state represents information about reality, and not reality directly. Wave function collapse is then possibly no more mysterious than a Bayesian update of a probability distribution given new data. **Peter G. Lewis et al** consider models for quantum systems with measurement outcomes determined by an underlying physical state of the system but where several quantum states are consistent with a single underlying state—i.e., probability distributions for distinct quantum states overlap. Significantly, they demonstrate by example that additional assumptions are always necessary to rule out such a model. DOI: http://dx.doi.org/10.1103/PhysRevLett.109.150404 **Distinct Quantum States Can Be Compatible with a Single**

**State of Reality Phys. Rev. Lett. 109, 150404 – Published 9 October 2012 Peter G. Lewis et al** Communication complexity of a quantum channel is the minimal amount of classical communication required for classically simulating a process of state preparation, transmission through the channel and subsequent measurement. It establishes and perpetuates a limit on the power of quantum communication in terms of classical resources. **Alberto Montina** shows that classical simulations employing a finite amount of communication can be derived from a special class of hidden variable theories where quantum states represent statistical knowledge about the classical state and not an element of reality. This special class has attracted strong interest very recently. The communication cost of each derived simulation is given by the mutual information between the quantum state and the classical state of the parent hidden variable theory. Finally, find that the communication complexity for single qubits is smaller than 1.28 bits. The previous known upper bound was 1.85 bits. DOI: http://dx.doi.org/10.1103/PhysRevLett.109.110501 **Epistemic View of Quantum States and Communication Complexity of Quantum Channels Phys. Rev. Lett 109, 110501 – Published 12 September 2012 Alberto Montina for models see elsewhere .** Stability analysis of what? Quantum state representing reality or our knowledge of reality? Does the quantum state represent reality or our knowledge of reality? In making this distinction precise, **Nicholas Harrigan, Robert W. Spekkens** are led to a novel classification of hidden variable models of quantum theory. They show that representatives of each class can be found among existing constructions for two-dimensional Hilbert spaces. Approach also provides a fruitful new perspective on arguments for the nonlocality and incompleteness of quantum theory. Specifically, **Nicholas Harrigan, Robert W. Spekkens** show that for models wherein the quantum state has the status of something real, the failure of locality can be established through an argument considerably more straightforward than Bell's theorem. The historical significance of this result becomes evident when one recognizes that the same reasoning is present in Einstein's preferred argument for incompleteness, which dates back to 1935. This fact suggests that Einstein was seeking not just any completion of quantum theory, but one wherein quantum states are solely representative of our knowledge. Hypothesis is supported by an analysis of Einstein's attempts to clarify his views on quantum theory and the circumstance of his otherwise puzzling abandonment of an even simpler argument for incompleteness from 1927. **February 2010, Volume 40, Issue 2, pp 125-157 Date: 09 Jan 2010 Einstein, Incompleteness, and the Epistemic View of Quantum States Nicholas Harrigan, Robert W. Spekkens** statistical model of the probabilistic description of physical reality has been studied by many authors. **Andrei Khrennikov posits** a contextual statistical model of the probabilistic description of physical reality. Here contexts (complexes of physical conditions) are considered as basic elements of reality. There is discussed the relation with QM. He proposes a realistic analogue of Bohr's principle of complementarity. In the opposite of the Bohr's principle, **Andrei Khrennikov** principle has no direct relation with mutual exclusivity for observables. To distinguish his principle from the Bohr's principle and to give better characterization, he changes the terminology and speaks about supplementarity, instead of complementarity. Supplementarity is based on the interference of probabilities. It has quantitative expression trough a coefficient which can be easily calculated from experimental statistical data. There is need not appeal to the Hilbert space formalism and noncommutativity of operators representing observables. Moreover, in our model there exists a pair of supplementary observables which cannot be represented in the complex Hilbert space. There are discussed applications of the principle of supplementarity outside quantum physics. **October 2005, Volume 35, Issue 10, pp 1655-1693 The Principle of Supplementarity: A Contextual Probabilistic Viewpoint to Complementarity, the Interference of Probabilities and Incompatibility of Variables in Quantum Mechanics Andrei Khrennikov Foundations of Physics** Description of the vacuum in Yang-Mills theory and arrive at a physical interpretation of the pseudoparticle solution and the attendant violation of symmetries has been done by **R. Jackiw and C. Rebbi.** It is here vaccuum stability forms the bastion, pillar, post and stylobate of the mechanisms of yang mills Theory and action of a particle. The existence of topologically inequivalent classical gauge fields gives rise to a family of quantum mechanical vacua, parametrized by a CP-nonconserving angle. The requirement of **vacuum stability** against gauge transformations renders the vacua chirally noninvariant. DOI: http://dx.doi.org/10.1103/PhysRevLett.37.172 **Vacuum Periodicity in a Yang-Mills Quantum Theory Phys. Rev.**

**Lett 37, 172 – Published 19 July 1976 R. Jackiw and C. Rebbi.** Three-dimensional Yang-Mills and gravity theories augmented by gauge-invariant mass terms are analyzed. These topologically nontrivial additions profoundly alter the particle content of the models and lead to quantization of a dimensionless mass-coupling-constant ratio. The vector field excitations become massive, with spin 1 (rather than massless with spin 0), and the mass provides an infrared cutoff. The gravitation acquires mass, mediates finite-range interactions, and has spin 2 (rather than being absent altogether); although its mass term is of third derivative order, there are no ghosts or acausalities. DOI: http://dx.doi.org/10.1103/PhysRevLett.48.975 © 1982 The American Physical Society **Three-Dimensional Massive Gauge Theories Phys. Rev. Lett 48, 975 – Published 12 April 1982 S. Deser et al** All scattering amplitudes in the maximally supersymmetric N=4 super-Yang–Mills theory possess a new, dual superconformal symmetry which extends the previously discovered dual conformal symmetry of MHV amplitudes. To reveal this property we formulate the scattering amplitudes as functions on the appropriate dual superspace. Rewritten in this form, all tree-level MHV and next-to-MHV amplitudes exhibit manifest dual superconformal symmetry. We propose a new, compact and Lorentz covariant formula for the tree-level NMHV amplitudes for arbitrary numbers and types of external particles. The dual superconformal symmetry is broken at loop level by infrared divergences. However, we provide evidence that the dual conformal anomaly of the MHV and NMHV superamplitudes is the same and, therefore, their ratio is dual conformally invariant. We show this explicitly for the six-particle amplitudes at one loop. Authors conjecture that these properties hold for all, MHV and non-MHV, superamplitudes in N=4 SYM both at weak and at strong coupling. **Nuclear Physics B Volume 828, Issues 1–2, 21 March 2010, Pages 317–374 Dual superconformal symmetry of scattering amplitudes in N=4 super-Yang–Mills theory J.M. Drummonda DOI: 10.1016/j.nuclphysb.2009.11.022.** By comparison with numerical results in the maximal Abelian projection of lattice Yang–Mills theory, it is argued that the nonperturbative dynamics of Yang Mills theory can be described by a set of fields that take their values in the coset space SU(2)/U(1). The Yang–Mills connection is parameterized in a special way to separate the dependence on the coset field. The coset field is then regarded as a collective variable, and a method to obtain its effective action is developed. It is argued that the physical excitations of the effective action may be knot solitons. A procedure to calculate the mass scale of knot solitons is discussed for lattice gauge theories in the maximal Abelian projection. The approach is extended to the SU (N) Yang–Mills theory. A relation between the large N limit and the monopole dominance is pointed out. **Physics Letters B Volume 458, Issues 2–3, 8 July 1999, Pages 322–330 An effective action for monopoles and knot solitons in Yang–Mills theory Sergei V. Shabanov DOI: 10.1016/S0370-2693(99)00612-7.** It is here the quintessential formulation of consciousness occurs. Like a man standing on the threshold of infinity trying to ponder what lies beyond the veil, which separates the seen from unseen a true seeker tries to lift the individual general ledger to the cosmic general ledger. Like in Darwinian evolution, mind also has to evolve from individual consciousness to cosmic consciousness. In Kaishmiri Shaivism one withdraws from the finite to the infinite, but one also goes on an outward journey from the infinite to the finite, because both the finite and the infinite have an intimate connection. The finite is not seen as unreal, but as a symbol of the infinite. There is no real distinction between them. Those two movements constitute Spanda, a key concept in Kaishmiri Shaivism. Spanda is the pulsation of the Absolute in different phases of being. There are no opposites like subject and object, unity or duality, absolute or relative. They are just different phases of the universal vibration of the Absolute. The goal is to realize or be (=) at once infinite and finite. One does not turn away from appearances (like in Advaita Vedanta), but one realizes that the Absolute manifests all things. Spanda, the eternal pulsation of the Absolute, oscillates between a passion to create and dispassion from the created. Through it the Absolute transform itself into all things and then returns back into the emptiness of its undifferentiated nature. In Kaishmiri Shaivism the Absolute is seen as pure consciousness (=being). The Absolute is an eternal all-pervasive principle, the highest reality, the nature of all entities eternally and blissfully at rest within its own nature. The Absolute is the nature of the Self (and thus of us all). The Absolute is divine, it is Shiva, the Lord of the Universe. It is full of conscious activity through which it generates the universe, and reabsorbs it into itself at the end of each cycle of creation. Thus we speak of monism, as everything resides within this one absolute consciousness. It sustains all things, it embraces all things, and it pervades all things. All things are appearances within the absolute

consciousness, but nevertheless real (in contrast to Advaita Vedanta where appearances are seen as unreal or illusionary). All things appear external (out there, outside ourselves), but they do not have a being on their own. They do not exist as separate entities on their own. Everything is contained within consciousness. What we see as objects are manifestations of consciousness. The event which constitutes the universe is always internal events happening within consciousness because their essential nature is consciousness itself. If a physical object were totally material, and independent or external to consciousness, it could never be experienced. The universe and consciousness are two aspects of a whole. The universe is an attribute of consciousness which bears consciousness as its substance. Doctrine of Vibration: S.G.Dyczkowski What you "see" is not always what you "get." Many people mistakenly take their own visions literally without "seeing through" the various possibilities that are outside their belief system, knowledge or skill base. Deeper reality is not remote in the physical sense but in a psychological sense. The archetypes of the collective unconscious are arrayed behind the scenes of current worldwide conditions, of crisis and confusion. They mirror our own states back at us, whether we perceive them as such or interpret them plausibly or not. The noise of ordinary consciousness and beliefs drowns out the signal. Unconsciousness is the background of our ordinary awareness. Our organism is very much at the center of such effects. The organismic source is our human bodies and the focus of human consciousness. The fantasy principle dethrones reality, but can be dissociative or compensatory. The human mind is a meme-Scape. Pre-conceived concepts vie with structures, concepts with images. Like scientists who ignore assumed truths, we leapfrog over our beliefs and personality deficits, claiming idiosyncratic imagination is literal reality. It couldn't be further from the truth and symbolism is utterly lost. The metaphor that might heal us enslaves us. Perhaps images like the holographic universe have an implicate order. Can we have a sense of the cosmos in the world without projecting myriad fantasies on it that we embrace literally? Has the world become so horrible it is unreasonable to be realistic? We may need to look at our drives and wishes, rather than the fantasy content. Psyche constructs reality. Our experience of so-called reality is always mediated by our image of it. Even if all the contents of the psyche are real, that doesn't mean they are realistic. That psyche is real is still a radical proposition, but psychic politics certainly color the self-image and ideas of everyone. We observe and participate with images It is not a question of nature or nurture (genes alone or experience alone). Rather, everything is both. We inherit the structures that make our experience what it is. But the structure itself is “empty,” and each human culture “fills” it with its own specific adaptations. It is difficult to define an archetype and set boundaries that distinguish it from others. In a hologram each part contains all the information but in lower resolution. Archetypes have this holographic quality. There are patterns within patterns within patterns. Some overlap with others, and some are nested inside others. Archetypal realities, passed on through DNA, are expressed in distinctive neuronal tracts in the brain. They include customs and laws regarding property, incest, marriage, kinship, and social status or roles; myths and legends; beliefs about the supernatural; gambling, adultery, homicide, schizophrenia, and the therapies to deal with them. A mythic and visionary language of immediate experience encompasses themes of deepest, highest, and ultimate concern. Most fantasy-based individuals are at a complete loss to coherently explain their own conventional behavior much less anomalous events and their deep meaning, much less the cultural unconscious or mythological unconscious matrix. But they try, and become utterly entrenched in their belief that they are right about the nature of the world and reality. We have pseudo-memories about our personal lives. Why not more so for our collective life? The subject matter often revolves around catastrophe, creation and the mythopoeic forces of mankind. Ignorant of such dynamics, interpretive mistakes and displaced psychic contents proliferate into errors of fact. Propaganda, media distortions, memes, and disinformation compound the social problem of misapprehension further. Shameless self-promotion by personalities of such ideas leads to cults. They make up myths about the myths of by-gone eras. Roiling unconscious images can be fatally confusing. Thought illusions culminate in projections and projections of mythology. Jung suggested symbols live only as long as they are pregnant with meaning. Philosophy arose from criticism of myth, from discussing and challenging it. In science, we criticize, reject and eliminate theories. At the edge of the abyss of the unknown, new signs and symbols emerge. Credible theories and paradigms must include biology, physics, and neurophysiology. One of the reasons people "see God", or a guru, or anomalies may be because our brains are constructed to see

reality through the eyes of others. There are heaps of mirror neurons which are there to make us feel the 'other'. Mirror neurons do for psychology what DNA did for biology. They provide a unifying framework and help explain a host of mental abilities. As in the psychochemical processes of empathy or falling in love, a complex feedback loop sustains a state of mind. But when we empathically transpose ourselves into someone else's position, we expose ourselves to that reality -- cognitively and emotionally. The unconscious complicates empathy, both ways. Mirror neurons might well play a role in bonding, language and self-awareness. Naively, we take too much as self-evident. But 'seeing' does not always 'believe', though many make this error or leap in logic and formulate their choices and future accordingly. Yet, there is only one way to learn what consciousness is. Experience. But we have no satisfactory explanatory edifice for consciousness. Would such a theory release in each of us our own inner knowledge of the creativity of our own consciousness, and its infinite possibilities? The problem is trying to define a verb, a dynamic, as if it were a noun. But we do recognize the effect of consciousness. It functions to mediate states of consciousness, high and low psychobiological arousal. Consciousness is the subconscious lifted up by the physical body. When the body fails, the consciousness collapses back into the subconscious. All our thoughts come from the subconscious which can see our intentions but not our world. This relates somehow to intention being imaginary and not of the physical frictionized world (King). Gerald Edelman postulates that the flows of information in the brain are mediated through 're-entrant' feedback loops. As evolution provides new cognitive functions, new re-entrant loops are established. Even language itself is an archetype -- a chaotic field of dynamic associations. A subtle net of tropes, grammar, symbols, and meaning, the program language begins in limbic resonance. Some phenomena generate their own language patterns, nomenclature, and internal coherence of meaning and representation. In a holographic universe, even time and space could no longer be viewed as fundamentals. Because concepts such as location break down in a universe in which nothing is truly separate from anything else, time and three-dimensional space would also have to be viewed as projections of this deeper order. At its deeper level reality is a sort of super hologram in which the past, present, and future all exist simultaneously. This suggests that given the proper tools it might even be possible to someday reach into the super holographic level of reality and pluck out scenes from the long-forgotten past. Or not. A fantasy of such penetration or phenomenon inside the head is not the same as that penetration. **Jung in the 21st Century: Synchronicity and science By John Ryan Haule Mind Control Countermeasures.** The lower domain of experience is what we do not directly observe, the quantum realm, while the higher level is what we ordinarily observe, the classical realm. Based purely on experiment, the formulation of quantum theory initially placed the great divide between the quantum and classical domains between the measuring apparatus and the particle (Herbert, 1985). However, in the early 1930s, John von Neumann, in his rigorous mathematical treatise on quantum mechanics, found no support for such a division, compelling him to conclude that the wave for was collapsed by consciousness. Von Neumann's treatise has been called the "quantum bible" and "the most influential book on quantum theory ever written" (Herbert, 1985). The von Neumann formulation leads us to conclude that the lower realms of quantum reality form a virtually limitless array of potentials that are inherently incapable of realization without the observation of a knowing entity, which we identify with consciousness (Kafatos and Nadeau, 1990). We thus have a correspondence between quantum theory, the hierarchical structure implied by the Holographic Principle, and the higher orders of experience, leading to the full expression of the Conscious Universe. David Bohm's theory of the implicate and explicate orders involves a holographic Principle that is fully consistent with the Holographic Principle discussed here (Bohm 1980; 1986; Bohm and Hiley, 1993). According to Bohm, there is an implicate order that represents the universal, holographic subtext of reality, and which unfolds in every moment to produce the explicate order that we all observe. Thomas Germinario (2004) has equated the implicate order with (eb) the unconscious process, and the explicate order with (eb) conscious process. He emphasizes the nature of dreams within (eb) the implicate order, and the importance of the dream work for maintaining a healthy mind through (e&eb) integration of the implicate order and our daily lives. Allan Combs and Mark Holland (1990) connected the implicate order or holomovement (Bohm, 1980) with (e&eb) Carl Jung is theory of synchronicity (Jung and Pauli, 1955), with (e&eb) the implicate order providing (eb) a holographic medium through which apparently disconnected individuals become connected. The principle of

synchronicity, the instantaneous connection of people and events beyond (e) the senses, has been equated with the quantum-physical principle of non-locality (Combs and Holland, 1990; Germine, 1991), and has been proposed to be the fundamental mechanism of conscious process (Germine, 1991).Applying the Holographic Principle theory of mind, subjects may be connected synchronously in the manner of two individuals conversing through cellular phones, where the radio signals are transmitted through (e&eb) a distant satellite. The difference here is that the radio signals travel at the speed of light, and thus the there is some miniscule time lapse between the sending and receiving ends. The Holographic Principle implies (eb) a much more distant and instantaneous communication pathway, with (e&eb) the analogue of the satellite being the holographic boundary of the Universe, as well as a much richer communication, potentially involving a transmission of thoughts and feelings. Our ego-consciousness seems to mask the universal relatedness implied by (e) the Holographic Principle, and it is perhaps only through transcendence of (e) the ego-consciousness that the higher orders of experience can become conscious. In the phenomenon of synchronicity, there seems to be a meaningful connection between (e&eb) individuals that breaks through the barrier of ego-consciousness. Such a connection is reported by many individuals in the course of dreams, when (e) the ego-consciousness has been suspended, at times informing (e&eb) the dreamer of something that has happened in the life of a meaningfully-connected individual. **The Holographic Principle Theory of Mind MARK GERMINE Institute for Psycho science** Impassioned-intense Connectionist neuroscience, in an effort to attain relevance to clinical practice, has recently given us a theory of the relationship of the unconscious to (e&eb) consciousness which is (=) dualistic (Viamontes and Beitman, 2007), an ad hoc theory of conclusions based on (e) assumptions, the most erroneous of which are that representations of (e) information are information, and that such representations elaborate themselves through (e&eb) a process of representations of representations in neuronal circuits (Bechtel and Abrahamsen,1991). As theory, the connectionist models are lacking (e) heuristic value and **are (=) inimical to the progress** of (e) basic and clinical neuroscience. This is not to denigrate or minimize the importance of neuronal connections, but these connections are rather like the wires in (eb) a radio that plays a symphony. The radio, in and of itself, is (=) incapable of playing (e&eb) the symphony. The radio doesn't do not write the symphony, nor does it broadcast the Passionate-raging symphony over the airways. Yet, if we loosen one wire, the symphony is (=) not manifested through the radio. The brain is (=) not subject to (e) the same vulnerability, in that it has (e) parallel and redundant networks of processing, and, it is in this sense that the connectivity of the brain becomes (eb) important. Connectionism, to the extent that it accepts (e) consciousness and the (e&eb) unconscious at all, attributes them to (e) separate groups of "circuits" in the brain, with (e&eb) consciousness-processing circuits having (e) a limited but detailed ability to analyze (e&eb) information, and unconscious-processing processing more information in less detail (Viamontes and Beitman, 2007). There is an element of truth in this view in that information that has importance and meaning resonates within the brain at higher levels of experience and at higher levels of holographic recursion. This resonance involves what we know, what we feel, and what we are capable and willing to consciously realize. The Universe, as William Blake noted, can be seen in a grain of sand. Indeed, the grain of sand requires (e) the Universe to exist. The human organism requires (e) the Self to exist. The Self is not a simple amalgam of (e) sensory experiences, as suggested in the connectionist model (Blinder, 2007). This amalgam view of the Self can be damaging when applied to (e&eb) psychotherapy, as the connectionists suggest (Blinder, 2007). The Self is ours forever and ever, from eternity to eternity. As Milton suggested, the mind is (=) its own place. Our deepest purpose is for our minds to resonate with (e&eb) the supra-conscious levels of experience, much as it is (=) purpose of the radio to play the symphony. If the symphony is not to our liking, (eb) we can turn the radio off. So, perhaps, the transpersonal consciousness can turn (e&eb) our own individual and collective radios off, or perhaps turn the volume down, if (e) we are not in harmony with the **Universal Mind**. Physics has given us many enigmas. Quantum theory itself is an enigma. But some of the enigmas seem to involve (e&eb) the limitation of (e) our own abilities of discernment. The thought experiment of (e) "Schrödinger's Cat," as it is often interpreted (e.g. Gribbin, 1984), assumes (eb) that the **cat is not conscious, and is (=) incapable of observation in the context of (e) collapse of (e) the wave function.** This is due, in large part, to the false dichotomy of (e) information and (e&eb) experience. Our observation of other animals indicates, to

those that are attuned to (e) the animal mind, who can "feel" the animal mind, that, at (eb) the very least, all mammals are (=) conscious. It would seem that the great neuropsychiatrist, Stanley Cobb (1948), may have been correct in his attribution of (e) levels of consciousness to a wide variety of organisms: "...lower animals with (e&eb) no cerebrum appear to be conscious...even plants such as the sunflower that turns towards strong light may have (e) a vague awareness or warmth and comfort. There are many degrees of consciousness and it is my contention that it is integrated at (e&eb) many levels like other important functions of (e) the central nervous system."With respect to the theory of mind, recent mainstream thinking remains (eb) classical and mechanistic. Such views of mind are advocated by our most in prominent neuroscientists (e.g. Changeux, 1985; Gazzaniga, 1985) as neuroscience research attempts to explain all mental function on the basis of (e) **pure mechanism and the localized function of specific areas of the brain.** Francis Crick, the distinguished Nobel Laureate, had a very influential second career as a neuroscientist. He concluded that we are basically soulless creatures, that all mental processes could be reduced to (e&eb) identifiable, neural correlates in (eb) the brain, and that, in fact, we are (=) nothing but a collection of neurons (Crick, 1995). Crick even went so far as to say that what we call the soul is (=) a group of neurons located in the prefrontal lobes of the brain. Another Nobel Laureate, Gerald Edelman (1987), has written a critically acclaimed book on "neural Darwinism," which purports that the brain is "circuitry" is constructed through a developmental process of "survival of the fittest" neurons and neuronal connections. At the same time the total purposelessness and lack of direction in evolution is expounded, again to critical acclaim, by Richard Dawkins (1986). The idea that there essentially no self or soul imbued with agency and self determination was expounded by Gilbert Ryle (1949/2002) with his parody of the "ghost in the machine," and this work continues to be influential among many scholars of the mind. Philosophers such as Patricia Churchland (1986), who coined the term "neurophilosophy," take an "eliminative materialist" view of (e) belief, free will, and consciousness. Other philosophers, such as Daniel Dennett (1991) would make (eb) experience "epiphenomenal," a by-product of (e) brain processes with (e&eb) no effect whatsoever on (eb) the function of mind. However, unlike neuroscience, which is still in its infancy, the philosophy of mind has had a long history, including such notable figures as Plato, Aristotle, Descartes, Kant, Locke, Hume, Kierkegaard, and James, to name just a few. Perhaps, as noted by Ervin Laszlo (1974): "in today's world, most of the traditional functions of (e) cognitive synthesis have atrophied (e) and are ignored and neglected." Hopefully, what we have presented here will be a step in the right direction for what Laszlo calls a "conceptual synthesis" to help fill the need for (e) meaningful engagement is such a world. The fundamental elements of such a synthesis have been described by (e) Laszlo as follows: Conceptual synthesis performs (e&eb) at least five basic functions in the guidance of affairs. They are the mystical, the cosmological, the sociological, the pedagogical or psychological, and the editorial functions. The mystical function inspires (eb) in man a sense of mystery and profound meaning related to (e&eb) the universe and of himself in it. The cosmological function forms (eb) images of the universe in accord with (e&eb) local knowledge and experience, enabling (eb) men to describe and identify (e&eb) the structure of (e) the universe and the forces of nature. The sociological function validates supports and enforces (eb) social order, representing (eb) it in accord with (e&eb) the nature of the universe, or as the natural or right form of (e) social organization. The pedagogical or psychological function guides (eb) individuals through stages of life, teaching ways of understanding themselves and others and presenting (eb) desirable responses to life is challenges and trials. Finally, the editorial function of (e) conceptual is to define (eb)some aspects of reality as important and credible and hence to be attended to, and other aspects unworthy of serious attention. **The Holographic Principle Theory of Mind MARK GERMINE Institute for Psycho science** Kant sums up this inversion and its spirit early in the Critique of Pure Reason: Up to now it has been assumed that all our cognition must conform to the objects; but all attempts to find something about them a priori through concepts that would extend our cognition have, on this presupposition, come to nothing. Hence let us once try whether we do not get farther with the problems of metaphysics by assuming that the objects must conform to our cognition, which would agree better with the requested possibility of an a priori cognition of them, which is to establish something about objects before they are given to us. This would be just like the first thoughts of Copernicus, who, when he did not make good progress in the explanation of the celestial motions if he assumed that the entire celestial host revolves around the observer, tried to see if he might not have greater success if he made the observer revolve and left the stars at rest. (B xvi) Onticology,

like all variations of object-oriented ontology, is realist in its orientation. In defending a realist ontology onticology holds that the vast majorities of objects, actants, beings, or entities are independent of humans and are what they are regardless of whether any humans regard them or register them. In short, onticology rejects any anthropomorphic, idealist, or anti-realist thesis to the effect that to be is to be the correlate of mind, spirit, the body, the human, and language or otherwise. While it is certainly the case that knowledge is necessarily dependent on the object to which it relates, the converse does not hold true. Objects are not dependent on being known, regarded, perceived, or spoken about. As such, and to put it in Aristotlean terms, knowledge is an accident of objects, not objects an accident of knowledge. As Althusser so nicely puts it, "[n]o doubt there is a relation between thought-about-the-real and thisreal, but it is a relation of knowledge, a relation of adequacy or inadequacy of knowledge, not a real relation, meaning by this a relation inscribed in that real of which the thought is the (adequate or inadequate) knowledge" (Reading Capital, 96). Althusser goes on to remark that "[t]he distinction between a relation of knowledge and a relation of the real is a fundamental one: if we did not respect it we should fall irreversibly into either speculative or empiricist idealism" (ibid.). Onticology categorically endorses Althusser's verdict. It is a fundamental necessity to distinguish between those relations that belong to the object and those that belong to knowledge. Contemporary philosophy continuously confuses these two very different sorts of relations. Naturally the question arises of how it is possible to surmount our relation to the object so as to determine whether objects themselves possess the properties we encounter in relating to objects. In other words, given that we can only ever relate to the object in relating to the object how is it possible to surmount this relation to get at the being of the object itself? Much more will have to be said about this later– and the answers will be surprising with respect to standard prejudices about realism –however, for the moment it can be said that onticology takes its epistemological inspiration from the transcendental realism of Roy Bhaskar. Among other things, Bhaskar sought to provide a transcendental grounding for the sciences. Insofar as onticology defends the thesis that the field of being is much more vast than the field of objects investigated by the natural sciences, it parts way with the thesis that the domain of being is exhausted by the domain of natural objects. However, the general form of Bhaskar's argument holds for our realist purposes. A transcendental argument seeks to elucidate the conditions under which certain acknowledged practices and forms of cognition are possible. Kant, for example, asked what must be the case for mathematical judgments to be possible. How is it both that we are able to extend our knowledge, as if by magic, through mathematical judgments and, more significantly, that these judgments are able to provide genuine knowledge of the world despite the fact that these forms of reasoning are not based on experience? Part of Kant's argument consisted in claiming that mind imposes the forms of space and time on the data of experience. In other words, space and time are not attributes of being itself but rather of the mind that regards being. Insofar, Kant argues, as mathematics is ultimately a rumination on the nature of space and time taken in their most abstract form and insofar as the mind imposes space and time on the manifold of sensation, it thus follows that a priori judgments about the nature of spatio-temporal relations are possible that anticipate the structure of actual-space times without directly experiencing these space-times. Why? Because any manifold of sensation must necessarily be structured by these forms imposed by intuition. **Onticology– A Manifesto for Object-Oriented Ontology Part I Posted by larval subjects under Object-Oriented Philosophy My Question Is This: You Say, "Granted There Are Deeper And Deeper And More Primitive Forms Of "Knowledge" Or Abstraction Into Mechanism, But I Wouldn't Say That The Atomic Level Brings A New Epistemic World Into View." But Doesn't Any Kind Of Knowledge, However Primitive, Imply Bringing Some Kind Of Epistemic World Into View Or Proto-View?** Joel: David, Yes Exactly. My Question Is, Just How Far Down The Complexity Gradient Can The Meaning Of 'Knowledge,' Or 'Semiotics, Or Specifically Representation (Maya) Be Stretched Without Breaking? I Can Comfortably Stretch It To The Genetic Level, As The Proto-Epistemic, Where Genetic Codes 'Represent' Phenotypic Structures Or Modes, And The Whole Of Evolution Learns From Its Mistakes And Successes As That Knowledge And History Is Encoded Into And As The Recapitulation Of The Embryogenesis Of The Organism. But An Atom, In My Model, While Certainly Reacting To Its Environment From Its Own Attractor, Essence, And 'Interiority' (Prehension) Isn't Making Representations Of It To Do So. But The Key To The Power And 'Essence' Of Representation (And Simultaneously Its Limitation) Is The

Capacity Of Abstraction And Choice. Evolution Does This At A Rudimentary Level (And Anthropomorphizing A Bit) With Its Sending Into The World Its 'Random' Variations, Organismic Ideas, In A Sense. Its 'Mind' Is Literally On The Outside As Our Living World And Biosphere. A 'Choice' Is Made In The Biospheric Interiority, And Knowledge For The Evolutionary/Embryogenetic Trajectory Is Gained By The "Differential Reproductive Success" Of These Individuals Lives Through Time, As The Increasing Intelligence, Representation, And Capacity For Choice Is Directly Injected Into The Flow Through Sexual Selection. But Where Is The Encoded/Abstracted (Enfolded) Representation Of Options And A Choice Among Them Made At The Atomic Level? I Am Working With The Most Cutting-Edge And Coherent Models For The Atom (Including My Own) That I Can Find Among The Heterodoxy, And I See No Evidence Of Representation And Choice At This Level, Or Rather No Real Way To Further Stretch The Meaning Of The Terms To This Level. Sure There Is Uncertainty From Infinite Difference And Immanent Causation (See Bohm On Infinite Causation), But Without Representation There Really Is No Choice (Although Crudely We Could Call A 'Bifurcation' A 'Choice' By The System As A Whole). So I Am Simply Saying That Representation At The Atomic Level Is Entirely Enfolded And These Are Really Just Ontic Level Phenomena. The Atomic Level, In My View, Is Pre- Not Yet Proto-Epistemic, Which I Place At The Genetic Level Where We Find The First Codes And (e&eb) Primitive Representation. This Just Means That There Is A Real Distinction Between Ontic And Epistemic (And A Gradient Between Them) And We Can't Collapse The Two Into Just The Ontic-Epistemic. And Yet, With The Distinction Intact, And The Gradient To Explore, They Remain Nondual, Univocal, ONE Or Ontic.What I Would Say Exists At All Levels (My Own Self-Aware Myth About The Given) Is What I Call The "Symbiogenesis Of Subject And Object". This Is The Spinozan Essence Of Dynamic Stability And Growth, And Leibnizian Prehension. Also Called The "Nucleation Of Observability" In Spinbitz. It Is From This Nucleus Of The Observer (Fuller) In A "Point-Free Geometry" (Whitehead) Of Pointless "Points Of View" As Conditions Of Boundary That Wilber's "All Is Perspective" Finds Its Ontic Grounding. And This Symbiogenesis Is Also A Rudimentary Or Deep-Level Native Or Primitive Intelligence, Evolution At The Involution Of The Subject-Object Interface. It Is An Exploration Of Creative Learning And Direct Immanent Awareness, Unmediated (Enfolded) By Representational Forms (Unfolded). It Just Is, In Waves (Cosmic Vertebrae) Before It Abstracts, Represents And Incarnates The Flesh Of What Could Be. Matter Is Enfolded Seed Of The Flower Of Its Abstraction. Maya Is This Depth And Recursion Of Brahma Into And Through The Boundary Conditions Of Abstract Relation. And Brahma Is The Span Of The Emergent Rigor Of The Cosmic Ergodic Spine That Opens The Recursions Into The Field Of Infinite Difference. In Immanent-Transcendent Waves, Literally And Empirically A Limit-Cycle In Ergodicity, It Enfolds And Unfolds From Anatom To Anatomy, From Infinite Intelligence Or Omnirelational Awareness Of Self, Through The Representational Depths Of Self As Self-Consciousness. Omni-Evolution, Trans-Dynamics, And The Anatom Posted On February 21, 2013A Discussion With Tom Huston, Part II The Following Is Part 2 Of An Excerpt From A Discussion Initiated By Tom Huston On His Facebook Page, Edited And Embellished For Clarity. Find Part One, Here. In This Part We Dig Into The Nature Of The "Anatom" And The "Ergodic Spine" Of An Electro-Fractal Cosmos, And Into The Trans-Dynamic Integration Between Being And Becoming, And The X-Interface−The Crossroads Of The Ontic-Epistemic (Brahma And Maya) And The Subject-Object Polarities. Univocal_Dynamics_V1 Tom Huston Is A Founding Member Of Integral Institute And A Former Editor Of Enlighten Next Magazine. His Webpage Is Tomhuston.Com. David Marshall Is A Writer And Editor Living In Chicago. No meaning, value, or normativity to be found in nature, that there is nothing **natural beings ought to be,** but that, rather, these judgments arise from us (See last post of larval subjects) which has led some to raise valuable questions about the coherence of these claims. The problem arises when the following three propositions are taken together: 1. there is nothing outside of nature.2. Beings have no intrinsic meaning, purpose, or value (in and of themselves, there's nothing they ought to be). 3. Value judgments about what beings and being ought to be arise from us and beings like us (bonobo apes, dolphins, institutions, birds of paradise, etc). The problem arises between proposition 1 and 3. How can it both be true that there is nothing outside of being and those normative judgments belong to us and other beings capable of making normative judgments, not nature? The problem arises from restricting these judgments to humans and beings capable of

making these judgments. In making such a claim it seems as if we're saying that there's something outside of nature, something that is beyond nature, thereby violating the first thesis and potentially reintroducing the nature/culture distinction. By contrast, if we say that normative judgments are the special domain of those living beings with the proper degree of sentience to make such judgments, then we seem to reintroduce the nature/culture distinction and fall back into the sorts of problems that I outlined in my last post (and that are so nicely critiqued by thinkers such as Latour). Is there a way out of this? I don't know. Great_Chain_of_Being_2_ (lighter) ASIDE: It seems to me that this is really what the debate between realism and anti-realism, realism and socio-linguistic constructivism surrounding the new materialisms and speculative realists has really been all about. It's very easy to treat this as an abstract, academic debate: "Are you a realist or are you an anti-realist?", as if it were just a matter of what happens to be true. But it seems to me that this debate has, in reality, always been about politics. As I outlined in my last post, we have perpetually seen how appeals to the real and natural have been used in the name of oppressive power, inscribing both the exploitation of nature and the oppression of various people in the very fabric of being itself. Theistic theology and realist ontology have perpetually been used in the name of what Deleuze called "State Philosophies" or philosophies that ontologize contingent orders of power and privilege (e.g., "the great chain of being" used to justify patriarchy, monarchy, serfdom, poverty, etc, and appeals to nature used to justify poverty and racial inequality (The Bell Curve), patriarchy (evolutionary psychology), heteronormativity, capitalism, etc). Because arrangements of power and inequality are always contingent in the sense that there's no marked difference in the capacities of peoples, power always looks for a transcendent supplement that would provide justification through ontological necessity. Antirealism– from the Greek atomists to present –became the radical and emancipatory gesture because it revealed the lie behind all of these forms of social organization or their inherit contingency or arbitrariness. Realism, by contrast, has all too often functioned as an apologetics for arbitrary power and social organizations. Here it's worth recalling what Foucault said about science: "…Even before we know to what extent something like Marxism or psychoanalysis is analogous to a scientific practice in its day-to-day operations, in its rules of construction, in the concepts it uses, we should be asking the question, asking ourselves about the aspiration to power that is inherent in the claim to being a science. The question or questions that have to be asked are: "What types of knowledge are you trying to disqualify when you say that you are a science? What speaking subject, what discursive subject, what subject of experience and knowledge are you trying to minorize when you say 'I speak this discourse, I am speaking a scientific discourse, and I am a scientist.' What theoretico-political vanguard are you trying to put on the throne in order to detach it from the massive, circulating, and discontinuous forms that knowledge can take?" (Society Must Be Defended, 10). All of these questions hold equally for claims to something being real. What is one trying to minorize when claiming something is real? What becomes privileged? What is excluded? It is these questions that have been at the heart of the realism debates, for as Spencer-Brown taught us, every distinction has a marked and unmarked space, draws attention to something to be included and pushes something into the unconscious or the domain of the invisible, hidden, or veiled. This is above all the case with evocations of the real. However, as I've tried to show antirealism leads to its own problems. First, so long as we exclude real beings from our ontological inventory, we are unable to fully understand how power functions ("such and such a set of cultural formations have been thoroughly debunked, yet people still live as if they believed them"). Not only do we not fully understand the sources of the problems due to too much focus on the discursive and semiotic, but we deny ourselves valuable sites of political intervention at the level of infrastructure. Second, we are prevented from addressing things such as cultural racism, such as that found in Heidegger's privileging of the West and the Greeks and Germans in particular. We do a good job addressing biological and theological racism, heteronormativity, and sexism by showing how it is a cultural construction, but when faced with racism such as Heidegger's where he argues that there's something "unique" about the Greek event, it's language, and about the German language, or Badiou/Zizek's racism with respect to the "Pauline Event", we really have no response. Here someone like Jared Diamond or Fernand Braud el is needed to explain global-geographical inequalities. Third, the tools of the cultural turn really do not provide us with the means of thinking the ecological as a site of the political. For this reason, I've tried to formulate a third way– which might be called "constructivist

realism" or "constructivist naturalism" –that retains the insights of the cultural turn, while also allowing a robust place for the material. I don't claim to be original in this. I think that many such as Manuel DeLanda, Deleuze and Guattari, Stacy Alaimo, Karen Barad, Jane Bennett, etc., are up to something similar. It's a vast project that requires the work of a multitude of voices, especially given the way in which the culturalism pervades humanities. **The Relational and the Non-Relational: Notes towards an Immanent and Pluralist Theory of Meaning Posted by larval subjects under uncategorized k-bigpic I have deleted or reformulated some sentences for which I beg pardon attributable to spatial constraints** What is correlation, what is causation? What is coordination? What is coincidence? These thoughts have bothered humans since long. I have been able to excavate some reliable material from Wikipedia which I would like to share. "All views are partial and contingent – that's the lesson of pluralism, and you say as much yourself when you say that we must be attentive to ways our own knowledge might contain superstitions. But it's not discourse that determines whether one view works and another doesn't – it's a confrontation with **non-discursive (non-human)** agencies." I wonder if part of the issue here is that we understand pluralism differently. For me this doesn't sound like pluralism at all. Rather, it just sounds like our **epistemological condition**. We [hopefully] come to understand those portions of existence we question or investigate. The reason we come to investigate them is largely contingent. Finally, the accounts of these features of existence we give can be mistaken. All of this is perfectly consistent with a monism. I'm not sure why what he outlines above is the lesson of pluralism. It seems to me that every realist knows this. It seems to me that ontological pluralism is something quite different. Ontological pluralism is the thesis that there are many different worlds inhabited by many different entities. Thus, for example, you would have one world; say that of Lucretius, that's only inhabited by atoms and their combinations. You would have another world; say the world of the Mongolian shaman, that's inhabited by spirits that do all sorts of things. The ontological pluralist is saying that these spirits are, that they exist, that they're real. This is something quite different than merely saying that there are partial and contingent points of view on the world. It's this saying that these entities are rather than that some person or groups of people believe that they are that is the nub of the issue. Now, I think part of the issue here is that there's an ambiguity in the term ontology. Ontology can be one of two things. On the one hand, ontology is a group or persons set of beliefs as to what is. Here it's trivially true that there are pluralities of ontologies and the realist readily recognizes this. This is the whole reason there are debates over ontology. Mongolian shamans have their ontology, Europeans theirs, Christian fundamentalists theirs, materialists theirs, etc. When striving to understand and communicate with others it's vital to understand these ontologies because, as rhetoricians like Burke point out, our **beliefs about what is** are among the things that motivate our action. Despite having never seen bacteria I was my hands because I believe there are bacteria and viruses on door handles and whatnot and don't want to get sick. My belief about a particular thing existing is what motivates my action. And who knows, perhaps this belief is as superstitious as the belief that the crops failed because God was displeased with my community. On the other hand, ontology is a theory **about what is.** It is making a claim that something exists. This is where the rubber hits the road. The ontological pluralist seems committed to the thesis that every group set of beliefs about what exists is sufficient for granting the existence of those entities. This is what I find objectionable in Latour. Obviously I'm not bothered by Latour's suggestion that we should take into account the role that nonhumans like speed bumps, rivers, microbes, etc., play in the form that social assemblages take. But this is not all that Latour claims. This is precisely where the philosopher might balk. Unlike the ethnographer, the philosopher is not interested in what people believe exists, but rather philosophers– at least of the realist variant –are trying to figure out what is. In other words, the realist philosopher begins with the premise that not all of these beliefs about what is are true. So for the philosopher, recognizing that Mongolian shaman's believe in the existence of shamans would only be the first step. The next step would consist in determining whether there's good reason for thinking such entities really do exist, i.e., whether there's good reason for believing these entities have mind and culture independent reality. Lest readers think that I'm just picking on the supernatural here, we can ask similar questions about strings, subatomic particles, galaxies, etc., etc., etc. I guess the question really comes down to what exactly we mean by ontological pluralism. When we talk about ontological pluralism are we defending the thesis that people have different theories of being? If so, then

ontological pluralism is trivially true. If this is what is meant, then I certainly share Jeremy's view that it's valuable to understand the different world's people believe in. Certainly when I was practicing as an analyst I didn't get in ontological debates with my patients and it was necessary to understand their theory of being or their ontology to properly attend to them. Or, when we talk about ontological pluralism are we defending the thesis that all these ontological theories are true and refer to really existing entities? That's quite a different claim and is not one I would defend or endorse. Now someone might object that "in both The Democracy of Objects and Onto-Cartography you defend the thesis that there are multiple worlds." This is true. Because I hold that not everything is related I'm led to the conclusion that there are diverse worlds. However, I also hold that however many worlds there might be, these worlds are nonetheless composed solely of material entities. Within the framework I propose I wouldn't suggest that there's one world where there are spirits and another world where there are souls and yet another where there are only material entities. My view is that there aren't spirits or souls in any of these worlds. One of the things I keep hearing in these discussions is that somehow the realist adopts a view from nowhere. I honestly don't understand this criticism. Investigation always occurs somewhere and requires all sorts of mediations involving technologies, experiments, etc. It's that labor of gathering evidence, conducting experiments, using technologies to observe the world, etc., that gradually gives us a body of data that allows us to say there's good reason to believe that such and such a thing exists and has these powers. Another charge seems to be that the realist refuses to recognize that their claims about the world are fallible. I find this charge particularly strange because it's precisely because the realist recognizes the difference between our theories of the world or what we say about the world and the world itself that fallibility is built into the core of his position. Realism doesn't mean one holds they have special access to the world, that they know all truths, or that they have the truth in hand, only that there are truths to be known and that we can be mistaken about things. Circling **Squares Philip** has a post responding to my quandaries about how to mesh realism and pluralism. He writes: Ontologically and metaphysically the idea of realist pluralism is no longer an issue. There are (appropriately) numerous variants but the basic idea that reality is itself pluralistic is well established. The question is political-discursive. It's what Stengers and Latour are getting at with their concepts of diplomacy and cosmopolitics. They grant, first, that all entities exist and, second, that to say that someone's cherished idol (or whatever disputed entity they hold dear) is non-existent is a **'declaration of war'** – 'this means war,' as Stengers often says. They thus shunt onto-political discourse off of the terrain of knowledge/belief in the sense of existence/non-existence. Their basic claim seems to be that 'respect for otherness,' i.e. political pluralism, can only come from granting the entities that others hold dear an ontology, even if you don't 'believe' in them. You are thus permitted to say 'I do not follow that god, he has no hold over me' but you are not permitted to say 'your god is an inane, infantile, non-existent fantasy, grow up.' And it's not just a question of politeness (although there's that too). The point is to grant others' idols and deities an existence – one needn't agree over what that existence entails, over what capacities that entity has or what obligations it impresses upon you as someone in its partial presence but to deny it existence entirely is to 'declare war' – to deny the possibility of civil discourse, of pluralistic co-existence. I believe that it was Richard Rorty who once quipped something like claims to reconcile realism and idealism always seem to end with a triumph of idealism. We don't, in fact, get realism through such approaches, but rather just get a pervasive anti-realism. I think this is also the problem with the "non-controversial pluralism" advocated by Stengers and Latour that Phillip defends here. Such pluralism is not realism but is, in fact, a thoroughgoing social constructivism. I think this is the central problem with Latour's argument in Irreductions (these days I regret having ever defended it). In rejecting both Enlightenment critique and what he calls "reduction" he wants to say something like "The Pentacostal really is filled with the Holy Spirit", that for the 19th working scientist heat really is a fluid and phlogiston really is what allows things to burn, and that for the Greek lightning really is an expression of Zeus's anger. Latour tells us that we aren't to reduce or explain away the entities posited by another group's "ontology" but are to develop explanations from within that ontology. It might not sound particularly sexy– and it certainly doesn't tell us what is worth thinking –but I can't help but believe that philosophy is the critical and reflective investigation of basic concepts that guide our investigation of the world about us, how we ought to live our lives, and what form of governance might be best. Compare two figures. A scientist might ask,

what causes depression? We can very well imagine a philosopher turning around and asking the scientist, what is causality? The scientist presupposes a concept of causality in her investigations. She uses this concept in her inquiry. Now she might have a sophisticated concept of causality or she might never have thought much about causality at all, using it in the sort of colloquial and unreflective way that Plato decried when, for example, people like Euthyphro talked about piety. A whole cascade of questions arise when we raise a question like "what is causality?" We can ask whether or not causality exists at all. We can ask how we distinguish between correlation and two events that merely accompany one another from genuine causation. This, for example, was Hume's question. But perhaps most importantly we can ask whether there is only one form of causality or many forms of causality. Is there only one-to-one causation; one cause and one effect? Is there many-to-one causation; or many events conspiring to produce an effect? Is there one-to-many causation; or one event producing a variety of different effects? We can even ask whether causality necessarily moves from past to present or whether there aren't forms of causality that move from future to past! **January 23, 2014 Pluralism and Realism:** multiple-worlds Over at Struggles Forever, Jeremy Trombley has an interesting post up on "the ontological turn" in anthropology or ethnography. I've been meaning to have a discussion with him about this as I think it's an issue many of us are struggling with. For example, the core project of The Democracy of Objects– a proj ect which I think many have missed –is to somehow reconcile some version of social constructivism with a realist ontology capable of making room for ecology (which requires realist and materialist positions as there's a fact of the matter where global warming is concerned) as well as the role played by objective agencies in social assemblages such as technologies, infrastructure, features of geography, local climates, the growth cycles of plants and animals, waste, etc. Maybe we can try to organize some cross-blog event to discuss these issues. I certainly think they're close to the heart of Jeremy, Michael of Archive Fire, Arran James, and a host of others. As an aside, I'm beginning to realize how the different sites of the political I've been outlining– semiopolitics, thermopolitics, oikopolitics (political economy), geopolitics, eropolitics (the politics of sex and desire), biopolitics, and chronopolitics (and I'm sure there are other political sites!) –are drawing me away from traditional Marxism. Assuming that classical Marxism holds that economics or the conditions and relations of production are determinative of all other sites of the political, the various sites of the political that I've been outlining would lead to the conclusion that there is not one determinative base of the political. This would not require committing Marx to flames, but rather of recognizing the phenomenon of overdetermination, or of a variety of different entangled sites of the political. But I digress. First, I find myself wondering what the ontological turn means in ethnography. Is it 1) the investigation of the different ontologies held/proposed by different cultures? E.g., the Aztecs believed that reality was structured in this way, while the Greeks in that way, and the ancient Chinese this way, etc? Or 2) is it an investigation of how real entities– independent of cultural beliefs –influence cultural formations? Or is it a combination of both? A position that I would favor. January 20, 2014 Thoughts on the Social and Political Implications of Correlationism .Lion Mirror Truth be told, as my thought has evolved the issue of correlationism had fallen off the radar for me. Somehow the debate had come to seem too "philosophical" to me, too "scholastic", too remote from what interests me: understanding why social assemblages are organized as they are, how power functions in social assemblages, and what we might do to address that power and change things. Somehow the question of whether or not we can get out of the correlation between thinking and being just came to seem remote from these sorts of issues. Somehow it seemed too epistemological. ASIDE: Numerous discussions over the years have led me to believe that the debate over correlationism is poorly understood (or maybe I just don't understand it). On countless occasions I've heard people say "of course we must relate to things in order to know them." Well yeah, of course! I don't think this is what the critic of correlationism is getting at. It seems to me that correlationism is something more robust than the theses that we must relate to something to know it. Correlationism instead seems to require the theses that thought and being are indiscernible. Put more concretely, the correlationist is someone who argues that we either a) can never tell whether being is merely a construction of our thought (weak correlationism), or b) who argues that thought actually constructs being (strong correlationism). In other words, correlationism is another name for idealism. One can hold that we must relate to something in order to know it without being a correlationist. As an

aside I should also add that I am a correlationist about some things. For example, I think money is something constructed by society and is therefore a strong correlationist when it comes to money. At any rate, for a long time I'd become rather indifferent to debates about correlationism and philosophies of access. I had learned the lessons of speculative realism– which I could have also learned, I think, from Deleuze and Guattari, the new materialist feminists, actor-network theorists such as Latour –and had moved on. However, occasionally you come across a tone of phrase that pitches something in a different light. In The Cut of the Real, Katerina Kolozova writes, …the political problem of contemporary philosophy identified by the 'new realists' is, in fact, the product of a more fundamental epistemic problem. In his book After Finitude, Quentin Meillassoux calls this problem 'correlationism' and identifies it as an essentially post-Kantian legacy, which continues to dominate and limit philosophy. As a matter of fact, correlationism lies at the heart of postmodern theory and consists in the premise that thought can only 'think itself,' that the real is inaccessible to knowledge and human subjectivity, and that there is nothing but discursive constructs that fully determine thinking and that are meth0dologically accounted for all the way down. (1 – 2) Thought thinking only itself: Thought only encountering itself. In the jargon of postmodern and poststructuralist lingo, this would be the thesis of infinite semiosis, where signs ("thoughts") only ever relate to other signs. Within this framework, discursivity comes to be the hegemonic framework defining all of being. At the level of politics and social theory more generally, if the correlationist thesis is true the consequences are clear: all social phenomena are discursive and all solutions to social and political problems will be discursive. The sole sphere of the political will be the discursive and all questions of politics will be questions of speech-acts and interpretation. The problem here is not that many theorists recognize that the discursive and semiotic plays an important role in the social and the political. It does and I've repeated this tirelessly. The problem is with what happens when thought or the semiotic becomes a hegemon, an "all", foreclosing our ability to recognize other forms of power. What I've wanted to say is that not all power functions discursively. In my last post and elsewhere I spoke of some other forms of politics: Thermopolitics: The politics surrounding **energy in the form of calories and fuels such as gasoline** and coal, and how our life and our very bodies are structured by energy dependencies and by being trapped in particular distributive networks that render these forms of energy available. I'm being quite literal when I speak of energy, talking about the effects, for example, of the absence of food in certain educational environments on cognition, for example; and am generally hostile to metaphorical extensions of the concept of energy which I see as erasing the dimension of real materiality. Geopolitics: The role that features of natural and built geography such as mountain ranges, rivers, oceans, soil conditions, roads, housing design, etc., play in the form that social relations take and how they impact individual bodies. Chronopolitics: The way in which the structurations of time organize what is possible for us. For example, the structuration of the working day, how much we can say and comprehend at any given time, the impact of things like the invention of the clock, etc. **Oikopolitics: This would be the domain of political economy described so well by Marxists**. So five different types of politics: Semiopolitics (or what currently dominates critical theory), thermopolitics, geopolitics, chronopolitics, and oikopolitics. No doubt there are other sites of the political or political struggle that we could speak of, but this is a good start. Also, it should be obvious that these aren't exclusive domains, but are entangled in all sorts of important ways. For example, something might take place at the level of semiopolitics (speech, law, rhetoric, norms, and communication) that has all sorts of impact at the level of thermopolitics. Congress might decide to cut programs that fund school meal programs. This, in turn, will have a thermodynamic impact on those students that go without the calories they need developmentally and cognitively to function in a particular way. There is an entanglement here of semiopolitical and thermopolitical domains. The young student here has been constrained both at the level of **semiotic phenomena and thermodynamic structures**. The point is that if true, semiotic intervention (speech-acts, protests, interpretations, deconstructions, etc) will not be an appropriate response to all political problems because social formations are not entirely structured by the semiotic. The child in that school does not suffer from a lack of the right signs, but from a lack of calories needed to run the engine of his thought and body. Certainly semiotic interventions might be needed to render that energy available, but it is the energy itself that is at issue and the absence of that energy that forms the spider web entangling him in his position. A correlationist perspective tends to

erase this as even being a site of the political. **Larval Subjects January 28, 2014 Ontological Anarché Posted by larval subjects under Uncategorized January 27, 2014 Roden on Pluralism Over at enemy industry Roden has an excellent post up on the pluralism discussion. Check it out here. January 25, 2014 Different Senses of Pluralism and Ontology Posted by larval subjects under Uncategorized Responding to one of my comments over at his blog, Jeremy Trombley:** I have deleted restricted, reformulated some sentences .Kindly bear with me. Intention is to make the discursive enucleation consistent with the subject in question**.** Recourse to epistemology of the quantum mechanics generic and fervor of string theory is a sine qua non for the further comprehension of the theory and ontology .How would the world appear to us if its ontology was that of classical mechanics but every agent faced a restriction on how much they could come to know about the classical state? We show that in most respects it would appear to us as quantum. The statistical theory of classical mechanics, which specifies how probability distributions over **phase space evolve** under Hamiltonian evolution and under measurements, is typically called Liouville mechanics, so the theory we explore here is Liouville mechanics with an epistemic restriction. The particular epistemic restriction we posit as our foundational postulate specifies two constraints. The first constraint is a classical analog of Heisenberg's uncertainty principle; the second-order moments of position and momentum defined by the phase-space distribution that characterizes an agent's knowledge are required to satisfy the same constraints as are satisfied by the moments of position and momentum observables for a quantum state. The second constraint is that the distribution should have maximal entropy for the given moments. Starting from this postulate, authors derive the allowed preparations, measurements, and transformations and demonstrate that they are isomorphic to those allowed in Gaussian quantum mechanics and generate the same experimental statistics. They argue that this reconstruction of Gaussian quantum mechanics constitutes additional evidence in favor of a research program wherein quantum states are interpreted as states of incomplete knowledge and that the phenomena that do not arise in Gaussian quantum mechanics provide the best clues for how one might reconstruct the full quantum theory. I: http://dx.doi.org/10.1103/PhysRevA.86.012103 **Reconstruction of Gaussian quantum mechanics from Liouville mechanics with an epistemic restriction Phys. Rev. A 86, 012103 – Published 10 July 2012 Stephen D. Bartlett, Terry Rudolph, and Robert W. Spekkens.** Deleuze in fact characterizes modes of existence, with their powers and capacities. The answer is this: Deleuze approaches modes of existence, ethically speaking, not in terms of their will, or their conscious decision making power (as in Kant), nor in terms of their interests (as in Marx, for example), but rather in terms of their drives. For Deleuze, conscious will and preconscious interest are both subsequent to our unconscious drives, and it is at the level of the drives that we have to aim our ethical analysis. Here, I would like to focus on two sets of texts on the drives taken, not from Nietzsche and Spinoza, but rather from Nietzsche and Leibniz (Leibniz being one of the first philosophers in the history of philosophy to have developed a theory of the unconscious). The first set of texts comes from Nietzsche's great early book entitled Daybreak, published in July 1881. Nietzsche first approaches the question of the drives by giving us an everyday scenario: "Suppose we were in the market place one day," he writes, "and we noticed someone laughing at us as we went by: this event will signify this or that to us according to whether this or that drive happens at that moment to be at its height in us—and it will be a quite different event according to the kind of person we are. One person will absorb it like a drop of rain, another will shake it from him like an insect, another will try to pick a quarrel, another will examine his clothing to see if there is anything about it that might give rise to laughter, another will be led to reflect on the nature of laughter as such, another will be glad to have involuntarily augmented the amount of cheerfulness and sunshine in the world—and in each case, a drive has gratified itself, whether it be the drive to annoyance, or to combativeness or to reflection or to benevolence. This drive seized the event as its prey. Why precisely this one? Because, thirsty and hungry, it was lying in wait" (D 119). This is the source of Nietzsche's doctrine of perspectivism ("there are no facts, only interpretations"), but what is often overlooked is that, for Nietzsche, it is our drives that interpret the world, that are perspectival—and not our egos, not our conscious opinions. It is not so much that I have a different perspective on the world than you; it is rather that each of us has multiple perspectives on the world because of the multiplicity of our drives—drives that are often contradictory among themselves. "Within ourselves," Nietzsche writes, "we can be egoistic or altruistic, hard-hearted, magnanimous, just, lenient, insincere,

can cause pain or give pleasure" (Parkes, pp. 291-292). We all contain such "a vast confusion of contradictory drives" (WP 259) that we are, as Nietzsche liked to say, multiplicities, and not unities. Moreover, these drives are in a constant struggle or combat with each other: my drive to smoke and get my nicotine rush is in combat with (but also coexistent with) my drive to quit. This is where Nietzsche first developed his concept of the will to power—at the level of the drives. "Every drive is a kind of lust to rule," he writes, "each one has its perspective that it would like to compel all the other drives to accept as a norm" (WP 481). DELEUZE AND THE QUESTION OF DESIRE: TOWARD AN IMMANENT THEORY OF ETHICS Daniel W. Smith. **In on a Possible Physical Metatheory of Consciousness Miroljub Dugic et al** show that the modern quantum mechanics, and particularly the theory of decoherence, allows formulating a sort of a physical metatheory of consciousness. Particularly, the analysis of the necessary conditions for the occurrence of decoherence, along with the hypothesis that consciousness bears (more-or-less) well definable physical origin, leads to a wider physical picture naturally involving consciousness. This can be considered as a sort of a psycho-physical parallelism, but on very wide scales bearing some cosmological relevance. Open Systems & Information Dynamics Volume 9, Number 2, 153 (2002) arXiv: quant-ph/0212128. On a broader perspective and background, the theoretical computation of the universe is justified.IN physics and cosmology, digital physics is a collection of theoretical perspectives based on the premise that the universe is, at heart, describable by information, and is therefore computable. Therefore, according to this theory, the universe can be conceived of as either the output of a deterministic or probabilistic computer program, a vast, digital computation device, or mathematically isomorphic to such a device. Digital physics is grounded in one or more of the following hypotheses; listed in order of decreasing strength. The universe or reality: is essentially informational (although not every informational ontology needs to be digital); is essentially computable (the pancomputationalist position); can be described digitally; is in essence digital; is itself a computer (pancomputationalism); is the output of a simulated reality exercise. Following Jaynes and Weizsäcker, the physicist John Archibald Wheeler wrote the following: [...] it is not unreasonable to imagine that information sits at the core of physics, just as it sits at the core of a computer. (John Archibald Wheeler 1998: 340) It from bit. Otherwise put, every 'it'—every particle, every field of force, even the space-time continuum itself—derives its function, its meaning, its very existence entirely—even if in some contexts indirectly—from the apparatus-elicited answers to yes-or-no questions, binary choices, bits. 'It from bit' symbolizes the idea that every item of the physical world has at bottom—a very deep bottom, in most instances—an immaterial source and explanation; that which we call reality arises in the last analysis from the posing of yes–no questions and the registering of equipment-evoked responses; in short, that all things physical are information-theoretic in origin and that this is a participatory universe. (John Archibald Wheeler 1990: 5) David Chalmers of the Australian National University summarised Wheeler's views as follows: Wheeler (1990) has suggested that information is fundamental to the physics of the universe. According to this 'it from bit' doctrine, the laws of physics can be cast in terms of information, postulating different states that give rise to different effects without actually saying what those states are. It is only their position in an information space that counts. If so, then information is a natural candidate to also play a role in a fundamental theory of consciousness. We are led to a conception of the world on which information is truly fundamental, and on which it has two basic aspects, corresponding to the physical and the phenomenal features of the world. (Wikipedia). **Participatory consciousness forms the bastion, pillar, post, stylobate and fulcrum of the redressal measures and corrective steps taken for circumvention of many problems, identified and unidentified, spread widest commonalty. Anne Baring's sententious and pithy article will be good to have a timely outlook.** I would like to open my contribution to this meeting with the words of Ken Wilber that I quoted in the first Millennium Symposium: 'The secret impulse of life is towards greater consciousness. Maybe the evolutionary sequence is from matter to body, to mind, to soul, to spirit - each transcending and including, each with a greater depth and greater consciousness and wider embrace' Ken Wilber, and with these from Rupert Sheldrake's contribution to the same meeting: "It is possible to make quite a good case that the sun could be conscious and if the sun, then why not the stars? Why not the galaxies? Why should there not be galactic minds? And if they communicate with each other then the distances are far too great for light to be the medium of transmission. We are into inter-galactic telepathy." After many decades of study, I know of only one

diagram in all religious traditions that can adequately explain these statements; that can give us both a blue-print of the many levels of consciousness and a route-map to help us reach beyond our present level of understanding. I would like to introduce a diagram of the four worlds of Kabbalah to illustrate the theme that spirit has brought these four worlds into being and that we, living in the fourth and manifest world know nothing of the existence of the other three invisible ones. It is the clearest diagram I know of that can explain to us the multi-levelled structure of consciousness and our place in the great chain of being. We are indeed, as Wilbur suggests, evolving from matter to body, to mind, to soul, to spirit but where have we come from? It is possible that, as participants in the emanation of spirit into manifest life, we have also come from spirit to soul, to mind, to body, to matter. Everything, from the immense galactic energies to the minute forms of matter participates in one life, is generated from one creative spirit. There is nothing outside this life, this spirit. This diagram suggests that truth is not an article of faith but the experience of participating in the undiscovered reality of what has brought us into being. We cannot know truth until we have entered into communion with that reality. Continuing in similar vein **Michel Bitbol dilates upon the dialectic deliberation and polemical argumentation between** hermeneutists and eliminativists. When he formulated the program of Neurophenomenology, Francisco Varela suggested a balanced methodological dissolution of the "hard problem" of consciousness. He shows that his dissolution is a paradigm which imposes itself onto seemingly opposite views, including materialist approaches. I also point out that Varela's revolutionary epistemological ideas are gaining wider acceptance as a side effect of a recent controversy between hermeneutists and eliminativists. Finally, he emphasizes a structural parallel between the science of consciousness and the distinctive features of quantum mechanics. This parallel, together with the former convergences, point towards the common origin of the main puzzles of both quantum mechanics and the philosophy of mind: neglect of the constitutive blindspot of objective knowledge. Phenomenology and the Cognitive Sciences Phenomenology and the Cognitive Sciences **Phenomenology and the Cognitive Sciences 2002, Volume 1, Issue 2, pp 181-224 Science as if situation mattered Michel Bitbol.** Caroline Williams investigated affective density of the political and its effect on our understanding of political subjectivity. Taking up Spinoza's challenge to think about affect beyond corporeal embodiment, he argues that there is a modality of affectivity that cannot simply be inscribed within the borders of subjectivity. theorising affect as an impersonal force anchored in a relational ontology that gives due recognition to the circulation of affects, as well as to their ambivalent structure in creating sites of identification, and he utilises this ontology to reflect on the dynamic of the political and the shape of political subjectivity. Williams argues that Spinoza's philosophy (through ideas of conatus and imagination) offers the conceptual resources to reconfigure the composition of affective subjectivity as a transindividual social bond and as an unconscious dynamic of ethico-political existence. **Subjectivity (2010) 3, 245–262. doi:10.1057/sub.2010.15 Affective processes without a subject: Rethinking the relation between subjectivity and affect with Spinoza Caroline Williams** Phenomenological Time bears ample evidence and infallible observatory and apodictic evidence albeit with a sense of chagrined circumvention and defeated disgruntlment circularity of both psychological and phenomenological interpretations of art which are sometimes insightful, but most often work on the logic of cyclical time and unavoidable drives and representations played out as a 'natural' order: where 'men can't avoid being boys' a circularity which valorizes a helplessness/truthfulness conundrum. One begins with a classical phenomenological fallacy with Duchamp's Fountain, does the urinal keep returning or do we keep returning to the urinal? Put in this way, it is a male question, something that phenomenology conveniently conceals when positing the "I", or indeed, the "we". But we must be careful not to simply assume that the phenomenological first person is white, heterosexual, able-bodied and male, providing a foundation upon which knowledge should be based as a universal condition. Deleuze's second critique of time is linked to Kant. This model of time extends not only to phenomenology but to the phenomenological interpretation of art and is seen as a straight line. In the Critique of Pure Reason, Kant cuts up circular time and reassembles it as a series of sensory experiences which memory processes and which constitute subjectivity. Whereas the circular model of time (and interpretation) inscribes fatalism into the subject and the world, the phenomenological method is in danger of solipsism, where the subject constitutes the world. Much of embodied philosophy and enactivist theories of cognition go down this line with a

plethora of art historical analyses by the likes of Rosalind Krauss, Michael Fried and others taking the lead from Merleau-Ponty. These approaches successfully challenge the **homunculus model of a Cartesian mind-body** dualism where there is a supervisor in the mind who processes experience and even consciousness and to whom all representations are directed. But phenomenology is mainly concerned with perceptual consciousness, the processing of colors, shapes, forms, motion in art and the world. The senses and sensorimotor processes are the fundamental pathways by which the interior is connected to the exterior in a chiasm of self-constitution in the world. The problem with this is that we explore the art through our sense perceptions, and what does art reveal to us?—our sense perceptions! Such solipsistic approaches have real problems relating complex and abstract conceptual production to bodily and sensorimotor contingencies, and they nearly always posit the subject as the site and object of discovery in the world. To be fair, phenomenology has moved on to rather more sophisticated models of intersubjectivity, taking into account the specificities of different kinds of bodies and ways of being beyond the white, male heterosexual as the common denominator, yet many naïve approaches to art perpetuate the notion that perceptual, rather than conceptual processes and evolutionary neural patterns are primary, fundamental ways to understand the art experience In fact, these aspects are only part of art production and reception, aspects which are often filtered out or made known to us in order for us to parody such trigger responses, as in much of Duchamp's work, which holds in contempt such retinal approaches to art. If we take the **subject of abstract expressionism** in the 1950s or later minimalist art, art history, teaching and interpretation still insists on putting the act of perceiving as fundamental to the constitution not only of the phenomenological first person but also of art itself. The male act of usually standing up, which consists of directing a stream with the hand(s) into a container or on a flat surface, sand or snow in order to leave patterns and traces (the usual child's prank) underlies the logic of macho 1950s Abstract Expressionist culture, especially that of Jackson Pollock, so effectively parodied by Carolee Schneeman's Vagina Scroll, 1975. Here, painting or writing is a kind of a discharge which invests the **object and space with** the phenomenological "I", disciplined by existential struggle and reflected back by the traces, signatures and gestures left behind in the work of art, rediscoverable as an element in the making of other subjectivities.It is precisely this kind of solipsism which is blind to the art event as something well in excess of re-establishing habits of subjective self-identity through the production and interpretation of art. The attempt to reinforce a phenomenological identity upon the event by personalizing sense experience is unable to see art as something which remodels the phenomenological 'I' and neural plasticity itself. We learn new technologies and redraw our neural pathways through such new and spontaneous experiences which draw us into intersubjective worlds. Duchamp's Fountain and Deleuze's Repetition and Différence Grégory Minis sale (I have made some changes) The relation of different groups to deconstruction is settled by its relation to groups. An American enthusiast--one who has had a role in the development of our affection for Derrida--notices that the motivational chains are phobically driven by concerns with social scale. Richard Rorty: "He wants to figure out how to break with the temptation to identify himself with something big. . . ."(10) And: "So I take Derrida's importance to lie in his having had the courage to give up the attempt to unite the private and the public, to stop trying to bring together a quest for private autonomy and an attempt at public resonance and utility. He privatizes the sublime, having learned from the fate of his predecessors that the public can never be more than beautiful."(11) Notice is made here of what is indeed the telling feature. Levinas is praised, for example, for his alarm when "the social will [is] sought in an ideal of fusion . . . the subject losing himself in a collective representation, in a common ideal. . . . It is the collectivity which says 'us,' and which, turned toward the intelligible sun, toward the truth, experience, the other at his side and not face to face with him. The tellingly titled D'un ton apocalyptique contains passages that are particularly excited: These people situate themselves outside the ordinary, but they have in common this: they describe themselves as having an immediate and intuitive relation with mystery. And they want to attract, to seduce, and lead others to the mystery, through mystery. This agogic function of the leader of men, of the duce, of the Führer, of the leader, places him above the crowd that he manipulates with the aid of a small numbers of adepts joined together in a sect with a secret language, a clique or a small party with its ritualized practices. The mystifiers pretend to have exclusive access to the privilege of a secret mystery. . . . The revelation or the unveiling of the secret is something that they jealously reserve for

themselves. Jealousy is here a major characteristic. There is not narcissism and non-narcissism; there are narcissisms that are more or less comprehensive, generous, open, extended. What is called non-narcissism is in general but the economy of a much more welcoming, hospitable narcissism, one that is much more open to the experience of the other as other. I believe that without a movement of narcissistic reappropriation, the relation to the other would be absolutely destroyed; it would be destroyed in advance. The relation to the other--even if it remains asymmetrical, open, without reapporpriation--must trace a movement of reappropriation in the image of oneself for love to be possible, for example. Love is narcissistic. Beyond that, there are little narcissisms, there are big narcissisms. . . .(23)

**Justice of the Pieces Deconstruction as a Social Psychology Douglas Collins** Sabine Wilke cogently argues for the rise of the aesthetic dimension in philosophical writing by showing how the metaphorical and poetic style of Heidegger, the principles of phrase configurations with Adorno, or the topographic writing of derrida replace through their very form something what traditionally would have been subject to discursive exposure The concern shifts from telling to showing, showing first and foremost that something remains untold. An exemplary book in this aesthetic drive is Avital Ronell's Telephone Book where the playful layout and explicit break with traditional design interrupt the sequential logic of the paragraphs thereby performing the very argument at stake: open up new circuits of meaning and signification. The engagement with form and publishing formats is only consequential since media theory as exposed by Walter J. Ong, Eric A. Havelock, Parry Miman or Marshall McLuhan besides many others made it clear that different communication technologies condition the mode of thought of a given culture The fractals of Benoit Mandelbrot or the butterfly shaped Lorenz attractor are the popular icons of this turn in the sciences. For communication studies and philosophy, Vilém Flusser was one of the pioneer thinkers reflecting the rise of what he called "techno-images." The photograph is the paradigmatic case which condenses the whole crises of Western thought and culture on its grainy surface. It subverts the linear logic of writing.6 Flusser's central diagnosis detects a shift from the grand narratives associated with the literate mind and Modernity towards creating and envisioning the world through image based practices. Writing himself in a succinct, visually inspired style, Flusser contents that the web of meaning, spun throughout the centuries by our textual practices, has become transparent. We discover the void between the letters through which the world escapes: "We are alienated from the world of text, precisely because we see through them as our own product."7 As a consequence we are confronted with a world of dissociated facts and information bits. Flusser hails this fragmented world as the advent of the zero dimensional point-universe that will provide the new raw material for our image based practices. Techno-images are nothing else than a conscious, inherently artistic drawing-together of point-elements as exemplified first and foremost by the silver-grains of the photograph but also by more recent pixel based techno-images. Similar to blueprints, designs or statistical curves, the weather map for example visualizes and contracts a tremendous amount of meteorological data from widely distributed stations into a single image of high- and low pressure
areas, temperatures, wind speeds, etc. Techno-images "compute" in this sense the universe of unimaginable, abstract information-bits into a concrete surface – the image. Echoing Paul Feyerabend, these image practices have to be understood as creative practices where the truth of scientific models has been replaced by its aesthetic quality

**Aesthetics, Writing, Networked Computers Jörg Muller Dissertation Submitted to the Division of Media and Communications of The European Graduate School in Candidacy for the Degree of Doctor of Philosophy**

**Lothar Schäfer (1) and Sisir Roy (2) in** Quantum Reality, the Importance of Consciousness in the Universe, the Discovery of a Non-Empirical Realm of Physical Reality, and the Convergence with Ancient Traditions of Indian and Western Philosophy investigate metaphysical aspects that include 1) the discovery of a non-empirical part of physical reality in a realm of potentiality; 2) the emanation of the empirical world out of a realm of non-material forms; 3) the discovery that the nature of physical reality is that of an indivisible Wholeness – the One; and 4) panpsychism: the possibility that the One is aware of its processes like a Cosmic Consciousness. The convergence of powerful traditions of seemingly disparate cultures is particularly important to point out in the present process of globalization, when a unifying view is needed to avoid controversy and conflict. To quote them in extenso : This is the nonlocality of the quantum world. Menas Kafatos and Robert Nadeau (1990) have drawn a remarkable conclusion from this phenomenon: If reality is nonlocal, the nature of the universe is that of an undivided wholeness.

Because our consciousness has emerged from this wholeness and is part of it, it is possible to conclude that an element of consciousness is active in the universe: a Cosmic Consciousness. Pre-reflective self-consciousness is pre-reflective in the sense that (1) it is an awareness we have before we do any reflecting on our experience; (2) it is an implicit and first-order awareness rather than an explicit or higher-order form of self-consciousness. Indeed, an explicit reflective self-consciousness is possible only because there is a pre-reflective self-awareness that is an on-going and more primary self-consciousness. Although phenomenologists do not always agree on important questions about method, focus, or even whether there is an ego or self, they are in close to unanimous agreement about the idea that the experiential dimension always involves such an implicit pre-reflective self-awareness. In line with Edmund Husserl (1959, 189, 412), who maintains that consciousness always involves a self-appearance (Für-sich-selbst-erscheinens), and in agreement with Michel Henry (1963, 1965), who notes that experience is always self-manifesting, and with Maurice Merleau-Ponty who states that consciousness is always given to itself and that the word 'consciousness' has no meaning independently of this self-givenness (Merleau-Ponty 1945, 488), Jean-Paul Sartre writes that pre-reflective self-consciousness is not simply a quality added to the experience, an accessory; rather, it constitutes the very mode of being of the experience: This self-consciousness we ought to consider not as a new consciousness, but as the only mode of existence which is possible for a consciousness of something (Sartre 1943, 20 [1956, liv]). The notion of pre-reflective self-awareness is related to the idea that experiences have a subjective 'feel' to them, a certain (phenomenal) quality of 'what it is like' or what it 'feels' like to have them. As it is usually expressed outside of phenomenological texts, to undergo a conscious experience necessarily means that there is something it is like for the subject to have that experience (Nagel 1974; Searle 1992). This is obviously true of bodily sensations like pain. But it is also the case for perceptual experiences, experiences of desiring, feeling, and thinking. There is something it is like to taste chocolate, and this is different from what it is like to remember what it is like to taste chocolate, or to smell vanilla, to run, to stand still, to feel envious, nervous, depressed or happy, or to entertain an abstract belief. Yet, at the same time, as I live through these differences, there is something experiential that is, in some sense, the same, namely, their distinct first-personal character. All the experiences are characterized by a quality of mineness or for-me-ness, the fact that it is I who am having these experiences. All the experiences are given (at least tacitly) as my experiences, as experiences I am undergoing or living through. All of this suggests that first-person experience presents me with an immediate and non-observational access to myself, and that consequently (phenomenal) consciousness consequently entails a (minimal) form of self-consciousness. To put it differently, unless a mental process is pre-reflectively self-conscious there will be nothing it is like to undergo the process, and it therefore cannot be a phenomenally conscious process. The mineness in question is not a quality like being scarlet, sour or soft. It doesn't refer to a specific experiential content, to a specific what; nor does it refer to the diachronic or synchronic sum of such content, or to some other relation that might obtain between the contents in question. Rather, it refers to the distinct givenness or the how it feels of experience. It refers to the first-personal presence or character of experience. It refers to the fact that the experiences I am living through are given differently (but not necessarily better) to me than to anybody else. It could consequently be claimed that anybody who denies the for-me-ness of experience simply fails to recognize an essential constitutive aspect of experience. Such a denial would be tantamount to a denial of the first-person perspective. It would entail the view that my own mind is either not given to me at all — I would be mind- or self-blind — or is presented to me in exactly the same way as the minds of others. There are also lines of argumentation in contemporary analytical philosophy of mind that is close to and consistent with the phenomenological conception of pre-reflective self-awareness. Alvin Goldman provides an example: [Consider] the case of thinking about x or attending to x. In the process of thinking about x there is already an implicit awareness that one is thinking about x. There is no need for reflection here, for taking a step back from thinking about x in order to examine it…When we are thinking about x, the mind is focused on x, not on our thinking of x. Nevertheless, the process of thinking about x carries with it a non-reflective self-awareness (Goldman 1970, 96). A similar view has been defended by Owen Flanagan, who not only argues that consciousness involves self-consciousness in the weak sense that there is something it is like for the subject to have the experience, but also speaks of the low-level self-consciousness

involved in experiencing my experiences as mine (Flanagan 1992, 194). As Flanagan quite correctly points out, this primary type of self-consciousness should not be confused with the much stronger notion of self-consciousness that is in play when we are thinking about our own narrative self. The latter form of reflective self-consciousness presupposes both conceptual knowledge and narrative competence. It requires maturation and socialization, and the ability to access and issue reports about the states, traits, dispositions that make one the person one is. To claim that every kind of self-consciousness is conceptual is overly cognitive. Bermúdez (1998), to mention one further philosopher in the analytic tradition, argues that there are a variety of **nonconceptual forms of self-consciousness** that are "logically and ontogenetically more primitive than the higher forms of self-consciousness that are usually the focus of philosophical debate" (1998, 274; also see Poellner 2003). This growing consensus across philosophical studies supports the phenomenological view of pre-reflective self-consciousness. (References: Stanford Encyclopedia, Post Modersnist Theories, Internet Encyclopedia of Philosophy) Korzybski pointed out that there are two ways to slice easily through life; to believe everything or to doubt everything. Both ways save us from thinking. So, from time to time you have to give yourself a Virtual Reality Check. The hallmark of postmodern philosophy has been disbelief or skepticism of all "metanarratives," or translations of reality. Postmodernism has even turned its profound skepticism on such important humanist concepts as "objective truth" and reason.Yet, for a deconstructionist postmodern society, individually we are still riddled with superstition and gullibility, and open to manipulation through our belief systems as any politician, philosopher, clergy, or salesperson will attest. Further, most people are painfully naive when it comes to even the simplest scientific understanding. Most of us don't have a clue about the fundamental nature of physical reality or our own psychological nature, and our ability to be fooled by our senses and mind, or methods of social and technological persuasion. We may think we know, but the penetration is deeper than we dare imagine. Nature rejects the naiveté that seeks absolute truth. We are beginning to realize, individually and culturally, that "realities" are all human constructions. The task becomes one of "catching ourselves in the act" of creating our own "reality" from the flow of events. Human truth is always an engagement of mind with experience.. We don't need to fear the collapse of our personalistic belief system (the "box" we live in), nor our belief in absolute truth. **A strong desire to engage in the "quest for uncertainty" complements our anxiety that perhaps there is no absolute, objective ground to reality**. The warrant of Truth is ever elusive when we deconstruct the foundational justifications of our convenient notions about the way the world works. It is easy to confuse what is actually the creation of beliefs with the "discovery of Truth," a common goal of science and theology. Science offers no ontological Scientific Picture of the World but substitutes a number of applied theories as meta-theory. It's close, powerful and useful: it 'works' and allows use to do work. But the still mysterious "true" metatheory may reveal the basis of consciousness. But who knows when we will lift that veil? Permeating the living reality of our culture are certain contagious notions, fads and trends that have the ability to influence the way we think about the nature of Reality and ourselves. Some of them are toxic and can consume you; others are just whims of pop culture infectious and benign as the common cold. **Jung** described the concept of psychic contagion by certain archetypal forces inherent in the human psyche, which manifest in our spiritual lives and belief systems. He spoke of both conscious and unconscious contamination. Notions like this range from simple superstitions to scientific concepts, to urban myths. Notions sweep through our culture and insinuate themselves within its fabric, as fads, whether they are "real" or not, they can be influential. They mood alter us, make us feel we belong. An analysis of these notions is useful in distinguishing a common human phenomenon from any potential "alien influence" which may or may not be exerted on us from an unknown source. **Strange Attractors: Transference, Holography, and an Archetype Burke, J. (2003). Strange Attractors: Transference, Holography, and an Archetype (Doctoral dissertation, Pacifica Graduate Institute, 2003).** In consideration to the Nobel Proze for Chemistry this year(2014), a stability nalysis of the proble in to single molexues would be a contemporaneous question that has to be taken cognisace of and given importance to . Determination of molecular structure by geometry optimization became routine only after efficient methods for calculating the first derivatives of the energy with respect to all atomic coordinates became available. Evaluation of the related second derivatives allows the prediction of vibrational frequencies if harmonic motion is estimated. More importantly, it allows for the

characterization of stationary points. The frequencies are related to the eigenvalues of the Hessian matrix, which contains second derivatives. If the eigenvalues are all positive, then the frequencies are all real and the stationary point is a local minimum. If one eigenvalue is negative (i.e., an imaginary frequency), then the stationary point is a transition structure. If more than one eigenvalue is negative, then the stationary point is a more complex one, and is usually of little interest. When one of these is found, it is necessary to move the search away from it if the experimenter is looking solely for local minima and transition structures. The total energy is determined by approximate solutions of the time-dependent Schrödinger equation, usually with no relativistic terms included, and by making use of the Born–Oppenheimer approximation, which allows for the separation of electronic and nuclear motions, thereby simplifying the Schrödinger equation. This leads to the evaluation of the total energy as a sum of the electronic energy at fixed nuclei positions and the repulsion energy of the nuclei. Notable exceptions are certain approaches called direct quantum chemistry, which treat electrons and nuclei on a common footing. Density functional methods and semi-empirical methods are variants on the major theme. **Orlando Alvarez** discusses Polyakov's quantization of the string in the presence of a boundary allowing for an arbitrary topology for the world sheet. In addition to the dynamical conformal factor discovered by Polyakov, there are a finite number of new degrees of freedom if the surface is more complicated than a sphere or a disc. The quantization of the Liouville theory in an arbitrary topology is also discussed. A one-loop calculation shows that the model is renormalizable if one performs a mass renormalization and an additive field renormalization. The renormalization group equations have a perturbative infrared unstable fixed point in all topologies. Copyright © 1983 Published by Elsevier B.V. **Nuclear Physics B Volume 216, Issue 1, 25 April 1983, Pages 125–184 Theory of strings with boundaries: Fluctuations, topology and quantum geometry Orlando Alvarez DOI: 10.1016/0550-3213(83)90490-X** Quantum geometry (the modern loop quantum gravity involving graphs and spin-networks instead of the loops) provides microscopic degrees of freedom that account for black-hole entropy. However, the procedure for state counting used in the literature contains an error and the number of the relevant horizon states is underestimated. In our paper a correct method of counting is presented by **Marcin Domagala and Jerzy Lewandowski.** Results lead to a revision of the literature of the subject. It turns out that the contribution of spins greater than 1/2 to the entropy is not negligible. Hence, the value of the Barbero–Immirzi parameter involved in the spectra of all the geometric and physical operators in this theory is different than previously derived. Also, the conjectured relation between quantum geometry and the black-hole quasi-normal modes should be understood again. **Marcin Domagala and Jerzy Lewandowski 2004 Class. Quantum Grav 21 5233 doi:10.1088/0264-9381/21/22/014 Black-hole entropy from quantum geometry** Quantum geometry predicts that a universe evolves through an inflationary phase at small volume before exiting gracefully into a standard Friedmann phase. This does not require the introduction of additional matter fields with ad hoc potentials; rather, it occurs because of a quantum gravity modification of the kinetic part of ordinary matter Hamiltonians. Authors draw the cognizance of application of the same mechanism that can explain why the present day cosmological acceleration is so tiny. DOI: http://dx.doi.org/10.1103/PhysRevLett.89.261301 **Inflation from Quantum Geometry Phys. Rev. Lett 89, 261301 – Published 12 December 2002 Martin Bojowald** The loop quantum cosmology of the closed isotropic model is studied with special emphasis on a comparison with traditional results obtained in the Wheeler-DeWitt approach by **Martin Bojowald** .Investigation includes the relation of the dynamical initial conditions to boundary conditions such as the no-boundary or the tunneling proposal and a discussion of inflation from quantum cosmology. **DOI: http://dx.doi.org/10.1103/PhysRevD.67.124023 Loop quantum cosmology, boundary proposals, and inflation Phys. Rev. D 67, 124023 – Published 23 June 2003 Martin Bojowald and Kevin Vandersloot** In loop quantum cosmology, the universe avoids a big bang singularity and undergoes an early and short super-inflation phase. During super-inflation, non-perturbative quantum corrections to the dynamics drive an inflaton field up its potential hill, thus setting the initial conditions for standard inflation. **Shinji Tsujikawa et al** show that this effect can raise the inflaton high enough to achieve sufficient e-foldings in the standard inflation era. They also analyse the cosmological perturbations generated when slow-roll is **violated after super-inflation** and show that loop quantum effects can in principle leave an **indirect signature** on the largest scales in the CMB, with some loss of power and

running of the spectral index. **Shinji Tsujikawa et al 2004 Class. Quantum Grav 21 5767 doi:10.1088/0264-9381/21/24/006 Loop quantum gravity effects on inflation and the CMB Quantum geometry from quantum information** is the study of **Florian Girelli and Etera R Livine .**Loop quantum gravity defines the quantum states of space geometry as spin networks and describes their evolution in time. We reformulate spin networks in terms of harmonic oscillators and show how the holographic degrees of freedom of the theory are described as matrix models. This allows us to make a link with non-commutative geometry and to look at the issue of the semi-classical limit of loop quantum gravity from a new perspective. This work is thought of as part of a bigger project of describing quantum geometry in quantum information terms. **Florian Girelli and Etera R Livine 2005 Class. Quantum Grav 22 3295 doi:10.1088/0264-9381/22/16/011 Reconstructing quantum geometry from quantum information: spin networks as harmonic oscillators.** Quantum geometric mechanics is the topic of study by **Dorje C. Brody Lane P. Hughston** establishing a clear relation between the quantum mechanics and quantum geometry and also the entangled states. The manifold of pure quantum states can be regarded as a complex projective space endowed with the unitary-invariant Fubini–Study metric. According to the principles of geometric quantum mechanics, the physical characteristics of a given quantum system can be represented by geometrical features that are preferentially identified in this complex manifold. Here we construct a number of examples of such features as they arise in the state spaces for spin, spin 1, spin and spin 2 systems, and for pairs of spin systems. A study is then undertaken on the geometry of entangled states. A locally invariant measure is assigned to the degree of entanglement of a given state for a general multi-particle system, and the properties of this measure are analysed for the entangled states of a pair of spin particles. With the specification of a quantum Hamiltonian, the resulting Schrödinger trajectories induce an isometry of the Fubini–Study manifold, and hence also an isometry of each of the energy surfaces generated by level values of the expectation of the Hamiltonian. For a generic quantum evolution, the corresponding Killing trajectory is quasiergodic on a toroidal subspace of the energy surface through the initial state. When a dynamical trajectory is lifted orthogonally to Hilbert space, it induces a geometric phase shift on the wave function. The uncertainty of an observable in a given state is the length of the gradient vector of the level surface of the expectation of the observable in that state; a fact that allows us to calculate higher orders corrections to the Heisenberg relations. A general mixed state is determined by a probability density function on the state space, for which the associated first moment is the density matrix. The advantage of a general state is in its applicability in various attempts to go beyond the standard quantum theory, some of which admit a natural phase-space characterization. **Journal of Geometry and Physics Volume 38, Issue 1, April 2001, Pages 19–53 Geometric quantum mechanics Dorje C. Brody Lane P. Hughston DOI: 10.1016/S0393-0440(00)00052-8** Expansion (unmesa) of [consciousness] or the creative intuition {pratibha) is (=) [experienced] in the interval which divides two [moments] of differentiated perception (vikalpa) It is here that they arise and disappear. The Sastras and Agamas proclaim with reasoned argument that it is (=) free of thought-constructs {Nirvikalpa) and precedes (e) Path to Liberation all mental representations of any object. None can deny that a gap exists between perceptions insofar as two moments of thought are (=) invariably divided. This [gap] is (=) the undifferentiated unity of all the countless manifestations. Similarly, in the outer more objective sphere, where change consists (e) of the alterations in the configurations of manifest appearances (Abhasa\ the transition from one to (e&eb) another corresponds to (e&eb) a phase of pure luminosity that marks (eb) the beginning of one form and the end of another. The world of manifestation and differentiated perceptions (Yikalpa) thus extends from one Centre to (eb) the next. **What is meant again is the realisation or the evolution of individual consciousness or general ledger to be the cosmic general ledger is the sole criterion for the Satchidananda state where all dualities cease** Although it is never in fact divorced from (e) the subject who resides there, the ignorant fail (e) to grasp this fact and so, cut off from (e) the Centre, the world of objectivity becomes (=) for them the sphere of Maya. Bhagavatotpala quotes the Light of Consciousness amvitprakdsa): This ever pure experience (suddhanubhava) is variegated by (e) each form [revealed within it] Even so it remains unstained (nirmala) when moving to another. Just as a cloth which is naturally white, once dyed, cannot (e) change colour without [first] becoming white again, similarly the pure power of awareness, (citi) once coloured by (e) form, is pure [again] at the Centre where that form is (=) abandoned and from whence it

proceeds to (e&eb) another. In his Essence of Vibration (Spandasarpdoha), K§Emaraja explains that the rise and fall of every individual perception in the field of awareness is (=) a specific pulsation of consciousness. This is exactly the individual general ledger transaction lifted to the transaction in cosmic general ledger. During The Initial Instant Of Perception, T Consciousness Is Manifestly Apparent And The Yogi, Participating In Its Plenitude, Observes The Outer World Without Being Attached To Any Particular Divine Body And Sacred Circle Of The Senses Singling It Out From Any Other, Like A Man Who Observes A City From A High Mountain Peak. He Sees The Outer World Reflected Within His Consciousness Free Of Thought-Constructs And So 'Stamps* The Outer On The Inner While Absorbing The Object And Means Of Knowledge In The Pure Subject Which Grasps Them As The Expansion Of His Own Nature. K§Emaraja Says: By Penetrating Into Bhairavimudra, The Yogi Observes The Vast Totality Of Beings Rising From, And Dissolving Into, The Sky [Of Conscious- Ness], Like A Series Of Reflections Appearing And Disappearing Inside A Mirror. Through The Practice Of Bhairavimudra The Yogi Realises That He Is The Substratum Consciousness (Adhisfhatr) Which Both Underlies And Is The Essence Of All Things. He Discovers That Phenomena Have No Independent Existence Apart From Him And So Are, In This Sense, Void. At The Same Time, He Realises That Because All Things Are Consciousness, They Are Far From Unreal. He Views The Outer World Yet Sees It Not. Beyond Both Voidness And Non-Voidness He Penetrates Into The Supreme Abode (Pararn Padam) Of Siva's Consciousness. [The Powers Of The Senses] Endowed With The Attributes Of The Great Union [Between Subject And Object] Whose Form Is The Wakening Of Man's Spiritual Potential (Kuntfalinf), Fill [With Consciousness] The Outer Clatter Of Diversity (Bhedatfambara) Born Of Its Intense Power And Are Then Established In The Unobscured Abode Of The Void Of Consciousness To Shine [There] Eternally. Thus Residing Beyond Being And Non-Being, The Sole Protector Of The Unity Which Is Tranquil And Expanding [Consciousness], Whose Glory Is All-Embracing And Form Unobscured, Is Called Bhairavimudra. Through The Practice Of Bhairavimudra, The Yogi Unites The Universal Vibration Of T Consciousness With The Individual Pulsation Of Objectivised 'This' Consciousness. The Two Aspects Of Consciousness Are Now In A State Of Equilibrium Like The Two Pans Of An Evenly Weighed Balance And The Yogi Experiences The Pure Knowledge (Suddhavidya) That: 'I Alone Am All Things'. Thus Becoming The Master Of The Wheel Of Energies He Is Free, Like Siva, To Create And Destroy. When [The Yogi] Is Well Established, Without Wavering, Solely In The Integral Egoity Of His Authentic Nature, The Spanda Principle, And Is Absorbed In Contemplation (Samavisfa), He Becomes One With It The Doctrine Of Vibration (Tanmaya). Then . . . Dissolving And Creating The Universe By Means Of His Introverted And Extroverted Absorption, He Destroys And Creates All Things Out Of Sarikara, His Innate Nature. [Thus] He Assumes The State Of The Universal Experiencer And Having Absorbed All That Is To Be Experienced From [The Grossest Level] — Earth — To (The Subtlest) — Siva — He Reaches The State Of The Supreme Subject By Progressively Recognising [His Identity With Him]. Thus, Introverted And Extroverted Absorption Both Lead To The Recognition Of The Pulsation (Spanda) Of One's Own Consciousness. This Is Where Individual General Ledger Becomes Cosmic General Ledger. At The Level Of Consciousness Corresponding To Siva's Basic State (Saryibhavdvastha), The Alternation From Inner To Outer Is Instantaneously Resolved Into The Vibration Of His Nature. When The Yogi Finally Comes To Be Constantly Aware Of This Reality, His Enlightenment Is Full And Perfect. Freed Of All Means (Anupaya) And Delighting In The Power Of His Bliss (Anandasakti), He Knows And Does Whatever He Pleases. The Yogi Seeking Self-Realisation Must Acquire Mastery Over This Movement. Ksemaraja Stresses That The Doctrine Of Vibration Teaches That Liberation Can Only Be Achieved By First Withdrawing All Sense Activity In Introverted Contemplation (Nimilanasamadhi) To Then Experience The 'Great Expansion' (Mahdvikdsd) Of Consciousness While Recognising This To Be A Spontaneous Process Within It. This Is Done Through The Practice Of Kramamudra. A Passage From The Now Lost Kramasiitra Explains: Although The Adept's Attention [May Be] Outwardly Directed, He Enjoys Contemplative Absorption Through The Introverted Aspect Of Kramamudra. Initially He Turns Inward From The Outside World And [Then] From Within [Himself] He Exits Into The Outer World Under The Influence Of His Absorption. There Is No Necessity Of Any Glorification Or Mortification Here. It Is Just Like Joining A Job And Retiring And Realising All That Happened Is Just By His Own Effort And Nothing Else. Thus The Sequence

(Krama) In This Attitude (Mudra) [Ranges Through] Both Inner And Outer. The Yogi Must Pervade The Surface Level Of Awareness (Yyutthana) With The Same Bliss He Experiences Plunging Into The Depths Of Con-Templative Absorption (Samadhi). Note Here That Samadhi Is A Death Like State And Still Being Matter And Information Stored You Experience Consciousness. Submerging Himself And Emerging Repeatedly From Samadhi, He Eventually Recognises That The Unity Of Consciousness Pervades Both States: The Best Of Yogi's, Who Has Achieved A State Of Complete Absorption Even When Risen From Meditation, [Inwardly] Vibrating Like A Drunkard In Blissful Inebriation From The After-Effects Of The Nectar Of Con- Templation, Sees All Things Dissolving In The Sky Of Consciousness Like A Cloud In The Autumn Sky. He Plunges Repeatedly Within Himself Divine Body And Sacred Circle Of The Senses And Becomes (=) Aware Of His Identity With (e&eb) Consciousness By (e) The Practice Of Introverted Contemplation. Thus Even When He Is Said To Have Risen From (e) Absorption, He Is (=) One With [His] Experience Of It. Ksemaraja Goes On To Explain That This Practice Is Called 'Mudra' Because (e) It Both Fills The Adept With (e&eb) Bliss (Mud) And Is Itself The Bliss Of (e) Consciousness. Moreover, It Dissolves (e) Away (Dra) All Bondage And 'Stamps' The Universe Of Experience With (e&eb)The Seal (Mudra) Of The Fourth State (Turiya) Of Enlightened Consciousness Beyond, And Including, The Three States Of Waking, Dreaming And Deep Sleep. It Is Called 'Krama' Because (e) It Is The Root Source Of (e) All Emanation And All Other Conscious Processes Which Succeed (e&eb) One Another In Ordered Sequence (Krama) And Is, (=)At The Same Time, Their Successive (Krama) Appearance As Well. By The Practice Of Kramamudra The Opposites Fuse And Siva And Sakti Unite. They Yogi Comes To Experience The Simultaneous Pervasion Of All The Lower, Grosser Categories Of Existence By The Higher And The Presence Of The Lower In The Higher. Commencing His Practice In A Low Form Of Bhairavimudra, The Yogi Conjoins The Outer With The Inner; Then, In Kramamudra, He Fills Both The Outer With (e&eb) The Inner And (e&eb) The Inner With (e&eb)The Outer. When He Achieves (eb) Perfection In This Two-Fold Movement, He Attains To (e) The Highest Form Of Bhairavimudra In Which The Two Merge Completely In (eb)The Experience Of (e) The Absolute (Anuttara), Free Of (e) All Differentiation And (e&eb) Polarities. If He Fails To Maintain Awareness Of This State, He Again Falls Into (eb) Kramamudra Until He Has Finally Completely Merged All The Highest States In (e&eb) The Lower And The Lower In (e&eb) The Higher. He Then No Longer Needs To (e) Resort To (e) Any Means (Anupaya) To (e) Achieve Liberation. All He Says Or Does Anything He Perceives Or Thinks, Instantly Occasions In (eb) Him The Highest Level Of Consciousness. Thus The Fruit Of Bhairavimudra Is (*=)The Wonder (Camatkara) Or Amazement (Vismaya) That Overcomes (e)The Yogi When (e) He Reaches The Plane Of (e) Union (Yogabhumika) Y Xsl Where (e) All Opposites Merge In (e&eb)The Radiance Of The Great Light Of Consciousness. The Stanzas On Vibration Teach: How Can One Who, As If Astonished, Beholds (e) His Own Nature As (=) That Which Sustains [The Existence Of Everything] Be Subject To (e&eb) This Painful Round Of Transmigration (Kusrti)? The Yogi, Recognising His True Nature To Be (=)The Supreme Subject, Is Astonished To (e) Suddenly Discover That The Individual He Thought He Was, Caught Up (e&eb) In The Trammels Of Thought And Living In A World Enmeshed In (e&eb) The Web Of Time And Space, Does Not Really Exist At All. He Experiences A 'Turning About' (Paravrtti) In (eb)The Deepest Seat Of The Doctrine Of Vibration Consciousness As He Penetrates His True Nature. The Sudden Eruption Of This Intuition (Pratibha) Arouses In Him A Cry Of Amazement As He Transcends All Thought-Constructs And, Perfectly Absorbed In His Own Nature, Is Liberated. The Path To Liberation Essentially, Spanda Doctrine Is Concerned With Two Matters. The First Is To Impart To Those Who Are Fit To Receive The Teachings A Deeper Understanding Of The Ultimate Goal Of Life (Upeya). When We Have Understood What Truly Benefits Us And Is Worth Attaining And What, On The Contrary, Is Of No Real Value But Stands In The Way Of This Attainment, We Can Begin To Make Progress Towards Our Goal. This Is Spanda Doctrine's Second Concern, Namely, To Show The Way In Which We Can Develop Spiritually Through Siva's Grace And The Right Application Of The Means To Realisation That It Teaches. When Both These Aspects Of The Teaching Have Been Correctly Understood And Applied, The Spanda Yogi Achieves A Clear And Permanent Realisation Of His Goal And Is Liberated, Thus Fulfilling The Ultimate Aim Of The Teaching. The Doctrine Of Vibration Is Not Meant For The Spiritually Dull. It Is Not For The Worldly Whose Consciousness,

Clouded By Ignorance, Is As If Dreaming, Even During The Waking State Of Daily Life, The Dream Of Its Own Thought-Constructs. The Teachings Are Meant For Those Who Are Awake (Prabuddha), Those Who, Full Of Faith And Reverence, Are Always Alert And Intent On Discerning The True Nature Of Ultimate Reality. This Reality Is Understood In Three Basic Ways. The First Is Purely Transcendental. The Stanzas Choose This Aspect As The One Which Formally Defines It Most Specifically. Ultimate Reality Transcends All The Opposites, Including Subject And Object. This Does Not Mean, However, That It Is An Unconscious Void, A Mere Absence Of All Existence. In Fact, This Negative Characterization Of Reality (Which Includes Also A Denial Of All That Is Unconscious) Implies A Positive Immanence In Which The Opposites Are United In The Oneness Of Pure Consciousness That Is Equally Siva And Spanda, His Universal Activity. These Two Seemingly Contrasting Aspects Are Reconciled In The Third, Namely, Reality Understood As The Essential Nature Of All Things. Although Universal And Everywhere The Same, It Is Understood To Be The Essential And Specific Nature Of Each Existent As Its 'Own Nature' (Svabhava). In The Case Of The Individual Soul It Is Even More Specific, More Personal As His Own 'Own Nature' (Svasvabhava). Note Here That Svabhava Is Not Svasbhava. In Fact We Made A Statement That Svabhava Is The Very Characterstics Of Space Time Like, Crime Zones, Silent Zones, School Zones We Classify Space In To. Belonging To None Other Than Oneself It Is The Pure Subjectivity That Perceives Experiences, Enjoys, Reflects, Thinks And Senses As Well As Being The Conscious Agent Who Creates Every Possible Form Of Experience In All The States Of Consciousness. The Liberating Knowledge Of Reality Thus Corresponds To Our Regaining Possession Of Ourselves (Svatmagraha). We Must Lay Hold Of Ourselves And Abide In Our Authentic Nature. Reality Coincides With Our Own Most Fundamental State Of Being (Svasthiti), Free Of All Contrasts And Contradictions. Once We Have Overcome The Negative Forces That Arise From Our Ignorance And Prevent Us From Abiding In Ourselves, We Are Liberated. To Do This, We Must Penetrate Through The Pulsing Fluctuations Of Objectively Experienced States And Perceptions At The Surface Level Of Consciousness And Gain Insight Into The Timeless Rhythm Of Our Own Nature Manifest In The Universal Arising And Falling Away Of All Things. We Are Not Freed Of The Trammels Of Perpetual Change By Setting It Aside; On The Contrary, We Must Gain Insight Into The Recurrent Cycles Of Creation, Persistence And Destruction, Or Else Be Bound By Our Ignorance. This Spiritual Ignorance Consists Essentially Of Our Contracted State Of Consciousness And So Can Only Be Effectively Countered By Expanding It To Reveal Our Own Authentic Nature As This Expanded State Itself, Which Is The Universal Vibration (Samanyaspanda) Of Consciousness. Vibration Of Strings Which Gives Shape To Objects Is Itself A Information, A Sine Qua Non Of Individual Consciousness. The Spanda- Yogi Treads The Path Of Consciousness Expansion. The Movement From The Contracted To The Expanded State Marks The Transition From Ignorance To Understanding, From The Dispersion And Incompleteness Of A Form Of Consciousness Entirely Centred On An Objectively Perceived And Discursively Represented Reality To A Direct, Intuitive Awareness Of The Unity And Integral Wholeness Of Our Own Absolute Spanda Nature. Along The Way To This Supreme Realisation Consciousness Develops, As Veil After Veil Is Lifted, Until It Becomes Full And Perfect In The Absolute Which Encompasses Within Itself All Possible Formats Of Experience. As Abhinava Says: [This Realisation] Is The Supreme Limit Of Plenitude And As Such There Can Be No Higher Attainment. Any [Other] Attainment [We Can] Conceive Issues From A State That Falls Short Of [This] Perfection. Once* [This] Uncreated Fullness Has Been Attained, Pray Tell, What Other Fruit Can There Be [Beyond It]? The Fettered Soul's Contracted State Of Consciousness Binds Him Because He Is Deprived Thereby Of The Subtle, Intuitive Insight Into The Underlying Unity Of Existence And His Attention Is Focused Instead On Its Gross, Outer Diversity Easily Apparent To Everybody, However Restricted His Consciousness May Be. However, Although The Fettered Soul In This State Is Ignorant Of This Unity, This Does Not Mean That His Knowledge Of Diversity Is False. Ignorance Entails A Form Of Knowledge Which, Although Quite Correct, Is Binding. We Are Not Absolutely Ignorant Of Reality For If We Were We Would Be Totally Unconscious. Spiritual Ignorance Is Always Linked With Some Degree Of Consciousness. Those Subject To The Round Of Birth And Death Are Not Inert Clods Of Earth. This Also Leads To Conclusion That Shakti Responsible Is Impure Because Of Maya Imbibed In It. Thus, Although Ignorance Obscures Consciousness, It Is Wrong To Think Of It, As Dualist Saivites Do, In Terms Of A Defiling Impurity That Shrouds

It Like A Cloth Covering Ajar. Spiritual Ignorance Can Be Nothing But Consciousness Itself, Albeit In A Limited State. Siva, Who Is Universal Consciousness, Is The Innate Nature Of Both Its Contracted And Expanded States, Both Of Which Are Forms Of Knowledge, Namely: 1) Supreme Knowledge (Parajnana) Defined As The Revelation Of One's Own Innate Nature As The One Reality Which Is The Being Of All Things. 2) Inferior Knowledge (Aparajndna) Which Jayaratha Explains Results From The Mental Activity (Yyapard) Of The Individual Subject Whose Consciousness Is Contracted. It Consists Of The Mental Representations (Vikalpa) He Forms Of Himself And His Object, Of The Type I Know This'. The Lower Knowledge Obscures The Higher And Binds The Soul By Breaking Up His Direct, Pervasive Awareness Of His Own Pure Consciousness Nature, Free Of Mental Representation. The Stanzas On Vibration Teach: Operating In The Field Of The Subtle Elements, The Arising Of Mental Representation Marks The Disappearance Of The Flavour Of The Supreme Nectar Of Immortality; Due To This [Man] Forfeits His Freedom. As We Have Already Seen, Three Factors Are Necessary For Perception And Thought To Be Possible, Namely, The Perceiving Subject, The Means Of Knowledge And The Object Perceived. Rajanaka Rama, In His Commentary On The Stanza Cited Above, Explains At Length That These Three Factors Correspond To Three Major Divisions In The Lower Thirty-One Categories Of Existence, Namely: 1) The Object. This Consists Essentially Of The Five Primary Sensations Which Are The Subtle Elements {Tanmatras) Of Smell, Taste, Sight, Touch And Sound Along With The Five Gross Elements — Earth, Water, Fire, Air And Ether — Of Which These Sensations Are The Perceivable Qualities. 2) The Means Of Knowledge. This Consists Of The Senses And The Inner Mental Organ. 3) The Subject. At This Level, The Subject Is The Individual Soul (Purusa) Whose Consciousness Is Contracted By The Five Obscuring Coverings (Kancukas) Of Limited Knowledge And Action, Attachment, Natural Law And Time Along With Maya, Their Source. All These Categories Belong To The Impure Creation (Asuddhasrsti), Which Is The Sphere Of Maya Where The Lower Order Of Knowledge Operates And Subject And Object Are Divided. Above Them Are Five More Categories Which Belong To The Pure Creation (Suddhasrsfi) Where Subject And Object Are Still United. The Highest Of Those Categories Are Siva And Sakti. Combined They Represent The State Of Pure T Consciousness And Its Sentient Subjectivity (Upalabdhrta), Respectively. The Next Category Is Called Sadasiva. Here Faint Traces Of Objectivity Appear In The Pervasive, Undivided Consciousness Of Siva And Sakti. Consciousness, Now Full Of The Power Of Knowledge (Jnanasakti), Views The All In A State Of Withdrawal (Nimesa), Shining Within, And At One With Its Own Nature. T Consciousness Predominates Over 'This' Consciousness Which It Encompasses In The Awareness That: 'I Am This [Universe]' (Aham-Idam). Next Comes The Category 'Isvara' Corresponding To The Awareness: 'This (Universe) Is Me' (Idam-Aham). 'This' Consciousness Takes The Upperhand Over T Consciousness And Unfolds Externally Full Of The Creative Power Of Action (Kriya&Akti). The All Now Becomes More Clearly Manifest As An Independent Reality. It Is Still Experienced As One With Consciousness But Is No Longer Fully Merged Within It. Finally, When Both Subjective And Objective Aspects Share An Equal Status In The Two-Fold Awareness That: 'I Am This (Universe) And This (Universe) Is Me (Ahamidam-Idamaham)\ Pure Knowledge (Fuddhavidya), The Last Of These Categories, Emerges. The Pure Categories Are The Experience Of The Impure Categories When They Are Recognised To Be One With Consciousness. They Are Experienced Within The Domain Of The Pure Universal Subject The Enlightened Yogi Realises Himself To Be. Mental Representations (Yikalpa) Emerge From This Pure Awareness And Subside Into It In Consonance With The Rhythm Of The Emanation And Withdrawal Of The Lower Categories. Impelled By The Universal Will, This Movement Is Spontaneous And Free. Free Of All Hopes And Fears The Enlightened Yogi Sees All Things As Part Of This Eternal Cosmic Game, Played In Harmony With The Blissful Rhythm Of His Own Sportive Nature At One With All Things. The Stanzas On Vibration Teach: Everything Arises [Out Of) The Individual Soul And He Is All Things. Being Aware Of Them, He Perceives His Identity [With Them]. Therefore There Is No State In The Thoughts Of Words Or [Their] Meanings That Is Not Siva. It Is The Enjoyer Alone Who Always And Everywhere Abides As The Object Of Enjoyment. Or, Constantly Attentive, And Perceiving The Entire Universe As Play, He Who Has This Awareness (Sawvitti) Is Undoubtedly Liberated In This Very Life. According To The Doctrine Of Vibration, Only Liberation In This Life (Jivanmukti) Is Authentic Liberation. Liberation After Death (Yideha- Mukti) In Some Form Of Disembodied State Free Of All

Perceptions And Notions Of The World Of Diversity Is Not The Ultimate Goal. Ksemaraja Stresses That Liberation Is Only Possible By Realising One's Own Identity With The Whole Universe, However Difficult This May Be. Similarly, He Maintains That The Suspension Of All Mental And Sensory Activity, Which Takes Place In The Introverted Absorption Of Contemplation With The Eyes Closed (Nimilanasamadhi) That Leads To Identification With Transcendent Consciousness Is Complemented And Fulfilled By The Cosmic Vision Had Through The Expansion Of Consciousness That Takes Place In Contemplative Absorption With The Eyes Open (Unmilanasamadhi). Consequently, Ksemaraja Explains That The First Of The Three Sections, Into Which He Divides The Stanzas, Deals With The Former Mode Of Contemplation And The Second Section With The Latter. Significantly, The Last Stanza Of The Second Section Ends With The Declaration That 'This Is The Initiation That Bestows Siva's True Nature'. In Other Words, This Realisation, Attained Through The Expanding Consciousness Of Contemplation With The Eyes Open, Initiates The Yogi Into The Liberated State, Which Is Identification With Siva Whose Body Is The Universe. In Order To Attain This Expanded State Of Liberated Consciousness, The Yogi Must Find A Spiritual Guide Because The Master (Guru) Is The Means To Realisation. The Master Is For His Disciple Siva Himself For It Is He Who Through His Initiation, Teaching And Grace, Reveals The Secret Power Of Spiritual Discipline. Instructing In The Purport Of Scripture He Does More Than Simply Explain Its Meaning: He Transmits The Realisation It Can Bestow. The Master Is At One With Siva's Divine Power Through Which He Enlightens His Disciple. It Is This Power That Matters And Makes The Master A True Spiritual Guide, Just As It Was This Same Power That Led The Disciple To Him In His Quest For The Path That Leads To The Tranquility That Can Only Be Found 'In The Abode Beyond Mind'. The Master Is The Ferry That Transports The Disciple Over The Ocean Of Thought — If, That Is, The Disciple Is Ready. The Disciple Must Be 'Awake' (Prabuddha) 2% Attending Carefully To The Pulse Of Consciousness. This Alert State Of Wakefulness Is At Once The Keen Sensitivity Of Insight As Well As The Receptivity Of One Who Has No Other Goal To Pursue Except Enlightenment. The Highest, Most Perfect Relationship The Disciple Can Have With His Master Is Such As It Is With Siva Himself: One Of Identity. The Exchange That Takes Place Between Them Is An Internal Dialogue Within Universal Consciousness, Their Common Identity (Svabhava). Limiting Itself To A Point Source (Anu) And Obscured By The Thought-Constructs Born Of Doubt And Ignorance, Consciousness Assumes The Guise Of The Disciple Who Seeks To Attain The Expanded Fullness Of His Master's Consciousness. The Master, On The Other Hand, Embodies The Aspect Of Consciousness Which Responds To The Inquiring Consciousness Of His Disciple. Free Of The Notions Of 'Self And 'Other', When The Disciple Is Liberated By His Grace, It Is The Master Who In Reality Liberates Himself. Although Ksemaraja Assures Us That The Master Can By Himself Enlighten His Disciple By The Initiation He Imparts To Him And The Other Means (Yukti) N He Adopts, Even So, He Is Not The Only Guide On The Path. Apart From The Master There Is Scripture And, Above All, One's Own Personal Experience, Because, As Abhinava Says: The Knowledge [Acquired] By Gradually [Coming To Understand The Meaning Of] The Scriptures And Following The Master [Who Knows Them] Leads, [When] Confirmed For Oneself, To The Realisation Of One's Own Identity With Bhairava. It Is Important To Know The Scriptures. God Reveals Himself Through Them; They Are One Of The Forms In Which He Is Directly Apparent In This World. They Teach Man What Is Worth Attaining And What Should Be Avoided And So Like A Boat Convey Him Across The Ocean Of Profane Existence (Samsara) To The Other Shore Where God's True Nature Is Revealed To Him. However, The Study Of The Scriptures Is Of Value Only If Accompanied By The Spiritual Knowledge That Results From Personal Experience. Mahesvarananda Writes: Being Well Versed In The Nature Of Deity Is One Thing, But Being Well Versed In The Sacred Scripture Is Another, Just As The Peace Of That Abode Is One Thing And What Worldly People Experience Is Another. Vasugupta, Who Found The Sivasutra, Knew The Means To Realisation (Yukti) As Well As The Scriptures And Had Fully Experienced The One Ultimate Reality. Therefore, Ksemaraja Declares Him To Be Amongst The Best Of Teachers. The Stanzas On Vibration (That Ksemaraja Attributes To Vasugupta) Accordingly Transmit The Secrets Of The Sivasutra In Accord With Scripture, Sound Reasoning And Personal Experience. The Latter Is Particularly Important For The Spanda Yogi; He Is Not Interested In Wasting His Time In Useless Discussion About The Experience Of Consciousness Expansion And Its Fruits, For That Can Only Be Known For

Oneself. The Yogi Can Achieve This Experience Either Through Faith In The Master Or Personal Insight (Svapratyayatah) Acquired By Unswerving Devotion To God. Ksemaraja Accordingly Quotes A Passage From The Bhagavadgita Where Krsna Says: Those I Deem To Be The Best Yogis Who Fix Their Thoughts On Me And Serve Me, Ever Integrated [In Themselves], Filled With The Highest Faith. But While The Yogi's Development Depends On Faith And Personal Experience Of The Higher States Of Consciousness, He Can, And Must, Strengthen His Conviction In The Light Of Reason .When Reason (Upapatti) And Direct Insight (Upalabdhi) Work Together, They Serve As A Means To Liberation. Reason Alone Cannot Help Us, But When It Is Based On An Intuitive Insight Of Fundamental Principles Along With A Direct Experience Of Reality, Error Is Eradicated And The Yogi Is Freed. In This Way The Awakened Yogi Realizes His Inherent Spiritual Power (Svabala) With Which He Exerts Himself To Distinguish Between The **Motions Of Individualized Consciousness And The Universal Vibration (Samanyaspanda)** Of The Collective Consciousness That Is Their Ultimate Ground And Firm Foundation. Note The Information Of Individual, Collective And Cosmic General Ledger Is Alone Responsible For Creation And Also Destruction. Thus, Although The Doctrine Taught In The Stanzas On Vibration Accords With Scripture, It Is Supported By Reason And Above All By Personal Experience. Thus, For Example, The Seventeenth Stanza Describes The Difference In The Manner In Which The Well Awakened And The Unawakened Experience Their Own Nature (Atmopalambha While The Eighteenth Describes The Experience Of The Well Awakened In The Three States Of Waking, Dreaming And Deep Sleep. Indeed, Rajanaka Rama, One Of The Commentators, Explains That The First Sixteen Stanzas Establish On The Basis Of Personal Experience (Svanubhava) That One's Own True Nature Is Independent Of The Body. Similarly, The Remaining Stanzas Also Discuss The Direct Experience Of One's Own Nature, But This Time As The Unity Of All Things. This Direct Experience, In Its Diverse Aspects, Is Both The Means By Which The Yogi Develops His Consciousness As Well As His Ultimate Goal. **Doctrine Of Vibration: Mark S.G.Dyczkowski Models See Next Paper** From the point of view of the object, the expansion (unmesa) of this pulse is represented(eb) by the initial desire to perceive (didrksa) a particular object, while the contracted (nimesha) phase is (=) the withdrawal of attention from the object previously perceived. From the point of view of the perceiving subjectivity, the phases are (=) reversed, so that the initial desire to perceive marks (eb) the contraction (nimesha) of subjective consciousness while the falling away of the previous perception is (=) its expansion (unmesa). At the higher level, where these two phases are experienced within (eb) consciousness, they represent (eb) the state of the categories of Isvara ('this universe is me') and Sadasiva ('I am this universe'). Utpaladeva says: Expansion (unmesa\ which is in (eb) the external manifestation [of objectivity], is (=) Kvaratattva while contraction (Niemba), which is in the internal manifestation [of subjectivity], is Sadasiva. **There is no glorification or mortification here. It just like joining the job and retiring the involution and evolution phases**.(italics mine)The Doctrine of Vibration At this level all the powers of consciousness fuse (e&eb) and both phases are (=) manifest as part of one reality. This unity is in fact apparent (eb) to everybody at each moment. However, within the domain of Maya, which is the sphere of differentiated perceptions (Yikalpa), it is clearly manifest (eb) only at the juncture (Madhya) between two cognitions. In this Centre resides (eb) the void (kha) of consciousness (free of (e) thought-constructs) which divested of (e) diversity, digests into (eb) itself all the psycho-physical processes that give (eb) life to the multiplicity of perceptions. **Here it is important to note that the when the cosmic general ledger zero reflects upon itself and it is projected on the individual consciousness you see a film an illusory animations wherein creation itself becomes subjective wherein the individual consciousness is both the perceiver the perceived being the image on the screen on the screen of individual consciousness**. The yogi moves from the particular vibrations of consciousness at its periphery to (e&eb) the universal throb of the Heart in the Centre. As Abhinava explains: The self-reflective awareness in (eb) the Heart of pure consciousness, present (eb) at the beginning and end of each perception, within (eb) which the entire universe is (=) dissolved away without residue, is (=) called in the scriptures, the universal vibration of consciousness (samanyaspanda) and is (=) the outpouring (uccalana) [of awareness] within one's own nature. All the categories of existence (tattvas) are united in (eb) the Heart of the Centre where the life-giving elixir of Siva's consciousness floods (e&eb) one's own inner nature. To reside in the Centre is to abide by (e) the law of totality (gramadharma) in a state which transcends (e) the

workings of the mind (unmana).Consciousness (jnana) with Light as its support, residing in (eb) the Centre between being and non-being is (=) known as the act of abiding in (eb) one's own abode as (=) the perceiving subjectivity (dratfrtva) free of (e) all obscuration. That which has been purified by (e) pure awareness **(suddhavijnana)** is called the transcendent (viviktavastu), said to be (=) the mode of being (v/7//) of the law of totality (gramadharma) through (e&eb) which everything is (=) easily attainable. The power in the Centre (madhyasakti) is (=) the eternal Present Beyond time it is the source of both past and future. To be established there is to abide without a break in Rama, the supreme enjoyer, in every action of one's life. Rama is Siva, the supreme cause Who pervades the fourteen aspects which embrace the entire universe of experience, namely, moving, standing, dreaming, waking, the opening and closing of the eyes, running, jumping, exertion, knowledge [born] of the power of the senses, the [three] aspects of the mind, living beings, names and all kinds of actions. This came as a natural development in Spanda doctrine not only for **this reason but also because the universal ego is experienced as the inner dynamics of absolute consciousness. To conclude our summarial exposition of the Divine Means, which is centred on the direct experience of this pure ego (and hence on Spanda in this form), we turn now to a brief description of its inner, cyclic activity. We shall do this** by examining Abhinava's esoteric exegesis of the symbolic significance of the word 'A HAM', which in Sanskrit means T, and symbolises by its form the ego's dynamic nature. “Dasein Does Not Fill Up A Track Or Stretch ‘Of Life’ — One Which Is Somehow Present-At-Hand — With The Phases Of Its Momentary Actualities. It Stretches Itself Along In Such A Way That Its Own Being Is Constituted In Advance As A Stretching-Along. The ‘Between’ Which Relates To Birth And Death Already Lies In The Being Of Dasein … It Is By No Means The Case That Dasein ‘Is’ Actual In A Point Of Time, And That, Apart From This, It Is ‘Surrounded’ By The Non-Actuality Of Its Birth And Death. Understood Existentially, Birth Is Not … Something Past In The Sense Of Something No Longer Present-At-Hand; And Death Is Just As Far From Having The Kind Of Being Of Something … Not Yet Present-At-Hand But Coming Along … Factical Dasein Exists As Born; And, As Born, It Is Already Dying, In The Sense Of Being-Towards-Death. As Long As Dasein Factically Exists, Both The ‘Ends’ And Their ‘Between’ *Are*, And They Are In The Only Way Possible On The Basis Of Dasein’s Being As *Care* … As Care, Dasein Is The ‘Between’.” **— Martin Heidegger (Thomas Piel - Traducteur), Being And Time** Path to Liberation On without knowledge that everything is manifest within consciousness is illusory or unreal in that sense alone. 78 Things are more real or more tangibly experienced according to their own essential nature (svabhava) to the degree in which we recognise that they are appearances (abhasa) within absolute consciousness As Jayaratha says: Just as images manifest in a mirror, for example, are essentially mere appearances, so too are [phenomena] manifest within conscious- ness. Thus, beause they are external, [phenomena] have no being (sattva) of their own. The Lord says this [not with the intention of saying anything about the nature of things] but in order to raise the level of consciousness of those people who are attached to outer things; thus everything in this sense is essentially a mere appearance. [Knowing this], in order to quell the delusion of duality, one should not be attached to anything external. 79 The ultimate experience is the realisation that everything is contained within consciousness. We can discover this in two ways. Either we merge the external world into the inner subject, or we look upon the outer as a gross form of the inner. In these two ways we come to recognise that all things reside within our own consciousness just as consciousness resides within them. This all-embracing inwardness is only possible if there is an essential identity between the universe and consciousness. The events which constitute the universe are always internal events happening within consciousness because their essential nature is consciousness itself. 80 We can only account for the fact that things appear if there is an essential identity between consciousness and the object perceived. If a physical object were really totally material, that is, part of a reality independent of, and external to, consciousness, it could never be experienced. Abhinava says: The existence or non-existence of phenomena within the domain of the empirical (iha) cannot be established unless they rest within consciousness. In fact, phenomena which rest within consciousness are apparent (prakasamana). And the fact of their appearing is itself their oneness {abheda) with consciousness because consciousness is nothing but the fact of appearing {prakasa). If one were to say that they were separate from the light of [that consciousness] and that they appeared [it would be tantamount to saying that] 'blue' is separate from its own nature. However, [insofar as it appears and is known as such] one says:

'this is blue 1 . Thus, in this sense, [phenomena] rest in conscious- ness; they are not separate from consciousness. The Doctrine of Vibration The universe and consciousness are two aspects of the whole, just as quality and substance constitute two aspects of a single entity. The universe is an attribute (dharma) of consciousness which bears (dharmiri) it as its substance. It is said that 'substance' is that resting in which this entire group of categories manifests and is made effective. Now, if you don't get angry [we insist that] this entire class of worlds, entities, elements and categories (tattva) rests in consciousness and [resting in it] is as it is. Thus consciousness contains everything in the sense that it is the ground or basis (adhard) of all things, their very being (satta) and substance from which they are made. But, unlike the Brahman of the Advaita Vedanta, it is not the real basis (adhisthand) of an unreal projection or illusion. Consciousness and its contents are essentially identical and equally real. They are two forms of the same reality. Consciousness is both the substratum and what it supports: The perceiving awareness and its object. In this respect, the Kashmiri Saiva is frankly and without reserve an idealist. Although he does not deny the reality of the object, his position is at odds with most commonly accepted forms of realism. The realist maintains that the content perceived is independent of the act of perception. The content is only accidentally an object of perception and undergoes no change in the process of being perceived. His contention, however, is essentially unverifiable; to verify it, we would have to know an object without perceiving it. This, from the Kashmiri Saiva point of view, is not possible. Objects of which we have no knowledge may indeed exist, but they are knowable as objects only if they are related to subjects who perceive them. In this sense, if there were no subjects, there could be no objects. 86 The subject, however, as opposed to the object is, in terms of the phenomenology of perception, apparent to himself. He is self-luminous (svaprakdsa). Thus, conscious- ness (the essence of subjectivity) is one s own awareness by virtue of which all things exist. The realist maintains that consciousness clearly differs from its object insofar as their properties are contrary to each other. The Saivite idealist. However, says that the object is a form of awareness (vijnanakara)™ The objective status of the object is cognition itself. Perception manifests its object and renders it immediately apparent (sphuta) to those who perceive it. It does not appear at any other time. If 'blue' were to exist apart from the cognition of k blue\ two things would appear: 'blue' and its cognition, which is not the case. It is the perception of the object which constitutes its manifest nature. An entity becomes an Integral Monism of Kashmiri Saivism object of knowledge not by virtue of the entity itself but by our knowledge of it. If objects had the property of making other objects appear, it would be possible for one object to make another appear in its own likeness. 'Blue' is perceived to be 'blue' because it is manifest as such to the perceiver. As Abhinava points out: The [nature of an] object of knowledge could not be established through a means of knowledge totally unrelated to it — a crow does not become white because a swan [sitting next to it] is white. Perception, on the other hand, is immediately apparent to conscious- ness. It is self-luminous in the sense that it is directly known without need of being known by any ulterior acts of perception and makes its object known at the same time. Adopting the Buddhist Yogacara doctrine that things necessarily perceived together are the same (sahopa- lambhaniyamavada), the Saivite affirms that because the perceived is never found apart from perception, they are in fact identical. Reality (satya) is the point where the intelligible and the sensible meet in the common unity of being; it cannot be said to exist in itself outside, and apart from, knowledge or vision. Bhagavatotpala in his commentary on the Stanzas on Vibration quotes: Once the object is reduced to its authentic nature, one knows [the true nature of] consciousness. What then [remains of] objectivity? What [indeed could be] higher than consciousness? Consciousness is essentially active. Full of the vibration of its own energy engaged in the act of perception, it manifests itself externally as its own object. When the act of perception is over, consciousness reabsorbs the object and turns in on itself to resume its undifferentiated inner nature. Knowledge (jnana) manifests internally and externally as each individual entity.... Once knowledge has assumed that form it falls back [into itself]. The Yogacara Buddhist similarly maintains that consciousness creates its own forms. But, according to him, because the perceived and perception are identical; there is no perceived object at all. The so-called outer world is merely a flux of cognitions, it is not real. He is firmly committed to a doctrine of illusion. The reality of consciousness from The Doctrine of Vibration his point of view is established by proving the unreality of the universe. "All this consists of the act of consciousness alone", says Vasubandhu, "because unreal entities appear, just

as a man with defective vision sees unreal hair or a moon, etc."He points to dreams as examples of purely subjective constructs which appear to be objective realities. The apparent reality dreams possess is not derived from any concrete, objective world, but merely from the idea of objectivity. While the Yogacara does not say that an idea has, for example, spatial attributes, it does have a form manifesting them. While he agrees with the Saiva idealist that appearances have no independent existence apart from their appearing to consciousness, he maintains that for this reason they are unreal. The creativity of consciousness consists in its diversification in many modes having apparent externality; it is not a creation of objects. While the Kashmiri Saivite agrees that the world is pure conscious- ness alone, he maintains that it is such because it is a real creation of consciousness. The effect is essentially identical with the cause and shares in its reality. Matter and the entire universe are absolutely real, as 'congealed' (sty ana) or 'contracted' (samkucita) forms of consciousness. "This God of consciousness", writes Ksemaraja, "generates the universe and its form is a condensation of His own essence (rasa)" m By boiling sugarcane juice it condenses to form treacle, brown sugar and candy which retains its sweetness. Similarly, consciousness abides unchanged even though it assumes the concrete material form of the five gross elements. The same reality thus abides equally in gross and subtle forms. Consequently no object is totally insentient. Even stones bear a trace (vasana) of consciousness, although it is not clearly apparent because it is not associated with the vital breath (prana) and other components of a psycho-physical organism. Somananda goes so far as to affirm that physical objects, far from being insentient, can only exist insofar as they are aware of themselves as existing. The jar performs its function because it knows itself to be its agent. Indeed, all things are pervaded by consciousness and at one with it and hence share in its omniscience. Thus, Siva, Who perceives Himself in the form of physical objects, is the one ultimate reality. "The jar knows because it is of my nature", writes Somananda, "and I know it because I am of the jar's nature. I know because I am of Sadasiva's nature and He knows because He is of my nature; Yajnadatta [knows] because he is of Siva's nature and Siva [knows] because He is of Yajnadatta's nature". Integral Monism of Kashmiri Saivism Everything in this sense is directly perceived by absolute conscious- ness, and this direct perception (pratyaksa) unifies the knowable into a single, undivided whole. This is the central concept behind a doctrine originally expounded by Narasimha called 'the non-dualism of direct perception' (pratyaksadvaita). This states that consciousness is essentially perceptive and that its perception of all things operates throughout the universe. Insofar as phenomena are clearly evident (sphufa) to us, everything is directly perceived by absolute consciousness, with which our individual consciousness is identical. This direct perception unfolds everywhere; the one true reality, it is alone and without companion or rival (nihsapatna). Even though it remains one, it can, by its very nature, perceive distinctions (bheda) between one entity and another, without this engendering any division within it. We distinguish between two entities in empirical terms on the basis of their mutual exclusion (anyonyabhava). The relative distinction {bheda) between them is essentially the perceived difference between their respective characteristics. Despite this difference they are united within the purview of a single cognition insofar as they are equally both manifest appearances. This cognition is the undivided essence (rasa) or 'own nature' (svabhava) of both. Encompassed by the 'fire of consciousness', there is no essential difference between them. Just as when an emerald and ruby reflect each other's light, the ruby is reddish-green and the emerald greenish-red, similarly everything is connected with everything else as part of the single variegated (vicitra) cognition of absolute consciousness. Mahesvarananda writes: The Supreme Lord's unique state of emotivity (asadharanabhava) is the outpouring of pure Being (mahasatta). It is manifest as the brilliance (sphuratta) of the universe which, if we ponder deeply, [is realized to be] the single flavour (ekarasa) of the essence of Beauty which is the vibration of the bliss of one's own nature. In this way all things are in reality one although divided from the one another sharing as they do the 'single flavour' (ekarasa) of the pure vibration of consciousness. Kashmiri Saiva Realism Kashmiri Saivism as a whole has been variously called a form of 'realistic idealism', 'monistic idealism', ‘idealistic monism' and 'concrete monism'. It is easy to understand why Kashmiri Saivism is The Doctrine of Vibration said to be 'idealistic' and 'monistic', but in what sense is it also 'realistic*? The answer to this question is of no small importance in trying to under- stand the central idea behind its metaphysics and the fundamental importance of the concept of Spanda, in this seemingly impossible marriage between monistic idealism and pluralistic realism. The Kashmiri Saiva approach understands

the world to be a symbol of the absolute, that is, as the manner in which it presents itself to us. Again we can contrast this view with that of the Advaita Vedanta. The Advaita Vedanta understands the world to be an expression of the absolute insofar as it exists by virtue of the absolute's Being. Being is understood to be the real unity which underlies empirically manifest separateness and as such is never empirically manifest. It is only transcendentally actual as 'being-in-itself. The Kashmiri Saiva position represents, in a sense, a reversal of this point of view. The nature of the absolute, and also that of Being, is onceived as an eternal becoming (satatodita), a dynamic flux or Spanda, 'the agency of the act of being'. It is dentified with the concrete actuality of the fact of appearing, not passive unmanifest Being. Appearance (abhasa) alone is real.ppearing (prakasamanatva) is equivalent to the fact of being (astitva). Ksemaraja writes in his commentary on the Stanzas on Vibration: Indeed, all things are manifest because they are nothing but manifestation. The point being that nothing is manifest apart from manifestation The absolutely unmanifest, from this point of view, can have as little existence as the space in a lattice window of a sky-palace. Nay, even less, because even that space can appear as an imagined mage manifest within consciousness. Everything is real according to the manner in which it appears. Even an illusion is in this sense real, insofar as it appears and is known in the manner in which it appears. The empirical and the real are identical categories of thought. As Abhinava says: Thus this is the supreme doctrine (upanisatf), namely that, when- ever and in whatever form [an entity] appears, that then is its particular nature. Perhaps at this stage a brief comparison with Heidegger's ideas might prove to be enlightening and not altogether out of place. According to Heidegger's phenomenology of Being, reality is intelligible in a two-fold Integral Monism of Kashmiri Saivism manner as 'phenomenon' and 'logos'. Heidegger defines what he means by 'phenomenon' as: "that-which-shows-itself. The manifest . . . phenomena are then the collection of that which lies open in broad daylight or can be brought to the light of day — what the Greeks at times implicitly identified as 'ta onta' (the things-which-are)". 127 In his later writings Heidegger drops the term 'phenomenon' in preference for the verbal form 'phainesthai' in order to emphasize even more the actuality or presentational property of Being. Explaining this new form of the term he writes: "Being disclosed itself to the ancient Greeks as 'physis'. The etymological roots 'phy-' and 'pha-' designate the same thing: 'phyein', the rising-up or upsurge which resides within itself as 'phainesthai', lighting-up, self-showing, coming-out, appearing-forth." Heidegger contrasted his notion of phenomenon with semblance (Schein) and with appearing (Erscheinung). In the case of semblance a thing can show itself as that which it is not, as when fool's gold shows itself to be gold. The ancients always allied semblance with non-being. Heidegger points out, however, that semblances are grounded in showings, and so does Abhinava. Both Heidegger and Abhinava consequently maintain that all semblances have a real basis and are to be treated as instances of phenomena along with the so-called real showing or manifestation of non-deceptive objects. So Heidegger states that: 'how- ever much seeming, just that much being'. Thus self-showing or appearing defines Being as phenomenon, but this definition of Being is as yet incomplete. Being is not only self-showing but 'logos' which Heidegger explains means 'discourse' (Rede) in the sense of 'apophansis': 'letting-be-seen'. Phenomenology, which according to Heidegger is the only correct study of Being, means 'letting-be-seen-that-which- shows-itself. This is true of Saiva Paramadvaita as well. The reality of the world demands recognition; we are forced to accept the direct presentation of the fact of our daily experience. As Abhinava says: "if practical life, which is useful to all persons at all times, places and conditions were not real, then there would be nothing left which could be said to be real." A thousand proofs could not make 'blue' other than the colour blue. The reality of whatever appears in consciousness cannot be denied. Objects appear; they do not cease to do so by a mere emphatic denial. The manifestation of an entity in its own specific form is a fact at one level of consciousness; it is real. The appearing of the same entity in the same form but recognised to be a direct representation of the absolute is also a fact, but at another level of consciousness. It is no more or less real than the first. 'As is the state of consciousness, so is the experience,' says Abhinava. Although the nature of the absolute is discovered at a higher level of consciousness, The Doctrine of Vibration nonetheless it presents itself to us directly in the specific form in which we perceive things; otherwise there would be no way in which we could penetrate from the level of appearing to that of its source and basis. Abhinava writes: Real is the entity iyastu) that appears in the moment of direct perception (sak$atkara), that is to say, within our experience of it. Once its own specific form has been clearly determined one

should, with effort, induce it to penetrate into its pure conscious nature. All things are known to be just as they present themselves. The concrete actuality of being known (pramiti), irrespective of content, is itself the vibrant (spanda) actuality of the absolute. Liberating knowledge is gained not by going beyond appearances but by attending closely to them. "The secret," Mahesvarananda says, "is that liberation while alive (jivanmukti) is the profound contemplation of Maya's nature." No ontological distinction can be drawn between the absolute and its manifestations because both are an appearing (dbhasa), the latter of diversity and the former of 'the true light of consciousness which is beyond Maya and is the category Siva'. Those who have attained the category of Pure Knowledge above Maya and have thus gone beyond the category of Maya, see the entire universe as the light of consciousness . . . Just as the markings [on a feather] are nothing apart from the feather, the feather [is nothing apart from] them, similarly, when the light of consciousness is manifest, the whole group of phenomena is manifest as the light of consciousness itself. Within the sphere of Maya, every entity's 'own nature' (svabhdva) corresponds to its specific manifest form. Accordingly it is defined as that which distinguishes it from all else and from which it never deviates. Above the sphere of Maya, that is, above the level of objectivity, is the domain of the subject. At this level, everything is realised to be part of the fullness of the experience r and hence no longer bound by the conditions which impinge on the object. Here the part is discovered to be the whole, that is, consciousness in toto. In this sphere beyond relative distinctions, the yogi realises that (all) the categories of existence are present in every single category. m The yogi experiences every individual particular as the sum total of everything else. He recognises that all things have one nature and that every particular is all things. This is the 'essence' (sard) or co-extensive unity (samarasya) of all things. Integral Monism of Kashmiri SaivismWe have established that reality is manifest according to how [and the degree in which] the freedom of consciousness reveals it and that [this freedom] is the womb of all forms. Just as 'sweetness' is present in its entirety in every atom of the sugarcane, so each and every atom [of the universe] bears within itself the emanation of all things. This is the level of consciousness in which the absolute reflects on itself realising to its eternal delight and astonishment (camatkara) its own integral nature. The reality of the world of diversity is not denied, but experienced in a new mode of awareness free of time and space in the eternal omnipresence of the Here and Now. [Phenomenal forms of awareness] such as 'this [exists]', born of the colouring [imparted to the absolute] by the limitations engendered by the diversifying power of time {kalakalana) also emanate within the Supreme Principle. There [at that level], Fullness {purriata) is the one nature [of all things] and so everything is omnipresent; otherwise, associated with division (khancjlana), the Fullness [of the absolute] would not be full. The content of absolute consciousness consists of diverse appearings (abhdsa) which, because they are manifest through it in this way, do not compromise the wholeness of consciousness. Everything we perceive is a momentary collocation of a number of such manifestations which combine together like 4 a row of altar lamps' (dipavalT) to form the single radiant picture of the universe. The individual objects which constitute the universe are specific collocations of such 'atomic' appearings. Together they form a single unified particular which appears according to its own defining features (svalaksand). A jar, for example, consists of a number of appearances such as 'round', 4 fat\ 'earthen', 'red', etc., which together discharge a single function (arthakriya), in this case, that of carrying the appearance 'water'. They unite with each other much as the scattered rays of a lamp come together when focused, or as the various currents of the sea together give rise to waves. Atomic appearings can combine in any number of ways, provided that they are not contrary to one another as established by the dictates of natural law (niyati). An appearance of 'form', for example, cannot combine with that of 'air'. Insofar as they share a common basis (samanyadhikaranya), a given cluster of appearances appears as a single whole. This common basis is the most prominent member of the group; the appearance 'jar' is such in the example quoted above. Any one appearance in a cluster may assume a more important or subordinate role. The result is a specific The Doctrine of Vibration awareness of an object of the form: 'here this is such.' While individual appearances do not lose their separate identity {svarupabheda) when they rest on a common basis, even so the particular object which appears according to its own characteristics (svalak$ana) is an individual reality in its own right. It is a different kind of appearance characterised by its association with the appearance of the specific location and time in which it is made manifest. The form of our experience is thus 'I now see this here'. But when we perceive each particular constituent

appearance separately, each assumes a separate fixed function. Abhinava cites the following colourful example to illustrate how the various combinations of appearances account for the variety of experience: Thus even though the appearance of the beloved may manifest externally, it is as if far away in the absence of another appearance, namely, that of 'embracing'. So when the [appearing of the beloved] is associated with another appearance [namely that of 'far away'] the power (arthakriya) it formerly had of giving pleasure appears as its contrary. The form our experience assumes depends, not only on the nature of the object perceived, but also on personal factors entirely peculiar to ourselves. This theory explains this in two ways. In one sense, the object remains the same, but one or other of its constituent appearances comes to the fore according to the inclinations of the perceiver. From another point of view, we can say that the perceived object is different for each perceiver according to the difference in the prominent appearance manifest to him. Abhinava, citing as an example a golden jar, illustrates how the same object appears differently to different perceivers according to the use they wish to make of it and to their state of mind: When a person who is depressed and feels that there is nothing [of value for him in the world] sees the jar, he merely perceives the appearance 'exists' [in the form of the awareness that] 'it is\ He is not conscious of any other [of its constituent appearances] at all. An individual who desires to fetch water [perceives] the appearance 'jar'. The man who simply wants something that can be taken somewhere and then brought back [perceives] the appearance 'thing'. The man who desires money [perceives] the appearance 'gold'. The man who desires a pleasing object [perceives] the appearance 'brightness' while he who wants something solid sees the appearance 'hardness'. These 'atomic events' or appearances emerge from the pure subject's consciousness and combine together to form a total event at each moment. Integral Monism of Kashmiri Saivism Daily life (vyavahara) goes on by virtue of this ever renewed flux of appearances. They are connected together and work towards a single unified experience because they appear within the field of consciousness of the universal subject. The aggregate of appearances arises in the [supreme] subject as do [sprouts in] a rice field. Even though each sprout germinates from its own seed, they are perceived as a collective whole. Appearances rest in this way within the universal subject. 'External- ity' is itself another appearance; it arises from a distinction between appearances and the individual subject. So, although all manifestation always occurs within the subject, it appears to be external due to the power of Maya which separates the individual subject from his object. This split must occur for daily life to be possible. Only externally manifest appearances can perform their functions; when they are merged within the subject and at one with him, they cannot do so. Daily life proceeds on the basis of the operation and withdrawal of the conditions necessary for fruitful action to be possible. Appearance in this sense represents the actualisation of a potential hidden in consciousness made possible by virtue of its dynamic, Spanda nature which is both the flow from inner to outer and back as well as the power that impels it. The emergence from, and submergence into, pure consciousness of each individual appearance is a particular pulsation (visesaspanda) of differentiated awareness. Together these individual pulsations constitute the universal pulse (samanyaspandd) of cosmic creation and destruction. Thus, every single thing in this way forms a part of the radiant vibration {sphuratta, sphurana) of the light of absolute consciousness. Light and Awareness: The Two Aspects of Consciousness Absolute consciousness understood as the unchanging ontological ground of all appearing is termed 'Prakasa'. As the creative awareness of its own Being, the absolute is called ' Vimarsa '. Prakasa and Vimarsa — the Divine Light of consciousness and the reflective awareness this Light has of its own nature — together constitute the all-embracing fullness (purnata) of consciousness. The Recognition (pratyabhijfia) school of Kashmiri Saivism develops this concept of the absolute which finds its fullest expression in Utpaladeva's Stanzas on the Recognition of God. Even though neither of these two key terms appear in the Stanzas on Vibration or the Aphorisms of Siva, they recur frequently in their commentaries. Thus, although the original formulation of the Doctrine of Vibration differs from the theology of Recognition in this respect, it was extended in the course of its development to accommodate this concept of the absolute as well. This was possible, and quite justified, insofar as the absolute understood in Pratyabhijfia terms does not, as we shall see, differ essentially from that of the Spanda school. We can, as Kashmiri Saivites themselves have done, explain one in terms of the other. The Doctrine of Vibration Prakasa: The Light of Consciousness Prakasa is the pure luminosity' (bhdna) or 'self-showing' that constitutes the essence and ultimate identity (atman) of phenomena. That things appear at all is due to the light of consciousness,

and their appearing (avabhasana) is itself this Light which bestows on all things their evident, manifest nature. Established in the light of consciousness everything appears there according to its own specific nature (svabhava). Anything that supposedly does not rest in this Light is as unreal as a sky-flower. 3 Thus, according to Rajanaka Rama, unlike the light of the sun, or any other light, this Light not only makes all things apparent, it is also their ultimate source. 4 Full of its divine vibration the Light makes all things manifest and withdraws them into itself. This supra- temporal activity characterises it most specifically; devoid of it, it would be no better than an inert physical phenomenon. At the same time, this light is the conjunction (slesa) or oneness (aikdtmya) of its countless manifest forms, 6 and the collective whole (sampina'ana) of all the categories of existence. The universe is nothing but the shining of the Light within itself. It is the radiant vibration (sphuratta) of this Light, the state (avast hand) in which consciousness becomes manifest.Although the Light shines as all things at all times and hence also makes their diversity manifest, 9 penetrating each object individually as well as collectively, it is not totally 'merged' (magna) or identified with the object so as to suffer any division within itself. Our experience of any object is of the form: T see this': it is not itself an object, but the manifest form the object assumes as a luminous principle of experience. The Light is ever revealed and can never be obscured; objectivity can never cast a shadow on the light of consciousness. The Stanzas on Vibration declare: That in which all this creation is established and from whence it arises is nowhere obstructed because it is unconditioned by [its very] nature. This Light is the highest reality (paramartha). It is the 'Ancient Light' (puranaprakasa) that makes all things new and fresh every moment. It is 'always new and secret, ancient and known to all'. It is the form of the Present (vartamanarupa), the Eternal Now. Time and space are relations between the contents of consciousness; they cannot impinge on the integrity of the absolute itself. 16 Neither space nor time can divide it, for they are one with the Light that illumines them Light and Awareness: Two Aspects of Consciousness and makes them known as elements of experience. But this Light is the shining of the absolute; it is not an impersonal principle. It is the living Light of God, indeed it is God Himself, and the Master Who instructs the entire universe. 18 Siva is this 'auspicious lamp', Who illumines all things. He is the Light of consciousness that reveals the presence of both the real and the unreal, of light' and 'darkness'. 20 Abhinavagupta writes: Thus Bhairava, the Light, is self-evident (svatahsiddha); without (e) beginning, He is the first and last of all things, the Eternal Present. And so what else can be said of Him? The unfolding of the categories of existence (tattva) and creation, which are the expansion of His own Self, He illumines, luminous with His own Light, in identity with Himself, and because He illumines Himself, so too He reflects on (e&eb) His own nature, without His wonder (camatkara) being in any way diminished. Since Zwicky's early observations, similar data from other galaxy clusters has yielded the same result – we consistently see that clusters of galaxies appear to by developing an awareness of the Centre, the yogi experiences the bliss of consciousness. 117 Through this gap he plunges into introverted absorption (nimilanasamadhi) and then emerges again to pervade the field of awareness between Centres and so experience the Cosmic Bliss (jagadananda) of the universal vibration of consciousness. 1 i8 He then recognises that this state pervades (e&eb) every aspect of experience. In this way the yogi's consciousness is no longer afflicted by (e&eb) the power which obscures (e) it, hemming the Centre in on (e&eb) both sides with (e&eb) thought-constructs that seemingly deprive it of (e) its fullness. As he realises directly his pure conscious nature as (=) the universal ego free of all mental representations, it expands out to embrace (e&eb) all things within (eb) itself. Thus the realisation the Divine Means leads to, and is directly based upon, is that this pure ego is in all things just as all things are within it. In the Spanda tradition, as recorded in the Stanzas on Vibration, no such ego is recognised. ' I9 Man's authentic nature is, however, understood in (eb) personal terms as (=) every individual's own 'own nature* (svasvabhava) which is (=) Siva, the universal vibration of pure subjectivity (upalabdhfta). It is not surprising, therefore, that later commentators found these two conceptions to be essentially the same and accordingly identified one's own inner nature with the pure ego. This came as a natural development in Spanda doctrine not only for **this reason but also because (e) the universal ego is experienced as (=) the inner dynamics of (e) absolute consciousness. To conclude our summarial exposition of (e) the Divine Means, which is centred on (e&eb) the direct experience of (e) this pure ego (and hence on (e&eb) Spanda in this form), we turn now to (e0 a brief description of its inner, cyclic activity. We shall do this** by examining Abhinava's esoteric exegesis

of the symbolic significance of the word 'A HAM', which in Sanskrit means T, and symbolises by (e) its form the ego's dynamic nature. The objective world of perceptions is (=), as we have seen, essentially a chain of thought-constructs (prapanca) closely linked to (e&eb) one another and woven into (e&eb) the fabric of diversity (vicitrata). This thought (vikalpa) is (=) a form of speech (vac) uttered internally by (e) the mind (citta), which is (=) itself an outpouring of consciousness. Consciousness also, in its turn, resounds with (e&eb) the silent, supreme form of speech {para vac) which is (=) the reflective awareness through which it expresses itself to (e&eb) itself. Consequently, the fifty letters of the Sanskrit alphabet, which are the smallest phonemic units into which speech can be analysed, are symbolic of (e) the principal elements of (e) the activity of consciousness. Letters come together to generate (eb) words and words go on to form (eb) sentences. In the same way the **fifty phases in the cycle of (e) consciousness** represent, in (eb) the realms of denoted meaning (vacya) the (=) sum total of its universal activity (kriya) corresponding to (e&eb) the principal forces (kala) which come together to form (eb)the metaphysical categories of 186 The Doctrine of Vibration experience, which in their turn appear in (eb) the grossest, most explicitly 'articulate' form as (=) the one hundred and eighteen world-systems (bhuvana). 'A', the first letter of both AHAM and the Sanskrit alphabet, is the point of departure or initial emergence of (e) all the other letters and (e&eb) hence denotes Anuttara — the absolute. 'Ha', is the final letter of (e) the alphabet and represents (eb) the point of (e) completion when (e) all the letters have emerged. It represents (eb) the state in which all the elements of (e) experience, in the domains of (e) both inner consciousness and (e&eb) outer unconsciousness, are fully displayed. It is (=) also the generative, emission (visarga) which, like the breath, casts the inner into (e&eb) the outer, and draws what is outside (e&eb) inward. The two letters 'A' and 'Ha' thus represent (eb) Siva, the transcendental source and (e&eb) Sakti, His cosmic outpouring that flows back into Him. The combined 'A-Ha' contains within itself all the letters of the alphabet — every phase of consciousness, both transcendental and universal. (For a graphic representation of this analysis, see figure 1.) M, the final letter of AHAM, is written as a dot placed above the letter which precedes it. It comes at the end of the vowel series and before the consonants and so is called 'anus vara '(lit. 'That which follows the vowels') and also 'bindu ' (lit. 'dot,' 'drop,' 'point' or 'zero'). While the consonant 'M' symbolises the individual soul (purusa), 'bindu 1 represents (eb)the subtle vibration of T, which is (=) the life force (jivakala) and essence of the soul's subjectivity manifest at (eb) the transcendental, supra-mental level (unmana). 12 ° It is (=) the zero-point in the centre between the series of negative numbers, in this case the vowels which represent (eb) the processes happening internally within (e&eb) Siva, and the series of (e) positive numbers — the consonants which symbolise the processes happening externally within (eb) Sakti. Bindu, as (=) a point without area, symbolises (eb) the non-finite nature of the pure awareness (pramitibhava) of AHAM. It is the pivot around which the cycle of energies from 'A' to (e&eb) 'Ha' rotates, the Void in the centre from (e) which all the powers emanate and into (e&eb) which they collapse. As such, it is the supreme power of action which holds (e&eb) subject, object and means of knowledge together in (eb) a potential state in (eb) the one Light that shines as (=) all three 121 containing (e) them in its repose 122 (visrdnti). Bindu is (=) the 'knower' (jndtr), who is (=) essentially consciousness that, though omniscient does not (e) manifest its intelligence, like a man who knows the scriptures but having no occasion to explain them to others silently bears this knowledge within himself. As such, it symbolizes (eb) the union of Siva and Sakti (sivasakti- mithunapina'a) 123 in (eb) a state of heightened potency in which they have not (e) yet divided to generate (eb) the world of diversity. It stands, in other words, at (eb) the threshold of differentiation in the stream of emanation still contained within (eb) Siva. Expansion Commences Bindu — The Individual Soul Withdrawal Commences Figure 1 188 The Doctrine of Vibration Then, to the degree in which that which is to be (=) accomplished by (e) the power of action residing within (eb) it [as a potential] penetrates into (e&eb) the absolute, it appears initially as (=) bindu, which is (=) the light of pure consciousness. 124 When outer objectivity is reabsorbed into its transcendent source, bindu is the point into which all the manifest powers of consciousness are gathered and fused together. The universal potency of all the letters is thus contained in bindu which, as the reflective awareness of supreme T consciousness, 125 gives them all life. Thus bindu also marks the beginning of Siva's internal movement back to the undifferentiated absolute and so stands at the threshold of both emission and absorption without being involved in either. The three aspects of AH AM together constitute (e) a

movement from the undifferentiated source of transcendental consciousness — 'A' — through (e&eb) the expansion or emission of its power — 4 Ha' — to the subject — 'M 1 — which contains (e) and makes manifest (eb) the entire universe of experience. The reverse of this movement, which of withdrawal (samhara), is represented by (e) M-Ha-A. AHAM and M-Ha-A alternate in the rotation (ghurnana) of the reflective awareness of T consciousness as immanent Sakti emerges from transcendental Siva to then merge back into Him. As Abhinava says: The universe rests within Sakti and She on the plane of the absolute (anutiara) and this again within Sakti ... for the universe shines within consciousness and [consciousness shines] there [within the universe by the power of] consciousness. These three poles, forming a couple and merging, make up the one supreme nature of Bhairava Whose essence is AHAM- 126 At the microcosmic level, fc A' represents the initial moment when the subject begins to rise out of himself to view the object. The movement from 'A to 'Ha' marks the emergence of sensation within the field of awareness, which is represented by the fifty letters of the alphabet symbolic of the fifty aspects of the flux of consciousness leading to objectified perception. 'Nl' is tne subject who, resting content within himself when he has perceived his object, merges through the inner flow of awareness into k A\ the absolute. Then from the absolute (A) its emission (Ha) flows back into the pure subject (M) set to perceive his object. Thus all the cycles of creation and destruction are contained within AHAM through which they are experienced simultaneously as the spontaneous play of the absolute. The yogi who recognises this recurrent pulse of awareness to be the movement of his own consciousness merges his limited ego with the Path to Liberation universal ego. **This is natures general ledger** Thus he realises that its power to create, sustain and destroy all things is his own inner strength (svabaia) that he exerts effortlessly in the same state of mystical absorption (turlya) in universal consciousness that the absolute itself enjoys. In this way he shares in the three-fold awareness Siva Himself has of His own nature which Abhinava describes as follows: 4 I make the universe manifest within myself in the Sky of Consciousness. I, who am the universe, am its creator! ' — this awareness is the way in which one becomes Bhairava. 'AH of manifest creation (sadadhvari) is reflected within me, I cause it to persist 1 — this awareness is the way in which one becomes the universe. The universe dissolves within me. I who am the flame of the [one] great and eternal fire of consciousness' — seeing thus one achieves peace. 127 The experience of the liberated thus coincides with the realisation of their own divine nature which, through its power, rules and guides the cosmic order. Thus this attainment (siddhi), which is liberation itself, is in the DoctrLie of Vibration technically called 'Mastery over the Wheel of Energies' (cakresvaratvasiddhi) because the liberated soul, identified with Siva, now governs, as does Siva, the cycle of the powers that bring about the creation and destruction of all things, Here we are talking of individual consciousness becoming cosmic general ledger and the liberationist realises all destruction and creation is by him. This clearly means that he is in his own world of illusion. Creation is subjective quintessentially The Empowered Means (Saktopaya) All the practices taught in the Stanzas on Vibration are internal. Whenever ritual is mentioned, it is invariably interpreted in terms of the dynamics of the inner processes the yogi experiences and implements in the course of his yogic practice. The Doctrine of Vibration, Ksemaraja affirms, 129 is concerned entirely with these inner disciplines centred, as it is, in one way or another, on consciousness or, at least, on the inner activity of the mind. Thus the Empowered Means which, like the other categories we have discussed, is entirely internal, includes an important part of Spanda practice. Spanda practice belonging to the Divine Means centres on one's own inherent nature (svasvabhava) as Siva, the universal perceiver and agent, that belonging to the Empowered Means on His power. Instead of arriving directly at the all-embracing emptiness of subjective conscious- ness, the yogi practising the Empowered Means realises his true nature through the fullness of its energy. Practising the Divine Means, the yogi plunges, as it were, straight into the fire of consciousness; practising the 190 The Doctrine of Vibration Empowered Means he merges with its rays. Either way the yogi is centred equally on ultimate reality. The power of consciousness is no less absolute than its possessor. To make this point Abhinava quotes the Matanga- tantra: This reality consists of the rays of [Siva's] power and is variously said to be the abode of the Lord's manifestation . . . That same [power] illumined [by Siva] is itself also luminous, unshaken and unmoving. That very [power] is the supreme state, subtle, omnipresent, the nectar of immortality, free of obscuration, peaceful, yearning for pure Being alone {vastumatra) and devoid of beginning and end. Perfectly pure, it is said to be the body [of ultimate reality].

110 The yogi concentrates on the powers operating in all of life's activities as particular pulsations (visesaspanda) in the universal rhythm (samanyaspanda) of the power of consciousness. In this way he rises progressively from the particular to the universal until he reaches pure Being (satta), the greatest of all universals (mahasamanyd) and the highest form of Siva's power. Thus the creative power of Maya, manifest through countless lesser powers, no longer causes the yogi to stray from Siva's consciousness but becomes the means through which it can be realised 13 1 in the illuminating brilliance (sphuratta) which is Siva's pure Being. Thus by discovering the true nature of Sakti, the yogi realises himself to be Siva, its possessor Who consists of all its countless powers. Thus practise belonging to this Means leads to the same pure consciousness free of thought-constructs realised through the Divine Means. Although the ultimate realisation is instantaneous, the yogi rises to it gradually by freeing his consciousness of the limitations imposed upon it by thought. Abhinava explains: The same occurs in the Empowered Means [as does in the Divine]. At the discursive level of consciousness {yaikalpikibhumi) [where the Empowered Means functions] knowledge and action, although evident, are, for the reasons explained previously, contracted. A blazing energy [is revealed within] the one who dedicates himself to removing the burden of this contraction. [This energy eventually] brings about the inner manifestation {antarabhasa) of pure consciousness he seeks. 132 Consciousness is individualized and its power of knowledge and action contracted by the thought-constructs born of ignorance. The arising of these mental representations, as the Stanzas on Vibration say, deprives the soul of its freedom and immortal life. 133 The practise of the Path to Liberation 191 Empowered Means is meant to free the fettered soul of this constriction on his consciousness. It operates within the mental sphere (cetasy™ and is designed to purify thought (vikalpasamskara) in order to. reveal the pure consciousness which is its ground and ultimate source. Thus, the Empowered Means is concerned with the second instant of perception, during which the subject forms mental representations of his object. Thought functions on the basis of an awareness of relative distinctions between specific particulars, distinguishing them from one another and thus seemingly fragmenting the essential unity of reality. 135 The vibrant vitality of (e) consciousness, universally manifest, is clouded (e&eb) like a mirror by a child's breath 136 and the soul is deprived of (e) the liberating intuition of (e) the one reality free of (e) thought-constructs (nirvikalpa). Abhinava writes: The [fettered soul] is like a dancing girl who although wishing to leave the dancehall is collared by (e) the doorkeeper of thought and thrown back onto the stage of Maya. 137 All thought is centred on objectivity and hence dislodges awareness from the plenitude of pure subjective consciousness. Thus, to regain the original state of rest (visranti) consciousness enjoys, the yogi must rid himself of thought. As thought-forms decrease, pure, thought-free awareness is strengthened 138 until the yogi is fully established in a state in which the relative distinctions (bheda) conceived between entities dissolve away. Everything appears to him as pure Being (sattdmdtra) l}9 and the entire universe shines before him pervaded by Siva's radiance. 140 His intuitive faculty (mati) thus purified, the yogi gains both the perfections (siddhi) of yogic practice and liberation (mukti). His consciousness is now like a well-polished mirror which reflects everything he desires and grants it to him. 141 Abhinava writes: Just as a man who has been ill for a long time forgets his past pain completely when he regains his health, absorbed as he is in the ease of his present condition, so too those who are grounded in pure awareness free of thought-constructs are no longer conscious of their previous [fettered] state. Consciousness, the sole truly existent reality, free of thought- constructs is made fully and evidently manifest by eliminating thesedifferentiated perceptions. The wise man should therefore exert himself to attend closely to this [state of awareness]. I42 The thought-constructs generated within consciousness do not in reality affect it at all. They can neither break up nor add anything to the Light which shines as all things. 143 They are in fact nothing but 192 The Doctrine of Vibration consciousness itself 144 which perceives, through its power of reflective awareness (yimarsa), the multitude of objects in diverse ways, and so assumes this form. 145 Although thought-constructs are mental representations of objects once seen or present, they are products of the power of consciousness and not of the objects they represent. 146 Thought is both analytic and synthetic; 147 it serves the useful purpose of separating individual elements of experience from others and linking together those that appear to be distinct from one another so that they can be better understood. 148 It does not consist merely of false mental constructs projected onto reality that need to be wholly rejected. Thought obscures consciousness and distracts it only when it appears in the form of doubt,

vacillating between alternatives. 149 Once this conflicting duality (dvaitddhivasa) 150 is eliminated, thought is purified and rests in itself as the 'thought-less thought' of pure consciousness. 151 By gradually eliminating the multitude of conflicting notions that agitate him, the yogi ultimately achieves the certainty (niscaya) corresponding to a direct awareness of his own divine nature. 152 Abhinava explains: Thought is in reality none other than pure consciousness. Even so, it serves as a means to liberation for the individual soul (anu) only when it takes the form of certainty (niscaya). 153 The yogi must eliminate every doubt and misguided notion that leads him to believe himself to be other than Siva. By developing the thought: 4 1 am Siva', it ultimately affirms itself directly as a pure awareness beyond thought without any intervening mental representations. Abhinava says: Just as the man who thinks intensely that he is a sinner becomes such, just so one who thinks himself to be Siva, and none other than He, becomes Siva. This certainty (ddrtfhya), which penetrates and affirms itself in our thoughts, coincides with an awareness free of thought- constructs engendered by a series of differentiated mental representa- tions, the object of which is our identity with Siva. 154 As thought is gradually purified, it becomes progressively clearer until its object becomes maximally apparent (sphufatama). 155 The stream of perceptive consciousness (pramana) progressively reveals each aspect of its object which, thus affirming itself with increasing clarity, reveals its ultimate nature. The yogi reflects repeatedly upon it as the object of his realisation and loving devotion, for all that is perceptible and need be known (j fie yd) is Siva alone. As Abhinava says: Path to Liberation 193 What should we say of those who before they are satisfied have to see their beloved again and again, caress her and think about her for a long time? 156 The yogi practising the Empowered Means is initiated into the Great Sacrificial Rite (mahayaga), eternally enacted at the interface between the inner and outer aspects of consciousness, by a direct infusion of awareness from his master who is the embodiment and outer symbol of the yogi's enlightened identity. ,57 The rite begins with ritual bathing (snana) which is in this case the immersion of the body of thought in the white ashes of the cosmic fuel of duality, burnt in the fire of consciousness. 158 The yogi then goes on to worship (puja) by uniting all that is pleasing to the senses in the oneness of consciousness. 159 The ritual formula (mantra) he recites is the eternal resonance of the awareness which is the pulsation of the Heart of his own consciousness. 160 Repetition (Japa) of the formula is every activity, perception, breath or thought which arises within him while plunged in the universal awareness of his true nature. 161 The mental image he visualises meditating (dhyana) on the Deity in the course of the rite, is whatever the yogi spontaneously imagines and contemplates as the outpouring of the universal creativity of consciousness. 162 Ritual gesture (mudra) is whatever bodily posture the yogi may assume when, fully absorbed in consciousness, he moves, staggering about (ghumita) as it were, drunk with the wine of self-realisation. 163 Oblation is performed by offering with devotion and awareness all the sensations which flow in through the channels of the senses to the fire of his subjectivity, which is thus inflamed (uddipita) and makes all things one with itself. 164 The outer ritual which commences in the sphere of the Individual Means thus leads naturally to the inner rite of the Empowered Means. When the yogi's practise (abhyasa) reaches fruition, the rite merges with the spontaneous activity of consciousness. This is fullness (purnata), the completion and reunification of the forces within consciousness which, through the power of ignorance, were formerly dispersed and divided. "Just as a horse driven here and there", writes Abhinava, "over plains, hills and dales follows the will of its rider, so also consciousness, driven by various expedients (bharigi), quiescent or terrific, abandoning duality, becomes Bhairava. Just as by looking repeatedly at one's own face in a mirror one comes to know that it is the same [as the image reflected], so also, [one sees] in the mirror of mental representations of meditation {dhyana), ritual (puja) and worship (area) one's own Self as Bhairava and so quickly identifies with Him. This identification is the realisation that takes place in the absolute (anuttara)." 165 194 The Doctrine of Vibration By ridding himself of the relative distinctions engendered by thought, the yogi practising the Empowered Means, illumined by the power of self-awareness of Pure Knowledge {suddhavidya), transcends the distinction between right and wrong, purity and impurity. He is led to the conviction that the pure consciousness, which is his true nature, is unaffected by whatever action he may do, whether conventionally accepted as good or bad. Abhinava quotes the Malinivijaya as saying: All here is enjoined and all prohibited. This alone, O Lord of the gods, is here prescribed as obligatory, namely that the mind be firmly applied to the true reality. It matters little how this is achieved. He whose mind is firmly established

in [this] reality, even if he eats poison, is as little affected by it as are lotus petals by water. 166 Impurity is a state of seeming separation from consciousness. 167 The yogi who has freed himself of all false notions comes to realise that the true nature of consciousness can never be sullied or limited by any object appearing within it. 168 This is the realisation the ancient sages achieved through a direct intuition of reality free of intruding thought-constructs {avikalpabhava), but kept secret in order not to confuse the worldly. 169 Similarly, in reality nobody is ever bound. It is ignorance to believe bondage exists and to contrast it with a conceived state of liberation. 170 If the Self is one with Siva, how can it be either bound or released? 171 Nothing essentially distinguishes those who are bound from those who are free. 172 The difference between their states is merely conceptual. 173 Pure consciousness abides free of all such distinctions. Thus Bhagavatotpala, <n his commentary on the Stanzas, repeatedly stresses that thought-constructs obscure consciousness and misguide the individual soul. 174 Those who are bound are convinced that they are dull witted, conditioned by Karma, sullied by their sin and helplessly impelled to action by some power beyond their control. He who manages to counter this conviction with its opposite achieves freedom. 175 He who considers himself to be free is free indeed, while he who thinks himself bound remains so. Thus at the highest level of realisation, as Abhinava says: Nothing new is achieved nor is that which in reality is unmanifest, revealed- [only] the idea is eradicated that the luminous being shines not. 176 Nothing is impure, all is perfect, including Maya and the diversity it engenders. To say that illusion exists and that ignorance must be Path to Liberation 195 eradicated implies that it has a separate existence apart from conscious- ness. If this is so, it has as little reality as the shadow of a shadow, but if not,then it must be consciousness itself. Thus, as Kallata says, bondage, the binder and the bound are in fact one. 177 It is Siva Himself Who freely obscures His own nature. Siva binds Himself by Himself. 178 Concealing and revealing Himself, Siva plays His timeless game. At the Divine (sambhava) level of pure Siva-consciousness, the Spanda yogi directly lays hold of the power inherent in his own conscious nature (svabala) which gives life to the psycho-physical organism and impels the senses and mind to action. 179 In this way every thought-construct, and with it the ego, is instantly annulled in the immediacy of the pure subjectivity that remains unaltered throughout every perception and state of consciousness. The same takes place at the Empowered level b> attending to the recurrent activity — Spanda — of the subject, that is, the flux of awareness through the cyclic movement of the powers of consciousness. 180 By attending (avadhana) to this movement the thought- constructs that emerge and subside in the course of perception are seen to be part of this universal process, and, in this way purified, are no longer binding. Thus, Ksemaraja says that the Spanda teachings are concerned most directly with the Empowered Means. m The yogi who is always alert to discern the pulse of Spanda quickly realises his own authentic state of being (nijarfi bhavam). 1 * 2 He is then truly awake, not only literally, but also in the deeper sense that he is awake to his authentic nature, its power and activity. When attention (avadhana) slackens, this movement takes place unconsciously and so the thought-constructs and perceptions generated through it appear to take on an autonomous existence of their own, just as happens when we dream. 183 The spontaneity of the movement that travels between subject and object and holds them together in the pure awareness of the universal subject's identity with his cosmic object devolves into the creative activity of waking and dreaming. Man, in other words, becomes a victim of his states of consciousness and the contents that they, by their very nature, generate within themselves. 184 The Spanda teachings are not only concerned with the structure of thought and its functions, but also with the powers and properties of its vehicle, namely, speech. Speech issues out of consciousness, develops into thought to then become articulated sound. A focal point of Spanda doctrine is thus the role speech plays in the formation of thought-constructs and their purification. Although this takes place at all levels of practise below the Divine (sambhava), the Spanda teachings, meant as they are for advanced yogis, ignore the outer forms of spiritual discipline to concentrate on practise in the Empowered (sakta) psychic sphere (cetas) and what lies beyond it where speech is the pure inner awareness (yimarsa) **Although Abhinava does not differentiate between individual consciousness and cosmic general ledger, one tends to comprehend that right interpretation is the transfor mentioned the individual consciousness to cosmic consciousness responsible for the creation and destruction which the sadhaka sees as his own power. Every one is rama and every one is Shiva. only thing that is essential is striving and concerted efforts and sustained struggle to reach.** The Doctrine of Vibration of the light of consciousness. The Doctrine of

Vibration identifies this, the highest level of speech (para vac), with the universal pulse of consciousness that resounds spontaneously within it as the inner flow of its own undifferentiated awareness. 185 Beyond the realms of language, it is the transcendental consciousness in which all language is rooted and pervades all that language denotes as its essential being. Utpaladeva writes: The Supreme Voice is consciousness. It is self-awareness spontaneously arisen, the highest freedom and sovereignty of the Supreme Lord. That pulsing radiance (sphuratta) is pure Being, unqualified by time and space. As the essence [of all things] it is said to be the Heart of the Supreme Lord. 186 When the intention arises within consciousness to discern its own brilliance manifest in the world of denotations and denoted meanings, speech turns from the supreme transcendental level to that of immanence and assumes the form of a pure intuitive awareness (pratibha) which perceives and comprehends its universal manifestation. This is the voice of intuition (pasyanti), which grasps the meaning inherent inwardly in all words and externally in all that they denote. Analogous to the non- discursive, instinctual knowledge animals possess, it is a pure generic perception not yet formed into language in which the act of denotation, its object and that which denotes it are indistinguishable. Illumined by the voice of intuition birds migrate in their due seasons, the cock crows at dawn and young mammals suck at the breast. 187 Infants similarly reflect and respond instinctively to their environment by virtue of this intuitive sense 188 and through it come to grasp the link between words and the objects they denote. As they learn to speak, they begin to form concepts and so the next two levels of speech develop. One is the outer corporeal speech (yaikhari) and the other the subtler, inner discourse (antah- saryijalpa) of thought that forms at the intermediate level (madhyama) where the ratiocinating mind stands between the higher levels of intuition and its outer verbal expression. In this way the development of speech in infancy reflects its progressive actualisation in every spoken word. A hymn to the Goddess quoted by Bhagavatotpala describes this process well: Therefore, O Supreme Goddess, the highest form of speech should be worshipped as the [universal] cause that establishes the existence of all things by insight {niscaya) into their nature (artha) brought about by their manifestation through the superimposition [of verbal designations]. O Mother, insight into the true nature [of things] is nothing but the Path to Liberation 197 act of intent of that [same speech], apart from which [speech itself and all that it expresses] could not attain to its own nature. Again, in that state [speech] is said to be the light of one's own nature. Free of division and succession it is attainable [only] by the yogi. Then from the state of intent, O Siva, speech [assumes] the nature of thought as the radiant pulse (sphuraria) of desire to speak of that which is in the domain of words. Then consisting of words, it bears a clearly expressed meaning, for if [speech] were not such, meaning could not be understood. 189 personal experience clearly proves that thought is invariably associated with speech. 190 Thought is a function of language. Through it we communicate to ourselves a mental image of the world about us and can construct complex ideas about ourselves. Language is the fabric from which our world of ideas is woven. Mental representation which orders the influx of sensation and presents us with a meaningful, picture of the world, memory, the elaboration of ideas and the shifting tides of emotion are all intimately connected with language and through it to the consciousness which underlies them. To think of language as nothing more than a system of denotation based on a commonly accepted convention (sanketd) fails to fully account forks inherent power to convey meaning (vacakasakti). In order to learn the convention we must be born with an innate ability to grasp meaning, and this ability is not itself learned nor found anywhere within the domain of convention. Lacking this ability we would be caught in an infinitely expanding system of denotation in which each element pointed to some other within it, without ever coming to rest anywhere. Unless we can couple the word 'jar' with the object it denotes, explaining that the word 'pot' is a synonym of the word 'jar' would leave us none the wiser. 191 The connection between word and meaning is only explicable if we postulate that it is an inherent property of the power of awareness to link one with the other. Language must be grounded in the pure cognitive awareness (prama) of consciousness which stands beyond, and yet illumines, the sphere of experience we define and understand through the medium of language. As Abhinava says: Someone may hear another person speak, but if his awareness (prama) is obscured, he is unable to rise, unconscious as he is, to the level of the experiencing subject [who understands] what has been said. He only grasps the outer successive (sound) of what the other person says and thus can only repeat it as would a parrot. An understanding of its meaning presupposes that he has caught hold of his own power

of awareness (prama) by attaining the autonomy [of the conscious, universal subject]. **Note that we have defined sachitananada as one which understand Truth and Untruth, Bliss and Unbliss, and Knowledge and ignorance where all these are equal. Truth=Untruth, Bliss=Unbliss and Ignorance=Knowledge** The Doctrine of Vibration Outer, articulate speech consists of a series of ordered phonemic elements produced and combined by the vocal organs to form meaningful words. In order for this to be possible, these elements must also be grounded in consciousness (prama). The articulated phonemes are merely outer, gross manifestations of the phonemic energies (yarnagrama) held in a potential state within consciousness. This 'mass of sounds* (sabdarasi) is the light of consciousness (prakdsd) which makes the universe manifest and contains all things within itself. In other words, it is the totality of consciousness expressed as the collective awareness symbolised by all the letters corresponding to the introverted subjectivity of Siva Himself. The power through which this potential actualises itself into speech and the world of denotation is technically called * Mdtrkd*. It is the reflective awareness (vimarsa) and radiance (sphuratta) of the supreme subject — the 'mass of sounds' (Jabdarasi) — and the undivided wonder Siva experiences when He contemplates the universe He gathers up into Himself in the form of countless words (vdcaka) and their meanings (vdcya). 193 Mdtfkdsakti is manifest in the second movement of consciousness after the primal vibration of the pure luminosity of the 'mass of sounds', as the state of pure potency which arises when its unsullied subjectivity begins to turn away from itself and is associated with faint traces of objectivity (dmrsya- cchdyd). 194 Mdtrkd contains within itself the various aspects of objectivity that, although not yet manifest, are ready to issue forth. Thus this power, at one with Siva, is called 'Mdtrkd' because she is the mother (mdtrkd) of the universe that she contains within herself as does a pregnant woman her child. 195 The circle of the powers of Mdtrkd (mdtrkdcakra) consists of the phonemic energies contained in AH AM, the universal ego. 196 When grasped in its entirety at its source, these energies elevate the consciousness of the enlightened, but when split up and dispersed give rise to the obscuring forces (kald) which lead the ignorant away from realisation. The fettered soul is ignorant of the pure egoity that is the source of speech and so it generates, through its powers, the many thought-constructs that deprive him of the awareness of unity and obscure Siva's universal activity. 197 The Stanzas on Vibration declare: He who is deprived of his power by the forces of obscuration {kala) and a victim of the powers arising from the mass of sounds (sabdardsi) is called the fettered soul. 198 The powers [of speech] are always ready to obscure his true nature as no mental representation can arise that is not penetrated by speech. 199 The rays of phonemic energies emanate from the light of Siva, the Path to Liberation 199 'mass of sounds' (sabdarasi) in eight groups. They constitute the powers of the inner mental organ and the five senses, figuratively arranged in a circle around the sacred shrine {pi(ha) of Matrka&akti who manifests externally as the body. 200 The eight classes and the names of the deities presiding over them are as follows: 201 Gutturals Brahman! Intellect (buddhi) Palatals Mahegvari Ego (ahankara) Cerebrals Kaumarl Mind (manas) Dentals Narayani Hearing Labials Varahi Touch Semivowels Aindri Sight Sibilants Camuntfa Taste Vowels Mahalak§ml Smell The yogi who grasps the true nature of the power of Matfka and its phonemic forces is liberated 202 by recognising that the activity of the senses and the discursive representations of the mind are in fact emanations of universal consciousness. Conversely, when ignorant, he is affected by its power in its multiple negative aspects known as 'Mahdghora' ('greatly terrible') and, unable to rest within himself free of the sense of diversity, he is constantly disturbed by the flux of extroverted perceptions. Abhinava explains: When the [phonemic energies] are not known to be [emanations of the Lord] they conceal the wonder (camatkara) of consciousness which is the one essential non-discursive awareness [present throughout perception] and even in discursive thought. They obscure it with thought-constructs constituted by the diverse configurations of phonemes and syllables which [although also] a form of the deity [are no longer benevolent but] most terrible. Inducing doubt and fear, they engender the fettered soul's state, bound by the shackles of transmigra tion. But once their true nature is understood correctly in this way, they bestow freedom in this very life This knowledge of their intimate being [at one with the absolute] consists of this, namely, that even in the midst of all these fluctuations, free at their inception of discursive representations, thought-constructs do not conjoin [individualised consciousness] with the wheel of energies consisting of the totality of phonemes, even though [these constructs] are coloured by the many diverse words generated by the aggregate of phonemes. Language has a powerful effect on us. A few words we may hear or read

can inspire us with joy, fear or sadness, and the constant inner 200 The Doctrine of Vibration dialogue of thought arouses intense feelings within us. This power hidden in language, which binds us through the thought-constructs it generates, can also be used to free us of them by channeling it through Mantra. Mantric practice begins at the Individual (driava) level where Mantras are recited in consonance with the rising and falling away of the breath. In this way they are charged with the vibration (spandd) of consciousness and, in their turn, make consciousness vibration (Doctrine of Vibration S.G.Dyczkowski) On without knowledge that everything is manifest within consciousness is illusory or unreal in that sense alone. 78 Things are more real or more tangibly experienced according to their own essential nature (svabhava) to the degree in which we recognise that they are appearances (abhasa) within absolute consciousness As Jayaratha says: Just as images manifest in a mirror, for example, are essentially mere appearances, so too are [phenomena] manifest within conscious- ness. Thus, beause they are external, [phenomena] have no being (sattva) of their own. The Lord says this [not with the intention of saying anything about the nature of things] but in order to raise the level of consciousness of those people who are attached to outer things; thus everything in this sense is essentially a mere appearance. [Knowing this], in order to quell the delusion of duality, one should not be attached to anything external. 79 The ultimate experience is the realisation that everything is contained within consciousness. We can discover this in two ways. Either we merge the external world into the inner subject, or we look upon the outer as a gross form of the inner. In these two ways we come to recognise that all things reside within our own consciousness just as consciousness resides within them. This all-embracing inwardness is only possible if there is an essential identity between the universe and consciousness. The events which constitute the universe are always internal events happening within consciousness because their essential nature is consciousness itself. 80 We can only account for the fact that things appear if there is an essential identity between consciousness and the object perceived. If a physical object were really totally material, that is, part of a reality independent of, and external to, consciousness, it could never be experienced. Abhinava says: The existence or non-existence of phenomena within the domain of the empirical (iha) cannot be established unless they rest within consciousness. In fact, phenomena which rest within consciousness are apparent (prakasamana). And the fact of their appearing is itself their oneness {abheda) with consciousness because consciousness is nothing but the fact of appearing {prakasa). If one were to say that they were separate from the light of [that consciousness] and that they appeared [it would be tantamount to saying that] 'blue' is separate from its own nature. However, [insofar as it appears and is known as such] one says: 'this is blue 1 . Thus, in this sense, [phenomena] rest in conscious- ness; they are not separate from consciousness. The Doctrine of Vibration The universe and consciousness are two aspects of the whole, just as quality and substance constitute two aspects of a single entity. The universe is an attribute (dharma) of consciousness which bears (dharmiri) it as its substance. It is said that 'substance' is that resting in which this entire group of categories manifests and is made effective. Now, if you don't get angry [we insist that] this entire class of worlds, entities, elements and categories (tattva) rests in consciousness and [resting in it] is as it is. Thus consciousness contains everything in the sense that it is the ground or basis (adhard) of all things, their very being (satta) and substance from which they are made. But, unlike the Brahman of the Advaita Vedanta, it is not the real basis (adhisthand) of an unreal projection or illusion. Consciousness and its contents are essentially identical and equally real. They are two forms of the same reality. Consciousness is both the substratum and what it supports: The perceiving awareness and its object. In this respect, the Kashmiri Saiva is frankly and without reserve an idealist. Although he does not deny the reality of the object, his position is at odds with most commonly accepted forms of realism. The realist maintains that the content perceived is independent of the act of perception. The content is only accidentally an object of perception and undergoes no change in the process of being perceived. His contention, however, is essentially unverifiable; to verify it, we would have to know an object without perceiving it. This, from the Kashmiri Saiva point of view, is not possible. Objects of which we have no knowledge may indeed exist, but they are knowable as objects only if they are related to subjects who perceive them. In this sense, if there were no subjects, there could be no objects. 86 The subject, however, as opposed to the object is, in terms of the phenomenology of perception, apparent to himself. He is self-luminous (svaprakdsa). Thus, conscious- ness (the essence of subjectivity) is one s own awareness by virtue of which all things exist. The realist maintains that

consciousness clearly differs from its object insofar as their properties are contrary to each other. The Saivite idealist. However, says that the object is a form of awareness (vijnanakara)™ The objective status of the object is cognition itself. Perception manifests its object and renders it immediately apparent (sphuta) to those who perceive it. It does not appear at any other time. If 'blue' were to exist apart from the cognition of k blue\ two things would appear: 'blue' and its cognition, which is not the case. It is the perception of the object which constitutes its manifest nature. An entity becomes an Integral Monism of Kashmiri Saivism object of knowledge not by virtue of the entity itself but by our knowledge of it. If objects had the property of making other objects appear, it would be possible for one object to make another appear in its own likeness. 'Blue' is perceived to be 'blue' because it is manifest as such to the perceiver. As Abhinava points out: The [nature of an] object of knowledge could not be established through a means of knowledge totally unrelated to it — a crow does not become white because a swan [sitting next to it] is white. Perception, on the other hand, is immediately apparent to conscious- ness. It is self-luminous in the sense that it is directly known without need of being known by any ulterior acts of perception and makes its object known at the same time. Adopting the Buddhist Yogacara doctrine that things necessarily perceived together are the same (sahopa- lambhaniyamavada), the Saivite affirms that because the perceived is never found apart from perception, they are in fact identical. Reality (satya) is the point where the intelligible and the sensible meet in the common unity of being; it cannot be said to exist in itself outside, and apart from, knowledge or vision. Bhagavatotpala in his commentary on the Stanzas on Vibration quotes: Once the object is reduced to its authentic nature, one knows [the true nature of] consciousness. What then [remains of] objectivity? What [indeed could be] higher than consciousness? Consciousness is essentially active. Full of the vibration of its own energy engaged in the act of perception, it manifests itself externally as its own object. When the act of perception is over, consciousness reabsorbs the object and turns in on itself to resume its undifferentiated inner nature. Knowledge (jnana) manifests internally and externally as each individual entity.... Once knowledge has assumed that form it falls back [into itself]. The Yogacara Buddhist similarly maintains that consciousness creates its own forms. But, according to him, because the perceived and perception are identical; there is no perceived object at all. The so-called outer world is merely a flux of cognitions, it is not real. He is firmly committed to a doctrine of illusion. The reality of consciousness from The Doctrine of Vibration his point of view is established by proving the unreality of the universe. "All this consists of the act of consciousness alone", says Vasubandhu, "because unreal entities appear, just as a man with defective vision sees unreal hair or a moon, etc."He points to dreams as examples of purely subjective constructs which appear to be objective realities. The apparent reality dreams possess is not derived from any concrete, objective world, but merely from the idea of objectivity. While the Yogacara does not say that an idea has, for example, spatial attributes, it does have a form manifesting them. While he agrees with the Saiva idealist that appearances have no independent existence apart from their appearing to consciousness, he maintains that for this reason they are unreal. The creativity of consciousness consists in its diversification in many modes having apparent externality; it is not a creation of objects. While the Kashmiri Saivite agrees that the world is pure conscious- ness alone, he maintains that it is such because it is a real creation of consciousness. The effect is essentially identical with the cause and shares in its reality. Matter and the entire universe are absolutely real, as 'congealed' (sty ana) or 'contracted' (samkucita) forms of consciousness. "This God of consciousness", writes Ksemaraja, "generates the universe and its form is a condensation of His own essence (rasa)" m By boiling sugarcane juice it condenses to form treacle, brown sugar and candy which retains its sweetness. Similarly, consciousness abides unchanged even though it assumes the concrete material form of the five gross elements. The same reality thus abides equally in gross and subtle forms. Consequently no object is totally insentient. Even stones bear a trace (vasana) of consciousness, although it is not clearly apparent because it is not associated with the vital breath (prana) and other components of a psycho-physical organism. Somananda goes so far as to affirm that physical objects, far from being insentient, can only exist insofar as they are aware of themselves as existing. The jar performs its function because it knows itself to be its agent. Indeed, all things are pervaded by consciousness and at one with it and hence share in its omniscience. Thus, Siva, Who perceives Himself in the form of physical objects, is the one ultimate reality. "The jar knows because it is of my nature", writes Somananda, "and I know it because I am of the jar's nature. I know

because I am of Sadasiva's nature and He knows because He is of my nature; Yajnadatta [knows] because he is of Siva's nature and Siva [knows] because He is of Yajnadatta's nature". Integral Monism of Kashmiri Saivism Everything in this sense is directly perceived by absolute conscious- ness, and this direct perception (pratyaksa) unifies the knowable into a single, undivided whole. This is the central concept behind a doctrine originally expounded by Narasimha called 'the non-dualism of direct perception' (pratyaksadvaita). This states that consciousness is essentially perceptive and that its perception of all things operates throughout the universe. Insofar as phenomena are clearly evident (sphufa) to us, everything is directly perceived by absolute consciousness, with which our individual consciousness is identical. This direct perception unfolds everywhere; the one true reality, it is alone and without companion or rival (nihsapatna). Even though it remains one, it can, by its very nature, perceive distinctions (bheda) between one entity and another, without this engendering any division within it. We distinguish between two entities in empirical terms on the basis of their mutual exclusion (anyonyabhava). The relative distinction {bheda) between them is essentially the perceived difference between their respective characteristics. Despite this difference they are united within the purview of a single cognition insofar as they are equally both manifest appearances. This cognition is the undivided essence (rasa) or 'own nature' (svabhava) of both. Encompassed by the 'fire of consciousness', there is no essential difference between them. Just as when an emerald and ruby reflect each other's light, the ruby is reddish-green and the emerald greenish-red, similarly everything is connected with everything else as part of the single variegated (vicitra) cognition of absolute consciousness. Mahesvarananda writes: The Supreme Lord's unique state of emotivity (asadharanabhava) is the outpouring of pure Being (mahasatta). It is manifest as the brilliance (sphuratta) of the universe which, if we ponder deeply, [is realized to be] the single flavour (ekarasa) of the essence of Beauty which is the vibration of the bliss of one's own nature. In this way all things are in reality one although divided from the one another sharing as they do the 'single flavour' (ekarasa) of the pure vibration of consciousness. Kashmiri Saiva Realism Kashmiri Saivism as a whole has been variously called a form of 'realistic idealism', 'monistic idealism', ‘idealistic monism' and 'concrete monism'. It is easy to understand why Kashmiri Saivism is The Doctrine of Vibration said to be 'idealistic' and 'monistic', but in what sense is it also 'realistic*? The answer to this question is of no small importance in trying to under- stand the central idea behind its metaphysics and the fundamental importance of the concept of Spanda, in this seemingly impossible marriage between monistic idealism and pluralistic realism. The Kashmiri Saiva approach understands the world to be a symbol of the absolute, that is, as the manner in which it presents itself to us. Again we can contrast this view with that of the Advaita Vedanta. The Advaita Vedanta understands the world to be an expression of the absolute insofar as it exists by virtue of the absolute's Being. Being is understood to be the real unity which underlies empirically manifest separateness and as such is never empirically manifest. It is only transcendentally actual as 'being-in-itself. The Kashmiri Saiva position represents, in a sense, a reversal of this point of view. The nature of the absolute, and also that of Being, is onceived as an eternal becoming (satatodita), a dynamic flux or Spanda, 'the agency of the act of being'. It is dentified with the concrete actuality of the fact of appearing, not passive unmanifest Being. Appearance (abhasa) alone is real.ppearing (prakasamanatva) is equivalent to the fact of being (astitva). Ksemaraja writes in his commentary on the Stanzas on Vibration: Indeed, all things are manifest because they are nothing but manifestation. The point being that nothing is manifest apart from manifestation The absolutely unmanifest, from this point of view, can have as little existence as the space in a lattice window of a sky-palace. Nay, even less, because even that space can appear as an imagined mage manifest within consciousness. Everything is real according to the manner in which it appears. Even an illusion is in this sense real, insofar as it appears and is known in the manner in which it appears. The empirical and the real are identical categories of thought. As Abhinava says: Thus this is the supreme doctrine (upanisatf), namely that, when- ever and in whatever form [an entity] appears, that then is its particular nature. Perhaps at this stage a brief comparison with Heidegger's ideas might prove to be enlightening and not altogether out of place. According to Heidegger's phenomenology of Being, reality is intelligible in a two-fold Integral Monism of Kashmiri Saivism manner as 'phenomenon' and 'logos'. Heidegger defines what he means by 'phenomenon' as: "that-which-shows-itself. The manifest . . . phenomena are then the collection of that which lies open in broad daylight or can be brought to the light of day — what the Greeks

at times implicitly identified as 'ta onta' (the things-which-are)". 127 In his later writings Heidegger drops the term 'phenomenon' in preference for the verbal form 'phainesthai' in order to emphasize even more the actuality or presentational property of Being. Explaining this new form of the term he writes: "Being disclosed itself to the ancient Greeks as 'physis'. The etymological roots 'phy-' and 'pha-' designate the same thing: 'phyein', the rising-up or upsurge which resides within itself as 'phainesthai', lighting-up, self-showing, coming-out, appearing-forth." Heidegger contrasted his notion of phenomenon with semblance (Schein) and with appearing (Erscheinung). In the case of semblance a thing can show itself as that which it is not, as when fool's gold shows itself to be gold. The ancients always allied semblance with non-being. Heidegger points out, however, that semblances are grounded in showings, and so does Abhinava. Both Heidegger and Abhinava consequently maintain that all semblances have a real basis and are to be treated as instances of phenomena along with the so-called real showing or manifestation of non-deceptive objects. So Heidegger states that: 'how- ever much seeming, just that much being'. Thus self-showing or appearing defines Being as phenomenon, but this definition of Being is as yet incomplete. Being is not only self-showing but 'logos' which Heidegger explains means 'discourse' (Rede) in the sense of 'apophansis': 'letting-be-seen'. Phenomenology, which according to Heidegger is the only correct study of Being, means 'letting-be-seen-that-which- shows-itself. This is true of Saiva Paramadvaita as well. The reality of the world demands recognition; we are forced to accept the direct presentation of the fact of our daily experience. As Abhinava says: "if practical life, which is useful to all persons at all times, places and conditions were not real, then there would be nothing left which could be said to be real" A thousand proofs could not make 'blue' other than the colour blue. The reality of whatever appears in consciousness cannot be denied. Objects appear; they do not cease to do so by a mere emphatic denial. The manifestation of an entity in its own specific form is a fact at one level of consciousness; it is real. The appearing of the same entity in the same form but recognised to be a direct representation of the absolute is also a fact, but at another level of consciousness. It is no more or less real than the first. 'As is the state of consciousness, so is the experience,' says Abhinava. Although the nature of the absolute is discovered at a higher level of consciousness, The Doctrine of Vibration nonetheless it presents itself to us directly in the specific form in which we perceive things; otherwise there would be no way in which we could penetrate from the level of appearing to that of its source and basis. Abhinava writes: Real is the entity iyastu) that appears in the moment of direct perception (sak$atkara), that is to say, within our experience of it. Once its own specific form has been clearly determined one should, with effort, induce it to penetrate into its pure conscious nature. All things are known to be just as they present themselves. The concrete actuality of being known (pramiti), irrespective of content, is itself the vibrant (spanda) actuality of the absolute. Liberating knowledge is gained not by going beyond appearances but by attending closely to them. "The secret," Mahesvarananda says, "is that liberation while alive (jivanmukti) is the profound contemplation of Maya's nature." No ontological distinction can be drawn between the absolute and its manifestations because both are an appearing (dbhasa), the latter of diversity and the former of 'the true light of consciousness which is beyond Maya and is the category Siva'. Those who have attained the category of Pure Knowledge above Maya and have thus gone beyond the category of Maya, see the entire universe as the light of consciousness . . . Just as the markings [on a feather] are nothing apart from the feather, the feather [is nothing apart from] them, similarly, when the light of consciousness is manifest, the whole group of phenomena is manifest as the light of consciousness itself. Within the sphere of Maya, every entity's 'own nature' (svabhdva) corresponds to its specific manifest form. Accordingly it is defined as that which distinguishes it from all else and from which it never deviates. Above the sphere of Maya, that is, above the level of objectivity, is the domain of the subject. At this level, everything is realised to be part of the fullness of the experience r and hence no longer bound by the conditions which impinge on the object. Here the part is discovered to be the whole, that is, consciousness in toto. In this sphere beyond relative distinctions, the yogi realises that (all) the categories of existence are present in every single category. m The yogi experiences every individual particular as the sum total of everything else. He recognises that all things have one nature and that every particular is all things. This is the 'essence' (sard) or co-extensive unity (samarasya) of all things. Integral Monism of Kashmiri SaivismWe have established that reality is manifest according to how [and the degree in which] the freedom of consciousness reveals it and that [this freedom] is the

womb of all forms. Just as 'sweetness' is present in its entirety in every atom of the sugarcane, so each and every atom [of the universe] bears within itself the emanation of all things. This is the level of consciousness in which the absolute reflects on itself realising to its eternal delight and astonishment (camatkara) its own integral nature. The reality of the world of diversity is not denied, but experienced in a new mode of awareness free of time and space in the eternal omnipresence of the Here and Now. [Phenomenal forms of awareness] such as 'this [exists]', born of the colouring [imparted to the absolute] by the limitations engendered by the diversifying power of time {kalakalana) also emanate within the Supreme Principle. There [at that level], Fullness {purriata) is the one nature [of all things] and so everything is omnipresent; otherwise, associated with division (khancjlana), the Fullness [of the absolute] would not be full. The content of absolute consciousness consists of diverse appearings (abhdsa) which, because they are manifest through it in this way, do not compromise the wholeness of consciousness. Everything we perceive is a momentary collocation of a number of such manifestations which combine together like 4 a row of altar lamps' (dipavalT) to form the single radiant picture of the universe. The individual objects which constitute the universe are specific collocations of such 'atomic' appearings. Together they form a single unified particular which appears according to its own defining features (svalaksand). A jar, for example, consists of a number of appearances such as 'round', 4 fat\ 'earthen', 'red', etc., which together discharge a single function (arthakriya), in this case, that of carrying the appearance 'water'. They unite with each other much as the scattered rays of a lamp come together when focused, or as the various currents of the sea together give rise to waves. Atomic appearings can combine in any number of ways, provided that they are not contrary to one another as established by the dictates of natural law (niyati). An appearance of 'form', for example, cannot combine with that of 'air'. Insofar as they share a common basis (samanyadhikaranya), a given cluster of appearances appears as a single whole. This common basis is the most prominent member of the group; the appearance 'jar' is such in the example quoted above. Any one appearance in a cluster may assume a more important or subordinate role. The result is a specific The Doctrine of Vibration awareness of an object of the form: 'here this is such.' While individual appearances do not lose their separate identity {svarupabheda) when they rest on a common basis, even so the particular object which appears according to its own characteristics (svalak$ana) is an individual reality in its own right. It is a different kind of appearance characterised by its association with the appearance of the specific location and time in which it is made manifest. The form of our experience is thus 'I now see this here'. But when we perceive each particular constituent appearance separately, each assumes a separate fixed function. Abhinava cites the following colourful example to illustrate how the various combinations of appearances account for the variety of experience: Thus even though the appearance of the beloved may manifest externally, it is as if far away in the absence of another appearance, namely, that of 'embracing'. So when the [appearing of the beloved] is associated with another appearance [namely that of 'far away'] the power (arthakriya) it formerly had of giving pleasure appears as its contrary. The form our experience assumes depends, not only on the nature of the object perceived, but also on personal factors entirely peculiar to ourselves. This theory explains this in two ways. In one sense, the object remains the same, but one or other of its constituent appearances comes to the fore according to the inclinations of the perceiver. From another point of view, we can say that the perceived object is different for each perceiver according to the difference in the prominent appearance manifest to him. Abhinava, citing as an example a golden jar, illustrates how the same object appears differently to different perceivers according to the use they wish to make of it and to their state of mind: When a person who is depressed and feels that there is nothing [of value for him in the world] sees the jar, he merely perceives the appearance 'exists' [in the form of the awareness that] 'it is\ He is not conscious of any other [of its constituent appearances] at all. An individual who desires to fetch water [perceives] the appearance 'jar'. The man who simply wants something that can be taken somewhere and then brought back [perceives] the appearance 'thing'. The man who desires money [perceives] the appearance 'gold'. The man who desires a pleasing object [perceives] the appearance 'brightness' while he who wants something solid sees the appearance 'hardness'. These 'atomic events' or appearances emerge from the pure subject's consciousness and combine together to form a total event at each moment. Integral Monism of Kashmiri Saivism Daily life (vyavahara) goes on by virtue of this ever renewed flux of appearances. They are connected together and work towards a single unified experience because they appear within

the field of consciousness of the universal subject. The aggregate of appearances arises in the [supreme] subject as do [sprouts in] a rice field. Even though each sprout germinates from its own seed, they are perceived as a collective whole. Appearances rest in this way within the universal subject. 'External- ity' is itself another appearance; it arises from a distinction between appearances and the individual subject. So, although all manifestation always occurs within the subject, it appears to be external due to the power of Maya which separates the individual subject from his object. This split must occur for daily life to be possible. Only externally manifest appearances can perform their functions; when they are merged within the subject and at one with him, they cannot do so. Daily life proceeds on the basis of the operation and withdrawal of the conditions necessary for fruitful action to be possible. Appearance in this sense represents the actualisation of a potential hidden in consciousness made possible by virtue of its dynamic, Spanda nature which is both the flow from inner to outer and back as well as the power that impels it. The emergence from, and submergence into, pure consciousness of each individual appearance is a particular pulsation (visesaspanda) of differentiated awareness. Together these individual pulsations constitute the universal pulse (samanyaspandd) of cosmic creation and destruction. Thus, every single thing in this way forms a part of the radiant vibration {sphuratta, sphurana) of the light of absolute consciousness. Light and Awareness: The Two Aspects of Consciousness Absolute consciousness understood as the unchanging ontological ground of all appearing is termed 'Prakasa'. As the creative awareness of its own Being, the absolute is called ' Vimarsa '. Prakasa and Vimarsa — the Divine Light of consciousness and the reflective awareness this Light has of its own nature — together constitute the all-embracing fullness (purnata) of consciousness. The Recognition (pratyabhijfia) school of Kashmiri Saivism develops this concept of the absolute which finds its fullest expression in Utpaladeva's Stanzas on the Recognition of God. Even though neither of these two key terms appear in the Stanzas on Vibration or the Aphorisms of Siva, they recur frequently in their commentaries. Thus, although the original formulation of the Doctrine of Vibration differs from the theology of Recognition in this respect, it was extended in the course of its development to accommodate this concept of the absolute as well. This was possible, and quite justified, insofar as the absolute understood in Pratyabhijfia terms does not, as we shall see, differ essentially from that of the Spanda school. We can, as Kashmiri Saivites themselves have done, explain one in terms of the other. The Doctrine of Vibration Prakasa: The Light of Consciousness Prakasa is the pure luminosity' (bhdna) or 'self-showing' that constitutes the essence and ultimate identity (atman) of phenomena. That things appear at all is due to the light of consciousness, and their appearing (avabhasana) is itself this Light which bestows on all things their evident, manifest nature. Established in the light of consciousness everything appears there according to its own specific nature (svabhava). Anything that supposedly does not rest in this Light is as unreal as a sky-flower. 3 Thus, according to Rajanaka Rama, unlike the light of the sun, or any other light, this Light not only makes all things apparent, it is also their ultimate source. 4 Full of its divine vibration the Light makes all things manifest and withdraws them into itself. This supra- temporal activity characterises it most specifically; devoid of it, it would be no better than an inert physical phenomenon. At the same time, this light is the conjunction (slesa) or oneness (aikdtmya) of its countless manifest forms, 6 and the collective whole (Sampina'ana) of all the categories of existence. The universe is nothing but the shining of the Light within itself. It is the radiant vibration (sphuratta) of this Light, the state (avast hand) in which consciousness becomes manifest. Although the Light shines as all things at all times and hence also makes their diversity manifest, 9 penetrating each object individually as well as collectively, it is not totally 'merged' (magna) or identified with the object so as to suffer any division within itself. Our experience of any object is of the form: T see this': it is not itself an object, but the manifest form the object assumes as a luminous principle of experience. The Light is ever revealed and can never be obscured; objectivity can never cast a shadow on the light of consciousness. The Stanzas on Vibration declare: That in which all this creation is established and from whence it arises is nowhere obstructed because it is unconditioned by [its very] nature. This Light is the highest reality (Paramartha). It is the 'Ancient Light' (puranaprakasa) that makes all things new and fresh every moment. It is 'always new and secret, ancient and known to all'. It is the form of the Present (vartamanarupa), the Eternal Now. Time and space are relations between the contents of consciousness; they cannot impinge on the integrity of the absolute itself. 16 Neither space nor time can divide it, for they are one with the Light that illumines them Light and

Awareness: Two Aspects of Consciousness and makes them known as elements of experience. But this Light is the shining of the absolute; it is not an impersonal principle. It is the living Light of God, indeed it is God Himself, the Master Who instructs the entire universe. 18 Siva is this 'auspicious lamp', Who illumines all things. He is the Light of consciousness that reveals the presence of both the real and the unreal, of light' and 'darkness'. 20 Abhinavagupta writes: Thus Bhairava, the Light, is self-evident (svatahsiddha); without beginning, He is the first and last of all things, the Eternal Present. And so what else can be said of Him? The unfolding of the categories of existence (tattva) and creation, which are the expansion of His own Self, He illumines, luminous with His own Light, in identity with Himself, and because He illumines Himself, so too He reflects on His own nature, without His wonder (camatkara) being in any way diminished. Although It Is Possible To Catch Glimpses Of The Highest Reality In Advanced States Of Contemplation Before Attaining Perfect Enlightenment, These States, However Long They Last, Are Transitory (Kadacitka) And When They End The Vision Of The Absolute Ceases With Them. The Highest Realisation, However, Persists In All States Of Consciousness. It Happens Once And Need Never Occur Again. A Passage From A Lost Tantra Declares: "The Self Shines Forth But Once, It Is Full [Of All Things] And Can Nowhere Be Unmanifest." All Spiritual Discipline Culminates In This Moment Of Realisation. Accordingly, Abhinava Stresses That The Goal Of All The Means To Realisation, Even The Individual Means, Is This Absolute Consciousness. Finally, It Is Worth Noting That Although Abhinava Affirms That The Teachings Concerning Anupaya Are Found In The Siddhayogesvarimata And The Malinxvijaya, Both Of Which, According To Abhinava, Are Major Tantras Of The Trika School, It Is In The Theology Of The School Of Recognition That It Is Best Exemplified. Abhinava Himself Refers To Somananda, The Founder Of This School, As Teaching It And Alludes To The Following Passage In The Vision Of Siva To Support His Own Exposition: When Siva, Who Is Everywhere Present, Is Known Just Once Through The Firm Insight Born Of Right Knowledge (Pramana), The Scripture And The Master's Words, No Means [To Realisation] Serves Any Purpose And Even Contemplation {Bhavana) [Is Of No Further Use]. Anupaya Is Therefore, According To Abhinavagupta, The Recognition Of One's Own Authentic Siva-Nature, Which All The Higher Tantric Traditions Teach Is (=) The Ultimate Realisation. This Is Also True Of (e) The Doctrine Of Vibration Whose Precedents Are (=) Clearly Traceable To (e) These Same Traditions. Thus, Although The Stanzas Themselves Never Refer Directly To (e) Enlightenment As (=) An Experience Of Recognition, There Can Be (=) Little Doubt That Spanda Practice Leads To (eb) This Same Realisation. Accordingly, Commentators Stress That We Realise The Vibration Of (e) Consciousness By Recognising Its Activity And That Liberation Depends On (=) The Recognition Of This As (=) One's Own Nature. Ksemaraja Describes (eb)What Happens In This Moment Of (e) Recognition According To (e&eb)The Doctrine Of Vibration Thus: At The End Of Countless Rebirths, The Yogi's [Psycho-Physical] Activity [Which Issues From Ignorance] Is (=) Suddenly Interrupted By (e) The Recognition Of His Own Transcendent Nature, Full Of (e) A Novel And Supreme Bliss. He Is (=) Like One Struck With Awe. And In This Attitude Of (e) Astonishment (Vismaya- Mudra) Achieves (eb) The Great Expansion [Of Consciousness] {Mahavikasa). Thus He, The Best Of Yogis, Whose True Nature Has Been Revealed [To Him] Is(=) Well Established [At The Highest Level Of Consciousness], Which He Grasps Firmly And His Hold Upon (e) It Never Slackens. Thus He Is (=) No Longer Subject To (e) Profane Existence (Pravftti), The Abhorrent And Continuing Round Of Birth And Death, Which Inspires (eb) Fear In All Living Beings, Because Its Cause, His Own Impurity, No Longer Exists. The Divine Means (Sambhavopava) In (eb) Anupaya The Yogi Does Not (e) Need To Deal With (e&eb)The World Of Diversity At All; Only Paramasiva Exists There. Beyond (e) Both Immanence And (e&eb)Transcendence, He Has (e) Nothing To Do With (e&eb) The World Of Practice And Realisation. Anupaya Is (=) The Experience Of The Undefinable (Qnakhya) Light Of Consciousness, Which Is (=)The Pure Bliss Beyond Even The Supreme State (Paratlta) Of Sivatattva. At A Slightly Lower Level, Corresponding To (e&eb) The Divine Means, A Subtle Distinction Emerges Between The Goal And (e&eb) The Path. The Yogi Now Practises Within (eb) The Domain Of The Outpouring Of (e) The Power Of Consciousness. From This Level He Penetrates Directly Into (e&eb) The Universal Egoity Of (e) Pure Consciousness By (e) The Subtle Exertion (Udyama) Of (e) Its Freedom (Svatantrya) And Reflective Awareness. The Yogi Who Practises The Divine Means Is (=) Not Concerned With (e&eb) Any Partial Aspect Of Reality But

Centers His Attention Directly On (eb) Its Abounding Plenitude. Hence This Means Is Based On (e) Siva's Own State (Sambhavavastha) In Which Only The Power Of Freedom Operates As (=) The Pure Being (Satta) Or Essence Of (e) All The Other Powers. This State Is (=) The Light Of Consciousness Which, Free Of (e) All Thought- Forms, Is The Basis Of (e) All Practice." The Yogi Who Recognizes That Pure Consciousness, Free Of Thought-Constructs (Nirvikalpa), Is His Basic State, Can Practice In (eb) Any Way He Chooses; Even The Most Common Mantra Will Lead Him (eb) Directly To The Highest State. Thus The Forms Of (e) Contemplative Absorption, Empowered (Sakta) And Individual (Anava), Which Are (=) The Fruits Of The Other Means To (e) Realisation Both, Attain Maturity In (eb) This Same Undifferentiated Awareness. This Awareness Is (=) The Pure Ego Manifest At The Initial Moment Of Perception (Prathamikalocana), When The Power Of The Will To Perceive Is Activated. (eb) It Is The Subtle State Of Consciousness That Reveals (eb) The Presence And Nature Of (e) Its Object Directly: That Which Shines And Is Directly Grasped In (eb) The First Moment Of Perception While (e) It Is Still Free Of (e) Differentiated Representations And Reflects (e&eb) Upon Itself Is [The Basis Of The Divine Means] Said To Be (=) The Will. Just As An Object Appears Directly To One Whose Eyes Are Open Without The Intervention Of Any Mental Cogitation (Anusaipdhana), So, For Some, Does Siva's Nature. The Movement Of Awareness At This Level Of Practice Attains Its Goal Quickly. While Consciousness Is Heightened Progressively In The Other Means, Here It Expands Freely To The Higher Levels, Unconfined By Any Intruding Thought-Constructs. The Divine Means Is A 'Thoughtless Thought', A 'Processless Process', That Occurs At The Juncture Between Being And Becoming. Abhinava Explains: When The Heart [Of Consciousness] Is Pure And [Free Of Thought- Constructs], It Harbours The Light Which Illumines The Radiant, Primordial Plane Ipragrabhumi) Together With All The Categories Of Existence. [The Yogi] Then Realises Through It His Identity With Siva Who Is Pure Consciousness The Yogi Must Catch The Initial Moment Of Awareness (Adiparamarsa) Just When Perception Begins^ He Must Not Move On From The First Pure Sensation Of The Object But Return To Its Original Source In His Own T Consciousness. Observing In This Way The Objective Field Of Consciousness Without Laboring To Distinguish Particulars, The Yogi Penetrates Into His Own Subjectivity Which, Vacuous And Divested Of All Outer Supports (Niralamba), Is Not Directed Anywhere Outside Itself (Ananyamukha- Preksin). Here He Can Hold Of The Power Inherent In His Own Consciousness Through Which He Discerns The True Nature Of Whatever Appears Before Him. Thus The Stanzas On Vibration Teach: Just As An Object, Which Is Not Seen Clearly At First Even When The Mind Attends To It Carefully, Becomes Later Fully Evident When Observed With The Effort Exerted Through One's Own [Inherent] Strength (Svabala), In The Same Way, When [The Yogi] Lays Hold Of That Same Power, Then Whatever [He Perceives Manifests To Him] Quickly According To Its True Nature, Whatever Be Its Form, Locus, Time Or State. Thus, Although The Practice Of This Divine Means Starts By Catching Hold Of The Will In The First Moment Of Awareness, It Also Concerns The Second And Third Moments In Which The Means Of Knowledge And The Object Are Made Manifest. When Practice At This Level Proceeds Smoothly And Without Interruption, The Three Powers Of Will, Knowledge And Action Fuse Into The Trident (Trisula) Of Power, Which Is The Subject Free Of All Obscuration (Niranjana), M At One With The Power Of Action In Its Most Powerful And Evident Form. The Kaula Schools Call This State The Stainless (Niranjanatattva). Equated In The Spanda Tradition With The Dawning Of The Vibration Of Consciousness {Spandodaya), It Is The Enlightenment The Spanda Yogi Seeks. Spanda Practice Is Based On The Experience Of Spanda Which, As We Have Seen, Is Defined As The Intent (Aunmukhya) Of Consciousness, Unrestricted To Any Specific Object And Hence Free Of Thought-Constructs. Spanda Can Therefore Be Experienced Directly When A Powerful Intention Develops Within Consciousness, Whatever Is Its Ultimate Goal Or Cause. We Have Already Noted That Intense Anger, Joy, Grief Or Confusion Is Such Occasions. Similarly, The Yogi Can Make Contact With The Omnipotent Will, Which He As Siva Possesses, Through Intense Prayer. Directing His Entire Attention To Siva, The Benefactor Of The World, Entreating Him Fervently And Without Break, His Will Merges With Siva's Universal Will, Which Is The Source Of Every Impulse And Perception. As He Looks About Him, The Yogi Realises That It Is Siva Himself, The Universal Consciousness And The Yogi's Authentic Identity, Who Ordains His Every Action, Thought And Perception. Thus The Yogi's Cognitive Intent On His Object Coincides With The Universal Will To Make That

Object Known To Him, Whether The Yogi Is Awake Or Dreaming. He Is Thus No Longer Like The Worldly Man Who Cannot Dream As He Wishes And Is Forced To Experience Whatever Spontaneously Happens In These States Of Consciousness. Ultimately The Yogi Manages, By Siva's Grace, To Maintain A Constant Awareness Of His Own Pure Perceptive Consciousness (Upalabdhrta) Divested Of All Obscuring Thought-Constructs In Deep Sleep As Well As In The Contemplative State (Turiya) Beyond It. When He Rises To The Higher Levels Of Contemplation In Which The Breath Is Suspended And All Sensory And Mental Activity Ceases, The Yogi Who Manages To Sustain This Pure, Undifferentiated Awareness Does Not Succumb To Sleep As Do Less Developed Yogis. Perfection In The Practice Of The Divine Means Thus Coincides With The Goal Of Spanda Practice, Namely, A Constant, Alert Attention To The Perceiving Subjectivity Which Persists Unchanged In Every State Of Consciousness Both As The Perceiver And Agent Of All That It Experiences. Another Important Spanda Practice Belonging To This Means Is Centering. The Spanda Yogi Seeks To Find The Centre (Madhya) Between One Cognition And The Next; For It Is There That He Discovers The Expansion (Unmesd) Of Consciousness Free Of Thought-Constructs From Whence All Differentiated Perceptions (Yikalpa) Emerge. 107 Abhinava Explains That This Pure Awareness Is Called: . . . The Expansion (Unme$a) Of [Consciousness] Or The Creative Intuition {Pratibha) [Experienced] In The Interval Which Divides Two [Moments] Of Differentiated Perception (Vikalpa). It Is Here That They Arise And Disappear. The Sastras And Agamas Proclaim With Reasoned Argument That It Is Free Of Thought-Constructs {Nirvikalpa) And Precedes All Mental Representations Of Any Object. None Can Deny That A Gap Exists Between Perceptions Insofar As Two Moments Of Thought Are Invariably Divided. This [Gap] Is The Undifferentiated Unity Of All The Countless Manifestations. Similarly, In The Outer More Objective Sphere, Where Change Consists Of The Alterations In The Configurations Of Manifest Appearances (Abhasa\ The Transition From One To Another Corresponds To A Phase Of Pure Luminosity That Marks The Beginning Of One Form And The End Of Another. The World Of Manifestation And Differentiated Perceptions (Yikalpa) Thus Extends From One Centre To The Next. Although It Is Never In Fact Divorced From The Subject Who Resides There, The Ignorant Fail To Grasp This Fact And So, Cut Off From The Centre, The World Of Objectivity Becomes For Them The Sphere Of Maya. Bhagavatotpala Quotes The Light Of Consciousness (Samvitprakdsa): This Ever Pure Experience (Suddhanubhava) Is Variegated By Each Form [Revealed Within It]; Even So It Remains Unstained (Nirmala) When Moving To Another. Just As A Cloth Which Is Naturally White, Once Dyed, Cannot Change (e&eb) Colour Without [First] Becoming White Again, Similarly The Pure Power Of Awareness, (Citi) Once Coloured By Form, Is Pure [Again] At The Centre Where That Form Is Abandoned And From Whence It Proceeds To Another. In His Essence Of Vibration (Spandasarpdoha), K§Emaraja Explains That The Rise And Fall Of Every Individual Perception In The Field Of Awareness Is A Specific Pulsation Of Consciousness. From The Point Of View Of The Object, The Expansion (Unmesa) Of This Pulse Is Represented By The Initial Desire To Perceive (Didrksa) A Particular Object, While The Contracted (Nimesa) Phase Is The Withdrawal Of Attention From The Object Previously Perceived. From The Point Of View Of The Perceiving Subjectivity, The Phases Are Reversed, So That The Initial Desire To Perceive Marks The Contraction (Nimesa) Of Subjective Consciousness While The Falling Away Of The Previous Perception Is Its Expansion (Unmesa). At The Higher Level, Where These Two Phases Are Experienced Within Consciousness, They Represent The State Of The Categories Of Isvara ('This Universe Is Me') And Sadasiva ('I Am This Universe'). Utpaladeva Says: Expansion (Unme$A\ Which Is In The External Manifestation [Of Objectivity], Is Kvaratattva While Contraction (Nime$A), Which Is In The Internal Manifestation [Of Subjectivity], Is Sadasiva. At This Level All The Powers Of Consciousness Fuse And Both Phases Are Manifest As Part Of One Reality. This Unity Is In Fact Apparent To Everybody At Each Moment. However, Within The Domain Of Maya, Which Is The Sphere Of Differentiated Perceptions (Yikalpa), It Is Clearly Manifest Only At The Juncture (Madhya) Between Two Cognitions. In This Centre Resides The Void (Kha) Of Consciousness (Free Of Thought-Constructs) Which, Divested Of Diversity, Digests Into Itself All The Psycho-Physical Processes That Give Life To The Multiplicity Of Perceptions. The Yogi Moves From The Particular Vibrations Of Consciousness At Its Periphery To The Universal Throb Of The Heart In The Centre. As Abhinava Explains: The Self-Reflective Awareness In The Heart Of Pure Consciousness, Present At The Beginning And End

Of Each Perception, Within Which The Entire Universe Is Dissolved Away Without Residue, Is Called In The Scriptures, The Universal Vibration Of Consciousness (Samanyaspanda) And Is The Outpouring (Uccalana) [Of Awareness] Within One's Own Nature. All The Categories Of Existence (Tattvas) Are United In The Heart Of The Centre Where The Life-Giving Elixir Of Siva's Consciousness Floods One's Own Inner Nature. To Reside In The Centre Is To Abide By The Law Of Totality (Gramadharma) In A State Which Transcends The Workings Of The Mind (Unmana). Consciousness (Jnana) With Light As Its Support, Residing In The Centre Between Being And Non-Being Is Known As The Act Of Abiding In One's Own Abode As The Perceiving Subjectivity (Dratfrtva) Free Of All Obscuration. That Which Has Been Purified By Pure Awareness (Suddhavijnana) Is Called The Transcendent (Viviktavastu), Said To Be The Mode Of Being (V/7//) Of The Law Of Totality (Gramadharma) Through Which Everything Is Easily Attainable. The Power In The Centre (Madhyasakti) Is The Eternal Present. Beyond Time It Is The Source Of Both Past And Future. To Be Established There Is To Abide Without A Break In Rama, The Supreme Enjoyer, In Every Action Of One's Life. Rama Is Siva, The Supreme Cause Who Pervades The Fourteen Aspects Which Embrace The Entire Universe Of Experience, Namely, Moving, Standing, Dreaming, Waking, The Opening And Closing Of The Eyes, Running, Jumping, Exertion, Knowledge [Born] Of The Power Of The Senses, The [Three] Aspects Of The Mind, Living Beings, Names And All Kinds Of Actions. By Developing An Awareness Of The Centre, The Yogi Experiences The Bliss Of Consciousness. Through This Gap He Plunges Into Introverted Absorption (Nimilanasamadhi) And Then Emerges Again To Pervade The Field Of Awareness Between Centres And So Experience The Cosmic Bliss (Jagadananda) Of The Universal Vibration Of Consciousness. 1 I8 He Then Recognises That This State Pervades Every Aspect Of Experience. In This Way The Yogi's Consciousness Is No Longer Afflicted By The Power Which Obscures It, Hemming The Centre In On Both Sides With Thought-Constructs That Seemingly Deprive It Of Its Fullness. As He Realises Directly His Pure Conscious Nature As The Universal Ego Free Of All Mental Representations, It Expands Out To Embrace All Things Within Itself. Thus The Realisation The Divine Means Leads To, And Is Directly Based Upon, Is That This Pure Ego Is In All Things Just As All Things Are Within It. In The Spanda Tradition, As Recorded In The Stanzas On Vibration, No Such Ego Is Recognised. ' I9 Man's Authentic Nature Is, However, Understood In Personal Terms As Every Individual's Own 'Own Nature* (Svasvabhava) Which Is Siva, The Universal Vibration Of Pure Subjectivity (Upalabdhfta). It Is Not Surprising, Therefore, That Later Commentators Found These Two Conceptions To Be Essentially The Same And Accordingly Identified One's Own Inner Nature With The Pure Ego. This Came As A Natural Development In Spanda Doctrine Not Only For This Reason But Also Because The Universal Ego Is Experienced As The Inner Dynamics Of Absolute Consciousness. To Conclude Our Summarial Exposition Of The Divine Means, Which Is Centred On The Direct Experience Of This Pure Ego (And Hence On Spanda In This Form), We Turn Now To A Brief Description Of Its Inner, Cyclic Activity. We Shall Do This By Examining Abhinava' S Esoteric Exegesis Of The Symbolic Significance Of The Word 'A HAM', Which In Sanskrit Means T, And Symbolises By Its Form The Ego's Dynamic Nature. The Objective World Of Perceptions Is, As We Have Seen, Essentially A Chain Of Thought-Constructs (Prapanca) Closely Linked To One Another And Woven Into The Fabric Of Diversity (Vicitrata). This Thought (Vikalpa) Is (=) A Form Of Speech (Vac) Uttered Internally By (e) The Mind (Citta), Which Is Itself An Outpouring Of Consciousness. Consciousness Also, In Its Turn, Resounds With The Silent, Supreme Form Of Speech {Para Vac) Which Is The Reflective Awareness Through Which It Expresses Itself To Itself. Consequently, The Fifty Letters Of The Sanskrit Alphabet, Which Are The Smallest Phonemic Units Into Which Speech Can Be Analysed, Are Symbolic Of The Principal Elements Of The Activity Of Consciousness. Letters Come Together To Generate Words And Words Go On To Form Sentences. In The Same Way The Fifty Phases In The Cycle Of Consciousness Represent, In The Realms Of Denoted Meaning (Vacya), The Sum Total Of Its Universal Activity (Kriya) Corresponding To (e&eb) The Principal Forces (Kala) Which Come Together To Form The Metaphysical Categories Of Experience, Which In Their Turn Appear In The Grossest, Most Explicitly 'Articulate' Form As The One Hundred And Eighteen World-Systems (Bhuvana). 'A', The First Letter Of Both AHAM And The Sanskrit Alphabet, Is The Point Of Departure Or Initial Emergence Of All The Other Letters And Hence Denotes Anuttara — The Absolute. 'Ha', Is The Final Letter Of The Alphabet And

Represents The Point Of Completion When Ail The Letters Have Emerged. It Represents The State In Which All The Elements Of Experience, In The Domains Of Both Inner Consciousness And Outer Unconsciousness, Are Fully Displayed. It Is Also The Generative, Emission (Visarga) Which, Like The Breath, Casts The Inner Into The Outer, And Draws What Is Outside Inward. The Two Letters 'A' And 'Ha' Thus Represent Siva, The Transcendental Source And Sakti, His Cosmic Outpouring That Flows Back Into Him. The Combined 'A-Ha' Contains Within Itself All The Letters Of The Alphabet — Every Phase Of Consciousness, Both Transcendental And Universal. (For A Graphic Representation Of This Analysis, See Figure 1.) M, The Final Letter Of AHAM, Is Written As A Dot Placed Above The Letter Which Precedes It. It Comes At The End Of The Vowel Series And Before The Consonants And So Is Called 'Anus Vara '(Lit. 'That Which Follows The Vowels') And Also 'Bindu ' (Lit. 'Dot,' 'Drop,' 'Point' Or 'Zero'). While The Consonant 'M' Symbolises The Individual Soul (Purusa), 'Bindu 1 Rep Resents The Subtle Vibration Of T, Which Is The Life Force (Jivakala) And Essence Of The Soul's Subjectivity Manifest At The Transcendental, Supra-Mental Level (Unmana). 12 ° It Is The Zero-Point In The Centre Between The Series Of Negative Numbers, In This Case The Vowels Which Represent The Processes Happening Internally Within Siva, And The Series Of Positive Numbers — The Consonants Which Symbolise The Processes Happening Externally Within Sakti. Bindu, As A Point Without Area, Symbolises The Non-Finite Nature Of The Pure Awareness (Pramitibhava) Of AHAM. It Is The Pivot Around Which The Cycle Of Energies From 'A' To 'Ha' Rotates, The Void In The Centre From Which All The Powers Emanate And Into Which They Collapse. As Such, It Is The Supreme Power Of Action Which Holds Subject, Object And Means Of Knowledge Together In A Potential State In The One Light That Shines As All Three Containing Them In Its Repose (Visrdnti). Bindu Is The 'Knower' (Jndtr), Who Is Essentially Consciousness That, Though Omniscient, Does Not Manifest Its Intelligence, Like A Man Who Knows The Scriptures But Having No Occasion To Explain Them To Others Silently Bears This Knowledge Within Himself. As Such, It Symbolises The Union Of Siva And Sakti (Sivasakti- Mithunapina'a) In A State Of Heightened Potency In Which They Have Not Yet Divided To Generate The World Of Diversity. It Stands, In Other Words, At The Threshold Of Differentiation In The Stream Of Emanation Still Contained Within Siva. The Absolute. Expansion Commences Bindu — The Individual Soul Withdrawal Commences Then, To The Degree In Which That Which Is To Be Accomplished By The Power Of Action Residing Within It [As A Potential] Penetrates Into The Absolute, It Appears Initially As Bindu, Which Is The Light Of Pure Consciousness. When Outer Objectivity Is Reabsorbed Into Its Transcendent Source, Bindu Is The Point Into Which All The Manifest Powers Of Consciousness Are Gathered And Fused Together. The Universal Potency Of All The Letters Is Thus Contained In Bindu Which, As The Reflective Awareness Of Supreme T Consciousness, Gives Them All Life. Thus Bindu Also Marks The Beginning Of Siva's Internal Movement Back To The Undifferentiated Absolute And So Stands At The Threshold Of Both Emission And Absorption Without Being Involved In Either. The Three Aspects Of AH AM Together Constitute A Movement From The Undifferentiated Source Of Transcendental Consciousness — 'A' — Through The Expansion Or Emission Of Its Power — 4 Ha' — To The Subject — 'M 1 —Which Contains And Makes Manifest The Entire Universe Of Experience. The Reverse Of This Movement, That Of Withdrawal (Samhara), Is Represented By M-Ha-A. AHAM And M-Ha-A Alternate In The Rotation (Ghurnana) Of The Reflective Awareness Of T Consciousness As Immanent Sakti Emerges From Transcendental Siva To Then Merge Back Into Him. As Abhinava Says: The Universe Rests Within Sakti And She On The Plane Of The Absolute (Anutiara) And This Again Within Sakti ... For The Universe Shines Within Consciousness And [Consciousness Shines] There [Within The Universe By The Power Of] Consciousness. These Three Poles, Forming A Couple And Merging, Make Up The One Supreme Nature Of Bhairava Whose Essence Is AHAM- At The Microcosmic Level, Fc A' Represents The Initial Moment When The Subject Begins To Rise Out Of Himself To View The Object. The Movement From 'A To 'Ha' Marks The Emergence Of Sensation Within The Field Of Awareness, Which Is Represented By The Fifty Letters Of The Alphabet Symbolic Of The Fifty Aspects Of The Flux Of Consciousness Leading To Objectified Perception. 'Nl' Is Tne Subject Who, Resting Content Within Himself When He Has Perceived His Object Merges Through The Inner Flow Of Awareness Into K A\ The Absolute. Then From The Absolute (A) Its Emission (Ha) Flows Back Into The Pure Subject (M) Set To Perceive His Object. Thus All The

Cycles Of Creation And Destruction Are Contained Within AHAM Through Which They Are Experienced Simultaneously As The Spontaneous Play Of The Absolute. The Yogi Who Recognises This Recurrent Pulse Of Awareness To Be The Movement Of His Own Consciousness Merges His Limited Ego With The Universal Ego. Thus He Realises That Its Power To Create, Sustain And Destroy All Things Is His Own Inner Strength (Svabaia) That He Exerts Effortlessly In The Same State Of Mystical Absorption (Turlya) In Universal Consciousness That The Absolute Itself Enjoys. In This Way He Shares In The Three-Fold Awareness Siva Himself Has Of His Own Nature Which Abhinava Describes As Follows: I Make The Universe Manifest Within Myself In The Sky Of Consciousness. I, Who Am The Universe, Am Its Creator! ' — This Awareness Is The Way In Which One Becomes Bhairava. 'AH Of Manifest Creation(Sadadhvari) Is Reflected Within Me, I Cause It To Persist 1 — This Awareness Is The Way In Which One Becomes The Universe. The Universe Dissolves Within Me. I Who Am The Flame Of The [One] Great And Eternal Fire Of Consciousness' — Seeing Thus One Achieves Peace. The Experience Of The Liberated Thus Coincides With The Realisation Of Their Own Divine Nature Which, Through Its Power, Rules And Guides The Cosmic Order. Thus This Attainment (Siddhi), Which Is Liberation Itself, Is In The Doctrlie Of Vibration Technically Called 'Mastery Over The Wheel Of Energies' (Cakresvaratvasiddhi) Because The Liberated Soul, Identified With Siva, Now Governs, As Does Siva, The Cycle Of The Powers That Bring About The Creation And Destruction Of All Things, The Empowered Means (Saktopaya) All The Practices Taught In The Stanzas On Vibration Are Internal. Whenever Ritual Is Mentioned, It Is Invariably Interpreted In Terms Of The Dynamics Of The Inner Processes The Yogi Experiences And Implements In The Course Of His Yogic Practice. The Doctrine Of Vibration, Ksemaraja Affirms, Is Concerned Entirely With These Inner Disciplines Centred, As It Is, In One Way Or Another, On Consciousness Or, At Least, On The Inner Activity Of The Mind. Thus The Empowered Means Which, Like The Other Categories We Have Discussed, Is Entirely Internal Includes An Important Part Of Spanda Practice. Spanda Practice Belonging To The Divine Means Centres On One's Own Inherent Nature (Svasvabhava) As Siva, The Universal Perceiver And Agent, That Belonging To The Empowered Means On His Power Instead Of Arriving Directly At The All-Embracing Emptiness Of Subjective Conscious- Ness, The Yogi Practising The Empowered Means Realises His True Nature Through The Fullness Of Its Energy. Practising The Divine Means, The Yogi Plunges, As It Were, Straight Into The Fire Of Consciousness; Practising The Empowered Means He Merges With Its Rays. Either Way The Yogi Is Centred Equally On Ultimate Reality. The Power Of Consciousness Is No Less Absolute Than Its Possessor. To Make This Point Abhinava Quotes The Matanga- Tantra: This Reality Consists Of The Rays Of [Siva's] Power And Is Variously Said To Be The Abode Of The Lord's Manifestation . . . That Same [Power] Illumined [By Siva] Is Itself Also Luminous, Unshaken And Unmoving. That Very [Power] Is The Supreme State, Subtle, Omnipresent, The Nectar Of Immortality, Free Of Obscuration, Peaceful, Yearning For Pure Being Alone {Vastumatra) And Devoid Of Beginning And End. Perfectly Pure, It Is Said To Be The Body [Of Ultimate Reality]. The Yogi Concentrates On The Powers Operating In All Of Life's Activities As Particular Pulsations (Visesaspanda) In The Universal Rhythm (Samanyaspanda) Of The Power Of Consciousness. In This Way He Rises Progressively From The Particular To The Universal Until He Reaches Pure Being (Satta), The Greatest Of All Universals (Mahasamanyd) And The Highest Form Of Siva's Power. Thus The Creative Power Of Maya, Manifest Through Countless Lesser Powers, No Longer Causes The Yogi To Stray From Siva's Consciousness But Becomes The Means Through Which It Can Be Realised In The Illuminating Brilliance (Sphuratta) Which Is Siva's Pure Being. Thus By Discovering The True Nature Of Sakti, The Yogi Realises Himself To Be Siva, Its Possessor Who Consists Of All Its Countless Powers. Thus Practise Belonging To This Means Leads To The Same Pure Consciousness Free Of Thought-Constructs Realised Through The Divine Means. Although The Ultimate Realisation Is Instantaneous, The Yogi Rises To It Gradually By Freeing His Consciousness Of The Limitations Imposed Upon It By Thought. Abhinava Explains: The Same Occurs In The Empowered Means [As Does In The Divine]. At The Discursive Level Of Consciousness {Yaikalpikibhumi) [Where The Empowered Means Functions] Knowledge And Action, Although Evident, Are, For The Reasons Explained Previously, Contracted. A Blazing Energy [Is Revealed Within] The One Who Dedicates Himself To Removing The Burden Of This Contraction. [This Energy Eventually] Brings About The Inner Manifestation {Antarabhasa) Of Pure

Consciousness He Seeks. Consciousness Is Individualized And Its Power Of Knowledge And Action Contracted By The Thought-Constructs Born Of Ignorance. The Arising Of These Mental Representations, As The Stanzas On Vibration Say, Deprives The Soul Of Its Freedom And Immortal Life. The Practise Of The Empowered Means Is Meant To Free The Fettered Soul Of This Constriction On His Consciousness. It Operates Within The Mental Sphere (Cetasy™ And Is Designed To Purify Thought (Vikalpasamskara) In Order To. Reveal The Pure Consciousness Which Is Its Ground And Ultimate Source. Thus, The Empowered Means Is Concerned With The Second Instant Of Perception, During Which The Subject Forms Mental Representations Of His Object. Thought Functions On The Basis Of An Awareness Of Relative Distinctions Between Specific Particulars, Distinguishing Them From One Another And Thus Seemingly Fragmenting The Essential Unity Of Reality. The Vibrant Vitality Of Consciousness, Universally Manifest, Is Clouded Like A Mirror By A Child's Breath And The Soul Is Deprived Of The Liberating Intuition Of The One Reality Free Of Thought-Constructs (Nirvikalpa). Abhinava Writes: The [Fettered Soul] Is Like A Dancing Girl Who Although Wishing To Leave The Dancehall Is Collared By The Doorkeeper Of Thought And Thrown Back Onto The Stage Of Maya All Thought Is Centred On Objectivity And Hence Dislodges Awareness From The Plenitude Of Pure Subjective Consciousness. Thus, To Regain The Original State Of Rest (Visranti) Consciousness Enjoys, The Yogi Must Rid Himself Of Thought. As Thought-Forms Decrease, Pure, Thought-Free Awareness Is Strengthened Until The Yogi Is Fully Established In A State In Which The Relative Distinctions (Bheda) Conceived Between Entities Dissolve Away. Everything Appears To Him As Pure Being (Sattdmdtra) L} And The Entire Universe Shines Before Him Pervaded By Siva's Radiance. His Intuitive Faculty (Mati) Thus Purified, The Yogi Gains Both The Perfections (Siddhi) Of Yogic Practice And Liberation (Mukti). His Consciousness Is Now Like A Well-Polished Mirror Which Reflects Everything He Desires And Grants It To Him. Abhinava Writes: Just As A Man Who Has Been Ill For A Long Time Forgets His Past Pain Completely When He Regains His Health, Absorbed As He Is In The Ease Of His Present Condition, So Too Those Who Are Grounded In Pure Awareness Free Of Thought-Constructs Are No Longer Conscious Of Their Previous [Fettered] State. Consciousness, The Sole Truly Existent Reality, Free Of Thought- Constructs Is Made Fully And Evidently Manifest By Eliminating These Differentiated Perceptions. The Wise Man Should Therefore Exert Himself To Attend Closely To This [State Of Awareness]. The Thought-Constructs Generated Within Consciousness Do Not In Reality Affect It At All. They Can Neither Break Up Nor Add Anything To The Light Which Shines As All Things. They Are In Fact Nothing But Consciousness Itself Which Perceives, Through Its Power Of Reflective Awareness (Yimarsa), The Multitude Of Objects In Diverse Ways, And So Assumes This Form. Although Thought-Constructs Are Mental Representations Of Objects Once Seen Or Present, They Are Products Of The Power Of Consciousness And Not Of The Objects They Represent. Thought Is Both Analytic And Synthetic; It Serves The Useful Purpose Of Separating Individual Elements Of Experience From Others And Linking Together Those That Appear To Be Distinct From One Another So That They Can Be Better Understood. It Does Not Consist Merely Of False Mental Constructs Projected Onto Reality That Need To Be Wholly Rejected. Thought Obscures Consciousness And Distracts It Only When It Appears In The Form Of Doubt, Vacillating Between Alternatives. Once This Conflicting Duality (Dvaitddhivasa) Is Eliminated, Thought Is Purified And Rests In Itself As The 'Thought-Less Thought' Of Pure Consciousness. By Gradually Eliminating The Multitude Of Conflicting Notions That Agitate Him, The Yogi Ultimately Achieves The Certainty (Niscaya) Corresponding To A Direct Awareness Of His Own Divine Nature. Abhinava Explains: Thought Is In Reality None Other Than Pure Consciousness. Even So, It Serves As A Means To Liberation For The Individual Soul (Anu) Only When It Takes The Form Of Certainty (Niscaya). This surprising result--that information capacity depends on surface area—has (e) a natural explanation if the holographic principle (proposed in 1993 by Nobelist Gerard't Hooft of the University of Utrecht in the Netherlands and elaborated by Susskind) is true. In the everyday world, a hologram is a special kind of photograph that generates a full three-dimensional image when it is illuminated in the right manner. When it com,es to structures,one cannot bypass Deleuze and his work. Levi Strauss Has Also Noticed That Signs Always Offer An Excess. The System Of Language, 'The Order Of The Known' Exceeds Actual Speech, Even Attempts At Totalization (48). Laws Pre-Exist Actual Cases. [So We Were Getting Close To A Role For Social Life, But Then

It Gets Metaphysical Again]. As LS Put It, The Universe Signified Long Before Human Beings Knew What It Was Signifying. By Contrast, The Domination Of Nature Proceeds Partially And Progressively, Step By Step Unlike Social Life Where All Its Goals And Possibilities Given At Once. We're Back With Two Series, This Time Conceived As Rhythms, Social And Natural. Both Technocrats And Dictators Attempt A False Synthesis Of These Two Rhythms. Levi Strauss Referred To 'The Floating Signifier' As A Creative Force And Deleuze Wants To Say It's 'The Promise Of All Revolutions' (49). There Are Also 'Floated Signifieds', Which Seem To Be Possibilities Which Have Not Yet Been Realized. These Can Fill The Gap Between Signifier And Signified [And Are Found In Common Sense Expressions Like 'Gadgets' Or 'Whatnot' – Maybe Connected To The Idea Of A Bricoleur?]. It Implies A Symbolic Content, But Does Not Attempt To Fill It With Specifics. Together, These Possibilities Constitute A Structure, Two Heterogeneous Series, One Signifying, One Signified, Interdependent, And Including Particular Events, Singularities, Emitted By A Differentiator. The Singularities Belong To Neither Series Exclusively And Thus Have No Coherent Identity –Each Is An Excess In One Series And A Lack In The Other. The Singularities Can React Back On The Series, So Structures And Events Are Interdependent [So Structures Need A Dynamic Element, And, D Argues, An Excess, An Empty Square Instead Of Total Systematic Closure. Addresses The Old Issue Of The Static Nature Of Structuralism]. The Signifying Series Contains A Series Of Ideal Events, An Internal History. Differentiators Articulate Series And This Produces A 'Tangled Tale' Overall (51). **Sense Can Be Found In Either Series. It Is Not Just Signification But The Relation That Produces Signifier And Signified**, [The Operation Of The Whole Structure In This Expanded Sense]. Alexander I. Stingl's Blog A Nomadic Scholarship Entity NOTES ON: Deleuze, G (1990) The Logic Of Sense, Trans Mark Lester, Edited By Constantin Boundas, New York: Columbia University Press Attributable And Ascribable To Constraints On Space Accomodation Model Is Provided In Some Paper, Notwithstanding The Generality And Commonalty Of The Observation And Its Concatenatability With The Other Modules.All the information describing the 3-D scene is encoded into the pattern of light and dark areas on the two-dimensional piece of film, ready to be regenerated. The holographic principle contends that an analogue of this visual magic applies to the full physical description of any system occupying a 3-D region It proposes that another physical theory defined only on the 2-D boundary of the region completely describes the 3-D physics. If a 3-D system can be fully described by a physical theory operating solely on its 2-D boundary, one would expect the information content of the system not to exceed that of the description on the boundary. Can we apply the holographic principle to the universe at large? The real universe is a 4-D system It has volume and extends in time. If the physics of our universe is holographic, there would be an alternative set of physical laws, operating on a 3-D boundary of spacetime somewhere, which would be equivalent to our known 4-D physics. We do not yet know of any such 3-D theory that works in that way. Indeed, what surface should we use as the boundary of the universe? One step toward realizing these ideas is to study models that are simpler than our real universe. A class of concrete examples of the holographic principle at work involves so-called anti-de Sitter spacetimes **.** The original de Sitter spacetime is a model universe first obtained by Dutch astronomer Willem de Sitter in 1917 as a solution of Einstein's equations, including the repulsive force known as the cosmological constant. De Sitter's spacetime is empty, expands at an accelerating rate and is very highly symmetrical. In 1997 astronomers studying distant supernova explosions concluded that our universe now expands in an accelerated fashion and will probably become increasingly like a de Sitter spacetime in the future. Now, if the repulsion in Einstein's equations is changed to attraction, de Sitter's solution turns into the anti-de Sitter spacetime, which has equally as much symmetry. More important for the holographic concept, it possesses a boundary, which is located "at infinity" and is a lot like our everyday spacetime. Using anti-de Sitter spacetime, theorists have devised a concrete example of the holographic principle at work: a universe described by superstring theory functioning in an anti-de Sitter spacetime is completely equivalent to a quantum field theory operating on the boundary of that spacetime Thus, the full majesty of superstring theory in an anti-de Sitter universe is painted on the boundary of the universe. Juan Maldacena, then at Harvard University, first conjectured such a relation in 1997 for the 5-D anti-de Sitter case, and it was later confirmed for many situations by Edward Witten of the Institute for Advanced Study in Princeton, N.J., and Steven S. Gubser, Igor R. Klebanov and Alexander M. Polyakov of

Princeton University. Examples of this holographic correspondence are now known for spacetimes with a variety of dimensions. This result means that two ostensibly very different theories--not even acting in spaces of the same dimension--are equivalent. Creatures living in one of these universes would be incapable of determining if they inhabited a 5-D universe described by string theory or a 4-D one described by a quantum field theory of point particles. (Of course, the structures of their brains might give them an overwhelming "commonsense" prejudice in favor of one description or another **Holographic Space-Time (Crystal Links, Tom Hanks and Wikipedia Francisco Di Biase** propose a quantum-informational holographic model of brain-consciousness-universe interactions based in the holonomic neural networks of Karl Pribram, in the holographic quantum theory developed by David Bohm, and in the non-locality property of the quantum field described by Hiroomi Umezawa. He considers this model an extension of the interactive dualism of Sir John Eccles, of an interconnection between brain and spirit by means of quantum microsites named dendrons and psychons. **Francisco Di Biase** proposes a dynamic concept of consciousness seen as a holoinformational flux interconnecting the holonomic informational quantum brain dynamics, with the quantum informational holographic nature of the universe. This self-organizing flux is generated by the holographic mode of treatment of neuronal information and can be optimized through practices of deep meditation, prayer, and others states of higher consciousness that underlie the coherence of cerebral waves. In brain mapping studies performed during the occurrence of these harmonic states we can see the spectral array of brain waves highly synchronized and perfectly ordered like a unique harmonic wave, as if all frequencies of all neurons from all cerebral centers played the same symphony. This highly coherent brain state generates the non-local holographic informational cortical field of consciousness that interconnect the human brain and the holographic cosmos. The comprehension of this holonomic quantum informational nature of brain-consciousness-universe interconnectedness allows us to solve the old mind-matter Cartesian hard problem, unifying science, philosophy, and spiritual traditions in a more transdisciplinary, holistic, integrated paradigm. In this new arrangement cosmovision, consciousness and transpersonal phenomena becomes part of Science and of the very holoinformational nature of the Holographic Conscious Multiverse. **DOI: 10.14704/nq.2009.7.4.259 Quantum-Holographic Informational Consciousness Francisco Di Biase NeuroQuantology** Holography suggests a considerable reduction of degrees of freedom in theories with gravity. However it seems to be difficult to understand how holography could be realized in a closed re-contracting universe. In this Letter we claim that a scenario which achieves that goal will eliminate all spatial degrees of freedom. This would require a different concept of quantum mechanics and would imply an intriguing increase of power for the natural laws. **Physics Letters B Volume 451, Issues 1–2, 1 April 1999, Pages 19–26 Richard Dawid doi:10.1016/S0370-2693(99)00205-1Vijay Balasubramanian, Per Kraus, Albion Lawrence, and Sandip P. Trivedi** describe probes of anti–de Sitter spacetimes in terms of conformal field theories on the AdS boundary. Basic tool is a formula that relates bulk and boundary states—classical bulk field configurations are dual to expectation values of operators on the boundary. At the quantum level they relate the operator expansions of bulk and boundary fields. Using our methods, we discuss the CFT description of local bulk probes including normalizable wave packets, fundamental and D-strings, and D-instantons. Radial motions of probes in the bulk spacetime are related to motions in scale on the boundary, demonstrating a scale-radius duality. They also discuss the implications of these results for the holographic description of black hole horizons in the boundary field theory.DOI: http://dx.doi.org/10.1103/PhysRevD.59.104021 **Holographic probes of anti–de Sitter spacetimes Phys. Rev. D 59, 104021 – Published 26 April 1999 Vijay Balasubramanian, Per Kraus, Albion Lawrence, and Sandip P. Trivedi** The existence of a fundamental scale, a lower bound to any output of a position measurement, seems to be a model-independent feature of quantum gravity. In fact, different approaches to this theory lead to this result. The key ingredients for the appearance of this minimum length are quantum mechanics, special relativity and general relativity. As a consequence, classical notions such as causality or distance between events cannot be expected to be applicable at this scale. They must be replaced by some other, yet unknown, structure. **LUIS J. GARAY, Int. J. Mod. Phys A, 10, 145 (1995) DOI: 10.1142/S0217751X95000085 QUANTUM GRAVITY AND MINIMUM LENGTH** A digital computer is generally believed to be an efficient universal computing device; that is, it is believed able to simulate any physical

computing device with an increase in computation time by at most a polynomial factor. This may not be true when quantum mechanics is taken into consideration. This paper considers factoring integers and finding discrete logarithms, two problems which are generally thought to be hard on a classical computer and which have been used as the basis of several proposed cryptosystems. Efficient randomized algorithms are given for these two problems on a hypothetical quantum computer. These algorithms take a number of steps polynomial in the input size, e.g., the number of digits of the integer to be factored. Copyright © 1997 Society for Industrial and Applied Mathematics **SIAM J. Comput., 26(5), 1484–1509. (26 pages) Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer ISSN (print): 0097-5397 ISSN (online): 1095-7111 Publisher: Society for Industrial and Applied Mathematics Peter W. Shor DOI: 10.1137/S0097539795293172** The classical dynamics of M-dimensional extended objects arising from stationary points of the world volume swept out in space time is discussed from various points of view. A introduction to the Hamiltonian mechanics of bosonic compact M (em) branes is given, emphasing the diversity of the different formulations and gauge choices. For moving hypersurfaces, a graph description—including its nonlinear realization of Lorentz invariance—and hydrodynamic formulations (in light-cone coordinates as well as when choosing the time coordinate of a Lorentz observer as the dependent variable) are presented. A matrix regularization for M = 2 (existing for all topologies) is explained in detail for the 2-sphere, as well as multilinear formulations for M > 2. The recently found dynamical symmetry that exists for all M and related reconstruction algebras are covered, just as some explicit solutions of the level-set equations. **Jens Hoppe 2013 J. Phys A: Math. Theor. 46 023001 doi:10.1088/1751-8113/46/2/023001 Relativistic membranes N Bodendorfer et al** rederive the results of our companion paper, for matching space–time and internal signature, by applying in detail the Dirac algorithm to the Palatini action. While the constraint set of the Palatini action contains second class constraints, by an appeal to the method of gauge unfixing, we map the second class system to an equivalent first class system which turns out to be identical to the first class constraint system obtained via the extension of the ADM phase space performed in our companion paper. Central to analysis is again the appropriate treatment of the simplicity constraint. Remarkably, the simplicity constraint invariant extension of the Hamiltonian constraint, that is a necessary step in the gauge unfixing procedure, involves a correction term which is precisely the one found in the companion paper and which makes sure that the Hamiltonian constraint derived from the Palatini Lagrangian coincides with the ADM Hamiltonian constraint when Gauß and simplicity constraints are satisfied. **N Bodendorfer et al** therefore have rederived new connection formulation of general relativity from an independent starting point, thus confirming the consistency of this framework. **N Bodendorfer et al 2013 Class. Quantum Grav 30 045002 doi:10.1088/0264-9381/30/4/045002 new variables for classical and quantum gravity in all dimensions: II. Lagrangian analysis** As K§emaraja points out, none of the practices taught in the Stanzas on Vibration belong to the Individual Means and so it does not, (e) strictly speaking, concern Spanda doctrine, if that is, we consider the Stanzas to be the basic text of the Spanda school. From K§emaraja's point of view, however, the third section of the Aphorisms of Siva (Sivasutra), which is both the last and most extensive, is (=) largely an exposition of this category of practice. The Stanzas and Aphorisms have been traditionally linked together (e&eb) and so, even though we feel that they should be distinguished sofar as the Stanzas rather than the Aphorisms teach the Doctrine of vibration as such, we are nonetheless justified in referring to the Aphorisms as (=) its major source. Our exposition of the Individual Means will therefore be largely based on (e) Ksemaraja's interpretation of (e) the third section of Aphorisms and we will present it, as he does, as an exposition of a possible mystical journey of individualised (anava) consciousness to realisation. We follow K§emaraja because he understood the practise taught in the Aphorisms in these terms, thereby not only illustrating for us how it fits into this scheme but also how he understood the basic categories of practice and their relationship to one another. According to K§emaraja, the first Aphorism of each section of the Sivasutra characterises the condition and nature of the Self at the corresponding three levels of practice. In other words, they indicate the yogi's basic state at each level in terms of his self-identification. This identification corresponds to his existential condition as a degree of self- realisation in the process leading to the authentic self-awareness of the liberated. The very first Aphorism starts directly with this, the highest state, by declaring that the Self is pure, dynamic and universal consciousness

(caitanya).This is true for the yogi who has awakened to his authentic nature at the Divine (sambhava) level of being. At the Individual (anava) level, however, the situation has changed. In this sphere of consciousness the intermediate processes of discernment, analysis and classification of perceptions, which bridge the gap in the flow of awareness from the universal subject to a specific object of knowledge, appear to take over the status of the perceiving subjectivity which underlies them. The universal Self recedes into the background as a pure, undefinable awareness, and the individual ego, consisting of the perceptions, thoughts and emotions generated by the contact between the universal perceiver and the perceived, emerges in the juncture between them. Thus at this level, as the Aphorisms say, the Self is the mind. This is the Self which moves (atati) from one state of being to another, from one body to the next carrying with it subtle traces left behind by its sensory and mental activity. Together these are said to constitute, and be caused by, the subtle body technically called the 'City of Eight' (puryasfaka) with which consciousness is identified and due to which it is subject to the constant alterations of pleasure, pain and inertia. The Stanzas teach: [The soul] is bound by the City of Eight (puryaffaka) that resides in the mind, intellect and ego and consists of the arising of the [five] subtle elements [of sensory perception]. He helplessly suffers worldly pleasure and pain (bhoga) which consists of the arising of mental representations born of that [City of Eight] and so its existence subjects him to transmigration. **Whereas consciousness itself is (=) the subject who practises the Divine Means (Sambhavopayd), the subject who practises the Individual Means is (=) the mind. Unlike the Empowered Means, however, the mind is not (e) directed inwards onto (e&eb) itself. At the Empowered level, enlivened by (e) the direct intuition (pratibha) consciousness has (e) of its own nature, mind ceases to (e) function merely in the paradigmatic, formative manner which gives rise to (eb) mental representations, but operates instead as (=) the subtle introverted activity of (e) reflective awareness (vimarsa), the power of consciousness (sakti).** This activity, as we have seen, is the essence of Mantra which, independent of the senses, is no longer restricted (e&eb) in any way. At the Individual lew, however, the creative powers of consciousness reflected through the eAtroverted mind are greatly attenuated. All that remains is the power to form thought-constructs and make determined resolutions (sankalpa) which go on to issue through the body into outer action to make the private creations of the mind apparent to others. The Individual Means, therefore, deals with the objectively perceived contents of consciousness and hence with the individual subject as a composite aggregate of objective elements, ranging from the subtle life force (prdrta) to the physical body and its outer environment. The practices belonging to this Means are thus of two types. One is concerned with the individual subject who resides in, and as, the psycho-physical organism; the other with external reality. What this implies essentially is that practice at this level is not concerned as much with the will or cognitive consciousness **(refer Hiedigger who proposes the same philosophy: italics mine**) as are the other two Means, but with the power of action applied, in the context of the practice taught in the Aphorisms, to (e&eb) the spiritual activity of Yoga. According to Ksemaraja, the Individual Means culminates in (eb) the Empowered state and hence leads to (eb) the levels of practice beyond it. 251 This is possible because (e) despite their differences, there is (=) an essential similarity between them. The aim of both the Individual and Empowered Means is (=) to purify the discursive representa- tions of (e) differentiated perceptions (vikalpasamskara) 252 and so lead (eb) the yogi to the expanded (yikasita) consciousness of (e) the Divine (sambhava) state. The other levels of practice therefore both sustain and complement (eb&eb+) it. The activity of individual consciousness can be fully perfected only when it operates through the flow of the conative and cognitive powers which together constitute the pure activity of universal consciousness beyond all means (anupaya). In fact, according to Ksemaraja, all three soteriological types function together in various ways, their corresponding states representing dimensions of the same experience. For example, the upsurge of consciousness (udyama) which is the supreme, **illuminating intuition (parapralibha)** of the Divine state (sambhavavastha) is concomitant with the gathering together of all the powers of consciousness in the Empowered state. The Divine Means, in other words, leads to the experience of Power (sakti) which in its turn, when fully affirmed, marks the attainment of a permanent contemplative consciousness (turiyatita) at the Divine level which persists unaltered in every state of consciousness. Consequently, Ksemaraja concludes his exposition of the first section of the Aphorisms which exemplifies, according to him, the Divine Means, by saying: Thus we have explained the first

expansion which starts with [the Aphorism] 'the Self is pure dynamic consciousness' **(note this happens in evolutionary pocess and in normal human being it remains as witness consciousness italics mine)** and expounds the nature of the realisation (prathana) attained through the Divine Means. It is the intuitive insight (samapatti) of Bhairava's nature which is, as we have said, the upsurge of consciousness that quells all bondage, namely, the ignorance of that freedom which makes it manifest. Transforming all things into the nectar of one's own innate bliss, it bestows every yogic accomplishment (siddhi) including mystic absorption in the vitality of Mantra, the highest of them all. Accordingly, we have, in the course of this exposition, explained the nature of Sakti in order to show that the Divine nature (Sambhavarupa) possesses [every] power. Another way in which the Means are related to (e&eb) one another is illustrated by the recurrence of the same Aphorism in different sections of the Sivasutra which indicates, according to Ksemaraja, that the same practice belongs to (e) more than one Means. Both times this happens, the Aphorism appears first in the section dealing with (e&eb) the Divine Means and then recurs in that concerned with (e&eb) the Individual Means. In one case, Ksemaraja tells us this is because practice at the Divine level requires no effort whereas at the Individual level, the yogi must exert himself to achieve the same state that at the Divine level dawns spontaneously. At the Empowered level also, as the Sivasutra says, 'effort achieves the goal'. Here, however, because as the Empowered Means is, according to Ksemaraja, predominantly concerned with the contemplation (anusamdhi) of the vitality of Mantra, the effort exerted is that required to bring the practice of Mantra to fulfillment. It is, as Ksemaraja says, 'the spontaneous effort exerted to grasp the initial expansion of intention to apply oneself to the contemplation [of Mantra]. It is this exertion which wins the favour of the gods of Mantra and identifies the adept with them.' The second case of the same practice being taught in different sections of the Aphorisms concerns the realisation of the Fourth State of contemplative consciousness (turiya) in the other three states of waking, dreaming and deep sleep. At the Divine level this takes place by **'violently digesting' (hafhapaka)** the three states in the Fourth. At the Individual level the Fourth state is first experienced at the junctures between the other three states and then induced gradually to spread out from these Centres to pervade the other states like oil extending slowly through a piece of cloth. The difference in this case between the levels of practice is not only that at the Divine level it reaches fulfillment spontaneously, but it is also sudden and complete, leading directly to the liberated state of consciousness Beyond the Fourth (turiyatita) At the Individual level, however, practice is gradual and even when the yogi manages to rise to states of contemplation, he must take care not to fall to lower levels of consciousness. Indeed, until the yogi attains the sudden and direct realisation of perfect enlightenment, whatever be his state of consciousness or level of practice, he is bound to rise and fall because his contemplative state is necessarily transitory (kadacitka) however long it may last. The yogi is more prone to these ups and downs the lower his basic state of consciousness. Consequently, the last section of the Sivasutra repeatedly instructs the yogi not only how to rise to higher levels of consciousness and maintain them, but also in what way he is liable to fall from them and how to regain them. Ksemaraja stresses that the rise from one level of consciousness to another is marked by the transition from a lower Means to a higher. Conversely, a fall from the higher level to the lower entails practice of a lower Means. The measure of the yogi's level of consciousness and that which sustains him in it allowing him to progress further, is his attentiveness (avadhana) to the higher realities he experiences in the more elevated states. Thus the last Aphorism of the second section of the Sivasutra warns the yogi that if his pure awareness (suddhavidya) of his oneness with all things slackens, he will fall from his awakened state to dream the dream of thought-constructs. From Ksemaraja's point of view this means that the negligent yogi must now resort to the Individual Means described in the next section to return to his former, higher Empowered practice in which he experiences this oneness. Ksemaraja expounds practice at the Individual level, as he sees it in the Aphorisms, as extending from one Means to the next. For example, practice at the Individual level diverts the flow of the vital breath (prana) from its more usual course and induces it to enter the Central Channel (susumna) along which it rises as a pure conscious energy (technically called 'kuritfalini'). This leads the yogi to the Empowered state in which he enjoys the pure awareness of unity. If he manages to make it truly his own and it becomes his basic state of being, he enters the Divine plane (sambhavapadd) of identity with Siva. The Individual Means is both a point of departure to higher levels of practice and the level to which the yogi returns if he falls. Thus although the practices taught in the

last section of the Aphorisms may belong to any one of the three Means, they are collectively treated as part of the Individual Means because they start from it and because it is the yogi's abiding standby if he falls. Let us turn now to the basic practice at the Individual level, as Ksemaraja understands it. This is essentially Yoga. According to the Classical Yoga system taught by Patanjali in the Aphorisms of Yoga (Yogasutra), Yoga is defined as 'the quelling of the fluctuations of the mind' (cittavrttinirodha). The aim is to sever the spiritual essence of the Person (purusa) from the defiling materiality of Nature (prakfti), even though the word 'Yoga' means to 'unite' or 'yoke together.' Here, however, Yoga combines both union and cessation. It is the act (kriya) of removing the latent traces (yasana) of differentiated perceptions (yikalpa) born of the impurities (mala) which contract consciousness **(like pravrutti and Nivrutti; italcis mine**). This is achieved by uniting all the elements of experience (tattva) together in the wholeness of the activity of consciousness. As Jayaratha explains: The [wise] consider Yoga to be the union of one thing with another/ thus, in accord with this dictum, Yoga is the [act] of uniting [all] the metaphysical principles together within consciousness . . . , Ksemaraja seeks initially to establish the best form of Yoga for the yogi to practice at the Individual level. His sources are two Tantras he knew well and considered to be amongst the most important, namely, The Tantra of (Siva's Third) Eye (Netratantra) and The Tantra of the Liberated Bhairava (Svacchandabhairavatantra). The basic model is that of the Eight-limbed Yoga (asfariga) taught by Patanjali which consists of: 1) The five restraints (yama\ namely, abstention from violence (ahirrisa), falsehood (satya), dishonesty (asteya), sexual intercourse (brahmacarya) and desire for more than the essential (aparigrahd) 2) The five disciplines (niyama), namely, cleanliness (sauca), contentment (santo$a), austerity (tapas\ study (svadhyaya) and reverence for God (Uvarapranidhand) 3) Posturing of the body (asana) in a manner conducive to the practice of meditation and physical health. 4) Regulation of the breath (pranayama) 5) Withdrawal of the senses from their objects (pratyahara) 6) Focusing of attention (dharana) 7) Meditation (dhyana), that is, steady, uninterrupted concentration. 8) Contemplation (samadhi). Ksemaraja **rejects** Patanjali's system because he believes it to be a form of Yoga that can, at best, lead only to limited yogic attainments (mitasiddhi). In the Netratantra, however, Siva teaches a different, higher form of the Eight Limbs of Yoga which lead to perfect penetration into the supreme, transcendental principle 280 of which the Netratantra says: Speech cannot express, nor the eye see, the ears hear, or the nose smell, the tongue taste, the skin touch or the mind conceive that which is eternal. Free of all colour and flavour, endowed with all colours and flavours, it is beyond the senses and cannot be objectively perceived. O goddess, those yogis who attain it become immortal gods! By great practice and supreme dispassion . . . one attains Siva, the supreme imperishable, eternal and unchanging reality. A necessary preliminary of all Tantric Yoga is a process technically called the 'purification of the elements' (bhutasuddhi), through which the body is homologized with the macrocosm and so made a fit vessel for the pure, conscious presence of the Deity within it. Ksemaraja equates this with the meditation (dhyana) which, according to the Mdlinivijayatantra, characterises the Individual Means. In order to practice this meditation the yogi must visualise the dissolving away of all the forces in the body. There are two ways in which this can be done. The first is called 'the contemplation of dissolution' (layabhdvana). Through it the progressive differentiation of consciousness from its causal, pre-cosmic form to its phenomenal manifestation is reversed As the Vijnanabhairava teaches: "One should meditate on the All in the form of the Paths of the world- orders etc. considered in their gross, subtle and supreme forms until, at the end, the mind dissolves away." Mediated by consciousness, the macrocosm rests in the microcosm which is emitted along with it successively in the emptiness of the individual subject, vital breaths, mind, psychic nerves (ndtfi), senses and external body. The yogi reproduces this process by visualising the totality of reality including the world-systems, metaphysical principles and cosmic forces along with the Mantras, letters and syllables which represent them, as arising successively throughout the psycho-physical body so as to constitute it. Deployed in this way they form the Cosmic Path along which the yogi ascends, absorbing as he does so, the lower elements into the higher, thus strengthening and extending his unifying awareness (anusamdhdnd) of the configuration of the Path. Thus, moving from the gross elements constituting the outer physical body, to pure sensations (tanmdtra), then to the senses and mind back to their primordial source, the yogi rises from the **embodied subjectivity** of the waking state to the Fourth State (turiya) of contemplation where he is one with the pervasive intent which initiates the creative vision of

consciousness. Abhinava writes: Once [the yogi] has known [this] Path in its completeness, he must then dissolve it into the deities who sustain it and these successively into the body, breath, mind [and emptiness] as before, and all these into his own consciousness. Once this is full and an object of constant worship, it destroys, like the fire at the end of time, the ocean of transmigration.Thus, the second method Ksemaraja teaches to dissolve away the diversity of sensory, mental and physical energies into the unity of consciousness is a meditation on the Fire of Consciousness (dahacintd) which the yogi visualises as burning away all division. At the Divine level (sambhavopaya) the yogi witnesses the sudden and violent withdrawal of all objectivity into the pure ego (aharri), like the pouring of fuel into a raging fire. He does not need to visualise this process but merely attend to it with a passive, receptive attitude. At the Individual level the yogi must exert his imagination to induce this process and so rise to the Divine level through the Empowered. The Vijnanabhairava teaches: Visualise the fortress [of your body] burning with the Fire of Time (kalagni) risen from (e) the Abode of Time; then at the end peace manifests. The Fire of Time {kalagni) resides underneath the hell worlds at the bottom of the Cosmic Egg (Brahman^a). It issues from Ananta — a form of Siva who presides over the lower regions. He floats on a boat in the causal waters supporting the Egg, his mind all the while fixed on Bhairava. The flames of the Fire of Time rise up to the hell-worlds heating them intensely and radiate its energy throughout the universe. At the end of each period of creation the flames rise higher and destroy the old cosmic order to make room for a new one. At the microcosmic level the yogi reproduces this process by mentally placing the letters of the alphabet, in the prescribed order, on the limbs of his body starting from the left toe to the top of the head. As his attention progresses upwards, he visualises the Fire of Time moving with it in such a way that his bodily consciousness, together with the universe of differentiated perceptions, is gradually burnt away leaving in its place the white ashes of the undivided light of consciousness. Ksemaraja considers this meditation (dhyana) to be a limb of a programme of yogic practice at the Individual level 291 of which the remaining limbs are as follows: Posture (A sana). The yogi fixes his attention on the centre between the inhaled and exhaled breath, absorbing in this way the flux of his awareness into the unfolding power of knowledge which rises initially as the upward flowing breath (udanaprana) in the Central Channel (su§umna) between the other two breaths. The Prank aspect of this flow disappears as it moves upward and the yogi experiences the spontaneous rise of the omniscience of consciousness within himself. The mind reverts back to its original, pervasive conscious nature and understands the infinite fact of Siva's omnipresence. This is the firm seat (asana) upon which the yogi sits to practice. Regulation of the Breath (Pranayama) .To regulate the movement of the breath, the yogi must first cleanse the right and left channels of the ascending and descending breath by blocking the left nostril while exhaling and the right while inhaling a few times. This ensures that the movement of the breath is firm and evenly distributed. Next, without attempting to control it in any way, he attends to the flow of his breathing. As the mind becomes steadier and in closer harmony with the rhythm of its movement, the duration of each inhalation and exhalation gradually alters until they become equal. At this stage they unite and merge in the upward flowing current of vitality in the Central Channel (Susumna). This is when true Pranayama begins. The yogi's mind pure and tranquil, he returns, as it were, to a prenatal state and the external breathing cycle is internalized, so that it no longer moves through the lungs but passes directly to the universal source of vitality. The yogi, now at the Empowered level of practice, experiences this movement as travelling from the Heart centre upwards to a point distant twelve fingers above the head where it merges in the void of consciousness. Free of its outer gross form, the breath moves freely through the Central Channel and soon transcends even this subtle movement to become one with the supreme vibration of consciousness. In this way, the yogi's breathing becomes one with the spontaneous rise and fall of energy from the bosom of the absolute. Abhinava quotes the Tantra of the Line of Heroes (Viravalitantra) as saying: When, by constantly merging the mind in Siva, Who is the pure conscious nature, the Sun and Moon [of the two breaths] have dissolved away and the Sun of Life, which is one's own consciousness, has reached the twelve-finger space, this is termed liberation. Breath control [at this stage] serves no useful purpose. Breath control which merely inflicts pain on the body is not to be practised. He who knows this secret is both themselves liberated and liberates others. **Focusing of Attention (Dhara nA):** Attention is fixed on the psychic centres in (eb) the body corresponding to (e&eb) the five gross elements In this way the vital breath is successively directed to (e) these centres from (e) the

Heart of consciousness to refresh and stimulate their activity. First it moves to the Earth centre in the throat which regulates (e&eb) the firmness of (e) the bones and flesh of the body; then to the Water centre in (eb) the glottis responsible for (e) the balance of (e) the bodily fluids. After this it travels to (e&eb) the navel which is the Fire centre dealing with (e&eb) digestion and anabolisms and catabolisms in general. It then moves to the Wind centre in the toe of the left foot which governs the movement of gases to and from the cells via the circulatory system. When the yogi has thus achieved control over these forces, the breath rises from the Heart to the top of the head and he becomes master of the Ether element and so attains every yogic power. **Meditation (Dhyana)** The highest form of meditation stills the flux of the qualities (guna) and induces the mind into a state of contemplative absorption. The object of this meditation is the supreme and pervasive divinity of the pure subject whose true nature is known to none but himself alone (svasatpvedya). The yogi attains him by merging into the constant flow of awareness that streams into the Light which illumines his own nature. Contemplation (Samadhi) .The yogi rises to the level of contemplation when the awareness he has of himself and the things around him become one and he realises his own identity with Siva, **(or Rama the omniscient: cosmic general ledger)** the sole reality. The aim of this Yoga in all its phases is to achieve the Fourth State of consciousness (turiya) beyond the three states of waking, dreaming and deep sleep and to then ultimately reach the liberated state Beyond the Fourth (turiyatita). These five states correspond to: (a) Siva's activity (vy apara), that is, His power of action; (b) Siva's Lordship (adhipatya), which is His power of knowledge; (c) the absence of these two, which corresponds to Siva's power of will; (d) His exertion (prerakatva), which contains all the cycles of creation and destruction and, (e) the rest Siva enjoys in His own nature, which is His power of consciousness. The first three states, when divorced from the last two, belong to the sphere of transmigratory existence. The Fourth and Beyond the Fourth on the other hand are higher, supramundane (alaukika) states of consciousness in which the yogi enjoys bliss and repose (visranti) in (e&eb) his own nature by penetrating (samavesa) into (e&eb) the universal consciousness of the Self, through which he ultimately becomes liberated (jivanmukta). Beyond the Fourth is the state of awareness ParamaSiva Himself enjoys when duality has entirely disappeared and everything is realised to be one with consciousness. The Fourth is the state of awareness of the yogi who, catching hold of the pure subjectivity (upalabdhrta) flowing through the lower three states, is still actively eliminating his sense of duality. While the former is the supreme subject as (=) T consciousness (aham), the latter is (=) the pure awareness (prama) or I-nesses (ahanta) of (e) the subject which encompasses the lower states, giving them (e&eb) life and uniting (e&eb) them together As such, the Fourth State is the reflective awareness of one's own nature shining in all three states at one with (e&eb) them. The fact that we recall that we slept well is proof that this state of consciousness persists (eb,eb+) even in deep sleep. Indeed, if the flow of Turiya could somehow be brought to a halt, all the other states of consciousness would come to an end in the absence of the pure subjectivity which makes them, and their contents, manifest. The states of waking, dreaming and deep sleep correspond to the form of awareness consciousness assumes (eb) when it predominantly manifest as the object, means of knowledge and individual subject, respectively. Turiya is the pure awareness (prama) that both transcends them and merges them (e&eb)all into itself. 300 As such, it appears as the triad of deed, means and agent in the pure act (vy apara) of consciousness unsullied by any outer reality. Abhinava explains: {Turiya) transcends the three aspects of *form\ 'sight' and T consisting as it does of the pure act of 'seeing'; therefore any means [by which this state could be realised] has merely a [provisional] instrumental value. It is, in other words, pure subjectivity of the nature of absolute freedom, independent of all external means. This is the state of consciousness called Turiya, luminous with its own light. Turiya is thus not just a psychological state but the supreme creative power (para sakti) of consciousness, the Goddess (samviddevi) who generates and withdraws the entire universe of subject, object and means of knowledge. In the Heart of Recognition Ksemaraja explains: Whenever the extroverted [conscious] nature rests within itself, external objectivity is withdrawn [and consciousness] is established in the inner abode of peace which threads through the flux of awareness in every [externally] emanated [state]. Thus Turiya, the Goddess of Consciousness, is the union of creation, persistence and destruction. **(Read as cosmic consciousness; Kshemaraja does not distinguish like we have done, individual, collective and cosmic and of three states)** She emanates every individual [cycle] of creation and withdraws it. Eternally full [of all things] and [yet] void [of diversity] She is

both and yet neither, shining radiantly as non-successive [consciousness] alone. The yogi is fully absorbed in this state of consciousness and takes possession of its power when he is able to rise from contemplation (samadhi) carrying with him the abiding awareness of Turiya throughout his waking, dreaming and deep sleep. When he achieves this constantly, he continues to experience these states individually, but they no longer obscure the insight (pratibha) he has acquired because he realises that they are all aspects of the bliss of Turiya. Thus, while the common man calls this state the 'Fourth' (turiya) because he cannot experience it directly and knows only that it is beyond the other three, the yogi calls it 'Beyond Form' (rupatita) because (e) it transcends the detachment of the state of deep sleep which, devoid of objective content, is the naked form' of the individual subject tending towards the fullness of consciousness. Those who are on the path of knowledge (jnaniri) call it the 'Whole* (pracaya) because, in this state, they see the entire universe gathered together in one place. 'Supra-mental Awareness' (manonmana) is the name given to the experience of Turiya in the waking state. The yogi in this state moves and lives in the world of waking experience free of all disturbing thoughts while abiding in the transcendental silence beyond the activities of the mind. 'Infinite' is the name of the experience of Turiya while dreaming because, free of the limitations imposed upon the body by time and space, the yogi enjoys the unlimited expanse of the Self. When Turiya is experienced in deep sleep, the yogi's state is called 'All things' (sarvdrtha) because in it he discovers his freedom from limitations in this, the most contracted state of human consciousness. The yogi who manages to maintain Turiya- consciousness comes to experience the three states of waking, dreaming and deep sleep as the constant flow of the bliss of consciousness in which all traces of the relative distinction between these states and their contents is eradicated. Following the stream of Turiya to its highest level (para katfha), he reaches the state Beyond the Fourth (turiydtita), which is the universal consciousness (caitanya) of the Self. Here the yogi comes to rest within his own nature. Plunged in the vast, waveless ocean of the consciousness and bliss (ciddnanda) of the state Beyond the Fourth, the yogi becomes Siva, the Free One (svacchanda), and thus wanders freely, practising the Yoga of Freedom. K§emaraja equates the Fourth State with the pure (suddha), innate (sahaja) knowledge that one's own conscious nature is all things. It is the Supra-mental State (unmana) in which Siva's pervasive presence is experienced once the Yoga practised at the Individual level attains fruition at the Empowered. What the yogi must do, once consciousness is elevated to grasp the Fourth State, is make it constant. He must forcefully lay hold of it within himself and not release his grip until it becomes permanent. Then he travels 'Beyond the Fourth' to enlighten- ment. Before this ultimate attainment the yogi inevitably falls. The forces operating within consciousness that limit and obscure it throw him down whenever they possibly can. The only way the yogi can defend himself against them is to maintain a constant attentive awareness of the Fourth State. He falls when he is distracted but when he attends carefully to his pure conscious nature, he realises that every aspect of his state of being, including the forces that lead him astray, are one with the pulsing flux of his own consciousness and so cannot affect him. These powers, which are the energies of Matrka we have already discussed, are not the only obstacles the yogi must overcome. He must, for example, also resist the temptation to rest content with the miraculous yogic powers (siddhi) he acquires in the course of his spiritual develop- ment. Again to do this he must practice Yoga. Similarly, in order to pervade the Fourth State gradually through the other states in the manner proper to practice at the Individual level, the method is the same. He must practise the higher yoga of the Tantras which, turning his mind inwards and freeing it from discursive representations, allows him to penetrate into the Supreme Principle. Once the yogi has attained this contemplative state, his main problem is to make it permanent. In the introvered state the gross external movement of the breath is suspended and with it the activity of the intellect, mind, individualised consciousness, powers of the senses and the ego. 313 When the yogi rises out of this state, he is liable to fall again into the lower order of creation generated by Maya if he does not maintain his awareness of the higher reality he has experienced and allows his awakened, illumined insight to be obscured by the dream-like vision of thought-constructs. 314 Naturally, the yogi must rise out of the introverted condition of suspension. It is inherent in the very nature of reality that it should move out of itself. 315 Pure, universal consciousness initially transforms itself into the vital breath 316 charged with the impression iyasana) of the power of awareness attained through introversion. By attending to the pulse (spanda) of the breath as it moves out of the absolute, the yogi can develop an intuitive sense of the inherent unity of all he will

perceive in the mental and physical spheres created by the outpouring of consciousness. In this way he realises that his own nature is everywhere present in all he perceives and that all things thus reside within him. Blessed with this insight his consciousness remains free and unlimited even at the individual level where the breath, mind, senses and body are active. If the yogi fails to do this, he finds himself once again beset by the strictures of his embodied existence and must, as before, try to pervade all his other states of consciousness with the aesthetic delight (rasa) and wonder (camatkara) of the Fourth State he experienced in contemplation. Again this means that he must strengthen his pure, empowered awareness that his universal nature manifests as all things. In this way he discovers Siva's presence in every sphere of individualised consciousness ranging from the breath to outer objectivity. The yogi's mind then becomes tranquil and undistracted because wherever it may wander, the yogi perceives only Siva, his authentic nature. Consciousness is thus freed of all external referents and the yogi's subjectivity is purified of all identification with the body or anything else that belongs to the objective sphere. The yogi then becomes detached from the opposites of pleasure and pain and is transcendentally free (kevalin). The yogi is again, however, liable to fall if he allows himself to get entangled in the play of opposites. This fall is more serious than the others because, although he is caught by the confining restrictions of individual- ised consciousness as before, he is now also affected by karma. Fleeing from pain in the pursuit of pleasure he is bound to act (karma) to minimise one and maximise the ( a **maximaization problem is mentioned elsewhere about the svabhava of human beings to enhance pleasure reduce deprivation)** other and so is thrown down to the lowest level of embodied subjectivity (sakala). In order to regain his lost state, he must ascend gradually, by Siva's grace, from one order of subjectivity to the next and so free himself progressively of the limitations of the lower levels to gain the greater freedom and expansion of the higher. As he progresses, the objective sphere also evolves from the grossest perceptions of physical objects outside the lowest order to subjectivity, through to the subtler inner, mental perceptions to finally reach the order of subjectivity that contains objectivity within itself and is free to externalise it at will. The degree to which this process develops depends, as before, on the yogi's awareness of the Fourth State. In consonance with the general principle that the remedy should suit the defect, the yogi is instructed to seek this higher state of consciousness in the wonder (camatkara) or delight (ananda) he feels in moments of intense physical pleasure. At first he experiences this subtle consciousness for an instant in the subjective sphere. If he manages to catch hold of it, it becomes more intense as the cognitive and objective spheres are also gradually pervaded and vitalised by it. Occasions for this practice are, for example, the sense of satisfaction one feels after a good meal or the aesthetic delight one experiences when listening to good music or the pleasure of sexual union with the Tantric consort or even solitary sexual excitation. In these moments of delight the yogi can penetrate momentarily into his own authentic Siva-nature (sambhavavesa) through the empowered contact (saktasparsa) m he makes with it in the freedom of the pure subjectivity of the Fourth State. If the yogi develops his awareness of this higher level of consciousness and maintains it, he eventually experiences it constantly. Clearly, what prevents the yogi from attending to his state of consciousness rather than the circumstances which induce it is the craving for pleasure (abhila§a) born of ignorance — the source of every impurity which clouds consciousness. Craving directs the yogi's attention towards outer, worldly things and so he is caught in the **net of thought- constructs**. To free himself of his worldly desires and reverse this binding extroversion, the yogi must eradicate its cause. To be freed of all the ups and downs of the path and no longer be tormented by the possibility of a fall, the yogi must see reality perfectly and completely. This insight is itself liberation and the moment it dawns the yogi is instantly freed. This sudden realisation is the goal of Tantric Yoga. Accordingly the Tantra declares: "He, who perceives reality directly, even for the brief moment it takes to blink, is liberated that very instant and never reborn again." Although the yogi's body and mind continue to function as before, they are like mere outer coverings which contain, but do not obscure, the mighty, universal consciousness which operates through them. The yogi's body is the universe, the senses the energies that vitalise it, his mind Mantra, the rhythm of his breath the pulse of time and his inner nature pure, dynamic consciousness. Raised above all practice, and hence all possibility of falling to lower levels, the yogi realises that he has always been free and that his journey through the dark land of Maya was nothing but a dream, a construct of his own imagination. **(Models are given in one of the papers of the series due to spatial constraints)**

**Relationship betwixt string theory (note the spade remarks above) and holographic principle has been enucleated and expatiated in literature**. Some exemplar work is sententiously mentioned in the following:The ratio of shear viscosity to volume density of entropy can be used to characterize how close a given fluid is to being perfect. Using string theory methods, we show that this ratio is equal to a universal value of $\hbar/4\pi kB$ for a large class of strongly interacting quantum field theories whose dual description involves black holes in anti–de Sitter space. **P K. Kovtun, D. T. Son, and A. O. Starinets** provide evidence that this value may serve as a lower bound for a wide class of systems, thus suggesting that black hole horizons are dual to the most ideal fluid **Viscosity in Strongly Interacting Quantum Field Theories from Black Hole Physics Phys. Rev. Lett. 94, 111601 – Published 22 March 2005 P K. Kovtun, D. T. Son, and A. O. Starinets** According to't Hooft the combination of quantum mechanics and gravity requires the three-dimensional world to be an image of data that can be stored on a two-dimensional projection much like a holographic image. The two-dimensional description only requires one discrete degree of freedom per Planck area and yet it is rich enough to describe all three-dimensional phenomena. After outlining't Hooft's proposal we give a preliminary informal description of how it may be implemented. One finds a basic requirement that particles must grow in size as their momenta are increased far above the Planck scale. The consequences for high-energy particle collisions are described. The phenomenon of particle growth with momentum was previously discussed in the context of string theory and was related to information spreading near black hole horizons. The considerations of this paper indicate that the effect is much more rapid at all but the earliest times. In fact the rate of spreading is found to saturate the bound from causality. Perception is not reality. It needs some hidden variables to posit the reality. (see earlier models).We Make Sense Of Objects, Events Or State Of Affairs When We Pin Them Down And Describe Them, State Propositions About Them. We Can Then Develop These Propositions Into Logical Forms To Draw Conclusions, Make Predictions Or Whatever. The Trouble Is, Neither Object As We See Them, Nor Propositions As We Commonly Use Them, Are At All Simple. Objects As We Perceive Them, For Example, Clearly Have All Sorts Of Characteristics As Well That We Don't Perceive. They Have A History, And A Potential For The Future, For Example. The Worst Problem Is That Objects As We See Them Are Actually Mixtures Of Things, The Affects Of Complex Causes: They Are 'Overdetermined', To Use The Phrase Associated With Both Freud And Marx. Deleuze Actually Draws On Resources In Classical Philosophy To Argue This, Especially The Work Of The Stoics, Who Saw Objects And Events As A 'Mixture Of Bodies', This Mixture Emanating From Somewhere In The Depths, Below Conscious Sense Making. Incidentally, This Reliance On The Stoics Also Makes Deleuze A Bit Stoical, In The Sense That He Thinks That These Mixed Bodies Are Part Of Some Unified Whole (He Was Going To Call This Being), And The Role Of Destiny Is Associated With The Whole. It Follows That Ethically And Conceptually, All Human Beings Can Do This Discover The Ways In Which Being Works. There's No Point Complaining About It Either. This Is the Basis of Badiou's Critique Of Deleuze. It Is Also The Basis Of Deleuze's Anti-Humanism: We Don't Understand Reality By Trying To Develop A Conscious Synthesis, Tracking Analogies Based On Some Notion Of A Shared Essence, Or Seeing God As An Ultimate Synthesiser For That Matter. Events And States Of Affairs Are Also More Complex Than They Look, And Deleuze Illustrates This Discussion With The Extraordinary Notion Of 'Compossibility' In Leibniz. Basically, The Idea Is That Singularities Can Produce Events In All Sorts Of Divergent Ways, All Of Them Equally Possible, And This Potential Is Shrunk If We Just Study The Actual Course Of Events. Objects Are More Complex Than They Look, With All Sorts Of Bits Of Their Being Unavailable To Us, But Language Is Equally Complex. Here, Deleuze Tries To Demonstrate This By Looking At People Who Use All Kinds Of Linguistic Anomalies Such As Paradoxes, Or, In The Case Of Lewis Carroll Especially, Portmanteau Words Or Esoteric Words. Although These Constructions Offend Logic, And Are Strictly Speaking Nonsense, They Still Make Sense. This Also Points To Some Powers Of Language That Are Not Immediately Available To Inspection. More Conventionally, We Know That The Use Of Particular Words Can Have Different Functions In Any Language System—They Can Denote, They Can Manifest Inner Thoughts, And They Can Signify (In The French Sense, That Is Indicate The Existence Of A Structured Language System Which Enables Us To Communicate With Each Other. Deleuze Proposes To Modify Classical Saussurian Semiotics By Adding A Moveable Element, An 'Empty Space' Or Floating Signifier'). Again Wordplay

Can Deliberately Confuse These Different Functions, As When I Use The Term 'It' To Denote A Specific Object, Only To Refer To A Whole Process As 'It', To Express Myself By Saying Something About 'It' And So On (Actual Easy Examples Are Thin On The Ground In Deleuze!). Then There Is One Of Those (Many) Philosophical Diversions Into Various Types Of Explanation For The Extra Elements That Are Not Available To Actual Objects Or Speech Acts. Platonists Thought The Extra Bits Referred To Some Universal Form Or Idea, Other Philosophers Like Husserl Or Kant Suggest There Was Some Transcendental Realm Beyond The Immediate. Deleuze Offers Objections To Both For Those Approaches, And Suggests That The Real Is Actually Divided Into Virtual And Actual Levels (Delanda's Commentary Is Invaluable Here). The Objects And Words Of Our Immediate Perception Are Actualised Are Condensed Out From Much More Complex Objects And Events At The Virtual Level. In This Book, About The Only Candidate For These Complex Objects Is The Singularity, Which Actualizes Itself In Various Partial Ways. Deleuze Goes On To Develop This Idea By Saying That Singularities Themselves, And The Events And Objects They Actualize, Are Produced In A Random Or Arbitrary Fashion, The Chance Is At The Heart Of Actualisation. We Learn That The Virtual Also Has Its Own Sense Of Time, Which Is Not The Usual One Which Is Highly Limited By The Insistence On The Present Tense In Human Operations. It Is A Very Abstract Discussion, Which Finally Comes To An Example I Could Understand—The Emergence Of Language In Freud. As Infants Develop, Their Biological Urges Produce Certain Infantile Concepts Like Part Objects. These Are Classically Mixed, Deriving From Contradictory Drives In The Depths Of The Unconscious. The Sounds The Infant Makes Our Initially Just Bodily Emanations, But They Gradually Come To Take On The Form Of Linguistic Units, Like Phonemes. There Are Elements Of Adult Language Available Too, Of Course, But These Are Initially Unintelligible. Then A Process Described As The Phantasm Manages To Combine Various Infantile Concepts Together Into A'disjunctive Unity', A Mixture Of Heterogeneous Elements Which Deleuze Thinks Characterises Most Objects. A Kind Of Primitive Narrative Develops In The Phantasm, Partly Driven By Biological Drives, And Partly Driven By Emerging Linguistic Competencies (Such As The Oedipal Scene). The Key To This Is The Freudian Notion Of The Phallus, Which Both Refers To Biological Organs Like Penises, And To Linguistic And Cultural Functions To Do With Authority And Value. The Phallus Is The Ideal Ambiguous Object (Or Empty Object As Deleuze Insists On Calling It), Able To Zigzag Between Bodies And Language, And Thus Make Divergent Series 'Resonate'. The Phantasm Therefore Develops An Energy Of Its Own, Which Permits A Relative Disengagement From Sexual Energy (Sublimation) And, In The Final Stages, The Proper Development Of Language In The Form Of Symbolisation. At Last We Have Language And Events Brought Together On The Surface, Which Is Of The Level Of Consciousness. If I Have Read The Appendices Correctly, This Phantasmic Form Is Then Generalised To Include All The Operations Of Making Sense As Describes Right At The Beginning. I Have Lots Of Reservations About This Whole Schema, In Fact, Ranging From The General Structure Of The Argument, Where Philosophical Issues Are Introduced As If They Were Necessary, Whereas Their Role Is Been Predetermined All Along (For Example, Deleuze Has Always Admired Stoics, And Here They Are As Offering The Best Account), Right Down To The Curious Insistence On The Phantasm As The Adult Form Of Making Sense—In My View, Dreams Are A Lot More Like A General Model Of Thinking, And They Include The More Sophisticated Linguistic Operations Of Metonymy (Condensation), And Metaphor (Displacement). Alexander I. Stingl's Blog A Nomadic Scholarship Entity NOTES ON: Deleuze, G (1990) The Logic Of Sense, Trans Mark Lester, Edited By Constantin Boundas, New York: Columbia University Press For Models Kindly See One Of The Papers In The Series. Attributable And Ascribable To Constraints On Space Accomodation Model Is Provided In Some Paper, Notwithstanding The Generality And Commonalty Of The Observation And Its Concatenatability With The Other Modules. Finally **Leonard Susskind** considers string theory as (=) a possible realization of't Hooft's idea. The light front lattice string model of Klebanov and Susskind is reviewed and its similarities with the holographic theory are demonstrated. The agreement between the two requires unproven but plausible assumptions about (e&eb) the nonperturbative behavior of string theory. Very similar ideas to those in this paper have long been held by Charles Thorn. © 1995 American Institute of Physics **The world as a hologram Leonard Susskind J. Math Phys. 36, 6377 (1995); http://dx.doi.org/10.1063/1.531249 Ofer Aharonya, , Steven S. Gubserb, , Juan Maldacenab, c, ,**

**Hirosi Oogurid, e, , Yaron Ozf,** review the holographic correspondence between field theories and string/M theory, focusing on the relation between compactifications of string/M theory on Anti-de Sitter spaces and conformal field theories. They review the background for this correspondence and discuss its motivations and the evidence for its correctness. They describe the main results that have been derived from the correspondence in the regime that the field theory is approximated by classical or semiclassical gravity. We focus on the case of the supersymmetric gauge theory in four dimensions, but discuss also field theories in other dimensions, conformal and non-conformal, with or without supersymmetry, and in particular the relation to QCD. We also discuss some implications for black hole physics. **Physics Reports Volume 323, Issues 3–4, January 2000, Pages 183–386 Large N field theories, string theory and gravity Ofer Aharonya, , Steven S. Gubserb, , Juan Maldacenab, c, , Hirosi Oogurid, e, , Yaron Ozf, , doi:10.1016/S0370-1573(99)00083-6 T. Banks, W. Fischler, S. H. Shenker, and L. Susskind** suggest and motivate a precise equivalence between uncompactified 11-dimensional M theory and the N=∞ limit of the supersymmetric matrix quantum mechanics describing D0 branes. The evidence for the conjecture consists of several correspondences between the two theories. As a consequence of supersymmetry the simple matrix model is rich enough to describe the properties of the entire Fock space of massless well separated particles of the supergravity theory. In one particular kinematic situation the leading large distance interaction of these particles is exactly described by supergravity. The model appears to be a nonperturbative realization of the holographic principle. The membrane states required by M theory are contained as excitations of the matrix model. The membrane world volume is a noncommutative geometry embedded in a noncommutative spacetime. DOI: http://dx.doi.org/10.1103/PhysRevD.55.5112 **M theory as a matrix model: A conjecture Phys. Rev. D 55, 5112 – Published 15 April 1997 T. Banks, W. Fischler, S. H. Shenker, and L. Susskind** These TASI lectures review the Holographic principle. The first lecture describes the puzzle of black hole information loss that led to the idea of Black Hole Complementarity and subsequently to the Holographic Principle itself. The second lecture discusses the holographic entropy bound in general space-times. The final two lectures are devoted to the ADS/CFT duality as a special case of the principle. The presentation is self contained and emphasizes the physical principles. Very little technical knowledge of string theory or supergravity is assumed. Report number: SU-ITP 99-14, KUL-TF-2000/03 Cite as: arXiv: hep-th/0002044 **TASI lectures on the Holographic Principle Daniela Bigatti, Leonard Susskind** The notion of a space-time uncertainty principle in string theory is clarified and further developed. The motivation and the derivation of the principle are first reviewed in a reasonably self-contained way. It is then shown that the nonperturbative (Borel summed) high-energy and high-momentum transfer behavior of string scattering is consistent with the space-time uncertainty principle. It is also shown that, as a consequence of this principle, string theories in 10 dimensions generically exhibit a characteristic length scale which is equal to the well-known 11 dimensional Planck length gs1/3 ℓs of M-theory as the scale at which stringy effects take over the effects of classical supergravity, even without involving D-branes directly. The implications of the space-time uncertainty relation in connection with D-branes and black holes are discussed and reinterpreted. Finally, we present a novel interpretation of the Schild-gauge action for strings from the viewpoint of noncommutative geometry. This conforms to the space-time uncertainty relation by manifestly exhibiting a noncommutativity of quantized string coordinates between, dominantly, space and time. We also discuss the consistency of the space-time uncertainty relation with S and T dualities. Copyright (c) 2000 Progress of Theoretical Physics **String Theory and the Space-Time Uncertainty Principle Tamiaki Yoneya Oxford Journals Science & Mathematics Progress of Theoretical Physics Volume 103, Issue 6Pp. 1081-112** In this Letter a recently proposed gravity dual of noncommutative Yang-Mills theory is derived from the relations between closed string moduli and open string moduli recently suggested by Seiberg and Witten. The only new input one needs is a simple form of the running string tension as a function of energy. This derivation provides convincing evidence that string theory integrates with the holographical principle and demonstrates a direct link between noncommutative Yang-Mills theory and holography.DOI: http://dx.doi.org/10.1103/PhysRevLett.84.2084 **Holography and Noncommutative Yang-Mills Theory Phys. Rev. Lett. 84, 2084 – Published 6 March 2000 Miao Li and Yong-Shi Wu Petr Hořava** suggests that M theory could be nonperturbatively equivalent to a local quantum field theory. More precisely, we present a

"renormalizable" gauge theory in eleven dimensions, and show that it exhibits various properties expected of quantum M theory, most notably the holographic principle of't Hooft and Susskind. The theory also satisfies Mach's principle: A macroscopically large space-time (and the inertia of low-energy excitations) is generated by a large number of "partons" in the microscopic theory. **Petr Hořava** argues that at low energies in large eleven dimensions, the theory should be effectively described by eleven-dimensional supergravity. This effective description breaks down at much lower energies than naively expected, precisely when the system saturates the Bekenstein bound on energy density. He shows that the number of partons scales like the area of the surface surrounding the system, and discuss how this holographic reduction of degrees of freedom affects the cosmological constant problem. proposal is put forth for the holographic field theory as a candidate for a covariant, nonperturbative formulation of quantum M theory. DOI: http://dx.doi.org/10.1103/PhysRevD.59.046004 **M theory as a holographic field theory Phys. Rev. D 59, 046004 – Published 26 January 1999 Petr Hořava** The recent proposal on the correspondence between the super-Yang–Mills theory and string theory in the Penrose limit of the AdS5×S5 geometry involves a few puzzles from the viewpoint of holographic principle, especially in connection with the interpretation of times. To resolve these puzzles, we propose to interpret the PP-wave strings on the basis of tunneling null geodesics connecting boundaries of the AdS geometry. Our approach predicts a direct and systematic identification of the S-matrix of Euclidean string theory in the bulk with the short-distance structure of correlation functions of super-Yang–Mills theory on the AdS boundary, as an extension of the ordinary relation in supergravity–CFT correspondence. Holography requires an infinite number of contact terms for interaction vertices of string field theory and constrains their forms in a way consistent with supersymmetry. Copyright © 2003 Elsevier B.V. All rights reserved. **Nuclear Physics B Volume 665, 18 August 2003, Pages 94–128 Holographic reformulation of string theory on AdS5×S5 background in the PP-wave limit Suguru Dobashi, Hidehiko Shimada, Tamiaki Yoneya doi:10.1016/S0550-3213(03)00460-7** The strongest adversary in quantum information science is decoherence, which arises owing to the coupling of a system with its environment1. The induced dissipation tends to destroy and wash out the interesting quantum effects that give rise to the power of quantum computation2, cryptography2 and simulation3. Whereas such a statement is true for many forms of dissipation, **Frank Verstraete1, Michael M. Wolf2 & J. Ignacio Cirac3** show here that dissipation can also have exactly the opposite effect: it can be a fully fledged resource for universal quantum computation without any coherent dynamics needed to complement it. The coupling to the environment drives the system to a steady state where the outcome of the computation is encoded. In a similar vein, they show that dissipation can be used to engineer a large variety of strongly correlated states in steady state, including all stabilizer codes, matrix product states4, and their generalization to higher dimensions5. **Letter abstract Nature Physics 5, 633 - 636 (2009) Published online: 20 July 2009 | doi: 10.1038/nphys1342 Quantum computation and quantum-state engineering driven by dissipation Frank Verstraete1, Michael M. Wolf2 & J. Ignacio Cirac3** Nonlocal twist operators are introduced for the O (n) and Q-state Potts models in two dimensions which count the numbers of self-avoiding loops (respectively, percolation clusters) surrounding a given point. Their scaling dimensions are computed exactly. This yields many results: for example, the number of percolation clusters which must be crossed to connect a given point to an infinitely distant boundary. Its mean behaves as $(1/33\sqrt{\pi})\,|\ln(pc-p)|$ as $p\to pc-$. As an application **John Cardy** computes the exact value $3\sqrt{/2}$ for the conductivity at the spin Hall transition, as well as the shape dependence of the mean conductance in an arbitrary simply connected geometry with two extended edge contacts. DOI: http://dx.doi.org/10.1103/PhysRevLett.84.3507 **Linking Numbers for Self-Avoiding Loops and Percolation: Application to the Spin Quantum Hall Transition Phys. Rev. Lett 84, 3507 – Published 17 April 2000 John Cardy Ferenc Iglói, Loïc Turban, Dragi Karevski, and Ferenc Szalma** consider the Ising model and the directed walk on two-dimensional layered lattices and show that the two problems are inherently related: zero-field thermodynamical properties of the Ising model are contained in the spectrum of the transfer matrix of the directed walk. The critical properties of the two models are connected to the scaling behavior of the eigenvalue spectrum of the transfer matrix which is studied exactly through renormalization for different self-similar distributions of the couplings. The models show very rich bulk and surface critical behaviors with nonuniversal critical exponents,

coupling-dependent anisotropic scaling, first-order surface transition, and stretched exponential critical correlations. It is shown that all the nonuniversal critical exponents obtained for the aperiodic Ising models satisfy scaling relations and can be expressed as functions of varying surface magnetic exponents. DOI: http://dx.doi.org/10.1103/PhysRevB.56.11031 **exact renormalization-group study of aperiodic Ising quantum chains and directed walks Phys. Rev. B 56, 11031 – Published 1 November 1997 Ferenc Iglói, Loïc Turban, Dragi Karevski, and Ferenc Szalma** Using exact expressions for the persistence probability and for the leading eigenvalue of the Fokker-Planck operator of a random walk in a random environment, we establish a fundamental relation between the statistical properties of anomalous diffusion and the critical and off-critical behavior of random quantum spin chains. Many exact results are obtained from this correspondence, including the space and time correlations of surviving random walks and the distribution of the gaps of the corresponding Fokker-Planck operator. In turn **Ferenc Iglói and Heiko Rieger** derives analytically the dynamical exponent of the random transverse-field Ising spin chain in the Griffiths-McCoy region. DOI: http://dx.doi.org/10.1103/PhysRevE.58.4238 **Anomalous diffusion in disordered media and random quantum spin chains Phys. Rev. E 58, 4238 – Published 1 October 1998 Ferenc Iglói and Heiko Rieger** To gain deeper insight into the dynamics of complex quantum systems we need a quantum leap in computer simulations. We cannot translate quantum behaviour arising from superposition states or entanglement efficiently into the classical language of conventional computers. The solution to this problem, proposed in 1982 (ref. 1), is simulating the quantum behaviour of interest in a different quantum system where the interactions can be controlled and the outcome detected sufficiently well. **A. Friedenauer1, H. Schmitz1, J. T. Glueckert, D. Porras & T. Schaetz** study the building blocks for simulating quantum spin Hamiltonians with trapped ions2. They experimentally simulate the adiabatic evolution of the smallest non-trivial spin system from paramagnetic into ferromagnetic order with a quantum magnetization for two spins of 98%. We prove that the transition is not driven by thermal fluctuations but is of quantum-mechanical origin (analogous to quantum fluctuations in quantum phase transitions3). We observe a final superposition state of the two degenerate spin configurations for the ferromagnetic order (|↑↑right fence+|↓↓right fence), corresponding to deterministic entanglement achieved with 88% fidelity. This method should allow for scaling to a higher number of coupled spins2, enabling implementation of simulations that are intractable on conventional computers. **Letter Nature Physics 4, 757 - 761 (2008) Published online: 27 July 2008 | doi:10.1038/nphys1032 Subject Categories: Quantum physics | Atomic and molecular physics Simulating a quantum magnet with trapped ions A. Friedenauer1, H. Schmitz1, J. T. Glueckert, D. Porras & T. Schaetz** This article gives a comprehensive description of the fractal geometry of conformally-invariant (CI) scaling curves, in the plane or half-plane. It focuses on deriving critical exponents associated with interacting random paths, by exploiting an underlying quantum gravity (QG) structure, which uses KPZ maps relating exponents in the plane to those on a random lattice, i.e., in a fluctuating metric. This is applied to critical models, like O(N) and Potts models, and to the Stochastic L\"owner Evolution (SLE). The multifractal (MF) function f(alpha, c) of the harmonic measure near any CI fractal boundary, is given as a function of the central charge c of the associated CFT. The Hausdorff dimensions D_{H} of a non-simple scaling curve or cluster hull, and D_{EP} of its external perimeter or frontier, are shown to obey the duality equation (D_{H}-1)(D_{EP}-1)=1/4, valid for any c. The universal mixed MF spectrum f(alpha,lambda;c) describing the local spiralling rate lambda and singularity exponent alpha of the potential near any CI scaling curve is given. The duality between simple and non-simple random paths is established via symmetry of the KPZ quantum gravity map. An extended dual KPZ relation is introduced for the SLE_{kappa}, which commutes with the kappa to kappa'=16/kappa duality. This gives the SLE exponents from simple QG rules, established from the general structure of correlation functions of arbitrary interacting random sets on a random lattice. Journal reference: {\it
Fractal Geometry and Applications: A Jubilee of Beno\^{i}t Mandelbrot} (M. L. Lapidus and M. van Frankenhuysen, eds.), Proc. Symposia Pure Math. vol. 72, Part 2, 365-482 (AMS, Providence, R.I., 2004) arXiv: math-ph/0303034(or arXiv: math-ph/0303034v2 for this version) **Conformal Fractal Geometry and Boundary Quantum Gravity Bertrand Duplantier** Hume, For Deleuze, Considers The Mind To Be A System Of Associations Alone, A Network Of Tendencies (ES 25): “We Are Habits, Nothing But Habits – The Habit Of

Saying 'I'. Perhaps There Is No More Striking Answer To The Problem Of The Self." (ES X.) The Mind, Affected By The Natural Principle Of Association, Becomes Human Nature, From The Ground Up: Empirical Subjectivity Is Constituted In The Mind Under The Influence Of The Principles Affecting It; The Mind Therefore Does Not Have The Characteristics Of A Preexisting Subject. (ES 29) These Associations Account Not Only for Experience in the Basic Sense, But up To the Highest Level of Social and Cultural Life: This Is The Basis For Hume's Rejection (E) Of A Social Contract Model Of Society (Such As Hobbes'), In Favour Of Convention Alone. Morals, Feelings, Bodily Comportment, All Of These Elements Of Subjectivity Are Explained, Not By Transcendental Structures, Such As Kant Will Propose, But By The Immanent Activity Of Association. Once This Habitual Structure Of The Self Is In Place, Deleuze Suggests, The Humean Concept Of Belief Comes Into Play , Which Is Resolutely A Central Part Of Human Nature. It Describes The Particularly Human Way Of Going Beyond The Given. When We Expect The Sun To Come Up Tomorrow, We Do Not Do So Because We Know That It Will, But Because Of A Belief Based On A Habit. This In Turn Reverses The Hierarchy Of Knowledge And Belief, And Results, For Deleuze, In A, "Great Conversion Of Theory To Practice." (PI 36) Every Act Of Belief Is A Practical Application Of Habit, Without Any Reference To An A Priori Ability To Judge. Not Only Is The Human Being Thus Habitual, On Deleuze's Reading, But Also Creative Even In The Most Mundane Moments Of Life. Finally, Deleuze Insists That One Of Hume's Greatest Contributions To Modern Philosophy Is His Insistence That All Relations Are External To Their Terms: This Is The Essence Of Hume's Anti-Transcendental Stance. Human Nature Cannot Unite Itself, There Is No 'I' Which Stands Before Experience, But Only Moments Of Experience Themselves, Unattached And Meaningless Without Any Necessary Relation To Each Other. A Flash Of Red, A Movement, A Gust Of Wind, Dashing Of A Woman Against You, Obscene Gestures, Spitting At You, Talking Extempore About You (My Addition) And These Elements Must Be Externally Related To Each Other To Create The Sensation Of A Tree In Autumn. In The Social World, This Externality Attests To The Always-Already Interested Nature Of Life: No Relation Is Necessary, Or Governed By Neutral Laws, So Every Relation Has A Localised And Passional Motive. The Ways In Which Habits Are Formed Attests To The Desires At The Heart Of Our Social Milieu. Subjectivity, As Deleuze Describes It Through His Reading Of Hume, Is A Practical, Passional, And Empiricist Concept, Immediately Located At The Heart Of The Conventional, Which Is To Say The Social. (Stanford Encyclopedia of Philosophy). The Buddha Always Used The Terms Void, No Rising And Falling, Calmness And Extinction To Explain The Profound Meaning Of Sunyata And Cessation. For Example, Sunyata And The State Of Nirvana Where There Is No Rising Nor Falling, Are Interpreted By Most People As A State Of Non-Existence And Gloom. They Fail To Realise That Quite The Opposite, Sunyata Is Of Substantial And Positive Significance. The Sutras Often Use The Word "Great Void" To Explain The Significance Of Sunyata. In General, We Understand The "Great Void" As Something That Contains Absolutely Nothing. However, From A Buddhist Perspective, The Nature Of The "Great Void" Implies Something Which Does Not Obstruct Other Things, In Which All Matters Perform Their Own Functions. Materials Are Form, Which By Their Nature, Imply Obstruction. The Special Characteristic Of The "Great Void" Is Non-Obstruction. The "Great Void" Therefore, Does Not Serve As An Obstacle To Them. Since The "Great Void" Exhibits No Obstructive Tendencies, It Serves As The Foundation For Matter To Function. In Other Words, If There Was Neither "Great Void" Nor Characteristic Of Non-Obstruction, It Would Be Impossible For The Material World To Exist And Function. The "Great Void" Is Not Separated From The Material World. The Latter Depends On The Former. We Can State That the Profound Significance of Sunyata and the Nature of Sunyata in Buddhism Highlight The "Great Void's" Non-Obstructive Nature. Sunyata Does Not Imply The "Great Void". Instead, It Is The Foundation Of All Phenomena (Form And Mind). It Is The True Nature Of All Phenomena, And It Is The Basic Principle Of All Existence. In Other Words, If The Universe's Existence Was Not Empty Nor Impermanent, Then All Resulting Phenomena Could Not Have Arisen Due To The Co-Existence Of Various Causes And There Would Be No Rising Nor Falling. The Nature Of Sunyata Is Of Positive Significance. Calmness And Extinction Are The Opposite Of Rising And Falling. They Are Another Way To Express That There Is No Rising And Falling. Rising And Falling Are The Common Characteristics Of Worldly Existence. All Phenomena Are Always In The Cycle Of Rising And Falling. However, Most People Concentrate On Living (Rising). They Think

That The Universe And Life Are The Reality Of A Continuous Existence. Buddhism On The Other Hand, Promotes The Value Of A Continuous Cessation (Falling). This Cessation Does Not Imply That It Ceases To Exist Altogether. Instead, It Is Just A State In The Continuous Process Of Phenomena. In This Material World, Or What We May Call This "State Of Existence", Everything Eventually Ceases To Exist. Cessation Is Definitely The Home Of All Existences. Since Cessation Is The Calm State Of Existence And The Eventual Refuge Of All Phenomena, It Is Also The Foundation For All Activities And Functions. Teachings In Chinese Buddhism (6) Sunyata (Emptiness) In The Mahayana Context (Wikipedia) Studies by Eric V. Linder about geometric dark energy bring out the quintessential prerequisites about the extreme physics ad dark matter**.** The acceleration of the expansion of the universe arises from unknown physical processes involving either new fields in high energy physics or modifications of gravitation theory. It is crucial for our understanding to characterize the properties of the dark energy or gravity through cosmological observations and compare and distinguish between them. In fact, close consistencies exist between a dark energy equation of state function w(z) and changes to the framework of the Friedmann cosmological equations as well as direct spacetime geometry quantities involving the acceleration, such as "geometric dark energy" from the Ricci scalar. We investigate these interrelationships, including for the case of super acceleration or phantom energy where the fate of the universe may be gentler than the Big Rip. DOI: http://dx.doi.org/10.1103/PhysRevD.70.023511 **Probing gravitation, dark energy, and acceleration Phys. Rev. D 70, 023511 – Published 28 July 2004 Eric V. Linder. In the same vein he continues to question the number of parameters that exists in dark energy** .How many dark energy parameters? For exploring the physics behind the accelerating Universe a crucial question is how much we can learn about the dynamics through next generation cosmological experiments. For example, in defining the dark energy behavior through an effective equation of state, how many parameters can we realistically expect to tightly constrain? Through both general and specific examples (including new parametrizations and principal component analysis) we argue that the answer is two. Cosmological parameter analyses involving a measure of the equation of state value at some epoch (e.g. w0) and a measure of the change in equation of state (e.g. w′) are therefore realistic in projecting dark energy parameter constraints. More elaborate parametrizations could have some uses (e.g. testing for bias or comparison with model features), but do not lead to accurately measured dark energy parameters. DOI: http://dx.doi.org/10.1103/PhysRevD.72.043509 **Phys. Rev. D 72, 043509 – Published 11 August 2005 Eric V. Linder and Dragan Huterer D.F. Mota, C. van de Bruck Oxford Univ** study the spherical collapse model in dark energy cosmologies, in which dark energy is modelled as a minimally coupled scalar field. We first follow the standard assumption that dark energy does not cluster on the scales of interest. Investigating four different popular potentials in detail, we show that the predictions of the spherical collapse model depend on the potential used. We also investigate the dependence on the initial conditions. Secondly, they investigate in how far perturbations in the quintessence field affect the predictions of the spherical collapse model. In doing so, we assume that the field collapses along with the dark matter. Although the field is still subdominant at the time of virialisation, the predictions are different from the case of a homogeneous dark energy component. This will in particular be true if the field is non--minimally coupled. **D.F. Mota, C. van de Bruck** conclude that a better understanding of the evolution of dark energy in the highly non--linear regime is needed in order to make predictions using the spherical collapse model in models with dark energy. Astrophys 421 (2004) 71-81DOI:10.1051/0004-6361:20041090arXiv: astro-ph/0401504 **on the spherical collapse model in dark energy cosmologies D.F. Mota, C. van de Bruck (Oxford Univ)** Studies by Eric V. Linder about geometric dark energy bring out the quintessential prerequisites about the extreme physics ad dark matter**.** The acceleration of the expansion of the universe arises from unknown physical processes involving either new fields in high energy physics or modifications of gravitation theory. It is crucial for our understanding to characterize the properties of the dark energy or gravity through cosmological observations and compare and distinguish between them. In fact, close consistencies exist between a dark energy equation of state function w(z) and changes to the framework of the Friedmann cosmological equations as well as direct spacetime geometry quantities involving the acceleration, such as "geometric dark energy" from the Ricci scalar. We investigate these interrelationships, including for the case of super acceleration or phantom energy where the fate of the universe may be gentler than the Big Rip. DOI: http://dx.doi.org/10.1103/PhysRevD.70.023511

**Probing gravitation, dark energy, and acceleration Phys. Rev. D 70, 023511 – Published 28 July 2004 Eric V. Linder. In the same vein he continues to question the number of parameters that exists in dark energy** .How many dark energy parameters? For exploring the physics behind the accelerating Universe a crucial question is how much we can learn about the dynamics through next generation cosmological experiments. For example, in defining the dark energy behavior through an effective equation of state, how many parameters can we realistically expect to tightly constrain? Through both general and specific examples (including new parametrizations and principal component analysis) we argue that the answer is two. All Existences Exhibit Void-Nature And Nirvana-Nature. These Natures Are The Reality Of All Existence. To Realise The Truth, We Have To Contemplate And Observe Our Worldly Existence. We Cannot Realise The Former Without Observing The Latter. Consider This Heart Sutra Extract, "Only When Avalokiteshvara Bodhisattva Practised The Deep Course Of Wisdom Of Prajna Paramita Did He Come To Realise That The Five Skandhas (Aggregates, And Material And Mental Objects) Were Void." Profound Wisdom Leads Us To The Realisation That All Existences Are Of Void-Nature. The Sutras Demonstrate That The Profound Principle Can Be Understood By Contemplating And Observing The Five Skandhas. We Cannot Realise The Truth By Seeking Something Beyond The Material And Mental World. The Buddha, Using His Perfect Wisdom, Observed Worldly Existence From Various Implications And Aspects, And Came To Understand All Existences. Teachings In Chinese Buddhism (6) Sunyata (Emptiness) In The Mahayana Context (Wikipedia) .But From The Profound Contemplation And Wisdom Of The Buddha And Mahabodhisattvas, We Know There Is No Such Reality. Instead, Egolessness (Non-Self) Is The Only Path To Understand The Reality Of The Deluded Life. All Existences Are Subject To The Law Of Causes And Conditions. These Include The Smallest Particles, The Relationship Between The Particles, The Planets, And The Relationship Between Them, Up To And Including The Whole Universe! From The Smallest Particles To The Biggest Matter, There Exists No Absolute Independent Identity. Egolessness (Non-Self) Implies The Void Characteristics Of All Existence. Egolessness (Non-Self) Signifies The Non-Existence Of Permanent Identity For Self And Existence (Dharma). Sunyata Stresses The Voidness Characteristic Of Self And Existence (Dharma). Sunyata And Egolessness Possess Similar Attributes. As We Have Discussed Before, We Can Observe The Profound Significance Of Sunyata From The Perspective Of Inter-Dependent Relationships. Considering Dharma-Nature And The Condition Of Nirvana, All Existences Are Immaterial And Of A Void-Nature. Then We See Each Existence As Independent Of Each Other. But Then We Cannot Find Any Material That Does Exist Independent Of Everything Else. So Egolessness Also Implies Void-Nature! From The Observation Of All Existences, We Can Infer The Theory Of Nirvana And The Complete Cessation Of All Phenomena. From The Viewpoint Of Phenomena, All Existences Are So Different From Each Other, That They May Contradict Each Other. They Are So Chaotic. In Reality, Their Existence Is Illusionary And Arises From Conditional Causation. They Seem To Exist On One Hand, And Yet Do Not Exist On The Other. They Seem To Be United, But Yet They Are So Different To One Another. They Seem To Exist And Yet They Do Cease! Ultimately Everything Will Return To Harmony And Complete Calmness. This Is The Nature Of All Existence. It Is The Final Resting Place For All. If We Can Understand This Reality And Remove Our Illusions, We Can Find This State Of Harmony And Complete Calmness. Teachings In Chinese Buddhism (6) Sunyata (Emptiness) In The Mahayana Context (Wikipedia) All Our Contradictions, Impediments And Confusion Will Be Converted To Equanimity. Free From Illusion, Complete Calmness Will Be The Result Of Attaining Nirvana. The Buddha Emphasised The Significance Of This Attainment And Encouraged The Direct And Profound Contemplation On Void-Nature. He Said, "Since There Is No Absolute Self-Nature Thus Every Existence Exhibits Void-Nature. Because It Is Void, There Is No Rising Nor Falling. Since There Is No Rising Nor Falling, Thus Everything Was Originally In Complete Calmness. Its Self-Nature Is Nirvana." Teachings In Chinese Buddhism (6) Sunyata (Emptiness) In The Mahayana Context (Wikipedia).The Scholars Of Tien Tai Called It The "Embodied Nature". (This Is The Buddha-Nature That Includes Both Good And Evil.) The Scholars Of Xian Shou Say, "It Is Arising From Primal Nature", And The Scholars Of Chan (Zen) Say, "It Is Nature That Causes The Rising Of Things". All Dharma Is Dharma-Nature. It Is Not Different From Dharma-Nature. Dharma And Dharma-Nature Are Not Two Separate Identities, "Phenomena" And "Nature" Is Also Not Distinguishable Either. In Other Words, There Is No

Difference Between Principle (Absolute) And Practice (Relative). Teachings In Chinese Buddhism (6) Sunyata (Emptiness) In The Mahayana Context (Wikipedia). In Other Words, All Dharma Arises From Causes And Conditions, And All Dharma Is Empty In Nature. The Law of Dependent Origination (Existence) And The Nature Of Emptiness Is Neither The Same Nor Different. They Exist Mutually. The Truth Of "Sunyata" And "Existence", And "Nature" And "Phenomena" Are Not In Conflict With Each Other. Unlike The Scholars Of The Dharmalaksana Sect Who Explain The Dharma Only From The Aspect Of Dependent Origination, Or The Scholars Of Dharma-Nature That Explain The Existence Of Dharma Only From The Aspect Of Dharma-Nature, The Scholars Of Madhyamika Explain The Truth Of The Dharma From Both Aspects. Hence This Is Called The Middle Path Which Does Not Incline To Either Side. Teachings In Chinese Buddhism (6) Sunyata (Emptiness) In The Mahayana Context (Wikipedia) **C S Lent, P D Tougaw, W Porod and G H Bernstein** formulate a new paradigm for computing with cellular automata (CAS) composed of arrays of quantum devices-quantum cellular automata. Computing in such a paradigm is edge driven. Input, output, and power are delivered at the edge of the CA array only; no direct flow of information or energy to internal cells is required. Computing in this paradigm is also computing with the ground state. The architecture is so designed that the ground-state configuration of the array, subject to boundary conditions determined by the input, yields the computational result. proposal is put forth for a specific realization of these ideas using two-electron cells composed of quantum dots. The charge density in the cell is very highly polarized (aligned) along one of the two cell axes, suggestive of a two-state CA. The polarization of one cell induces a polarization in a neighboring cell through the Coulomb interaction in a very non-linear fashion. Quantum cellular automata can perform useful computing. The authors show that AND gates, OR gates and inverters can be constructed and interconnected. **C S Lent et al 1993 Nanotechnology 4 49 doi:10.1088/0957-4484/4/1/004 Quantum cellular automata C S Lent, P D Tougaw, W Porod and G H Bernstein** The basic features of the dynamics of open quantum systems, such as the dissipation of energy, the decay of coherences, the relaxation to equilibrium or non-equilibrium stationary state, and the transport of excitations in complex structures are of central importance in many applications of quantum mechanics. The theoretical description, analysis and control of non-Markovian quantum processes play an important role in this context. While in a Markovian process an open system irretrievably loses information to its surroundings, non-Markovian processes feature a flow of information from the environment back to the open system, which implies the presence of memory effects and represents the key property of non-Markovian quantum behaviour. **Heinz-Peter Breuer** review recent ideas developing a general mathematical definition for non-Markovianity in the quantum regime and a measure for the degree of memory effects in the dynamics of open systems, which are based on the exchange of information between system and environment. **Heinz-Peter Breuer** further study the dynamical effects induced by the presence of system–environment correlations in the total initial state and design suitable methods to detect such correlations through local measurements on the open system. **Heinz-Peter Breuer 2012 J. Phys. B: At. Mol. Opt. Phys. 45 154001 doi:10.1088/0953-4075/45/15/154001 Foundations and measures of quantum non-Markovianity** A fundamental step towards atomic- or molecular-scale spintronic devices has recently been made by demonstrating that the spin of an individual atom deposited on a surface1, or of a small paramagnetic molecule embedded in a nanojunction2, can be externally controlled. An appealing next step is the extension of such a capability to the field of information storage, by taking advantage of the magnetic bistability and rich quantum behaviour of single-molecule magnets3, 4, 5, 6 (SMMs). Recently, a proof of concept that the magnetic memory effect is retained when SMMs are chemically anchored to a metallic surface7 was provided. However, control of the nanoscale organization of these complex systems is required for SMMs to be integrated into molecular spintronic devices8, 9. Here we show that a preferential orientation of Fe4 complexes on a gold surface can be achieved by chemical tailoring. As a result, the most striking quantum feature of SMMs—their stepped hysteresis loop, which results from resonant quantum tunnelling of the magnetization5, 6—can be clearly detected using synchrotron-based spectroscopic techniques. With the aid of multiple theoretical approaches, **M. Mannini et al** relate the angular dependence of the quantum tunnelling resonances to the adsorption geometry, and demonstrate that molecules predominantly lie with their easy axes close to the surface normal. Findings of **M. Mannini et al prove** that the quantum spin dynamics

can be observed in SMMs chemically grafted to surfaces, and offer a tool to reveal the organization of matter at the nanoscale. **Quantum tunnelling of the magnetization in a monolayer of oriented single-molecule magnets M. Mannini et al** Nature 468, 417–421 (18 November 2010) doi: 10.1038/nature09478 Supposing That There Were Indeed An "Energy Of Contraction" Constant In All Centers Of Force Of The Universe, It Remains To Be Explained Where Any Difference Would Ever Originate. It Would Be Necessary For The Whole To Dissolve Into An Infinite Number Of Perfectly Identical Existential Rings And Spheres, And We Would Therefore Behold Innumerable And Perfectly Identical Worlds COEXISTING [Nietzsche Underlines This Word Twice] Alongside Each Other. Is It Necessary For Me To Admit This? Is It Necessary To Posit An Eternal Coexistence On Top Of The Eternal Succession Of Identical Worlds? The Eternal Return: Genesis And Interpretation BY PAUL D'IORIO"The Cyclical Hypothesis, So Heavily Criticized By Nietzsche (VP II 325 And 334), Arises In This Way." In Fact, Nietzsche Was Not Criticizing The Cyclical Hypothesis But Only The Particular Form Of That Hypothesis Presented In Vogt's Work. All Of Nietzsche's Texts Without Exception Speak Of The Eternal Return As The Repetition Of The Same Events Within A Cycle Which Repeats Itself Eternally. If Deleuze's Interpretation Holds That The Eternal Return Is Not A Circle, Then What Is It? A Wheel Moving Centrifugally, Operating A "Creative Selection," "Nietzsche's Secret Is That The Eternal Return Is Selective" Says Deleuze: The Eternal Return Produces Becoming-Active. It Is Sufficient To Relate The Will To Nothingness To The Eternal Return In Order To Realize That Reactive Forces Do Not Return. However Far They Go, However Deep The Becoming-Reactive Of Forces, Reactive Forces Will Not Return. The Small, Petty, Reactive Man Will Not Return. Affirmation Alone Returns, This That Can Be Affirmed Alone Returns, Joy Alone Returns. Everything That Can Be Denied, Everything That Is Negation, Is Expelled Due To The Very Movement Of The Eternal Return. We Were Entitled To Dread That The Combinations Of Nihilism And Reactivity Would Eternally Return Too. The Eternal Return Must Be Compared To A Wheel; Yet, The Movement Of The Wheel Is Endowed With Centrifugal Powers That Drive Away The Entire Negative. Because Being Imposes Itself On Becoming, It Expels From Itself Everything That Contradicts Affirmation, All Forms Of Nihilism And Reactivity: Bad Conscience, Ressentiment..., We Shall Witness Them Only Once. [...] The Eternal Return Is The Repetition, But The Repetition That Selects, The Repetition That Saves. Here Is The Marvelous Secret Of A Selective And Liberating Repetition. There Is No Need To Remind The Reader That Neither The Image Of A Centrifugal Movement Nor The Concept Of A Negativity-Rejecting Repetition Appears Anywhere In Nietzsche's Writings, And Indeed Deleuze Does Not Refer To Any Text In Support Of This Interpretation. Further, One Could Highlight That Nietzsche Never Formulates The Opposition Between Active And Reactive Forces, Which Constitutes The Broader Framework Of Deleuze's Interpretation. For Some Years, Marco Brusotti Has Called Attention To The Fact That Deleuze Introduced A Dualism That Does Not Exist In Nietzsche's Writings. To Be Sure, The German Philosopher Describes A Certain Number Of "Reactive" Phenomena (For Example, In The Second Essay Of The Genealogy Of Morality, § 11, He Talks About "Reactive Affects" [Reaktive Affekte], "Reactive Feelings" [Reaktive Gefühlen], Reactive Men [Reaktive Menschen]); But These Are Nonetheless The Result Of Complex Ensembles Of Configurations Of Centers Of Forces That Remain In Themselves Active. Neither The Word Nor The Concept Of "Reactive Forces" Ever Appears In Nietzsche's Philosophy. The Eternal Return: Genesis And Interpretation BY PAUL D'IORIO Some Parts Are Deleted Due To Spatial Constraints. Please Pardon Me For The Deletion. Deleuze Opposes The Historical Course Of The Hegelian Notion That Confronts Struggles And Finally Dialectizes The Negative And Results In A Consoling Teleology Leading To The Triumph Of The Idea Or The Liberation Of The Masses With The Centrifugal Movement Of The Wheel, Which Simply Ejects The Negative. It Is Still A Case Of A Consoling And Optimistic Teleology, Which, Instead Of Confronting The Weight Of History, The Grief And The Negative, Makes It Disappear In One Centrifugal Stroke Of A Magic Wand. There Is Reason To Worry That This Be A Case Of Repression, Which, Unable To Dialectize Or Accept The Negative, Simply Seeks To Exorcise It In One Gesture Of "Creative Selection." But Exorcism Is A Feat Of Magic And Not Of Philosophy: It Is Unfortunately Not Enough To Make The Negative Disappear. In All Probability, The Negative Will Come Back With A Vengeance. In Contrast To Deleuze's "Affirmation Of Affirmation", Which Affirms Only Affirmation, Nietzsche Conceives Of The Eternal Return From

A Rigorously Non-Teleological Perspective As The Accomplishment Of A Philosophy Strong Enough To Accept Existence In All Its Aspects, Even The Most Negative, Without Any Need To Dialecticize Them, Without Any Need To Exclude Them By Way Of Some Centrifugal Movement Of Repression. It Denies Nothing And Incarnates Itself In A Figure Similar To The One Nietzsche, In Twilight Of The Idols, Draws Of Goethe: Such A Spirit, Who Has Become Free Stands In The Middle Of The World With A Cheerful And Trusting Fatalism In The Belief That Only The Individual Is Reprehensible, That Everything Is Redeemed And Affirmed In The Whole-He Does Not Negate Anymore. Such A Faith However, Is The Highest Of All Possible Faiths: I Have Baptized It With The Name Of Dionysus. The Eternal Return: Genesis and Interpretation BY PAUL D'IORIO The Most Complete Theory Of Mind That Comports With The Holographic Principle Of Mind, As Developed Here, Is The Microgenetic Theory Of Neurologist Jason Brown. Brown (2005) Presents An Evolutionary Theory Of Values, Morals, And Ethics, Building On The Microgenetic Theory And Process Theory Of His Earlier Works (1988, 1991, 1996, 1998, And 2000). Brown Is Work Has Been A Progression That Began With The Study Of Brain Processes In Neurological Subjects With Brain Injuries Or Lesions. In His Earlier Work (Brown, 1988; 1991) He Described The Process Of Microgenesis As A Process Of Elaboration Of Mental Contents In An Evolutionary And Developmental Hierarchy Of Brain Structures Within The Process Of The Genesis Of The Mental State, And Related The Hierarchy Of The Genesis Of The Mental State To Disturbances Of Language Comprehension Or Expression (Aphasia), Of Knowledge (Agnosia), And Or Purposeful Movement (Apraxia). The Fundamental Neurology Of These Disturbances Had Been Described In Brownís Previous Work (1972, 1988), And In The Seminal Work Of The Russian Neurologist And Neuropsychologist, Alexander Luria (1966). Brown (1988, 1991, 1996, 1998) Created A Process Theory Of Mind On The Basis Of His Neurological Observations, And Went On From His Neurological Work To Incorporate Process Theory In Philosophy, As Developed Principally By Henri Bergson (1911/1998). And Alfred North Whitehead (1925/1953, 1929/1978). Bergson Had Pioneered The Concept Of An Irreducible Duration Of Experience. This Concept Was Elaborated By Whitehead, In Keeping With Discontinuities Of Particles, As "Actual Entities" Of A Rudimentary Sort, As They Make "Quantum Leaps" Along Their Trajectories, Halting Or Persisting At A Given Location For A Short Period Of Time Or Duration Before Leaping To The Next. Whitehead Employed These "Quantum Leaps" To The Process By Which Experience Arises In "Actual Entities" At The Most Fundamental Level Of Actuality, Through The Epochal Or Halting Duration Of A Discontinuous Process (Whitehead, 1925/1953, 1929/1978). This Halting Allows Actual Entities To Participate In A Process Of Internal Relations, Relations On The "Inside" Of Things. According The Holographic Principle, This "Inside" Is In The Holographic Surface Of The Relevant Region Of Spacetime Of The Actual Entity, And Within Other Surfaces That Are Non-Locally Connected With That Surface. The Holographic Surface Is Thus The Locus Of Non-Local Information And Experience, In Our Interpretations Of Whitehead Is Metaphysics. Internal Relations Occur Through Feeling, Or "Prehension," Which Is Nothing More Than The Subjective Quality Of Non-Local Experience. The Internal Relations Of The Becoming Entity Occur In A Seriality Of Prior Becomings Of The Actual Entity, With The Entity Inheriting All Of Its Causal Past As Well As Its Relevant Feelings Or Prehensions From Prior "Occasions" Or Quanta Of Experience. This Process Is, In The Holographic Model, The Process By Which Former Surfaces, Now-Pasts, Are Elaborated On To The Now-Present. Prehensions Also Occur With Other Actual Entities Throughout The Universe, With Which They Are Non-Locally Connected, And With Neo-Platonic "Eternal Objects," Existing In An Eternal Heaven. The "Ingression" Of Eternal Objects Is Fundamental To Each Duration Of Becoming Of The Entity, And, On This Basis, Whitehead Formulated The Notion Of An Eternal Heaven In Constant Intercourse All Of Its Creation. There Are Clear Parallels Between Whitehead's Cosmology And The Cosmology Introduced Here According To The Holographic Principle, In That The "Heaven" May Be Regarded As The Holographic Boundary, In Constant Intercourse With The Relevant Volume Of Spacetime. This Interconnection Of Feelings Or Prehensions, According To Their Relevance, Produces Intensities And Contrasts Which Give Rise To The Creative Advance, Which, For Whitehead (1929/1978) Is The Process Which Impels The Movement From The Physical To The Mental Pole Of Process. With Each Halting, Internally-Timeless Duration There Is A Concrescence Or Integration Of These Feelings, Which, When Complete, Constitutes A New Actual Entity Or

Actual Occasion (They Are The Same Thing) At A New Locus In Spacetime. This Movement To A New Locus, Or Transition, Is, Fundamentally, The Quantum Leap. MICROGENESIS AND THE HOLOGRAPHIC PRINCIPLE The Holographic Principle Theory Of Mind MARK GERMINE Institute For Psycho Science If the Holographic Principle is true, than it must be the fundamental principle of mind. The brain has no way around the Holographic Principle. For the reductionist, the Holographic Principle is the ultimate reduction. It applies to the most minuscule level of what we can observe, and beyond. For the Universalist, the Holographic Principle gives us the ultimate universal. It extends to the limit of our Universe, the Universal holographic boundary, and beyond. For the Phenomenalist, the Holographic Principle gives us the ultimate ground of our phenomenal perception, our grounding in the Universal "now," in the now-present of the Universal holographic boundary, moving outward from now-pasts to now-futures. Wheeler (1988) has said that all of reality is information, and that other physical quantities are "mere incidentals." Information monism is gaining popularity in physics. By breaking the dichotomy of between information and experience, we find a deep connection between Wheeler is monism and the experiential monism of Whitehead, sometimes called panexperientialism, which relates to the later theological concept of his student, Charles Hartshorne, panentheism, God inside of everything (Hartshorne, 1964). There thus seems to be a convergence of the concepts of information, experience, and spirituality. There is not one universe, but many parallel universes, or, more accurately, a vast superposition of universes called the multiverse (Penrose, 2004). However, as explained in our treatment of the double-slit experiment, in the observation of events on the quantum level, the individual observer sees only one of these vast superpositions, not a summation of a vast number of potentials. This has been explained in terms of the many worlds or many minds theories, in which there are multiple copies of the same observing individual, but this particular issue remains unresolved, leaving physics ungrounded in reality (Penrose, 2004). The concept of One Mind is not only more parsimonious than that of many minds, but also lets us out of the bizarre and counterintuitive idea that there are multiple copies of our own selves, which exist as mere potential, and are thus not actual. There are many possible universes, but there is only One Mind, which determines events on the quantum level, and thus creates (eb)our Universe. As we had discussed previously, the quantum level provides (eb) the essential ground of the Holographic Principle, such that (e) quantum-level holographic surfaces are (=) elaborated at higher levels, manifesting higher orders of information from the quantum "world." In this process, there is a reduction of the wave function or potentialities of quantum fields. At the level of consciousness, this entails freedom to choose which observations we make (Stapp, 1997), which gives us the capacity to think and to make decisions. These capacities are the basis of individuality, self-determination, judgment, and values. The progressive evolution of the manifestation of (e) Mind through higher orders of experience, leading to (eb) higher orders of consciousness, is entailed by (e) the Holographic Principle. Again, we are dealing here with (e&eb) levels of description, with the multiverse of all potentials fundamentally supporting (eb) the single Universe we collectively observe. The multiverse is (=) the wave function of the Universe (Penrose, 2004). The recursive integration of (e) nested hierarchies of holographic surfaces brings out a single actuality in consciousness from a wave function that is unconscious. Consciousness thus gives us information at a level of experience that is causal. In this sense, we partake in the creation of the Universe. We participate in creation, and this participation, when fully realized, leads us to higher levels of consciousness and of realization. As a system, the biosphere that we live in has a holographic surface, creating a deep sense of ecology as we collectively move toward a planetary consciousness. It is only when the Universal Holographic Boundary reaches the information storage and processing capacity needed for the requisite biological and biochemical complexity that consciousness evolves in living things. This evolution is (=) natural and spontaneous, since consciousness gives rise to what is actual, as opposed to what is merely potential. In this sense, consciousness is still in the process of creating our Universe, and levels of higher consciousness will continue to evolve. We are all in the same "now," and that now is defined by the present Universal Boundary. This assures us that our experience is Universal, and does not pass with time. This is the fundamental basis of memory and of cognition. The identity of mind and brain is a myth. We have a continual, internal or non-local relation with the Universe, as it has with us. Once this mystery is resolved, the myth is no longer needed, and there is a confluence of science and spirituality.Experience is primary, information is secondary.

We can only gather information from experience, whether it is in the laboratory or in life. We cannot measure the information on the surfaces of systems. The physicists that have formulated the Holographic Principle for all systems are quite aware of this, or else the principle would have been established or discredited. We cannot measure what we experience. It is intangible, yet it is all we know to be actual. Everything else is inferred. Because it cannot be measured, it has been fundamentally disregarded by mainstream science. Materialism is considered scientific, while idealism is considered unscientific. But arenít ideas, fundamentally, made of information? Brain science has mistaken the representation of information for information itself, and has tied those representations to matter and energy. Consciousness, the highest order of information, has generally been regarded as superfluous, something that needs to explained away, or altogether ignored. Yet it is the only "thing" that reaches our awareness. Consciousness comes at a price. For everything that becomes conscious, there must be something that becomes conscious. Consciousness is certainty, and its complement, unconsciousness uncertainty. Consciousness is the particle nature of experience. It has definiteness about it. The unconscious is the wave nature of experience, it is like the metaphysical cloud of unknowing. If we are the most conscious of animals, then we must also be the most unconscious. Perhaps this is the predicament of humankind. The Holographic Principle of Mind leads us naturally from the most fundamental experiences, existing as quantum potentials from the conformations of proteins down through the fields of electrons, through their manifestation upward through a recursion or successive applications of the same holographic process, through higher levels of experience, to the emergence of consciousness as higher and higher orders of experience. The quantum Holographic Principle Mind does not require anything more quantum than is obviously present at the microscopic and submicroscopic levels, as it represents successive orders of manifestations from these levels. Recursion also applies here in the sense that it is used in computer science, in which the function of the part depends on the function of the whole. A program, as a part, cannot work without a functioning whole, the operating system. Recursive wholes which are, for us, supra-conscious, are on the group, species, planetary, and universal levels. As individuals, our consciousness cannot function without recursion to the Universal Consciousness, even though we may be unaware that such Universal Consciousness exists. Reaching upward to these supra-conscious levels is a spiritual process, making transcendence the ultimate solution to the unconscious human predicament. What was once supra-conscious becomes unconscious through a process of conditioning? We are born as creatures of the Earth and of the Universe, as evidenced by the beliefs and practices of "primitive peoples." There is evidence from cave paintings that our hominid ancestors experienced a kind of holographic perception (Combs, 1996), which could constitute our early connection with the holographic subtext of reality, and which we might then propose existed in our animal ancestors, and in extant animal species. If this is the case, than microgenesis would entail the recapitulation of this holographic experience as it progresses through our ancestral past. The supra-conscious mind seems to envelope perinatal experiences, and Stanislav Grof (1994) has developed techniques to gain access to these experiences, as well as to the earlier experiences of our human and animal ancestral lineage, and of our universal history. Grof (1994) concludes: "Our consciousness seems to have the amazing capacity to directly access the earliest history of the universe ñ witnessing dramatic sequences of the Big Bang, the formation of the galaxies, the birth of the solar system, and the early geophysical processes on this planet billions of years ago." The Holographic Principle Theory Of Mind MARK GERMINE Institute For Psycho Science. There Are Similarities Between The Patterns Of Holography And Of Psychological Transference, Where Holography Is The Process Of Recording And Reconstructing Holograms. Employing A Theoretical Perspective Using A Hermeneutic Method, This Dissertation Parallels Holography With Transference, Offering Another Way To Encounter Transference By Showing Similarities Between The Processes Of Each And The Results Of Each. Though Complex, Infinitely Varying, And Unique, Their Patterns Are Clearly Identifiable. Thus They Are A Metaphorical Fit To The Concept Of Strange Attractors In Physics And A More Literal Fit To The Concept Of Archetypes In Depth Psychology Or Dynamic Psychology, Psychology Which Attends To The Living, Autonomous Unconscious. This Study Explains How Holography Models Transference, What A Hologram Is And How It Works, And How Depth Psychology Understands Of The Interaction Between Consciousness And The Unconscious Is Related To The Hologram. It Describes Transference And Related Psychological Processes As Understood In Six

Different Schools Of Depth Psychological Thought. It Shows That The Underlying Pattern Or Strange Attraction Between Transference And Holography Extends To Other Processes Both Within And Outside The Field Of Psychology, Processes Such As Projection, Projective Identification, Splitting, Memory, Biology, Creative Discovery, Theology, Synchronicity, Chaos, And Nonlocality. By Identifying The Similar Patterns Of These Processes, This Study Demonstrates The Existence Of An Underlying Holographic Archetype In Which Essential Qualities Of The Whole Are Present In Each Of The Parts Of The Whole: The Visual Image Of The Overall Hologram Is Present In Each Component Part Of The Hologram, The Autonomy Of The Overall Human Is Present In Each Conscious And Unconscious Component Part Of The Human Psyche. By Noting Differences As Well As Similarities In These Processes, It Suggests An Inventory Of The Qualities Of The Holographic Archetype. This Study Furthers Understanding Of The Pervasiveness, Force, And Autonomy Of The Unconscious Acting Through Transference And Projection By Identifying A Group Projection Of Domestic Violence Lying At The Core Of The Christian Myth. This Study Also Furthers Understanding Of The Concept Of Transference By Providing A Reflection Hologram Of The Human Psyche As An Artistic Work And As A Visual Metaphor Of Transference. Strange Attractors: Transference, Holography, And An Archetype Burke, J. (2003). Strange Attractors: Transference, Holography, And An Archetype (Doctoral Dissertation, Pacifica Graduate Institute, 2003). Onticology, Like All Variations Of Object-Oriented Ontology, Is Realist In Its Orientation. In Defending A Realist Ontology Onticology Holds That The Vast Majorities Of Objects, Actants, Beings, Or Entities Are Independent Of Humans And Are What They Are Regardless Of Whether Any Humans Regard Them Or Register Them. In Short, Onticology Rejects Any Anthropomorphic, Idealist, Or Anti-Realist Thesis To The Effect That To Be Is To Be The Correlate Of Mind, Spirit, The Body, The Human, And Language Or Otherwise. While It Is Certainly The Case That Knowledge Is Necessarily Dependent On The Object To Which It Relates, The Converse Does Not Hold True. Objects Are Not Dependent On Being Known, Regarded, Perceived, Or Spoken About. As Such, And To Put It In Aristotelean Terms, Knowledge Is An Accident Of Objects, Not Objects An Accident Of Knowledge. As Althusser So Nicely Puts It, "[N]O Doubt There Is A Relation Between Thought-About-The-Real And This Real, But It Is A Relation Of Knowledge, A Relation Of Adequacy Or Inadequacy Of Knowledge, Not A Real Relation, Meaning By This A Relation Inscribed In That Real Of Which The Thought Is The (Adequate Or Inadequate) Knowledge" (Reading Capital, 96). Althusser Goes On To Remark That "[T] He Distinction Between A Relation Of Knowledge And A Relation Of The Real Is A Fundamental One: If We Did Not Respect It We Should Fall Irreversibly Into Either Speculative Or Empiricist Idealism" (Ibid.). Onticology Categorically Endorses Althusser's Verdict It Is A Fundamental Necessity To Distinguish Between Those Relations That Belong To The Object And Those That Belong To Knowledge. Contemporary Philosophy Continuously Confuses These Two Very Different Sorts Of Relations. Naturally The Question Arises Of How It Is Possible To Surmount Our Relation To The Object So As To Determine Whether Objects Themselves Possess The Properties We Encounter In Relating To Objects. In Other Words, Given That We Can Only Ever Relate To The Object In Relating To The Object How Is It Possible To Surmount This Relation To Get At The Being Of The Object Itself? Much More Will Have To Be Said About This Later– And The Answers Will Be Surprising With Respect To Standard Prejudices About Realism –However, For The Moment It Can Be Said That Onticology Takes Its Epistemological Inspiration From The Transcendental Realism Of Roy Bhaskar. Among Other Things, Bhaskar Sought To Provide A Transcendental Grounding For The Sciences. Insofar As Onticology Defends The Thesis That The Field Of Being Is Much Faster Than The Field Of Objects Investigated By The Natural Sciences, It Parts Way With The Thesis That The Domain Of Being Is Exhausted By The Domain Of Natural Objects. However, The General Form of Bhaskar's Argument Holds for Our Realist Purposes. A Transcendental Argument Seeks To Elucidate The Conditions Under Which Certain Acknowledged Practices And Forms Of Cognition Are Possible. Kant, For Example, Asked What Must Be The Case For Mathematical Judgments To Be Possible. How Is It Both That We Are Able To Extend Our Knowledge, As If By Magic, Through Mathematical Judgments And, More Significantly, That These Judgments Are Able To Provide Genuine Knowledge Of The World Despite The Fact That These Forms Of Reasoning Are Not Based On Experience? Part of Kant's Argument Consisted in Claiming That Mind Imposes the Forms of Space and Time on

the Data of Experience. In Other Words, Space And Time Are Not Attributes Of Being Itself But Rather Of The Mind That Regards Being. Insofar, Kant Argues, As Mathematics Is Ultimately A Rumination On The Nature Of Space And Time Taken In Their Most Abstract Form And Insofar As The Mind Imposes Space And Time On The Manifold Of Sensation, It Thus Follows That A Priori Judgments About The Nature Of Spatio-Temporal Relations Are Possible That Anticipate The Structure Of Actual-Space Times Without Directly Experiencing These Space-Times. Why? Because Any Manifold Of Sensation Must Necessarily Be Structured By These Forms Imposed By Intuition Onticology– A Manifesto For Object-Oriented Ontology Part I Posted By Larval Subjects Under Object-Oriented Philosophy. ……. it is necessary to distinguish the being of objects from the manifestation of objects. While objects are acts, these acts are not identical with their performance in either nature (events where no humans are about to perceive them) or with their performance for humans. Rather, the proper being of the object is not its performance or manifestation, but the generative mechanism that serves as the condition under which these performances or manifestations are possible. As Graham Harman will argue– though in a very different theoretical constellation –the being of objects is essentially withdrawn or hidden. No one has ever perceived a single object, but we do perceive all sorts of effects of objects. Traditional epistemology has confused these effects with the objects themselves. Fortunately we do occasionally manage to cognize objects through a sort of detective work that infers these generative mechanisms from their effects; without, for all this, ever exhausting the infinity of a single object. At any rate, if objects were not withdrawn in this way, the practice of experiment would be unintelligible. This leads to Bhaskar's answer to the first question: What must the world be like for science to be possible? Note, this is not a question about mind or culture, but about the world itself, regardless of whether or not humans exist. Once again, knowledge is an accident of objects, not objects an accident of consciousness or cognition. If science is to be possible– and I would argue, if any human practice is to be possible –then the world must be structured and differentiated. The world must have joints or, as Harman puts it, the world must be composed of "chunks". Why is this case? Let us return to the question of experimentation and the conditions under which experiment is possible. We will adopt two possible hypotheses pertaining to the ontological nature of the world independent of humans: 1) As certain mystics and contemporary crypto-mystics would have it, the world is an undifferentiated One-All that is only subsequently segmented or partitioned into discrete beings by some form of human agency whether this be through cognition or language (in the case of language we might think of Saussure's and Hjelmslev's undifferentiated "sonorous matter"). 2) Entities are the sum totality of their relations to all other entities in the universe. The first hypothesis is easily dispatched on two grounds: First, this hypothesis fails to register the contradiction in its own utterance. At the level of explicit content, it claims that the world is an undifferentiated One-All that is only subsequently segmented into discrete beings, yet what it misses at the unconscious level of its own utterance is that it registers at least one structured differentiation that is not undifferentiated within this One-All: Namely, the agency through which the One-All subsequently comes to be differentiated. Certain anti-realist, transcendental philosophers will, in a gesture that is all too cute, claim that the agency by which the world is segmented cannot properly be said to exist, thus attempting to resolve this contradiction through a sleight of hand. However, as Meillassoux has shown in After Finitude, all attempts on the part of transcendental anti-realist philosophies to treat the transcendental subject as a non-existing or non-objectile agency that does not itself exist end up, all too clearly, attaching that conception of finitude and the segmentary work with which it is charged to the body (a differentiated being or generative mechanism). Second, suppose the anti-realist transcendental philosophers were to convince us through his appeals to the non-existence of the transcendental, would we still encounter problems? Like Atlas, transcendental subjectivity is charged with the monumental task of segmenting the formless Apeiron of the One-All from out of primal chaos into a supremely segmented world. But this world appears to us to be too slippery for even a titan like Atlas to grasp. Were the world, prior to and independent of humans genuinely a formless Apeiron it would contain no differences providing hand-holds for Atlas to grasp in his segmenting activity. Consequently, no segments could ever come into being. Yet everywhere we encounter segments, so the world must not be a formless One-All, but must rather be structured and differentiated even if structure and differentiation are transformed in their encounter with the human. Onticology– A

Manifesto For Object-Oriented Ontology Part I Posted By Larval Subjects Under Object-Oriented Philosophy. The neural networks of the human brain act as very efficient parallel processing computers co-ordinating memory related responses to a multitude of input signals from sensory organs. Information storage, update and appropriate retrieval are controlled at the molecular level by the neuronal cytoskeleton which serves as the internal communication network within neurons. Information flow in the highly ordered parallel networks of the filamentous protein polymers which make up the cytoskeleton may be compared to atmospheric flows which exhibit long-range spatiotemporal correlations, i.e. long-term memory. Such long-range spatiotemporal correlations are ubiquitous to real world dynamical systems and are recently identified as signature of self-organized criticality or chaos. The signatures of self-organized criticality i.e. long-range temporal correlations have recently been identified in the electrical activity of the brain. A recently developed non-deterministic cell dynamical system model for atmospheric flows predicts the observed long-range spatiotemporal correlations as intrinsic to quantum-like mechanics governing flow dynamics. The model visualizes large scale circulations to form as the result of spatial integration of enclosed small scale perturbations with intrinsic two-way ordered energy flow between the scales. Such a concept maybe applied for the collection and integration of a multitude of signals at the cytoskeletal level and manifested in activation of neurons in the macroscale. The cytoskeleton networks inside neurons may be the elementary units of a unified dynamic memory circulation network with intrinsic global response to local stimuli. Cite as: arXiv: chao-dyn/9809003.The neural network of the human brain responds as a unified whole memory bank to a multitude of input signals from the environment and functions with a high degree of robustness and stability. The three aspects of neural networks memory bank are, storage, real-time update and retrieval. The memory is believed to be embedded in the strength of the numerous connections or synapses in the network. Sensory inputs (electrical) produce particular patterns of activity in groups of neurons which then trigger optimal response to the input signal. The cooperative response of millions of neurons to a multitude of input signals has been compared to a very efficient parallel processing computer with neurons and their synaptic connections as fundamental units of information processing, like switches within computers. However, recent studies by Hameroff et al [1,2] and Rasmussen et al [3] show that neurons and synapses are extremely complex and resemble entire computers, rather than switches. The interiors of neurons (and other eucaryotic cells) are now known to contain highly ordered parallel networks of filamentous protein polymers collectively termed the cytoskeleton. Information storage, update and appropriate retrieval are controlled at the molecular level by the neuronal cytoskeleton which serves as the internal communication network within neuron. Organization of information at the molecular level in the cytoskeletal network contributes to the overall response of each neuron and the collective activity pattern of neurons then governs the response to the environmental stimuli. The general awareness or consciousness of the individual to environment may also be governed by the overall background activity pattern of the neurons and their cytoskeletal networks. Coherent signal flow patterns in neural networks may form the basis for general consciousness and response to stimuli (external or internal). Inputs signals trigger spontaneous appropriate coherent pattern formation in the activity of the neurons with implicit spatial correlations in the activity pattern. The time variation of electrical activity of the brain as recorded by the Electro Encephalogram (EEG) exhibits fluctuations on all scales of time, i.e. a broadband spectrum of periodicities (frequencies) contribute to the observed fluctuations [4]. Power spectral analysis which is used to resolve the component frequencies (f) and their intensities shows that the intensity (power) of the component frequencies follows the inverse power law form 1/fB where B is the exponent. Inverse power law form for power spectra of temporal fluctuations imply long-range temporal correlations, i.e. long - term memory of short - term fluctuations. The signatures of short - term fluctuations are carried as internal structures of long - term fluctuations. Time variation of spatial activity pattern in neural networks therefore has inbuilt long - term memory. Neural network activity patterns therefore exhibit long - range spatial and temporal correlations. Such non-local connections in space and time are ubiquitous to time evolution of spatially extended dynamical system in nature and are recently identified as signature of self-organized criticality [5]. Examples of dynamical systems, i.e. systems which change with time include atmospheric flows, electrical activity of the brain, heart rhythms, stock market price fluctuations, etc. Extended dynamical systems in nature have self similar fractal geometry. Self similarity implies

that submits of a system resemble the whole in shape. The world fractal coined by Mandelbrot [6] means fractional or broken Euclidean geometry appropriate for description of non-Euclidean structure generic to natural phenomena. The fractal dimension D is given by dlnM/dlnR where M is the mass contained within a distance R from a point within the extended object. A constant value for D implies uniform stretching on logarithmic scale for length scale range R. Objects in nature exhibit multifractal structure, i.e. the fractal dimension D varies with length scale R. Fractal architectures generic to nature support functions which exhibit fluctuations on all time scale, i.e. the fluctuations are irregular (nonlinear) and apparently chaotic. The association of fractal structures with chaotic dynamics has been identified in all dynamical systems in nature. Fractals, chaos and nonlinear dynamics or Chaos Science is now an area of intensive research in all branches of science [7]. Incidentally, Chaos Science began in 1963 with identification of sensitive dependence on initial conditions resulting in chaotic solution for computer realizations of deterministic nonlinear mathematical model of atmospheric flows and named appropriately deterministic chaos .The computed trajectory of time evolution exhibits fractal geometry. The discipline of nonlinear dynamics and chaos began with investigation of universal characteristics of deterministic chaos in nonlinear mathematical models of dynamical systems in all branches of science. In mathematics, the Cantorian fractal space-time is now associated with reference to quantum mechanical objects [8,9,10]. Further, El Naschie has shown that fractal structures (space-time) incorporate the golden mean equal to (1+Ö 5)/2 =1.618 in their architecture signifying ordered signal /information flow in the fractal network. The golden mean is incorporated in the fractal architecture of the cycloskeleton network [11] which plays a very important role in sub-consciousness to consciousness process integration [12,13]. Surprisingly similar chaotic behavior in space and time was found to be exhibited by all real world dynamical systems. Fractal structure to the spatial pattern concomitant with chaotic (irregular) dynamics has now been identified to be intrinsic to physiological and biological systems [14,15]. The branching interconnecting networks of neurons and intra-neuronal cytoskeleton networks are fractal structures which generate electrical signal pattern with self-similar fluctuations on all scales of time characterised by 1/fB power law behavior for the power spectrum. Such inverse power law form for spectra of temporal fluctuations implies long-range temporal correlations, i. e., long term memory of short term fluctuations or events. Fractal architecture of neural networks supports and coordinates information (fluctuations) flow on all time and space time scales in a state of dynamic equilibrium, now identified as self-organized criticality, is ubiquitous to natural phenomena (living and non-living) and is independent of the exact details of the dynamical processes governing the space-time evolution. The physics of self-organized criticality or deterministic chaos is not yet identified. The physical mechanism governing self-organized criticality should be universally applicable to diverse biological, physical, chemical and other dynamical systems. In this paper a universal cell dynamical system model for self-organized criticality applicable to neural networks of the brain is summarised [16, 17, and 18]. This model was originally developed to explain the observed self-organized criticality in atmospheric flows [19, 20, and 21]. Therefore a brief description of the model with respect to atmospheric flows in first described followed by application to neural networks. Atmospheric flows exhibit self-organized criticality or long-range spatiotemporal correlations manifested in the self similar fractal geometry to the global cloud cover pattern concomitant with inverse power law form 1/fB for power spectrum of temporal fluctuations in meteorological variables such as temperature, pressure, etc. documented by Tessier et al [22]. The co-operative existence of fluctuations ranging in size (duration) from the turbulence scale of millimetres (seconds) to the planetary scale of thousands of kilometres (years) contribute to coherent weather pattern in atmospheric flows. Townsend [23] postulated that large eddies (waves) form in atmospheric flows as a chance configuration (envelope) of enclosed turbulent (small scale) eddies. A hierarchical continuum of eddies is therefore generated with larger eddies enclosing smaller eddies. Since large eddy is but the integrated mean of enclosed turbulent eddies, atmospheric eddy energy (kinetic) distribution follows normal distribution characteristics according to the Central Limit Theorem in Statistics. The eddy kinetic energy represented by square of eddy amplitude then represents the probability density. Such a result that the additive amplitudes of eddies, when squared, represent probability densities is observed in the subatomic dynamics of quantum system such as the electron or photon. Atmospheric eddy energy spectrum therefore follows quantum-like mechanical laws [19,20,21].

Condensation of water vapour in updraft regions of large eddies give rise to cloud formation while adjacent downdraft regions are associated with evaporation and cloud dissipation, thereby accounting for the discrete cellular structure to cloud geometry.\ Under the head of applications authors' describe the following: Frohlich [24] had described analogous self-organization of vibrational modes of all frequencies triggering coherent activity in biological functions. Insinna [25] has summarized Froehlich's coherent excitation concept as follows. More than 20 years ago Frohlich [24,26-28] introduced the concept of cooperative vibrational modes between proteins. Coherent oscillations in the range of 1010- 1012 Hz involving cell membranes, DNA and cellular proteins could be generated by interaction between vibrating electric dipoles contained in the proteins as a result of nonlinear properties of the system. Through long-range effects proper to Froehlich's nonlinear electrodynamics a temporospatial link, is, in fact, established between all molecules constituting the system. Single molecules may thus act in a synchronized fashion and can no longer be considered individual. New unexpected features arise from such a dynamic system, reacting as a unified whole entity [25]. Coherent Frohlich oscillations may be associated with the dynamical pattern formation of intraneuronal cytoskeletal architecture which coordinates and integrates information flow into the neuron and generates output signal. Hameroff and colleagues [1,29-31] have simulated such interaction in their cellular automata model. Conclusion they draw in is herein below: Fractal architecture to information flow path results in spatiotemporal integration of signals so that the fractal system responds as a unified whole to a multitude of input signals. Two disparate examples for such self-organized information flow networks are atmospheric flows and the neural networks of the human brain. **Cantorian Fractal Spacetime and Quantum-like Chaos in Neural Networks of the Human Brain A. M. Selvam (for reference s please see the original article. Models are given elsewhere in the papers series .** If The Holographic Principle Is True, Than It Must Be The Fundamental Principle Of Mind. The Brain Has No Way Around The Holographic Principle. For The Reductionist, The Holographic Principle Is The Ultimate Reduction. It Applies To The Most Minuscule Level Of What We Can Observe, And Beyond. For The Universalist, The Holographic Principle Gives Us The Ultimate Universal. It Extends To The Limit Of Our Universe, The Universal Holographic Boundary, And Beyond. For The Phenomenalist, The Holographic Principle Gives Us The Ultimate Ground Of Our Phenomenal Perception, Our Grounding In The Universal "Now," In The Now-Present Of The Universal Holographic Boundary, Moving Outward From Now-Pasts To Now-Futures. Wheeler (1988) Has Said That All Of Reality Is Information, And That Other Physical Quantities Are "Mere Incidentals." Information Monism Is Gaining Popularity In Physics. By Breaking The Dichotomy Of Between Information And Experience, We Find A Deep Connection Between Wheeler Is Monism And The Experiential Monism Of Whitehead, Sometimes Called Panexperientialism, Which Relates To The Later Theological Concept Of His Student, Charles Hartshorne, Panentheism, God Inside Of Everything (Hartshorne, 1964). There Thus Seems To Be A Convergence Of The Concepts Of Information, Experience, And Spirituality. There Is Not One Universe, But Many Parallel Universes, Or, More Accurately, A Vast Superposition Of Universes Called The Multiverse (Penrose, 2004). However, As Explained In Our Treatment Of The Double-Slit Experiment, In The Observation Of Events On The Quantum Level, The Individual Observer Sees Only One Of These Vast Superpositions, Not A Summation Of A Vast Number Of Potentials. This Has Been Explained In Terms Of The Many Worlds Or Many Minds Theories, In Which There Are Multiple Copies Of The Same Observing Individual, But This Particular Issue Remains Unresolved, Leaving Physics Ungrounded In Reality (Penrose, 2004). The Concept Of One Mind Is Not Only More Parsimonious Than That Of Many Minds, But Also Lets Us Out Of The Bizarre And Counterintuitive Idea That There Are Multiple Copies Of Our Own Selves, Which Exist As Mere Potential, And Are Thus Not Actual. There Are Many Possible Universes, But There Is Only One Mind, Which Determines Events On The Quantum Level, And Thus Creates Our Universe. As We Had Discussed Previously, The Quantum Level Provides The Essential Ground Of The Holographic Principle, Such That Quantum-Level Holographic Surfaces Are Elaborated At Higher Levels, Manifesting Higher Orders Of Information From The Quantum "World." In This Process, There Is A Reduction Of The Wave Function Or Potentialities Of Quantum Fields. At The Level Of Consciousness, This Entails Freedom To Choose Which Observations We Make (Stapp,

1997), Which Gives Us The Capacity To Think And To Make Decisions. These Capacities Are The Basis Of Individuality, Self-Determination, Judgment, And Values. The Progressive Evolution Of The Manifestation Of Mind Through Higher Orders Of Experience, Leading To Higher Orders Of Consciousness, Is Entailed By The Holographic Principle. Again, We Are Dealing Here With Levels Of Description, With The Multiverse Of All Potentials Fundamentally Supporting The Single Universe We Collectively Observe. The Multiverse Is The Wave Function Of The Universe (Penrose, 2004). The Recursive Integration Of Nested Hierarchies Of Holographic Surfaces Brings Out A Single Actuality In Consciousness From A Wave Function That Is Unconscious. Consciousness Thus Gives Us Information At A Level Of Experience That Is Causal. In This Sense, We Partake In The Creation Of The Universe. We Participate In Creation, And This Participation, When Fully Realized, Leads Us To Higher Levels Of Consciousness And Of Realization. As A System, The Biosphere That We Live In Has A Holographic Surface, Creating A Deep Sense Of Ecology As We Collectively Move Toward A Planetary Consciousness. It Is Only When The Universal Holographic Boundary Reaches The Information Storage And Processing Capacity Needed For The Requisite Biological And Biochemical Complexity That Consciousness Evolves In Living Things. This Evolution Is Natural And Spontaneous, Since Consciousness Gives Rise To What Is Actual, As Opposed To What Is Merely Potential. In This Sense, Consciousness Is Still In The Process Of Creating Our Universe, And Levels Of Higher Consciousness Will Continue To Evolve. We Are All In The Same "Now," And That Now Is Defined By The Present Universal Boundary. This Assures Us That Our Experience Is Universal, And Does Not Pass With Time. This Is The Fundamental Basis Of Memory And Of Cognition. The Identity Of Mind And Brain Is A Myth. We Have A Continual, Internal Or Non-Local Relation With The Universe, As It Has With Us. Once This Mystery Is Resolved, The Myth Is No Longer Needed, And There Is A Confluence Of Science And Spirituality.Experience Is Primary, Information Is Secondary. We Can Only Gather Information From Experience, Whether It Is In The Laboratory Or In Life. We Cannot Measure The Information On The Surfaces Of Systems. The Physicists That Have Formulated The Holographic Principle For All Systems Are Quite Aware Of This, Or Else The Principle Would Have Been Established Or Discredited. We Cannot Measure What We Experience. It Is Intangible, Yet It Is All We Know To Be Actual. Everything Else Is Inferred. Because It Cannot Be Measured, It Has Been Fundamentally Disregarded By Mainstream Science. Materialism Is Considered Scientific, While Idealism Is Considered Unscientific. But Aren't Ideas, Fundamentally, Made Of Information? Brain Science Has Mistaken The Representation Of Information For Information Itself, And Has Tied Those Representations To Matter And Energy. Consciousness, The Highest Order Of Information, Has Generally Been Regarded As Superfluous, Something That Needs To Explained Away, Or Altogether Ignored. Yet It Is The Only "Thing" That Reaches Our Awareness. Consciousness Comes At A Price. For Everything That Becomes Conscious, There Must Be Something That Becomes Conscious. Consciousness Is Certainty, And Its Complement, Unconsciousness Uncertainty. Consciousness Is The Particle Nature Of Experience. It Has Definiteness About It. The Unconscious Is The Wave Nature Of Experience, It Is Like The Metaphysical Cloud Of Unknowing. If We Are The Most Conscious Of Animals, Then We Must Also Be The Most Unconscious. Perhaps This Is The Predicament Of Humankind. The Holographic Principle Of Mind Leads Us Naturally From The Most Fundamental Experiences, Existing As Quantum Potentials From The Conformations Of Proteins Down Through The Fields Of Electrons, Through Their Manifestation Upward Through A Recursion Or Successive Applications Of The Same Holographic Process, Through Higher Levels Of Experience, To The Emergence Of Consciousness As Higher And Higher Orders Of Experience. The Quantum Holographic Principle Mind Does Not Require Anything More Quantum Than Is Obviously Present At The Microscopic And Submicroscopic Levels, As It Represents Successive Orders Of Manifestations From These Levels. Recursion Also Applies Here In The Sense That It Is Used In Computer Science, In Which The Function Of The Part Depends On The Function Of The Whole. A Program, As A Part, Cannot Work Without A Functioning Whole, The Operating System Recursive Wholes Which Are, For Us, Supra-Conscious, Are On The Group, Species, Planetary, And Universal Levels. As Individuals, Our Consciousness Cannot Function Without Recursion To The Universal Consciousness, Even Though We May Be Unaware That Such Universal Consciousness Exists. Reaching Upward To These Supra-Conscious Levels Is A Spiritual Process,

Making Transcendence The Ultimate Solution To The Unconscious Human Predicament. What Was Once Supra-Conscious Becomes Unconscious Through A Process Of Conditioning? We Are Born As Creatures Of The Earth And Of The Universe, As Evidenced By The Beliefs And Practices Of "Primitive Peoples." There Is Evidence From Cave Paintings That Our Hominid Ancestors Experienced A Kind Of Holographic Perception (Combs, 1996), Which Could Constitute Our Early Connection With The Holographic Subtext Of Reality, And Which We Might Then Propose Existed In Our Animal Ancestors, And In Extant Animal Species. If This Is The Case, Than Microgenesis Would Entail The Recapitulation Of This Holographic Experience As It Progresses Through Our Ancestral Past. The Supra-Conscious Mind Seems To Envelope Perinatal Experiences, And Stanislav Grof (1994) Has Developed Techniques To Gain Access To These Experiences, As Well As To The Earlier Experiences Of Our Human And Animal Ancestral Lineage, And Of Our Universal History. Grof (1994) Concludes: "Our Consciousness Seems To Have The Amazing Capacity To Directly Access The Earliest History Of The Universe Ñ Witnessing Dramatic Sequences Of The Big Bang, The Formation Of The Galaxies, The Birth Of The Solar System, And The Early Geophysical Processes On This Planet Billions Of Years Ago." The Holographic Principle Theory Of Mind MARK GERMINE Institute For Psycho Science There Are Similarities Between The Patterns Of Holography And Of Psychological Transference, Where Holography Is The Process Of Recording And Reconstructing Holograms Employing A Theoretical Perspective Using A Hermeneutic Method, This Dissertation Parallels Holography With Transference, Offering Another Way To Encounter Transference By Showing Similarities Between The Processes Of Each And The Results Of Each. Though Complex, Infinitely Varying, And Unique, Their Patterns Are Clearly Identifiable. Thus They Are A Metaphorical Fit To The Concept Of Strange Attractors In Physics And A More Literal Fit To The Concept Of Archetypes In Depth Psychology Or Dynamic Psychology, Psychology Which Attends To The Living, Autonomous Unconscious. This Study Explains How Holography Models Transference, What A Hologram Is And How It Works, And How Depth Psychology Understands Of The Interaction Between Consciousness And The Unconscious Is Related To The Hologram. It Describes Transference And Related Psychological Processes As Understood In Six Different Schools Of Depth Psychological Thought. It Shows That The Underlying Pattern Or Strange Attraction Between Transference And Holography Extends To Other Processes Both Within And Outside The Field Of Psychology, Processes Such As Projection, Projective Identification, Splitting, Memory, Biology, Creative Discovery, Theology, Synchronicity, Chaos, And Nonlocality. By Identifying The Similar Patterns Of These Processes, This Study Demonstrates The Existence Of An Underlying Holographic Archetype In Which Essential Qualities Of The Whole Are Present In Each Of The Parts Of The Whole: The Visual Image Of The Overall Hologram Is Present In Each Component Part Of The Hologram, The Autonomy Of The Overall Human Is Present In Each Conscious And Unconscious Component Part Of The Human Psyche. By Noting Differences As Well As Similarities In These Processes, It Suggests An Inventory Of The Qualities Of The Holographic Archetype. This Study Furthers Understanding Of The Pervasiveness, Force, And Autonomy Of The Unconscious Acting Through Transference And Projection By Identifying A Group Projection Of Domestic Violence Lying At The Core Of The Christian Myth. This Study Also Furthers Understanding Of The Concept Of Transference By Providing A Reflection Hologram Of The Human Psyche As An Artistic Work And As A Visual Metaphor Of Transference. **Strange Attractors: Transference, Holography, And An Archetype Burke, J. (2003). Strange Attractors: Transference, Holography, And An Archetype (Doctoral Dissertation, Pacifica Graduate Institute, 2003).** A host of observed, but very basic human phenomena, including consciousness itself,have eluded rigorous scientific description by all disciplines of science. This is true, not because of insufficient evidence for a particular phenomenon’s existence, but rather forlack of a theoretical construct, which could fit within the prevailing paradigms of science. For the past century eminent men and women of science have accumulated thousands of pages of data on mind/mind and mind/matter interactions. Many of the most telling experiments have been criticized, perfected and repeated numerous times during the past five decades, using increasingly sophisticated technologies. Meta analysis of these experiments produce accumulated probabilities against chance occurrences exceeding trillions to one (Radin, 1997). It has required, however, that quantum science mature for seventy-five years and during that period, test, validate and synthesize a

number of seemingly outrageous physical concepts arising from quantum theory, before testable theories could arise which offer hope that anomalous mind and consciousness data can be explained (Mitchell, 1996).The missing concepts that prevented the earliest investigators of consciousness from succeeding in their quest were 1) a generalized theory of information, and 2) quantum science itself, with the associated phenomena of non-locality, the zero point energy field and the quantum hologram. These associated phenomena are still not well understood butare sufficiently validated today by both theory and experiment to provide a basis for postulating a necessary condition for the existence of consciousness phenomena, as experienced in the observable four dimensional space/time universes. A third concept, chaos theory, is also necessary to understand the nonlinear evolutionary processes that caused consciousness to evolve toward the anthropic consciousness experienced by humans. In particular, chaos theory maps far from equilibrium systems and demonstrates the irreversibility of nonlinear processes and thus the irreversibility of time in the macro-scale universe. Another class of phenomena, including normal sensory perception and evolution, to cite but two, have explanatory theories in classical science, but which in view of current developments in late quantum physics and in chaos theory may be incomplete approximations to the correct theory. Information concepts have been examined by Weiner, von Neuman and Shannon in well-known seminal works and by Frieden more recently (1998) to produce theories useful to physics, to computation and to communications technologies. These theories, although accurate and mathematically useful in their domains, fall short of being sufficiently encompassing when considering the problem of consciousness, its evolution and its associated phenomena. Even relatively simple perceptual organisms utilize patterns of energy, that is, information, not completely described by existing mathematical theories. Theory and experimental evidence for the zero point energy field has been published by many authors, but I shall cite Haisch, Rueda and Puthoff, (1997, 1998), as the most contemporary and relevant work for this paper. Theory and experimental evidence concerning the quantum hologram has been developed by Schempp (1992, 1993) andMarcer (1996, 1997, 1998), separately and jointly, based upon a new understanding ofquantum mechanics. (See previous work by Cramer [1986], Berry [1988], Anandan [1992] and Resta [1997]).Non-locality, although predicted by the earliest work in quantum theory and decisively demonstrated by Aspect in 1982, has been thought to be a curious property of particle physics but of little relevance to macro-scale reality until discovery of the quantum hologram. Further, it is widely believed that non-local quantum information represented by entanglement of particles could not be recovered locally as useable information (Eberhard's theorem). However recent work both in theory and experiment (e.g. see Nature, 1997, 11th December, vol. 390, Sudbury T pp551-552, and Bouwmeester D. et alp 575-579) is in line with the work by Berry, Resta, Schempp and Marcer and makes it clear that this is not the general case for quantum information processing andcommunication.It has been widely accepted in science, until recently, particularly in the field of artificialintelligence, that the brain was likely a complex classical computer, incapable of supporting quantum processes. The work by Hammeroff (1994) and Penrose in isolating and describing microtubules in brain tissue have caused a re-examination of this dogma, and renewed interest in uncovering the quantum processes involved.Based upon this earlier work I postulate and examine the evidence in this paper for thefollowing theories: 1. The basis of subjective experience is rooted in the quantum attribute of naturecalled non-locality. I will use the word "perception" in its most generic sense to denote a basic subjective experience at all levels of complex matter. Thus the non-local quantum correlation between entangled quantum particles is considered the root cause of the phenomenon experienced as perception in more complex matter, but the non-local quantum hologram is the non-local carrier of information for molecular and larger scale matter. Thus, perception is not an object but rather the label for a nonlinear process involving an object, a percipient and information.2. The experience of humans is that they sometimes, perhaps often, perceive information from or about physical objects that is not available through normal, local, sensory mechanisms, nor classical space/time information. Objective testing data in overwhelming abundance provides evidence that this is true, though an explanatory mechanism has until contemporary times remained elusive. I shall call this intuitive information or intuitive perception. I postulate that a quantum hologram is the source of this intuitive perception and that the percipient is at that time in phase-conjugate-adaptive-resonance (pcar) with the entity or object associated with the quantum hologram.3. The phenomenon of "learning" in humans is a

subjective process that involves perception, memory, intentionality, and evaluation of outcome and behavior change. This may be viewed as a classical nonlinear feedback loop. Although we cannot know precisely the subjective experience of another entity, presumably in the successful training of animals, an analogous subjective process is in effect. Sheldrake (1981) has published a successful theory of morphic resonance related to animal learning based upon non-local information. Marcer has published papers (1996, 1997) theorizing a mechanism by which the quantum hologram causes learning to take place in both DNA molecules and prokaryote cells as an adaptation process of environmental resonance, rather than mutation and adaptation solely by random processes. I postulate that Marcer's concept can be generalized to nature alarge and that the quantum hologram is the information structure suitable to explain Sheldrake's morphic resonance. The non-local quantum correlations observed in particles, and the non-local quantum hologram associated with molecular and larger scale objects, serve the purpose of providing information at all scale sizes to guide evolutionary processes. That is to say, that quantum non-locality is the basis of perception, and thus fundamental and necessary to the complex organizations of matter and information in the universe. Further, since learning is an observed property of complex systems such as animals and, via the quantum hologram, is theorized to be a property of simple cells and molecules, one can also postulate the generalization that nature evolves through a learning process rather than because of random mutations.4. Marcer (1997) has proposed that the condition of phase-conjugate-adaptive-resonance (pcar) is a necessary condition for an object in three-dimensional reality to be perceived as it really is. That is, resonance requires a virtual path mathematically equal but opposite to the incoming sensory information about the object. Further, that it is the incoming electromagnetic (space/time) information (visual, acoustic, etc), which decodes the information of the quantum hologram and establishes the condition of pcar so that accurate three-dimensional perception is possible. That is to say, both quantum information and space/time information aroused in the act of perception by organisms having sensory preceptors. I propose that the two equal but opposite paths required by the pcar condition are the mathematical equivalent of perception and attention (or intention). (I shall distinguish between attention and intention in following pages.) Discussion The anecdotal evidence for humans perceiving non-local information dates to prehistory. The data were sufficiently robust that both experiences and philosophers, from Plato and Aristotle forward, accepted that both physical and non-physical realms of reality must exist. Non-physical was thought to explain the subtle, ephemeral and mystical subjective experiences ubiquitously reported in human culture. After Descartes and Newton, however, classical western science rapidly discarded the non-physical hypothesis and systematically began to ignore all evidence for perception of non-local information Field theories and point particles were created to preserve the concept of physical contact between particles and to explain obvious examples of "spooky action at a distance" such as gravitation and electromagnetic interactions. Information, broadly defined as patterns of energy, reemerges however, in non-local form in the mysterious quantum spin correlations of double slit experiments, although it has been widely believed that such non-local information could not be recovered and utilized by sensory systems. With validation of theory and experiments concerned with the non-local quantum hologram, information, including non-local information, suddenly acquires a more important status in physical theory, a status as important as energy itself. This is true because information is the basis of the cognition and knowing by which creatures perceive reality, and non-local information can now be seen as a ubiquitous and useful property of the cosmos, rather than a unique attribute of particles (and human animals). It is likely that most, if not all, subtle, ephemeral and unexplained phenomena associated with subjective experience are connected, directly or indirectly, with the phenomenon of non-locality. The brain is clearly a quantum computer (Schempp & Marcer, 1996) which utilizes both quantum and space/time information. This discovery alone almost certainly sets a necessary, but not sufficient condition, for intelligent life to have arisen in the cosmos, wherever environmental conditions permit. Many volumes have been written in this century by scientists experimenting with remote viewing, ESP, telepathy, clairvoyance, precognition, etc. Police agencies routinely use "psychics" to assist in criminal cases often with success. Intelligence agencies of governments have clandestinely utilized the findings to successfully gain information about an adversary. Many reports of these activities have been recently declassified and printed in open professional journals, even though no explanatory physical mechanism has yet been reported which is acceptable to mainstream science. The most

succinct modern summary of this activity and analysis of results have been published by Radin (1997). Quantum Holography Non-locality and the non-local quantum hologram provide the only testable mechanism discovered to date which offer a possible solution to the host of enigmatic observations and data associated with consciousness and such consciousness phenomena. Schempp (1992) has successfully validated the concept of recovery and utilization of non-local quantum information in the case of functional Magnetic Resonance Imaging (fMRI) using quantum holography. Marcer (1995) has made compelling arguments that a number of other chemical and electromagnetic processes in common use have a deeper quantum explanation that is not revealed by the classical interpretation of these processes. Hammeroff (1994) and Penrose have presented experimental data on microtubules in the brain supporting quantum processes. The absorption/re-emission phenomena associated with all matter is well recognized. That such re-emissions are sufficiently coherent to be considered a source of information about the object is due to the theoretical and experimental work of Schempp and Marcer, based upon the transactional interpretation of quantum mechanics of Cramer (1986), the Berry geometric phase analysis of information (Berry, 1988; Anandan, 1992) and the ability of quantum phase information to be recovered and utilized (Resta, 1997). The mathematical formalism appropriate to these analyses is consistent with standard quantum mechanical formalism and is defined by means of the harmonic analysis on the Heisenberg nilpotent Lie group G, algebra g and nilmanifold (see Schempp (1986) for a full mathematical treatment). The information carried by a quantum hologram encodes the complete event history of the object with respect to its three dimensional environment. It evolves overtime to provide an encoded non-local record of the "experience" of the object in the four-dimensional space/time of the object as to its journey in space/time and the quantum states visited. The question of the brain's ability, as a massively parallel quantum processor, to decode this information is addressed by Marcer and Schempp in "Model of the Neuron Working by Quantum Holography" (1997) and "The Brain as a Conscious System" (1998). They argue that an organism's ability to perceive objects as they are and where they actually are in three-dimensional reality requires the phase conjugate relationship provided by quantum holography. It is not sufficient for the incoming electromagnetic illumination (or acoustic signal) carrying object information to present tothe brain a wave front in the manner presented to a flat photographic plate. Rather, a virtual signal as mapped by the phase conjugation of quantum holographic formalism is required to decode the information in order for perception and cognition to exist as we experience it in three dimensional realities. The percipient and the source of information are in a resonant relationship for the information to be accurately perceived. Many investigators have proposed a holography mechanism as a basis for brain functioning, beginning with Pribram, and indeed, others have proposed holography as a construct forth universe itself, but discovery of the non-local quantum hologram created by the absorption/remission phenomenon and characteristic of all physical objects provides the first quantum physical mechanism compatible with macro-scale three dimensional worlds we experience it. The existence of a quantum hologram associated with each physical object provides each physical object with the non-local waveform predicted by quantum theory's wave/particle duality and extends quantum theory to all physical matter. It allows, for the first time, possible approach for understanding the mysterious world of consciousness. Postulating that this is globally true, we inhabit a quantum world where non-local effects should be expected at all levels of functioning, not just as a curious artifact of the subatomic level of reality. Existence of the non-local quantum hologram suggests that nature has utilized non-local information from the big bang forward, throughout its evolutionary history; and long before planetary environments self organized to permit living matter and complex space/time sensory systems to evolve. The papers of Marcer and Schempp on learning inherent in DNA and prokaryote cells using quantum holography, when generalized, helps explain the ubiquitous appearance in nature across distances, scale sizes and species, of similar processes, organs and sensory systems. This certainly conforms to the fractal geometry of chaos theory. Certainly the similarities, of DNA, cell structure, organs and brains across species are easier to reconcile with a non-local learning process than with a theory of localized random mutation and natural adaptation. It is important to observe that in standard particle physics experiments the object is to discover the quantum characteristics of the individual types of particles, and the conditions under which they split and recombine. In quantum holography the object is to treat the entire group of re-emitted quanta as a whole, and as in laser holography, to examine the

information carried in the interference pattern and phase relationships. These represent two quite different levels of approach to quantum information. In particle experiments, it is considered that the eigenvalues of the applicable matrix represent measurable values; and that information is lost during measurement due to decoherence of the particles and energy exchange. But in the quantum holographic formalism, the information is carried in the phase relationships, which are represented by off-diagonal terms in the matrix, and the information is recoverable under the proper conditions as Berry and Resta have predicted, and as Schempp has demonstrated with fMRI. The quantum mathematics is consistent with standard quantum theory in both cases. In decoding the quantum holographic information, however, the energy exchange is insignificant. The similarity of the mathematical treatment in these various experiments is important tothe thesis of this paper. In examining the quantum non-locality of particles it is spin numbers and/or polarization that are the parameters of interest. A standard technique of analysis is to use the Fourier transform to map the state of the particles into the frequency domain. In the formalism of the quantum hologram, mapping into the frequency domain is also fundamental; however, the requirement for pcar assures that the phase relationships are matched so that the percipient (sensory system) is able to decode the information carried in the phase relationships. It is precisely the pcar requirement that permits the encoded holographic information to be decoded by the percipient. Mathematically, decoding is simply reversing the rotation of the phase vector in phase space. Physically, it is matching the frequencies and phase of the information such that resonance results. Frequency, phase matching and resonance are an operational characteristic of every type receiver technology. Pribram's earliest proposal that the brain stored information encoded as in a hologram and mapped by the Fourier transform is in complete agreement with the evidence presented by quantum holographic mathematical formalism. It is the spin and polarization attributes of particles (both are mapped by wave mathematics) that represent the puzzling non-local property of subatomic matter. It is the phase relationships that carry the information in holography (again mapped by waveform mathematics). And it appears that the brain stores and manages information not as a classical digital machine, but rather as an analog device using non-local properties of thequantum hologram, which can be analyzed by wave form mathematics (harmonic analysis on the Heisenberg Lie group). In the cosmological evolutionary scheme of things this similarity of appropriate mapping techniques is too bizarre a coincidence to be ignored Asa cosmic accident. Thus there is ample evidence that the **non-local attribute of** nature is much more than just a curious artifact of subatomic particle interactions, but rather is a more fundamental phenomenon that appears at all scale sizes and is, in particular, associated with the utilization of information in nature, and associated with the fact that information has a causal effect independent of distance. It is precisely information, however, that is the basis of the phenomena of perception, cognition, memory, learning etc, that is to say, consciousness and the subjective experience. Though the evidence is quite ample to postulate that non-locality is the unique, universal basis for perception and the subjective experience, the evidence though compelling is not sufficient to be conclusive that such is indeed the case. The next steps are to validate more completely with experimental evidence that non-locality plays a major role at all scale sizes and that all physical objects are quantum objects and thus interconnected by information in this strange way. **Non-Locality in Nature**: There is experimental evidence to strongly suggest that simple organisms perceive and respond to information non-locally as well. Cleve Backster was perhaps the earliest to experiment with plants and simple life forms in electromagnetic isolation in late 1960'sand early 1970's. His work was not confirmed through replication by others at that time. Other investigators have had mixed results replicating non-local information perception by simple organisms and living tissues. In the area of human experimentation, results likewise have been mixed and controversial for three-quarters of a century. However Meta analysis by Radin (1997) and independently by Utts (1991) across a large and appropriate spectrum of experiments demonstrates compelling statistics that the **perception of non-local information exists and is real.** Perhaps was there a larger body of experimental evidence for simple life forms, similar Meta patterns would emerge. Failure to replicate results in well constructed experiments does not, in the case of subtle consciousness phenomena, prove that the phenomenon is missing but rather that a **hidden mechanism** below the threshold of classical measurement is operating. For example, the most telling experimental evidence to explain the sometimes inconsistent results relates to direct non-local observer and/or experimenter effects. Gertrude Schmiedler isolated the

**"sheep/goat "**effect in human experimentation decades ago (1972). Experimenters and/or participants in a human telepathy (or similar non-local) experiments exhibited results statistically above or below chance results depending upon their subjective bias toward the experiment. (In other words, 100% wrong answers would be as statistically significant as 100% correct answers in such tests, and in addition betrays the mind set or intention of the subject; whereas only chance results would be inconclusive.) More recently, a series of experiments by Marilyn Schlitz (1997) investigating "intentionality" clearly demonstrated that experimenter bias (intentionality) affected the outcome even of double blind experiments. Thus, in the subtle realms of mind and consciousness studies, bias, belief and intention clearly have an effect. The lack of an existing theoretical structure in classical science to support any type perception of non-local information, much less to support bias, belief or intention as having a non-local effect, when in fact it does have a non-local effect, is quite sufficient to account for anomalous results in many scientific experiments. Further validation and acceptance of the non-local thesis will have strong positive repercussions for the prevailing scientific paradigm and particularly the theory of measurement. **Attention and Intention:** A powerful and telling series of experiments conducted by Dean Radin (1997) at University of Nevada at Las Vegas following a decade long set of equally significant experiments by Brenda Dunne and Robert Jahn at Princeton University (1988) provide insight as to the subtleties involved in this level of mind/brain functioning. Jahn and Dunne provided overwhelming evidence that subjects could intentionally produce statistically skewed results in mechanical processes normally thought to be driven by random processes. Radin went further; he discovered that audiences watching a stage performance would skew the output of nearby random number generators during periods of high emotional content in the stage performance. Further, in a wide-ranging audience participation experiment, he recorded the output of computer random number generators during the television broadcasts of the O.J. Simpson murder trial. Most television media reported this event for weeks on end and tens of millions of humans were watching the results. Again, the results of the random number generators were skewed corresponding to emotional peaks during the trial drama and corresponding to the number of people watching television. The thesis in the Princeton experiments was that participant intentionality created a non-random effect to bias the skewed distribution. In the Radin experiments the results were not intentional, as the participants were unaware of the experiment, but the hypothesis was that attention (in particular, rapt attention) drove the system away from chaos (randomness) and toward greater order (reduced entropy). These results suggest that attention and intention provide closely correlated outcomes, further, that randomness may not be a general property of nature, but that what is perceived as random noise in a system may be information (a pattern of energy) that is not in resonance at that moment with the particular perceptual system. William Tiller, emeritus professor at Stanford also has performed experiments (1997) that are consistent with these results, though his interpretation of the operating mechanism is somewhat different. These different types of mind/mind, mind/matter experiments have been rigorously and routinely conducted for decades with statistically compelling results but just as routinely dismissed or ignored by main stream science because the implications of non-local action are so foreign to the classical paradigm. However, if we consider that the condition of phase-conjugate-adaptive-resonance is necessary to completely specify the act of perception as described in the mathematical formalism of the non-local quantum hologram by Marcer, then we may also consider the perceived object and the percipient's perceptual system as locked in a resonant feedback loop. The incoming wave front carrying information may be labeled as "perception" from the point of view of the percipient, and the return path required by the resonant relationship may be labeled "attention" (or for subsequent discussion, "intention"). It is a well established principle in the meditative practices of esoteric disciplines that prolonged focused attention on a object of meditation causes the percipient and the object to appear to merge so that a deeper level of information about the object is obtained; information such as history or internal functioning, that would not be available through classical space/time information. The concept of the quantum hologram adequately and completely describes how this phenomenon might take place. Further, it is accepted that the mind/brain is a massively parallel processor, capable of performing many tasks simultaneously and subconsciously (in the right, intuitive part of the brain). Attention (meaning conscious, focused attention) is a unique and singular task that must take place sequentially, mostly in the left cognitive part of the brain. The condition of attention deficit disorder

(ADD) is precisely the problem of a percipient being unable to maintain a singular focus for sufficient time to complete a desired task or observation. Thus, the action of focusing attention by apercipient may be construed as a necessary condition for pcar to be established with the perceived object. Non-Locality, Near and Far Marcer has presented the case for the pcar requirement in normal sensory perception (visual and acoustic). A frequent modality used by psychic sensitive individuals to gain information is to physically touch an object. Touching an object satisfies the pcar requirement and presumably allows the percipient access to information about the object not available from space/time information. Police agencies frequently use this modality with psychic sensitives to gain information about a crime scene, much as they utilize bloodhound to track the scent of an individual, often with considerable success. If, as in the theory of the quantum hologram, the object has been in the presence of the individual about whom information is desired, the event history of the object and that of the individual intersect. The Berry phase information of the object contains its journey in three dimensional space and time, as well as the quantum states through which it has passed on this journey. The sensitive individual, with a honed talent, seems often able to decode useful Berry phase information from the object about the individual sought. It may also be the case with the blood hound, which additional non-local information has been gained about the subject, even though the classical explanation is that the animal is operating only with heightened olfactory sensing. Although perception in the three dimensional world requires and utilizes pcar, most humans, however, do not bring to conscious awareness non-local information when weare routinely operating in three-dimensional reality. We perceive objects as presented by space/time information, that is, shape, color, function (tree, chair, table, etc) but are not usually aware of the additional non-local information. It takes training as provided by many of the esoteric traditions and/or certain naturally sensitive individuals to routinely perceive the non-local holographic information associated with a particular object. There is massive evidence to suggest, however, that the brain has these latter capabilities at birth. Suppression by cultural conditioning in childhood and subsequent lack of practice cause the natural ability for conscious, intuitive perceptions to atrophy. Particularly in western tradition, educational interest has been on the left brain, rational functions rather than right brain, intuitive functions. However, mystic adepts and natural psychics' routinely demonstrate that non-local information is perceptible from physical objects by focusing attention, quieting the left brain and allowing intuitive perceptions to appear. It is the left brain cognitive ability in humans that provides canonical labeling of the intuitive and artistic processes taking place in the right brain. The fact that with training and practice, individuals can recover, deepen and label their individual cognitive access to intuitive, on-local information demonstrates that learning is taking place within the whole brain itself and involves enhanced coherence and coordination between the hemispheres. This process is different and distinct from the left brain function of extending and extrapolating factual data and logical deduction to leap to an "intuitive" conclusion, while omitting the intermediate steps leading to that conclusion. Experimental protocols for remote viewing normally provide clues to the location of the object such as adescription, a picture, or location by latitude and longitude, that is to say, an icon representing the object. These clues seem to be sufficient for the percipient to establish a resonance with the object. Normal space/time information (visual, acoustic, tactile) about the object is not being directly perceived by the percipient, nor does the object usually appear at its physical location in space/time like a photograph or map in the mind. Rather, the information is perceived and presented as internal information and the percipient must associate the perceptions with his/her internal data base of experience in order to cognize and to describe the object's perceived attributes. In the case of complex objects being remotely viewed, the perceived information is seldom so unambiguous as to be instantly recognizable as correct. Sketches, metaphors and analogies are usually employed to cognize and communicate the non-local information. A considerable amount of training, teamwork and experience are necessary to reliably and correctly extract complex non-local information from a distant location. The information appears to the percipient as sketchy, often dream-like, and wispy, subtle impressions of the remote reality. Very skilled individuals may report the internal information as frequently vivid, clear and unambiguous. The remote viewing information, being strictly non-local, and in this hypothesis, the information perceived by quantumholography, is missing the normal space/time components of information necessary to completely specify the object. It has been demonstrated that this intuitive mode of perception can be

trained in most individuals. Perhaps additional training and greater acceptance of this capability will allow percipients to develop greater detail, accuracy and reliability in their skill. In principle, training will not only increase the skill and accuracy but should cause the appropriate neural circuitry to become more robust as well. In the absence of space/time (electromagnetic) signals to establish the pcar condition and to provide a basis for decoding the quantum hologram, an icon representing an object seems to be sufficient to allow the brain to focus on the object and to establish the pcar condition. However, a reference signal is also required to provide decoding of the encoded holographic phase dependent information. Marcer (1998) has established, using Huygen's principle of waves and secondary sources, which any waves reverberating through theuniverse remain coherent with the waves at the source, and are thus sufficient to serve as the reference to decode the holographic information of any quantum hologram emanating from remote locations. The Zero Point Field: The results of the Michelson/Morley experiment banished the concept of aether from early twentieth century physics. However, it left a void as the nature of interstellar space and nothing for propagating waves to wave in. Quantum physics reincarnated the aether as the zero point fields, a seething cauldron of quantum potential and unmanifest energy where particles and antiparticles spontaneously arise and then disappear. The very fabricant structure of space/time itself is again in question; its structure and its metric under intense investigation with far more questions than answers having emerged to date. For the purposes of this paper the relevant issues are two: 1) the emission/absorption phenomenon, and 2) the structure and mechanics of non-locality. Zero point (zero degrees Kelvin) emission and absorption of quanta from all physical objects is a well established phenomenon. It is our view that the zero point fields are the plenum (or cauldron) which supports this absorption and re-emission, and makes the phenomenon of the quantum hologram possible at all temperatures. Although particle experiments are carried out under rigid conditions of temperature and pressure, Schempps experimental work with the fMRI requires no such constraints. There are deep and difficult questions yet to be answered about how the information of thequantum hologram maintains its integrity and is propagated, about how resonance takes place at extremely large distances. There is considerable evidence that intuitively perceived information is truly non-local. It does not obey the inverse square law for space/time energy propagation; it is time independent and cannot be shielded by electromagnetic shielding. Such characteristics are the mark of non-locality. But understanding the mechanics of non-locality (or a visual picture) is missing from standard models. Some physicists turn to superluminal speed of propagation, others to the zero point as a zero dimension, which is resonant with all parts of the universe simultaneously. The issue of instantaneous communication (or at least superluminal communication) of non-local effects on a cosmic scale remains a problem, even though the phenomenon itself is well validated. Perhaps it is a problem of topology. What shape can the universe have such that one point can be in simultaneous contact with all other points? In this regard it is clear that certain problems between quantum mechanics, special and general relativity remain inexistence. Haisch, Puthoff and Rueda continue to investigate the **metrics of the zero point fields** with regard to better defining the unanswered questions about, mass, gravitation and inertia. Perhaps these investigations will also bring answers for how phase related information is propagated non-locally, likely within the zero point fields, and thereby unveil the mechanics of the resonance phenomenon. Further, new investigations reported by Van Flandern (1998) on measurements from orbiting Global Positioning System (GPS)clocks indicate the predictions from Lorentzian relativity to be approximately four times more accurate (0.7% opposed to 3%) than predictions of special relativity. If these measurements are further validated, it implies that Lorentzian relativity with a Hubble absolute rest reference frame, an aether (zero point field), and instantaneous propagation of non-local effects may be the preferred one. If this is the case, then many questions about non-locality would be resolved. Intentionality I have argued that by establishing pcar between a percipient and an object, the phase conjugate (equal but opposite) paths connecting the two can be labeled "perception" and "attention". In the case where the object is a simple physical object (rock, flower, etc.), our interest is on the non-local information perceived by the percipient about the object. However, from the point of view of the object, information about the percipient is also available to the object. The pcar condition is a reciprocal relationship, mathematically. Quantum holographic formalism predicts that the history of events of quantum objects is carried in the quantum hologram, thus we must conclude that the **"attention" focused upon the object causes**

**that event to be recorded in that object's quantum hologram.** Although we cannot query the object about its experience perhaps an experiment such as one utilizing the Aharonov-Bohm effect would detect a phase shift in the object's holographic field. (In this discussion I use anthropic labeling as we are discussing human perception. The phenomena however, are rooted in natural (and primitive) non-local physical processes, which are fundamental. The evolved complexities of perception, cognition, etc, associated with a brain obviously, as yet, have no analogous label to describe the experience of simple objects.)Once the pcar condition is established, the percipient can change its mind state with regard to the object. The perceived information can be operated upon by the brain's function so that cognition occurs with respect to the perceived information and meaning assigned. Cognition and meaning require finding a relationship between the perceived informationand the information residing in the percipient's memory. The percipient can then form intent with respect to the object. In such case the path I have labeled "attention" could Figure 1 In self aware animals (those with a brain) cognition, meaning and intent with respect to an object can often be described in simple terms, for example: enemy, fight or flight; food, eat; friend, greet, etc. The non-local component of information, although present and creating effect, is operating below the level of conscious perception in humans and results in "instinctual" subconscious behaviors in all animals. The brain as a massive parallel computer is simultaneously performing numerous tasks to accomplish the desired intention. Classical modelings of this autonomous activity describe it only in terms of classical information and energy flow in the central nervous system and the brain. However, if non-locality is operating at all levels of activity, as this theory suggests, certainly there are resonances involving non-local information operating throughout the body of an organism in parallel with classical space/time functions. Subsequent experimental work will surely uncover these quantum processes where non-local resonance is involved in the functioning of an animal's internal processes. In the case of non-local effects at a distance, outside the body, simple spin correlations of entangled particles is the most basic. The spin coherence is reciprocal. Action on one particle creates an effect on other entangled particles. The non-local information is causal of affects at large distances. It is no less important for macro-scale objects. Sheldrake (1995) proposed and others conducted experiments with dogs whereby the animals correctly anticipated their owners' departure from work to return home. He proposed other successful experiments where rats learning a new maze benefited non-locally from the experience of others that had previously learned the maze, in the total absence of classical space/time information. It is not surprising then, that human's exhibit an even wider range of reactions to non-local information. The evidence suggests that humans can perceive, cognize, and give meaning to non-local information across a range of complexity, from inanimate objects, simple organisms, animals and other humans. The existence of quantum holography provides an adequate informational structure to permit a theory for the observed results. The case is a classic case in phenomenology, where results are repeatedly observed over time that fall outside the prevailing paradigm, and must await new developments in science before an explanation is forthcoming. The results for intentional effects of non-locality should be no more difficult to accept than the results for perception. The pcar relationship implies symmetry, that is, information flows in both directions between object and percipient such that each is object and each is percipient. Only the complexity of the more ordered perceiving system suggests a non-symmetrical relationship. We humans have great difficulty in accepting that thoughts, specifically intentionality, can cause action at a distance. Yet, it has been observed for centuries and in recent decades subjected to scientific scrutiny. Were not prayer to have produced some positive results, religion would have been abandoned centuries ago. That cause was ascribed to supernatural agency rather than non-locality is simply, again, phenomenology needing to wait while science caught up. Modern studies by Dossey (1993), Byrd (1988) plus many others have attempted to document the efficacy of prayer, particularly healing prayer. The results in most cases are very suggestive of non-local effects, and some claim they establish the case for healing prayer. However, the difficulties of controlling all variables in such clinical studies leave many avenues for valid criticism. The fact that Radin's several studies (1997) demonstrated that attention alone produced non-local results in machines, i.e., reduced randomness (increases order) does confirm that information has non-local effect and may be correctly formulated asnegentropy. These results apply directly to healing prayer as well.The case for pcar conditions to create remote effects by transfer of non-local information between equally complex percipients, humans for example, is not

difficult to understand. Indeed, hundreds of successful experiments establish the case. In these cases no energy transfer is required, only non-local information, as each percipient/object has access to its own energy source. The case for intentionally creating remote physical effects in inanimate objects is more puzzling. Even though teleportation of quantum states has been successfully accomplished for particles, and numerous studies (Radin, 1997; Dunne and Dunne, 1988) show that macro-scale objects can also be changed or moved, the energy transfer mechanism by which the classical states of a remote object are affected remains elusive. Conclusions The case for mind/mind and mind/matter interactions is impressively well documented over many decades as studies in phenomenology, with staggering probabilities against chance having produced the results. The discovery of the non-local quantum hologram, which is theoretically sound and experimentally validated in at least one application, thé fMRI, is sufficient to postulate that the quantum hologram is a solution to the foregoing enigma. Further, recognition that the quantum hologram is a macro-scale, non-local, information structure described by the standard formalism of quantum mechanics extends quantum mechanics to all physical objects including DNA molecules, organic cells, organs, brains and bodies. The discovery of a solution which seems to resolve so many phenomena, and also that points to the fact that in many instances classical theory is incomplete without including the subtle non-local components involved, suggests a major paradigm change must be forthcoming. The papers already published by Marcer and Schempp proposing a learning model both for DNA and prokaryote cells, which uses quantum holography, suggests that evolution in general is driven by a learning feedback loop with the environment, rather than by random mutations. This solution to biological evolution was proposed by Lamarck in 1809 but discarded for the mechanistic solution of random mutations by the colleagues of Darwin. The fact that non-local correlations and non-local quantum information can now be seen as ubiquitous in nature leads to the conclusions that the **quantum hologram can properly be labeled as "nature's mind"** and that the intuitive function we label in humans as the "sixth sense" should properly be called the "first sense". The perception of non-local information certainly preceded and helped to shape, through learning feedback, the sensory systems that evolved in planetary environments, and which we currently label as the five normal senses. We must conclude that evolved, complex organisms, which can form intent, can produce and often **do produce non-local causal effects associated with that intent.** Further, that attention alone produces coherence in nature that in some measure reduces randomness. Finally, I conclude that the cited experiments and current understanding of non-locality in nature is sufficient to postulate that **non-locality is the antecedent attribute of energy and matter which permits perception and is the root of the consciousness which manifests in the evolved organisms existing in three dimensional realities**. ReferencesAnandan, J. "The Geometric Phase"; Nature, 360,26, 307-313 (1992)Berry, M. V. "The Geometric Phase" Scientific American, December, 26–32 (1988)Byrd, R. C. "Positive therapeutic effect of intercessory prayer in a coronary care population"Southern Medical Journal 81 (7): 826–829 (1988)Dossey, L. "Healing Words" Harper San Francisco, (1993)Dunne, B. J., Nelson, R. D., and Jahn, R. G. 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That Kind Of Eternal-Universal 'Unconsciousness'. If Consciousness Is Intrinsic to Physical Phenomena (As in Penrose-Hameroff's Microtubule-Based Quantum Consciousness) or Is Intrinsic to Information Integration (As in Tononi-Koch's IIT), We Still Have the Same Kind Of Mind-Body Problem. A 'Body' Which Is A Statistical Function Rather Than A Literal Form In Space Is Still Falls Short Of Explaining Why And How There Is Any Such Thing As Consciousness. In My View, Only The Idealist Monist View Or What I Call Pansensitivity Makes Sense Ultimately As The Parent Of Both Physics And Information. Just As We Learn To Count On Our Fingers, All Forms Of Information Are Representations Of Experiences Which Have An Aesthetic Foundation – A Seeing, Feeling, Touching, Thinking, Etc. Without That Sensory-Motive Context From The Start, There Would Nothing To Evolve; Only Abstractions In The Dark (Or Not Even Dark). Once We Can Get Over Ourselves As A Species And Recognize That Consciousness Doesn't Begin And End With Us, I Think That Awareness Will Be Seen As The Container Of Relativity Itself, With Quantum Mechanics And Evolutionary Biology As A Consequence Of Deeper Stories Rather Than Their Originator. Multisense Realism A Cosmology Of Sense, Essence & Existence For Models See One Of The Papers In The Series Procrastinated Due To Constraints On Space. Events Occur In Time, But There Is No Time Except That Which Has Been Produced By The Big Bang. It Makes No Difference Whether The Big Bang Is The Only Beginning Of The Only Universe Or Just The Beginning Of One Of Many Universes, Either Way It Doesn't Explain The Beginning Of Existence. Some Say That Everything Following The Big Bang Is Random, Not Designed, But Why Is 'Designed' Even An Option? If That Word Means Anything, And Anything Has Ever Been Designed By Anything, Then What Difference Does It Make Whether That Capacity Of Intention Came Early On In The Universe Or More Recently? How Do We Know The Difference Between What We Design And What Is Random? How Do We Know That Randomness Even Exists? We Can't Generate True Randomness Computationally, We Can Only Grab Onto Some Pattern That Seems Nearly Random By Some Arbitrary Duration Of Measurement And Say That It Is Good Enough. Random Is A Concept, and the Difference between Random and Intentional May Ultimately A Matter of Perspective. Some Interesting Possibilities Arise When We Consider The Observation That The More Something Seems Intentional, The More It Is 'Like Me' And The More Unlike Me Something Appears, The More It Seems Mechanistic. In A Way That Is Similar To How Any Object Appears As A Dot Or Smudge If It Is Too Small Or Distant For Us To See, The Notion That Perceptual Relativity Dictates The Quality Of Intention And Unintention Is A Provocative Hypothesis Of MSR. Random Events Cannot Follow A Series Nor Have Results. That Doesn't Mean That There Has To Be An Entity Making Everything Happen, But It Suggests That The Universe Is A Phenomenon Of Appearance, And Part Of Appearance Is Oscillation Between Intentional And Unintentional Attributes. The Universe Speaks In Both Entropy And Significance. Randomness Doesn't Do Anything, Can't Be Anywhere Or Feel Anything. Randomness Is An Abstraction Derived From An Expectation Of The Contrary. By Random, All We Mean Is That It Lacks Pattern And Intent – Both Of Which Must Be Implicitly Present Before They Can Be Hypothetically Absent. There Is Sense And There Is Non-Sense, Order And Dis-Order, It Is Not Non-Randomness And Dis-Chaos. Many People Believe That Physical Law Excludes The Possibility Of Free Will Because Of Strong Causal Closure. By This, What Is Meant Is That There Is No Room In Physics For Any Force That Causes A Physical Effect That Has Not Been Accounted For, Whether Or Not This Is Actually True Or Not Is Questionable In The First Place. The Sudden Addition Of Dark Matter And Dark Energy In The 1990s, Which Together Must Account For 95.1% Of Mass-Energy Of The Universe, Doesn't Seem To Have Encountered Nearly As Much Resistance From The Scientific Community Than Conscious Intention Has. Even If It Were The Case, However, And The Typing And Reading Of These Words Will Be Someday Explained By Purely Neurochemical Mechanisms, The Question Of Why The Feeling Of Intention Exists, And How It Can Be Produced Will Remain Unanswered. My Free Will Demands That Your Free Will Admit That There Is No Free Will. In All Of The Contemporary Debates On Free Will There Seems To Be A Blind Spot, In Which Free Will Does Not Exist, Except Where It Concerns The Application Of The Results Of Scientific Experiments. There Is Always A Call For The Educated And Enlightened To Voluntarily Change Their Own Minds, To Be Persuaded By The Argument Of Their Own Free Will. The Hypocrisy Is Hard To Overlook Once You See It. It Seems That The Anti-Free Will Assertion Makes An Exception For People Who Are 'Right'. Being Right Seems To Give Us A Right To Have

Opinions Which Others Cannot, Due To Their Enslavement To Physical Causes. An Issue Which Often Comes Up In Free Will Debates Is How Our Stance On Free Will Impacts Criminal Prosecution. Again, The Blind Spot Of The Anti-Free Will Philosopher Projects That While Criminals May Not Be Held Liable For Their Actions Because Their Neurology Is To Blame, Then We Can't Allow Society To Take Credit Or Blame For Its System Of Justice Either. If Guilt Or Innocence Is Irrational, Then Believing That We Can Modify Our Own Attitudes Toward Guilt And Innocence Must Also Be Irrational. What Seems More Irrational Is To Codify Law Into A Mechanistic Formula Which Diminishes Our Capacity For Thought And Feeling In Consideration Of The Fate Of Others (Or Others Considering Our Fate).Zero-Tolerance Determinism: Another Pillar Of The Anti-Free Will Position Is That It Must Be An All-Or-Nothing Phenomenon. This Seems To Be More Of A Hasty Generalization Than An Honest Look At The Phenomena. In A More Tolerant Analysis, It Should Be Clear That Free Will Doesn't Have To Be Absolutely Free. The Fact That Our Conscious Mind Thinks That It Was Presented With Certain Options Still Requires That We Choose Freely From Among Those Options. It's True That We Are Always, In a Sense, Only Able to Choose What Our Mind Thinks Is the 'Best' Option (Even If That Choice Is Based on Bad Advice from Our Stomach or Emotions), But Still There Is an Intentional Participation Which Cannot Be Explained by Statistical Functions. No Matter How Constrained We Are By The Rules Of The Road, Our Car, The Limits Of Our Driving Skill, Etc, There Is Still A Difference Between Driving And Being Asleep At The Wheel, Or Between Driving And Being Forced To Drive Somewhere At Gunpoint. We Should Not Look Only At How Our Freedom Seems To Vanish Upon Thorough Inspection; We Must Also Look At How It Appears In The First Place – Unbidden And Self-Evident. Look At The Universal Appeal Of Freedom And The Universal Disparagement Of That Which Is Unfeeling, Robotic, And Mechanistic. One Thing That Is Seldom Mentioned In These Endless Debates Is Creativity. Were The Egyptians Destined To Build Pyramids And Not Megalithic Circles? Was It Inevitable That The Star Spangled Banner Was Written In Reference To The United States? Did Deterministic Processes Have No Choice But To Create The Illusion Of Free Will? Why? The Work of Benjamin Libet Is Frequently (Compulsively?) Cited As Well, Work Which Even Libet He Later Made Clear Did Not Show That Free Will Didn't Exist. The Fact That A Neurological Signal Can Be Detected Before The Various Parts Of Us (The Self Who Makes The Decision, The Self Who Knows They Make The Decision, The Self Who Reports That They Make The Decision) Can Arrive At A Consensus Does Not Mean That The Initial Impulse Doesn't Correspond To Conscious Experience. There Is Also The Matter Of Focusing On Repetitive, Reflex Actions Which Minimize Free Will And Maximize Predictive Expectation. These Kinds Of Experiments Are Like Proving That Chefs Lack Imagination By Studying The Fry Cooks At Mcdonalds. Since Physics And Neuroscience Has No Theory At All As To The Origin Or Utility Of Consciousness, We Cannot Give Inanimate Instruments The Benefit Of The Doubt When It Comes To Our Subjectivity. Because We Are A Single Zygote Which Has Reproduced Itself, In Some Sense, Every Cell In Our Body Is 'Us' Just As Much As Any Organ Or Process In Our Body. We Are Complex, But Not In The Way That A Machine Is Complex. We Are Not Assembled From Specialized Parts; We Are A Single Whole, Divided Into Relative Specialization. Just Because Every Part Of Us Doesn't Know What Every Other Part Of Us Is Doing At All Times, Doesn't Mean That It Isn't All 'Us' Doing It Pansensitive Relativity: If The Perceptual Relativity Hypothesis Is Correct, And Pansensitivity Is The Engine Of Both Physics And Subjectivity, Then Cause Itself Is Relativistic. Intention Is Real And Primitively Creative In The First Person, And Unreal/Recombinant-Derivative In The Third Person…But Third Person Is Only Real Because There Is A First Person Participant Present. **Free Will, Determinism, And the Big Bang: Multi Realism Blog: Wikipedia http://multisenserealism.com/2012/08/27/deleuzes-the-logic-of-sense-part-i/.** From A Whiteheadian Perspective, However, There Are At Least Two Key Sources Of Misunderstanding Latent In Some Of These "Returns" To "The Body", "The Object" And "Materiality". There Is A Rather Too Hasty Dismissal Of The Concept Of Subjectivity As Such And There Is A Related Tendency To "Flatten Out" Any Would-Be Distinctions Between Human And Non-Human Entities. Such Positions Thus Risk A Return To A Bleak Anti-Subjectivism That Mocks Those Who Might Cling To The Idea That "Humans Are Very Different From Knives Or Paper" (Harman, 2002, Cited In Thrift, 2008). With Respect To The Latter, For Example, Nigel Thrift Invokes The ANT Principle Of The Democracy Of

Things And States That In His Theory "Things [And Human Beings] Are Given Equal Weight, And I Do Mean Equal" (Thrift, 2008, P. 9). Much Of This Is Done In The Name Of A Whiteheadian Inspiration Since The Book Begins With Strong References To Whitehead, Including Key Concepts Such As The "Actual Occasion". The Point Concerning The Dismissal Of Subjectivity Can Be Brought Out Most Starkly By Contrasting One Of Thrift's Statements About His Non-Representational Theory With A Statement From Whitehead Himself. "Thus Things", Writes Thrift, "Are Not Just Bound By Their Brute Efficacy To The Visible Termini Of Humans In Some Form Of Latent Subjectivism Such As 'Concern' Or 'Care'". Now, It Is Certainly True That For Whitehead A "Thing" Is, By Definition, "Describable Without Reference To Its Entertainment" In An Occasion Of Experience (Whitehead, 1933/1935, P. 226). In This Sense, Things Are Not "Bound By Their Brute Efficacy To The Visible Termini Of Humans". But It Is Certainly Problematic To Imply That His Work Is Based Upon A Move Away From The Subject–Object Relation As The Fundamental Structure Of Experience And Hence From Concepts Such As Concern And Care. On The Contrary, On This Matter Whitehead's Thinking Is Quite Comparable To That Of His Contemporary Martin Heidegger In Being Grounded In The Concept Of Concern. The Centrality Of This Concept Is Due To The Way In Which It Brings Together Subject And Object As Relative Terms In The Unity Of What He Calls An Actual Occasion Of Experience: Thus The Quaker Word "Concern", Divested Of Any Suggestion Of Knowledge, Is More Fitted To Suggest This Fundamental Structure. The Occasion As Subject Has a "Concern" For the Object. And The "Concern" At Once Places The Object As (=) A Component In The Experience Of The Subject, With (e&eb) An Affective Tone Drawn From (e) This Object And Directed Towards It. With This Interpretation, The Subject–Object Relation Is (=) The Fundamental Structure Of Experience. (Whitehead, 1933/1935, P. 226) Granted, Thrift Does State That He Wishes To (e) "Temper… The More Extreme Manifestations Of This Lineage, Which Can End Up By Positing A Continuity Of And To Experience About (e&eb) Which I Am Sceptical" (2008, P. 6) However, One Wonders What Is Left Of Whiteheadian Cosmology If This Radical Extension Of (e) The Concept Of Experience And Hence Of Empiricism Is Omitted. It Is A Little Like Being A Marxist Without (e) The Dialectical Materialism. It Is Also Difficult Not To Be Sceptical About The Remaining Self-Consciously "Inhuman" Theoretical Framework "In Which Individuals Are Generally Understood As (=) Effects Of The Events To (e) Which Their Body Parts (Broadly Understood) Respond And In Which They Participate" (Thrift, 2008, P. 60). In Recent "Radical" Social Theory, It Seems The Baby Of Subjectivity Is At Risk Of Being Thrown Out With The Bathwater Of Representationalism, Leaving Only The Hollow Remainder Of A Reactive Ensemble Of "Body Parts". Some Clarity Is Therefore Needed If We Are Not Simply Going To Recruit Whitehead Into Concerns Alien To His Own. His Chief Problem Was Not the Notion of the Subject/Object Structure of Experience Itself, But It's Too Rapid Identification with the Difference between Knower and Known (Whitehead, 1933/1935, P. 225). It Is This Conflation Of The Deeper Subject/Object Relation With The More Superficial Distinction Between Knower And Known That Gives Rise To A "Representational" Style Of Thinking And Its Interminable Debates. I Will Argue That It Is Precisely By Way Of The Subject/Object Relation Of The Actual Occasion That We Are Best Positioned To Conceive Of "The Cumulation Of The Universe And Not A Stage-Play About It" (Whitehead, 1927–1928/1985, P. 237). **Palgrave Communications | Palgrave Macmillan Journals Journal Home > Archive > Original Articles > Full Text Original Article Subjectivity (2008) 22, 90–109 Doi:10.1057/Sub.2008.4 A.N. Whitehead And Subjectivity Paul Stenner1** 1University Of Brighton, Brighton, UK Correspondence: Paul Stenner, School Of Applied Social Science, University Of Brighton, Falmer Campus, Mayfield House, Brighton BN9 1PH, UK. E-Mail: P.Stenner@Brighton.Ac.Uk (For Models Please See One Of The Papers In The Series Attributed To Spatial Restraints It Is Accommodated Where It Could Be). In Sum, Whitehead Offers A Relational Process Ontology That Promises To Deepen The Constructivist Insights Associated With The Turn To Textuality, But Without Reducing The Universe To "Discourse" And "Materiality". In This Ontology, Things (Whether Occasions Or Assemblages) Are Definable As Their Relevance To Other Things And In Terms Of The Way Other Things Are Relevant To Them. Things Have Relational Essences. Likewise, Things Do Not Exist Independently Of Temporality But Are Constituted By The History Of Their Specific And Situated Encounters. Every Actual Thing Is Thus "Something By Reason Of Its Activity" (Whitehead, 1927/1985, P. 26). Importantly, This Talk Of "Things"

Need Not Incline One Towards Denying The Relevance Of Subjectivity. I Have Thus Taken Issue With A Tendency Illustrated In The Work Of Thrift (2008), Who Appears To Define Cutting-Edge Social Theory As Concerned With "Flow" And "Play" Rather Than With Stability (Since "Non-Foundational Theory Takes The Leitmotif Of Movement", P. 5 And "Privileges Play", P. 7); As "A Means Of Going Beyond Constructivism" (P. 5); And As "Trading" In "Modes Of Perception Which Are Not Subject-Based" (P. 7). I Have Tried To Show That, In Fact, The Concept Of Process Is As Much About Stability As About Change. Stability Is To Be Thought Of As An Achievement Resulting From Particular Ways Of Actualizing Potential And Of Patterning Occasions Into Spatial And Temporal Co-Assemblies. Nevertheless, I Have Also Stressed That Becoming Is An Inherently Self-Creative Process, Albeit A Self-Creation Grounded In The Facticity Of A Concrete Inheritance. Whitehead's Category Of Subjective Unity Is Thus An Exemplary Statement Of Constructivism, Which States That: "Self-Realization Is The Ultimate Fact Of Facts. An Actuality Is Self-Realizing, And Whatever Is Self-Realizing Is An Actuality" (Whitehead, 1927–1928/1985, P. 222). The "Capture Of Intensity" And The "Clutch At Vivid Immediacy" Are Thus The Defining Characteristics Of Life (Whitehead, 1927–1928/1985, P. 105). Finally, I Have Suggested That It Is Only On The Basis Of A Deep Extension Of The Concept Of Experience Throughout Nature That Whitehead Is Able To Resoundingly Affirm His Reformed Notion Of The Subjectivist Principle: "That Apart From The Experiences Of Subjects There Is Nothing, Nothing, Nothing, Bare Nothingness" (1927–1928/1985, P. 167). "Scientific Reasoning Is Completely Dominated By the Presupposition That Mental Functionings Are Not Properly Part of Nature" (Whitehead, 1938/1966, P. 156). (Ibid). Lewis Carroll Is Wonderful In Exploring Some Of The Paradoxes Of Logic. Following Some Of These Paradoxes Will Lead To The Important Role Of 'Sense' In Understanding. [NB Weird Titles Are Deleuze's Own]. First Series Of Paradoxes Of Pure Becoming: We Need To Explore The Nature Of An Event. Events Assume Becoming, Since They Refer To States In The Past And The Future In A Way Which 'Eludes The Present'. This Is Paradoxical But Still Makes Sense. Plato Tried To Distinguish Between Limited Fixed Things And Pure Becoming, But The Two Cannot Be Separated. Instead, A Dualism Is Hidden In Material Bodies. We Need To Introduce The Notion Of A Simulacrum Which Is Neither Copy Nor Model [Massumi Has A Useful Article On This—Roughly, The Issue Is That What We Take To Be Material Reality Is Actually A Simulacrum Of The Virtual, A Limited Condensation 'Beneath Things' (2)]. It Also Avoids The Problems Of Subsuming Reality Under The Idea. Sometimes, This Is Indicated By A Summer The Peculiarities Of Language, Which Can Seem To Flow Over Specific Referents. This Provides A Clue That There Is Some Dimension To Language Which Serves To Come To The Aid Of More Specific Attempts To Name And Describe. There Are Implications For Identity. Fully Grasping Becoming Means That Identities Are Infinite, Incorporating Future And Past, Active And Passive, And Even Cause And Effect. Language Attempts To Limit This Infinity, But Still Often Alludes To It [With Open-Ended Statements Or Generalisations]. As With The Alice Stories, This Also Disrupts The Conventional Notion Of The Personal Identity. Normally, This Is Maintained By Some Underlying Commonsense Or Knowledge, As When 'The Personal Self Requires God And The World In General' (3). Becoming Threatens This Stability With The Paradox Of Events, Which Can Penetrate Even Commonsense—It Is Not Just A Doubt About Reality, But A Clear Indication Of The 'Objective Structure Of The Event Itself, Insofar As It Moves In Two Directions At Once, And Insofar As It Fragments The Subject Following This Double Direction' (3). Second Series Of Paradoxes Of Surface Effects: Stoics Divided Things Into Bodies And States Of Affairs, 'Actions And Passions' (4). There Is Also Some Cosmic Unifying Quality, Always In The Present. Bodies Can Interact and Cause Effects in Each Other, But These Effects Are Incorporeal, 'Logical or Dialectical Attributes… Not Things or Facts but Events' (5). They Have The Kind Of Subsidiary Existence, Acting As Verbs, And They Are Infinitives—The Example Is A Cut Inflicted On The Body, Which Is Seen As An Incorporeal Surface Effect, Compared To The Actuality Of Bodies And Their Mixtures. This Argument Had Important Implications For Understanding The Causal Relation. Specific Bodily Causes Produce Other Bodies, Linked By Some Cosmic Unity Or Destiny. Similarly, Effects Can Be Seen As Having Bonds Between Them, But Effects Can Never Be Causes In Themselves. They Can Only Be '"Quasi-Causes' Following Laws Which Perhaps Express In Each Case The Relative Unity Or Mixture Of Bodies On Which They Depend For Their Real Causes'

(6). These Combinations and Bonds to Provide For Some Emergent Qualities, Which Means That Destiny Can Be Avoided. An Alternative Is Offered By The Epicurean Classification Of Different Kinds Of Causes Which Are Relatively Independent [And So Can Interact], And This Is A Kantian Idea Too. There Is A Reference Back To The Capacities Of Language To Offer 'A Declension Of Causes... [And]... A Conjugation of Effects' (6). Stoic Philosophy Introduces The Notion Of A Something Behind Both Specific Material Beings And Incorporeal Events. The Idea Must Belong To 'This Impassive Extra-Being Which Is Sterile, Inefficacious, And On The Surface Of Things: The Ideational Or The Incorporeal Can No Longer Be Anything Other Than An "Effect"' (7). This In Turn Leads To A Change Of Metaphor From Surface/Depth To Just Surface, To A Series Of Effects Which Are Manifestations And Are Of Different Types. We Have A Notion of Possibilities, Of Ideality Itself, Rather Than the Platonic Idea, But With No 'Causal and Spiritual Efficacy' (7). The Simulacrum Now Appears On The Surface, Rather Than Being Hidden In The Depths. Events As Effects Combine Past And Present, Active And Passive, All Of Which Are Located Elsewhere As Causes. The Relation Between Events Can Only Be Quasi-Causes. Stoics Saw Dialectical Analysis Has Explorations Of These Combinations, Once They Had Been Expressed In Propositions—Dialectics As Conjugation. Language Also Enables Us To Go Beyond Events Into The Possible Or Becoming. The Relation Between Propositions And Specifics Is Itself Still Paradoxical—'Chryssipus Taught "If You Say Something It Passes Through Your Lips, So If You Say "Chariot", A Chariot Passes Through Your Lips' (8). It Is Deliberate Nonsense In The Anglo American Sense, Or Humourous Play On The Surface, As Opposed To An Ironic Exploration Of Depths And Heights. Lewis Carroll Did Something Similar In Alice. [A Commentary On Alice Ensues, Stressing The Surface Rather Than The Underground World, And Picking Up The Disdain Lewis Carroll Felt For Boys Who Did Not Like To Operate At The Surface. Left Handers And Stutterers Can Sometimes Remind Us Of The Paradoxes Of The Surface, However, Which Can Defeat Commonsense Understandings]. Third Series Of The Proposition: Describing Events As Propositions Raises The Question Of How Best To Analyse Surface Events. There Are Three Possibilities: Denotation [Roughly, A Direct Connection Between Words And Images Which Represent States Of Affairs, As In Indexical Signs. Here, Propositions Are Either True Or False, And May Be True In All Cases]; Manifestation -- A Relation Between The Proposition And The Person Expressing It, Statements Of Desire And Belief. These Are Causal Relations: 'Desire Is The Internal Causality Of An Image With Respect To The Existence Of The Object Or The Corresponding State Of Affairs' (13). Belief Anticipates Production Of An Effect By A Cause. Manifestation Includes Denotation, Makes It Possible. '"I" Is the Basic Manifester' (13). Manifestation Is 'The Domain Of The Personal, Which Functions As The Principle Of All Possible Denotation' (13). The Issues Here Turn On Veracity Other Than Truth And Falsehood, The Avoidance Of Illusion. Signification, The Relation Of A Word To Universal Or General Concepts, And Connections To Implications Which Have To Follow The Rules Of Syntax. Again, Signifying Involves Conceptual Implications Referring To Other Propositions, As In Premises Or Conclusions. This Involves A Certain Indirect Process, Implication Or Assertion, Instead Of Truth Or Veracity, Which Remain As Possible In Certain Conditions. However, It Is Not Just Formal Logical Operations That Are Involved, But Notions Of Probability Or Even Moral Terms Such As Promise Or Commitment. Error Produces Not Falsehood But Absurdity [Looks Really Close To Habermas and the Three Validity Claims Here]. Signification May Not Be Primary, Since All Language Begins From The Standpoint Of The 'I', But There Is An Assumption That Propositions Must Be Understood By Others And Have A General Force. This Implies That Manifestation Has Primacy. But Signification Is Implied, And [In Social Relations] Would Be The Basis Of Manifestation. It Is The Difference Between Langue And Parole. Particular Utterances Only Make Sense Against The Background Of Constant Concepts. This Is Extended To Particular Desires And Beliefs, As Opposed To 'Simple Opinions' Which Would Not Signify (16). Actual Utterances Often Involve Truth Claims And General Signification [And Other Presuppositions, Which Are Technically Infinite]. This Is The Paradox Facing Pure Logic, Solved By A Form Of [Smuggling]: 'Implication Never Succeeds In Grounding Denotation Except By Giving Itself A Ready- Made Denotation, Once In The Premises And Again In The Conclusion' (16). So Actual Propositions Feature Circular Relations between Signification, Manifestation, and Denotation. There Might Even Be A Fourth Dimension—Sense, But Introducing

This Will Depend On Making Relations Theoretically Consistent—It Is 'Not Simply A Question Of Fact' (17). To Approach The Issue, We Ask Whether Sense Might Be Located In One Of The Existing Three. Denotation Concerns Itself With Truth And Falsity, Which Is Too Narrow. The Mere Relation Between Words And Denoted Things Is To Paradoxical To Always Makes Sense, As In The Example Of Speaking The Term 'Chariot'. Instead, Denotation Presupposes Sense. Manifestation Does Involve Some Manifesting Subject Which Initiates, So May Be Sense Is Itself A Subjective Matter Of Beliefs And Desires Of Persons—But Subjects Only Possess This Ability To Speak Because Of A General System Of Signification In Language. It Looks Like Sense Must Be Identified With Signification—But Signification Is Linked In A Circular Relation With Denotation And Manifestation. Perhaps it's Necessary to Think Of Different Forms of Possibility of Propositions—Logical, Physical, Syntactic and So On. This Might Serve As Foundations For Sense, But This Would Be An External Foundation, Independent Of Speech [I Think The Problem Is The Connections Between Any Foundations And Actual Act Of Speech, Whether Anything Would Escape The Foundations]. The Concept Of Truth In Particular Implies Independence From Form. This Independence, Separate From Conceptual Possibilities In Signification, Is What Constitutes Sense. This Is The Fourth Dimension. It Is 'An Incorporeal, Complex, And Irreducible Entity, At The Surface Of Things' (19). There Is Philosophers Have Discovered And Rediscovered This Quality. It Is The Idea Of A Something Again, Beyond The Propositions And The Terms And The Objects Which Are Denoted, Beyond The Subjective I And Things Which Are Expressed. Sense Is Irreducible To Propositions, and It Is and Must Be '"Neutral," Altogether Indifferent To Particular and General, Singular and Universal, Personal and Impersonal' (19). There Is Been Little Agreement About This Possible Fourth Dimension, Whether It Exists Simply In The Form Of Some Enquiry. It Is Not Even Immediately Useful Because It Is Neutral. It Can Only Be Inferred Indirectly, By Questioning Characteristics Of Propositions As Above—This Is 'Inspired In Its Entirety By Empiricism… [Avoiding Notions of Essence or Idea, and Knowing]… Have To Track Down, Invoked And Perhaps Produce A Phantom At The Limit Of A Length Or Unfolded Experience' (20). It Might Be What Husserl Called 'Expression', Lingering In Terms Such As The Noema, As Pure Appearance, Outside Denotation Or Manifestation, Linked In Complex Ways To Appearances. In The Same Way, Sense Does Not Exist Outside Propositions Exactly, But 'Inheres or Subsists' (21). It Is Not Just An Expression, But An Attribute, Not Just Of The Proposition, But 'Of The Thing Or State Of Affairs' (21), [The Potential, 'To Be Able To Be Green' Rather Than Just The Denotation 'Green' Is The Example Here]. It Is Said Of A Thing, So It Depends On Propositions Which Express It And Is Therefore Not Separate From The Proposition. It Is Something Else, Both the Expressible, And the State Of Affairs: 'It Turns One Side towards Things and One Side towards Propositions' (22). It Is What Joins Propositions And Things. [It Is A Becoming]. It Operates On The Surface, Rather As Mathematics Does, Or The Nonsense Of Carroll. It Is The Operation Of Sense That Produces [Meaningful] Paradox. Fourth Series Of Dualities: Important Dualities Exist Between Causes And Effects, And 'Corporeal Things And Incorporeal Events' (23). This Is Extended To A Duality Between Things And Propositions, Bodies And Language. This Is Expressed In Lewis Carroll As A Duality Between Eating And Speaking—The Former Is A Matter Of Bodies Actions And Passions, And The Latter Movements Of The Surface And 'Ideational Attributes Or Incorporeal Events' (23) [Lots Of Examples From Alice About Being Presented To Food And Having Food Presented To You]. The Normal Relationship Can Be Distorted By 'Verbal Hallucinations… Unrestricted Oral Behaviour… And Various Disorders of the Surface' As Bodily Matters Intrude—Stuttering, Left Handedness (24). Sense Is Always Expressed In Propositions, But It Lies In States of Affairs, It Happens To Things. In This Sense, Bodies And Language Are United In The Production Of Sense, Existing 'On The Two Sides Of The Frontier Represented By Sense', Which Constantly Articulates The Differences (24)—Things Include 'Ideational Logical Attributes Which Indicate Incorporeal Events', And Propositions Include Both Denotations And Expressions, Names And Adjectives, And Verbs [The Latter Indicating Becoming And Chains Of Events] (24) [Illustrated With Words By Humpty Dumpty]. This Duality In Propositions Represents Two Dimensions, The 'Denotation Of Things And The Expression Of Sense' (25) [Here, It Is Sense That Is Being Expressed Not Subjectivity, However]. This Means The Duality Is Inherent In Propositions As Well As Between Propositions And Things. [More From Lewis Carroll Page 26, Turning On The Universal Denotation Implied By

The Term ‘It’, And Also The Ability Of The Term To Summarise The Sense Of An Earlier Proposition]. The Two Dimensions Can Emerge In an Esoteric Word, Or In A ‘Non Identifiable Aliquid’ (26). The Example Given Is The Word ‘Snark’ [Both A Limited Thing And A Representation Of Anything That Is Too Exotic To Exist?]. [The Section ends with an Extended Quotation from the Gardener’s Song in Sylvie And Bruno—Pass] Alexander I. Stingl's Blog A Nomadic Scholarship Entity NOTES ON: Deleuze, G (1990) The Logic Of Sense, Trans Mark Lester, Edited By Constantin Boundas, New York: Columbia University Press For Models Kindly See One Of The Papers In The Series. Attributable And Ascribable to Constraints on Space Accomodation Model Is Provided in Some Paper in the series Notwithstanding the Generality and Commonalty of the Observation and Its Concatenatability with the Other Modules. The Scholars Of Tien Tai Called It The "Embodied Nature". (This Is The Buddha-Nature That Includes Both Good And Evil.) The Scholars of Xian Shou Say, "It Is Arising From Primal Nature", And the Scholars of Chan (Zen) Say, "It Is Nature That Causes the Rising of Things". All Dharma Is Dharma-Nature. It Is Not Different From Dharma-Nature. Dharma And Dharma-Nature Are Not Two Separate Identities, "Phenomena" And "Nature" Is Also Not Distinguishable Either. In Other Words, There Is No Difference Between Principle (Absolute) And Practice (Relative). Teachings in Chinese Buddhism (6) Sunyata (Emptiness) In the Mahayana Context (Wikipedia). The reason for becoming, according to Whitehead, is always the actual entity itself. He called this the ontological principle. The entity "enjoys" its relations and is "satisfied" by its own becomings. Whiteheads theory of process and relations can be viewed as an experiential alternative view of quantum physics which incorporates a theory of mind. However, Whitehead’s master work, Process and Reality (1929/1978), is nearly impossible to fully understand, has its own idiosyncratic vocabulary, and is not entirely self-consistent. This relative opacity of Whitehead’s work has impeded the generalization of his quantum process theory, or metaphysics, to other areas of discourse. In his independent formulation of the microgenesis of the mental state, Jason Brown discovered a process theory of mind that had many similarities to Whiteheadian process. He took the concepts of duration and discontinuities of process and applied them to the duration of the mental state as it recapitulates the contents of prior mental states. Through the introduction of novelty, and through the process of microgenesis during the progression of mental states, the Self gives rise to a percept of self, the ego, in the object world, at the termination of the microgenetic process. This termination is the conscious instant, or instantiation of the microgenetic process. It is all that we consciously see. We see the ego, not the Self, and thus mistake the ego for self, replacing self with selfhood. The process itself is obscured from our consciousness, but can be inferred from those neurological deficit syndromes, which expose lower "layers" in the becoming. This parallels the "layers" of the nested hierarchy of holographic surfaces in the brain itself. Brown is theory of microgenesis involves a progression in the genesis of the mental state in a developmental sequence that recapitulates the organism is evolutionary development (phylogeny), embryologic development (ontogeny), and history of personal and social experience and development, in a movement in an upward direction through structures of increasing phylogenetic and ontogenic recency (Brown, 1988, 1991, 1996, 1998, 2005). He developed a view that considered the evolution of mental states to be a continuum from depth to surface, with leaps or saltation between mental states at intervals that are temporally irreducible units of the duration of microgenetic process (Brown, 1996, 1998). The process of neoteny, through which earlier features of development and evolution are brought to the present, was described (Brown, 1996) as a change in the development of some aspect of the mental state, which is attenuated at some stage of elaboration. In neurological and psychiatric deficit syndromes, this attenuation corresponds with the locus of a lesion or functional deficiency. The signature of the attenuated stage is carried through the remaining process of microgenesis, leading to expression of the attenuated phase at the endpoint of the duration of microgenesis, coupled with full elaboration of other elements of microgenesis. The intellectual pedigree of the theory of microgenesis dates back to the principle of superposition of Sir Charles Lyell, according to which younger strata lie on top of older strata. Lyell, a geologist, was to have great influence on evolutionary theorists, including Darwin and Wallace. The principle of superposition was taken up by Herbert Spencer, a scholar of many fields, who was a contemporary of Charles Darwin, and who had developed an evolutionary theory that perhaps rivaled Darwin is in popularity during the late 1800ís and early 1900ís. An important element of Spencer is theory was his observation of the brain, which

he fashioned to be layered, with the newer layers added on to older layer during the course of evolution. Looking at the deeper layers was, for Spencer, like peeling away the layers of an onion. Spencer is idea was taken up by the renowned British neurologist Hughlings Jackson (Kennard and Swash eds., 1989), who developed the notion of hierarchies of brain process and the concept of neurological deficit syndromes as disturbances in the hierarchical structure and function of the brain. The concepts of microgeny and microgenesis, derived from hierarchies of brain function, were later developed by Arieti (1962) and Werner (1956). Later, the concept of a hierarchy of adaptive ego mechanisms, including lower order defenses such as denial and reaction formation, and higher order defenses such as altruism, was developed by Vaillant (1971), with the more mature defense mechanisms bearing on higher order values. The process theory of the evolution of the brain is extraordinarily important to Brown is process theory of mind. We can place these theories in perspective with the process theory of Whitehead, into which the theory of Brown partially merged (Brown, 1996; 1998; 2005). The materialist perspective of evolution, was, according to Whitehead (1925/1953), inconsistent with the process by which organisms become: "...in truth, a thoroughgoing evolutionary philosophy is inconsistent with materialism. The aboriginal stuff, or material, from which a materialistic philosophy starts, is incapable of evolution. This material is in itself the ultimate substance. Evolution, in the materialistic theory, is reduced to the role of being another word for the description of the changes of the external relations between portions of matter. There is nothing to evolve, because one set of external relations is as good as any other set of external relations. There can merely be change, purposeless and unprogressive." Whitehead is (1929/1978) realm of eternal objects, apprehended in the unconscious, can be viewed as a basic element in the evolution of organisms and of mind. Whitehead (1929/1978), in his view of eternal objects, hearkens is back to Plato: "...eternal objects, as in Gods primordial nature, constitute the Platonic world of ideas." Whitehead is realm of eternal objects may include Carl Jung is (1934/1967) archetypes of the collective unconscious (Griffin, ed., 1989), connecting Whitehead is cosmology to the corpus of literature on depth psychology. Jung is archetypes were largely based on mythological, religious, and alchemical constructs. Mythology embodies the spiritual and social history of the species, as well as a variety of metaphysical metaphors (Campbell, 1969/1976; Neumann, 1954/1970). The physical nature of the brain state, including the notion of binding of the mental state into a coherent whole, is also critical to Brown is theory of microgenesis. Recent evidence indicates that the brain is a chaotic or dynamical system (Freeman, 1987; Combs, 2004; Abraham, 2004). The self-organization of dynamical systems, which include all living organisms as well as the brain, has been described in detail by Ilya Prigogine (1980), who won the Nobel Prize in chemistry in 1977 for his work on dynamical systems. Prigogine defines an "internal time" or duration of the states of dynamical systems. Physical time, he states, is secondary, and internal time is primary (Prigogine, 1986). The dynamical interval of internal time, or duration, is governed by the baker transformation (Prigogine, 1980), which can be envisioned as the time it takes the baker to knead or fold two different colors of dough successively, until it seems to be uniform in color. The brain, as a dynamical system, thus "deposits" time in the sense of Brown (1996). Here we have a parallel with the "layering" of process time according to the Holographic Principle. We can also detect, remotely, the undertones of Lyellís principle of superposition, albeit in a different form, and in a sense that Lyell could never have imagined. The work of Jason Brown in process neurology and psychology stands in sharp contrast to the prevailing scientific and philosophical nihilism in these fields regarding the self, free will, and the process of experience. In the sense of Brown is theory of microgenetic process of the brain, and in the sense of the duration of process discontinuities of Whitehead as it relates to the mind, it is important for the purposes of this paper to explore the scientific basis of the brain state. Mental process involves successive durations of such states, each with a "temporal thickness" (Brown, 1996). In recent years the "brain electrical microstate" has been described in detail as patterns of electrical field potential which endure for a period a period of time, typically around 100 milliseconds or a tenth of a second, and are punctuated by rapid transitions to the next state (Fingelkurts and Fingelkurts, 2001; Koenig and others, 2002). Brain electrical microstates are prime candidates for the brain states of microgenetic process and for the dynamical interval of internal time. The application of the theory of discontinuous information states, with the complete repetition of former states, as applied to the Holographic Principle of Mind, requires an information theory that includes time and

the duration of mental states (e.g. Germine, 1993).The process of microgenesis, as it applies to brain evolution, can be seen in purely classical way, related to the recapitulation theory of vertebrate brain evolution (Agoutis, Montiel, and Lopez, 2002), which implies that "sensory projection sites and sites and processing circuits have been conserved [recapitulated] in reptiles and mammals" in the course of evolution. Again, there is an element of predictability in evolution which seems to belie the purposelessness of random natural selection. In keeping with the Holographic Principle, the development of the organism is a product of its evolutionary history, and is in continuous intercourse with that history. Microgenesis is readily related to the Holographic Principle of Mind, whereby the past remains actual, and, in the course of each mental state or mental process, there is a literal recapitulation of the organisms' entire past. The depth to surface principle entails a direction of mental process which is essential to microgenesis, and which is explained naturally by the nested hierarchy of surfaces in the brain according to the Holographic Principle. In the Holographic Principle of Mind we have a universal principle of microgenesis, whereby the entire relevant history of the organism is repeated during the course of each timeless and eternal moment. In this way, microgenesis can extend back to the very inception of the Universe. MICROGENESIS AND THE HOLOGRAPHIC PRINCIPLE the Holographic Principle Theory of Mind MARK GERMINE Institute for Psycho Science. Aristotle and others (including, especially, Leibniz) have argued that time does not exist independently of the events that occur in time. This view is typically called either "Reductionism with Respect to Time" or "Relationism with Respect to Time," since according to this view; all talk that appears to be about time can somehow be reduced to talk about temporal relations among things and events. The opposing view, normally referred to either as "Platonism with Respect to Time" or as "Substantivalism with Respect to Time" or as "Absolutism with Respect to Time," has been defended by Plato, Newton, and others. On this view, time is like an empty container into which things and events may be placed; but it is a container that exists independently of what (if anything) is placed in it. Time First published Mon Nov 25, 2002; substantive revision Fri Jan 24, 2014Markosian, Ned, "Time", The Stanford Encyclopedia of Philosophy (Spring 2014 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/spr2014/entries/time/>. The question of whether there could be time without change has traditionally been thought to be closely tied to the question of whether time exists independently of the events that occur in time. For, the thinking goes, if there could be a period of time without change, then it follows that time could exist without any events to fill it; but if, on the other hand, there could not be a period of time without change, then it must be that time exists only if there are some events to fill it. Time First published Mon Nov 25, 2002; substantive revision Fri Jan 24, 2014Markosian, Ned, "Time", The Stanford Encyclopedia of Philosophy (Spring 2014 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/spr2014/entries/time/>."Whoever thou mayest be, beloved stranger, whom I meet here for the first time, avail thyself of this happy hour and of the stillness around us, and above us, and let me tell thee something of the thought which has suddenly risen before me like a star which would fain shed down its rays upon thee and every one, as befits the nature of light. - Fellow man! Your whole life, like a sandglass, will always be reversed and will ever run out again; - a long minute of time will elapse until all those conditions out of which you were evolved return in the wheel of the cosmic process. And then you will find every pain and every pleasure, every friend and every enemy, every hope and every error, every blade of grass and every ray of sunshine once more, and the whole fabric of things which make up your life. This ring in which you are but a grain will glitter afresh forever. And in every one of these cycles of human life there will be one hour where, for the first time one man, and then many, will perceive the mighty thought of the eternal recurrence of all things:- and for mankind this is always the hour of Noon". Notes on the Eternal Recurrence - Vol. 16 of Oscar Levy Edition of Nietzsche's Complete Works (in English). Physical time is public time, the time that clocks are designed to measure. Biological time, by contrast, is indicated by an organism's circadian rhythm or body clock, which is normally regulated by the pattern of sunlight and darkness. Psychological time is different from both physical time and biological time. Psychological time is private time. It is also called phenomenological time, and it is perhaps best understood as awareness of physical time. Psychological time passes relatively swiftly for us while we are enjoying an activity, but it slows dramatically if we are waiting anxiously for the pot of water to boil on the stove. The slowness is probably due to focusing our

attention on short intervals of physical time. Meanwhile, the clock by the stove is measuring physical time and is not affected by any person's awareness or by any organism's biological time. When a physicist defines speed to be the rate of change of position with respect to time, the term "time" refers to physical time, not psychological time or biological time. Physical time is more basic or fundamental than psychological time for helping us understand our shared experiences in the world, and so it is more useful for doing physical science, but psychological time is vitally important for understanding many mental experiences. Psychological time is faster for older people than for children, as you notice when your grandmother says, "Oh, it's my birthday again." That is, an older person's psychological time is faster relative to physical time. Psychological time is slower or faster depending upon where we are in the spectrum of conscious experience: awake normally, involved in a daydream, sleeping normally, drugged with anesthetics, or in a coma. Some philosophers claim that psychological time is completely transcended in the mental state called nirvana because psychological time slows to a complete stop. There is general agreement among philosophers that, when we are awake normally, we do not experience time as stopping and starting Time First published Mon Nov 25, 2002; substantive revision Fri Jan 24, 2014Markosian, Ned, "Time", The Stanford Encyclopedia of Philosophy (Spring 2014 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/spr2014/entries/time/>. But From The Profound Contemplation And Wisdom Of The Buddha And Mahabodhisattvas, We Know There Is No Such Reality. Instead, Egolessness (Non-Self) Is The Only Path To Understand The Reality Of The Deluded Life. All Existences Are Subject To The Law Of Causes And Conditions. These Include The Smallest Particles, The Relationship Between The Particles, The Planets, And The Relationship Between Them, Up To And Including The Whole Universe! From The Smallest Particles To The Biggest Matter, There Exists No Absolute Independent Identity. Egolessness (Non-Self) Implies The Void Characteristics Of All Existence. Egolessness (Non-Self) Signifies The Non-Existence Of Permanent Identity For Self And Existence (Dharma). Sunyata Stresses The Voidness Characteristic Of Self And Existence (Dharma). Sunyata And Egolessness Possess Similar Attributes. As We Have Discussed Before, We Can Observe The Profound Significance Of Sunyata From The Perspective Of Inter-Dependent Relationships. Considering Dharma-Nature and the Condition of Nirvana, All Existences Are Immaterial and Of A Void-Nature. Then We See Each Existence As Independent Of Each Other. But Then We Cannot Find Any Material That Does Exist Independent Of Everything Else. So Egolessness Also Implies Void-Nature! From The Observation Of All Existences, We Can Infer The Theory Of Nirvana And The Complete Cessation Of All Phenomena. From The Viewpoint Of Phenomena, All Existences Are So Different From Each Other, That They May Contradict Each Other. They Are So Chaotic. In Reality, Their Existence Is Illusionary And Arises From Conditional Causation. They Seem To Exist On One Hand, And Yet Do Not Exist On The Other. They Seem To Be United, But Yet They Are So Different To One Another. They Seem To Exist And Yet They Do Cease! Ultimately Everything Will Return To Harmony And Complete Calmness. This Is The Nature Of All Existence. It Is The Final Resting Place For All. If We Can Understand This Reality And Remove Our Illusions, We Can Find This State Of Harmony And Complete Calmness. Teachings in Chinese Buddhism (6) Sunyata (Emptiness) In the Mahayana Context (Wikipedia) for models please see one of the papers in the series. "The Difference Between Supermind And Big Mind (If We Take Big Mind To Mean The State Experience Of Nondual Suchness, Or Turiyatita) Is That Big Mind Can Be Experienced Or Recognized At Virtually Any Lower Level Or Rung. Magic to Integral. In Fact, One Can Be At, Say, The Pluralistic Stage, And Experience Several Core Characteristics Of The Entire Sequence Of State-Stages (Gross To Subtle To Causal To Witnessing To Nondual), Although, Of Course, The Entire Sequence, Including Nondual Suchness, Will Be Interpreted In Pluralistic Terms. This Is Unfortunate In Many Ways—Interpreting Dharma In Merely Pluralistic Terms (Or Mythic Terms, Or Rational, And So On)—Because It Is So Ultimately Reductionistic; But It Happens All The Time, Given The Relative Independence Of States And Structures At 1st And 2nd Tier. Supermind, On The Other Hand, As A Basic Structure-Rung (Conjoined With Nondual Suchness) Can Only Be Experienced Once All The Previous Junior Levels Have Emerged And Developed, And As In All Structure Development, Stages Cannot Be Skipped. Therefore, Unlike Big Mind, Supermind Can Only Be Experienced After All 1st-, 2nd-, and 3rd-Tier Junior Stages Have Been Passed Through. While, As Genpo Roshi Has Abundantly Demonstrated, Big Mind State

Experience Is Available To Virtually Anybody At Almost Any Age (And Will Be Interpreted According To The View Of Their Current Stage), Supermind Is An Extremely Rare Recognition. Supermind, As The Highest Structure-Rung To Date, Has Access To All Previous Structures, All The Way Back To Archaic—And The Archaic Itself, Of Course, Has Transcended And Included, And Now Embraces, Every Major Structural Evolution Going All The Way Back To The Big Bang. (A Human Being Literally Enfolds And Embraces All The Major Transformative Unfoldings Of The Entire Kosmic History—Strings To Quarks To Subatomic Particles To Atoms To Molecules To Cells, All The Way Through The Tree Of Life Up To Its Latest Evolutionary Emergent, The Triune Brain, The Most Complex Structure In The Known Natural World.) Supermind, In Any Given Individual, Is Experienced As A Type Of "Omniscience"—The Supermind, Since It Transcends And Includes All Of The Previous Structure-Rungs, And Inherently Is Conjoined With The Highest Nondual Suchness State, Has A Full And Complete Knowledge Of All Of The Potentials In That Person. It Literally "Knows All," At Least For The Individual." ― Ken Wilber, The Fourth Turning: imagining the Evolution of an Integral Buddhism (fopr models please see one of the papers in the series). "The Movement Of Descent And Discovery Begins At The Moment You Consciously Become Dissatisfied With Life. Contrary To Most Professional Opinion, This Gnawing Dissatisfaction With Life Is Not A Sign Of "Mental Illness," or An Indication Of Poor Social Adjustment, or A Character Disorder. For Concealed Within This Basic Unhappiness With Life And Existence Is The Embryo Of A Growing Intelligence, A Special Intelligence Usually Buried Under The Immense Weight Of Social Shams. A Person Who Is Beginning To Sense The Suffering Of Life Is, At The Same Time, Beginning To Awaken To Deeper Realities, Truer Realities. For Suffering Smashes To Pieces The Complacency Of Our Normal Fictions About Reality, And Forces Us To Become Alive In A Special Sense—To See Carefully, To Feel Deeply, To Touch Ourselves And Our Worlds In Ways We Have Heretofore Avoided. It Has Been Said, And Truly I Think, That Suffering Is The First Grace. In A Special Sense, Suffering Is Almost A Time Of Rejoicing, For It Marks The Birth Of Creative Insight. But Only In a Special Sense. Some People Cling To Their Suffering As A Mother To Its Child, Carrying It As A Burden They Dare Not Set Down. They Do Not Face Suffering With Awareness, But Rather Clutch At Their Suffering, Secretly Transfixed With The Spasms Of Martyrdom. Suffering Should Neither Be Denied Awareness, Avoided, Despised, Not Glorified, Clung To, And Dramatized. The Emergence Of Suffering Is Not So Much Good As It Is A Good Sign, An Indication That One Is Starting To Realize That Life Lived Outside Unity Consciousness Is Ultimately Painful, Distressing, And Sorrowful. The Life Of Boundaries Is A Life Of Battles—Of Fear, Anxiety, Pain, And Finally Death. It Is Only Through All Manner Of Numbing Compensations, Distractions, And Enchantments That We Agree Not To Question Our Illusory Boundaries, The Root Cause Of The Endless Wheel Of Agony. But Sooner Or Later, If We Are Not Rendered Totally Insensitive, Our Defensive Compensations Begin To Fail Their Soothing And Concealing Purpose. As A Consequence, We Begin To Suffer In One Way Or Another, Because Our Awareness Is Finally Directed Toward The Conflict-Ridden Nature Of Our False Boundaries And The Fragmented Life Supported By Them." ― Ken Wilber, No Boundary: eastern and Western Approaches to Personal Growth for models please see one of the papers in the series. "To The Extent That You Actually Realize That You Are Not, For Example, Your Anxieties, Then Your Anxieties No Longer Threaten You. Even If Anxiety Is Present, It No Longer Overwhelms You Because You Are No Longer Exclusively Tied To It. You Are No Longer Courting It, Fighting It, Resisting It, Or Running From It. In The Most Radical Fashion, Anxiety Is Thoroughly Accepted As It Is And Allowed To Move As It Will. You Have Nothing To Lose, Nothing To Gain, By Its Presence Or Absence, For You Are Simply Watching It Pass By. Thus, Any Emotion, Sensation, Thought, Memory, Or Experience That Disturbs You Is Simply One With Which You Have Exclusively Identified Yourself, And The Ultimate Resolution Of The Disturbance Is Simply To Dis-Identify With It. You Cleanly Let All Of Them Drop Away By Realizing That They Are Not You--Since You Can See Them, They Cannot Be The True Seer And Subject. Since They Are Not Your Real Self, There Is No Reason Whatsoever For You To Identify With Them, Hold On To Them, Or Allow Yourself To Be Bound By Them. Slowly, Gently, As You Pursue This Dis-Identification "Therapy," You May Find That You is Entire Individual Self (Persona, Ego, and Centaur), Which Heretofore You Have Fought to Defend and Protect, Begins to Go Transparent and Drop Away. Not That It Literally Falls Off And You Find Yourself Floating,

Disembodied, Through Space. Rather, You Begin To Feel That What Happens To Your Personal Self—Your Wishes, Hopes, Desires, Hurts—Is Not A Matter Of Life-Or-Death Seriousness, Because There Is Within You A Deeper And More Basic Self Which Is Not Touched By These Peripheral Fluctuations, These Surface Waves Of Grand Commotion But Feeble Substance. Thus, Your Personal Mind-And-Body May Be In Pain, Or Humiliation, Or Fear, But As Long As You Abide As The Witness Of These Affairs, As If From On High, They No Longer Threaten You, And Thus You Are No Longer Moved To Manipulate Them, Wrestle With Them, Or Subdue Them. Because You Are Willing To Witness Them, To Look At Them Impartially, You Are Able To Transcend Them. As St. Thomas Put It, "Whatever Knows Certain Things Cannot Have Any Of Them In Its Own Nature." Thus, If The Eye Were Colored Red, It Wouldn't Be Able To Perceive Red Objects. It Can See Red Because It Is Clear, Or "Redless." Likewise, If We Can But Watch Or Witness Our Distresses, We Prove Ourselves Thereby To Be "Distress-Less," Free Of The Witnessed Turmoil. That Within Which Feels Pain Is Itself Pain-Less; That Which Feels Fear Is Fear-Less; That Which Perceives Tension Is Tensionless. To Witness These States Is To Transcend Them. They No Longer Seize You From Behind Because You Look At Them Up Front." ― Ken Wilber, No Boundary: eastern and Western Approaches to Personal Growth (for models please see one of the papers in the series). "The Difference Between Supermind And Big Mind (If We Take Big Mind To Mean The State Experience Of Nondual Suchness, Or Turiyatita) Is That Big Mind Can Be Experienced Or Recognized At Virtually Any Lower Level Or Rung. ……In Difference and Repetition (1968, English 1994), Deleuze Develops His Project in Multiple Directions. His Work, He Says, Stems From The Convergence Of Two Lines Of Research: The Concept Of Difference Without Negation, And The Concept Of Repetition, In Which Physical And Mechanical Repetitions Are Masks For A Hidden Differential That Is Disguised And Displaced. His Major Focus Is A Thoroughgoing Critique Of Representational Thinking, Including Identity, Opposition, Analogy, And Resemblance (Deleuze 1994, 132). For Deleuze, "Appearances Of" Are Not Representations, But Sensory Intensities Free Of Subjective Or Objective Identities (Deleuze 1994, 144). Without These Identities, Appearances Are Simulacra of A Non-Apparent Differential He Calls the "Dark Precursor" Or "The In-Itself of Difference" (Deleuze 1994, 119). This Differential Is The Non-Sensible Being Of The Sensible, A Being Not Identical To The Sensible, Or To Itself, But Irreducibly Problematic Insofar As It Forces Us To Encounter The Sensible As "Given." Furthermore, Any Move Against Representational Thinking Impinges Upon The Identity Of The Subject. Where Kant Founds The Representational Unity Of Space And Time Upon The Formal Unity Of Consciousness (Kant 1964, 135-137), Difference Re-Distributes Intuitions Of Past, Present, And Future, Fracturing Consciousness Into Multiple States Not Predicable Of A Single Subject. Intensive Qualities Are Individuating By Themselves, Says Deleuze, And Individuality Is Not Characteristic Of A Self Or An Ego, But Of A Differential Forever Dividing Itself And Changing Its Configuration (Deleuze 1994, 246, 254, 257). In Nietzschean Fashion, The "I" Refers Not To The Unity Of Consciousness, But To A Multitude Of Simulacra Without An Identical Subject For Whom This Multitude Appears. Instead, Subjects Arise And Multiply As "Effects" Of The Intensive Qualities Saturating Space And Time. This Leads Deleuze to Postulate Multiple Faculties for Subjectivity, Which Are Correlates of the Sensible Insofar As It Gives Rise to Feeling, Thought, and Action. "Each Faculty, Including Thought, Has Only Involuntary Adventures," He Says, And "Involuntary Operation Remains Embedded In The Empirical" (Deleuze 1994, 145). Subjectively, The Paradox Of The Differential Breaks Up The Faculties' Common Function And Places Them Before Their Own Limits: Thought Before The Unthinkable, Memory Before The Immemorial, Sensibility Before The Imperceptible, Etc. (Deleuze 1994, 227). This Fracturing And Multiplying Of The Subject, He Notes, Leads To The Realization That "Schizophrenia Is Not Only A Human Fact But Also A Possibility For Thought" (Deleuze 1994, 148), Thus Expanding The Term Into A Philosophical Concept, Beyond Its Clinical Application. Postmodernism First Published Fri Sep 30, 2005 Copyright © 2005 By Gary Aylesworth Geaylesworth@Eiu.Edu Open Access To The SEP Is Made Possible By A World-Wide Funding Initiative for models kindly see one of the papers in the series, procrastinated due to restrictions on space. The Dissolution Of The Subject And Its Implications For Society Is The Theme Of Anti-Oedipus: Capitalism And Schizophrenia, Which Deleuze Published With Félix Guattari In 1972 (English 1983). The Book, In Large Part, Is Written Against An Established Intellectual

Orthodoxy Of The Political Left In France During The 1950s And 1960s, An Orthodoxy Consisting Of Marx, Freud, And Structuralist Concepts Applied To Them By Louis Althusser And Jacques Lacan. Deleuze And Guattari Argue That This Mixture Is Still Limited By Representational Thinking, Including Concepts Of Production Based Upon Lack, And Concepts Of Alienation Based Upon Identity And Negation. Furthermore, The Oedipus Concept In Psychoanalysis, They Say, Institutes A Theater Of Desire In Which The Psyche Is Embedded In A Family Drama Closed Off From The Extra-Familial And Extra-Psychic Forces At Work In Society. They Characterize These Forces As "Desiring Machines" Whose Function Is To Connect, Disconnect, And Reconnect With One Another Without Meaning Or Intention. The Authors Portray Society As A Series Of "Territorializations" Or Inscriptions Upon The "Body Without Organs, "Or The Free-Flowing Matter Of Intensive Qualities Filling Space In Their Varying Degrees. The First Inscriptions Are Relations Of Kinship And Filiation Structuring Primitive Societies, Often Involving The Marking And Scarring Of Human Bodies. As An Interruption and Encoding of "Flows," The Primitive Inscriptions Constitute a Nexus of Desiring Machines, Both Technical and Social, Who's Elements Are Humans and Their Organs. The Full Body Of Society Is The Sacred Earth, Which Appropriates To Itself All Social Products As Their Natural Or Divine Precondition, And To Whom All Members Of Society Are Bound By Direct Filiation (Deleuze 1983b, 141-42). These First Inscriptions Are Then De-Territorialized And Re-Coded By The "Despotic Machine," Establishing New Relations Of Alliance And Filiation Through The Body Of The Ruler Or Emperor, Who Alone Stands In Direct Filiation To The Deity (Deleuze 1983b, 192) And Who Institutes The Mechanism Of The State Upon Pre-Existing Social Arrangements. Finally, Capitalism De-Territorializes The Inscriptions Of The Despotic Machine And Re-Codes All Relations Of Alliance And Filiation Into Flows Of Money (Deleuze 1983b, 224-27). The Organs Of Society And The State Are Appropriated Into The Functioning Of Capital, And Humans Become Secondary To The Filiation Of Money With Itself. Postmodernism First Published Fri Sep 30, 2005 Copyright © 2005 By Gary Aylesworth Geaylesworth@Eiu.Edu Open Access To The SEP Is Made Possible By A World-Wide Funding Initiative For Models Kindly See One Of The Papers In The Series, Procrastinated Due To Restrictions On Space. Deleuze And Guattari See In The Capitalist Money System "An Axiomatic Of Abstract Quantities That Keeps Moving Further And Further In The Direction Of The Deterritorialization Of The Socius" (Deleuze 1983a, 33), Which Is To Say That Capital Is Inherently Schizophrenic. However, Because Capital Also Re-Territorializes All Flows Into Money, Schizophrenia Remains Capitalism's External Limit. Nevertheless, It Is Precisely That Limit Against Which Thinking Can Subject Capitalism To Philosophical Critique. Psychoanalysis, They Say, Is Part Of The Reign Of Capital Because It Re-Territorializes The Subject As "Private" And "Individual," Instituting Psychic Identity Through Images Of The Oedipal Family. However, The Oedipal Triangle Is Merely A Representational Simulacrum Of Kinship And Filiation, Re-Coded Within A System Of Debt And Payment. In This System, They Insist, Flows Of Desire Have Become Mere Representations Of Desire, Cut Off From The Body Without Organs And The Extra-Familial Mechanisms Of Society. A Radical Critique Of Capital Cannot Therefore Be Accomplished By Psychoanalysis, But Requires A Schizoanalysis "To Overturn The Theater Of Representation Into The Order Of Desiring-Production" (Deleuze 1983b, 271). Here, The Authors See A Revolutionary Potential In Modern Art And Science, Where, In Bringing About The "New," They Circulate De-Coded And De-Territorialized Flows Within Society Without Automatically Re-Coding Them Into Money (Deleuze 1983a, 379). In This Revolutionary Aspect, Anti-Oedipus Reads As A Statement Of The Desire That Took To The Streets Of Paris In May Of 1968, And Which Continues, Even Now, To Make Itself Felt In Intellectual Life. Postmodernism First Published Fri Sep 30, 2005 Copyright © 2005 By Gary Aylesworth Geaylesworth@Eiu.Edu Open Access To The SEP Is Made Possible By A World-Wide Funding Initiative For Models Kindly See One Of The Papers In The Series, Procrastinated Due To Restrictions On Space. The Term "Deconstruction," Like "Postmodernism," Has Taken On Many Meanings In The Popular Imagination. However, In Philosophy, It Signifies Certain Strategies For Reading And Writing Texts. The Term Was Introduced Into Philosophical Literature In 1967, With The Publication Of Three Texts By Jacques Derrida: Of Grammatology (English 1974), Writing And Difference (English 1978), And Speech And Phenomena (English 1973). This So-Called "Publication Blitz" Immediately Established Derrida As A Major Figure In The New Movement In Philosophy And The Human Sciences Centered In Paris, And Brought

The Idiom "Deconstruction" Into Its Vocabulary. Derrida And Deconstruction Are Routinely Associated With Postmodernism, Although Like Deleuze And Foucault, He Does Not Use The Term And Would Resist Affiliation With "-Isms" Of Any Sort. Of The Three Books From 1967, Of Grammatology Is The More Comprehensive In Laying Out The Background For Deconstruction As A Way Of Reading Modern Theories Of Language, Especially Structuralism, And Heidegger's Meditations On The Non-Presence Of Being. It Also Sets Out Derrida's Difference With Heidegger Over Nietzsche. Where Heidegger Places Nietzsche Within The Metaphysics Of Presence, Derrida Insists That "Reading, And Therefore Writing, The Text Were For Nietzsche 'Originary' Operations," (Derrida 1974, 19), And This Puts Him At The Closure Of Metaphysics (Not The End), A Closure That Liberates Writing From The Traditional Logos, Which Takes Writing To Be A Sign (A Visible Mark) For Another Sign (Speech), Whose "Signified" Is A Fully Present Meaning. This Closure Has Emerged, Says Derrida, With The Latest Developments In Linguistics, The Human Sciences, Mathematics, And Cybernetics, Where The Written Mark Or Signifier Is Purely Technical, That Is, A Matter Of (e) Function Rather Than Meaning. Precisely The Liberation Of Function Over Meaning Indicates That The Epoch Of (e) What Heidegger Calls The Metaphysics Of Presence Has Come To Closure, Although This Closure Does Not Mean Its Termination. Just As In The Essay "On The Question Of Being" (Heidegger 1998, 291-322) Heidegger Sees Fit To Cross Out The Word "Being," Leaving It Visible, Nevertheless, Under The Mark, Derrida Takes The Closure Of Metaphysics To Be Its "Erasure," Where It Does Not Entirely Disappear, But Remains Inscribed As (=) One Side Of A Difference, And Where The Mark Of Deletion Is (=) Itself A Trace Of The Difference That Joins And Separates (e&eb, e(e&eb))This Mark And What It Crosses Out. Derrida Calls This Joining And Separating Of Signs Différance (Derrida 1974, 23), A Device That Can Only Be Read And Not Heard When (e) Différance And Différence Are Pronounced In French. The "A" Is A Written Mark That Differentiates Independently Of The Voice, The Privileged Medium Of Metaphysics. In This Sense, Différance As The Spacing Of Difference, As Archi-Writing, Would Be (eb) The Gram Of Grammatology. However, As Derrida Remarks: "There Cannot Be (=) A Science Of Difference Itself In Its Operation, As It Is Impossible To (e) Have A Science Of The Origin Of Presence Itself, That Is To Say Of A Certain Non-Origin" (Derrida 1974, 63). Instead, There Is (=) Only The Marking Of (e) The Trace Of Difference, That Is, Deconstruction. Because (e) At Its Functional Level All Language Is A System Of Differences, Says Derrida, All Language, Even When Spoken, Is Writing, And This Truth Is Suppressed When Meaning Is Taken As An Origin, Present And Complete Unto Itself. Texts That Take Meaning Or Being As Their Theme Are Therefore Particularly Susceptible To Deconstruction, As Are All Other Texts Insofar As They Are Conjoined With These. For Derrida, Written Marks Or Signifiers Do Not Arrange Themselves Within Natural Limits, But Form Chains Of Signification That Radiate In All Directions. As Derrida Famously Remarks, "There Is No Outside-Text" (Derrida 1974, 158), That Is, The Text Includes The Difference Between Any "Inside" Or "Outside." A Text, Then, Is Not A Book, And Does Not, Strictly Speaking, Have An Author. On The Contrary, The Name Of The Author Is A Signifier Linked With Others, And There Is No Master Signifier (Such As The Phallus In Lacan) Present Or Even Absent In A Text. This Goes For The Term "Différance" As Well, Which Can Only Serve As A Supplement For The Productive Spacing Between Signs. Therefore, Derrida Insists That "Différance Is Literally Neither A Word Nor A Concept" (Derrida 1982, 3). Instead, It Can Only Be Marked As A Wandering Play Of Differences That Is Both A Spacing Of Signifiers In Relation To One Another And A Deferral Of Meaning Or Presence When They Are Read. How, Then, Can Différance Be Characterized? Derrida Refuses To Answer Questions As To "Who" Or "What" Differs, Because To Do So Would Suggest There Is A Proper Name For Difference Instead Of Endless Supplements, Of Which "Différance" Is But One. Structurally, This Supplemental Displacement Functions Just As, For Heidegger, All Names For Being Reduce Being To The Presence Of Beings, Thus Ignoring The "Ontological Difference" Between Them. However, Derrida Takes The Ontological Difference As One Difference Among Others, As A Product Of What The Idiom "Différance" Supplements. As He Remarks: "Différance, In A Certain And Very Strange Way, (Is) 'Older' Than The Ontological Difference Or Than The Truth Of Being" (Derrida 1982, 22). Deconstruction, Then, Traces The Repetitions Of The Supplement. It Is Not So Much A Theory About Texts As A Practice Of Reading And Transforming Texts, Where Tracing The Movements Of Différance Produces Other Texts Interwoven With The

First. While There Is Certain Arbitrariness In The Play Of Differences That Result, It Is Not The Arbitrariness Of A Reader Getting The Text To Mean Whatever He Or She Wants. It Is A Question Of Function Rather Than Meaning, If Meaning Is Understood As A Terminal Presence, (eb)And The Signifying Connections Traced In Deconstruction Are First Offered By The Text Itself. A Deconstructive Reading, Then, Does Not Assert Or Impose Meaning, But Marks Out Places Where The Function Of The Text Works Against Its Apparent Meaning, Or Against The History Of Its Interpretation. Hyperreality is closely related to the concept of the simulacrum: a copy or image without reference to an original. In postmodernism, hyperreality is the result of the technological mediation of experience, where what passes for reality is a network of images and signs without an external referent, such that what is represented is representation itself. In Symbolic Exchange and Death (1976) (English 1993), Jean Baudrillard uses Lacan's concepts of the symbolic, the imaginary, and the real to develop this concept while attacking orthodoxies of the political Left, beginning with the assumed reality of power, production, desire, society, and political legitimacy. Baudrillard argues that all of these realities have become simulations, that is, signs without any referent, because the real and the imaginary have been absorbed into the symbolic Postmodernism First Published Fri Sep 30, 2005 Copyright © 2005 By Gary Aylesworth Geaylesworth@Eiu.Edu Open Access To The SEP Is Made Possible By A World-Wide Funding Initiative For Models Kindly See One Of The Papers In The Series, Procrastinated Due To Restrictions On Space. Magic to Integral. In Fact, One Can Be At, Say, The Pluralistic Stage, And Experience Several Core Characteristics Of The Entire Sequence Of State-Stages (Gross To Subtle To Causal To Witnessing To Nondual), Although, Of Course, The Entire Sequence, Including Nondual Suchness, Will Be Interpreted In Pluralistic Terms. This Is Unfortunate In Many Ways—Interpreting Dharma In Merely Pluralistic Terms (Or Mythic Terms, Or Rational, And So On)—Because It Is So Ultimately Reductionistic; But It Happens All The Time, Given The Relative Independence Of States And Structures At 1st And 2nd Tier. Supermind, On The Other Hand, As A Basic Structure-Rung (Conjoined With Nondual Suchness) Can Only Be Experienced Once All The Previous Junior Levels Have Emerged And Developed, And As In All Structure Development, Stages Cannot Be Skipped. Therefore, Unlike Big Mind, Supermind Can Only Be Experienced After All 1st-, 2nd-, and 3rd-Tier Junior Stages Have Been Passed Through. While, As Genpo Roshi Has Abundantly Demonstrated, Big Mind State Experience Is Available To Virtually Anybody At Almost Any Age (And Will Be Interpreted According To The View Of Their Current Stage), Supermind Is An Extremely Rare Recognition. Supermind, As The Highest Structure-Rung To Date, Has Access To All Previous Structures, All The Way Back To Archaic—And The Archaic Itself, Of Course, Has Transcended And Included, And Now Embraces, Every Major Structural Evolution Going All The Way Back To The Big Bang. (A Human Being Literally Enfolds And Embraces All The Major Transformative Unfoldings Of The Entire Kosmic History—Strings To Quarks To Subatomic Particles To Atoms To Molecules To Cells, All The Way Through The Tree Of Life Up To Its Latest Evolutionary Emergent, The Triune Brain, The Most Complex Structure In The Known Natural World.) Supermind, In Any Given Individual, Is Experienced As A Type Of "Omniscience"—The Supermind, Since It Transcends And Includes All Of The Previous Structure-Rungs, And Inherently Is Conjoined With The Highest Nondual Suchness State, Has A Full And Complete Knowledge Of All Of The Potentials In That Person. It Literally "Knows All," At Least For The Individual." ― Ken Wilber, The Fourth Turning: imagining the Evolution of an Integral Buddhism (fopr models please see one of the papers in the series). As Lyotard Argues, Aesthetic Judgment Is The Appropriate Model For The Problem Of Justice In Postmodern Experience Because We Are Confronted With A Plurality Of Games And Rules Without A Concept Under Which To Unify Them. Judgment Must Therefore Be Reflective Rather Than Determining. Furthermore, Judgment Must Be Aesthetic Insofar As It Does Not Produce Denotative Knowledge About A Determinable State Of Affairs, But Refers To The Way Our Faculties Interact With Each Other As We Move From One Mode Of Phrasing To Another, I.E. The Denotative, the Prescriptive, the Performative, the Political, the Cognitive, the Artistic, Etc. In Kantian Terms, This Interaction Registers As An Aesthetic Feeling. Where Kant Emphasizes The Feeling Of The Beautiful As A Harmonious Interaction Between Imagination And Understanding, Lyotard Stresses The Mode In Which Faculties (Imagination And Reason,) Are In Disharmony, I.E. The Feeling of the Sublime. For Kant, The Sublime Occurs When Our Faculties Of Sensible Presentation Are Overwhelmed By Impressions Of

Absolute Power And Magnitude, And Reason Is Thrown Back Upon Its Own Power To Conceive Ideas (Such As The Moral Law) Which Surpass The Sensible World. For Lyotard, However, The Postmodern Sublime Occurs When We Are Affected By A Multitude Of Unpresentables Without Reference To Reason As Their Unifying Origin. Justice, Then, Would Not Be A Definable Rule, But Ability To Move And Judge Among Rules In Their Heterogeneity And Multiplicity. In This Respect, It Would Be More Akin To The Production Of Art Than A Moral Judgment In Kant's Sense. Postmodernism First Published Fri Sep 30, 2005 Copyright © 2005 By Gary Aylesworth Geaylesworth@Eiu.Edu Open Access To The SEP Is Made Possible By A World-Wide Funding Initiative. The Scholars Of Tien Tai Called It The "Embodied Nature". (This Is The Buddha-Nature That Includes Both Good And Evil.) The Scholars of Xian Shou Say, "It Is Arising From Primal Nature", And the Scholars of Chan (Zen) Say, "It Is Nature That Causes the Rising of Things". All Dharma Is Dharma-Nature. It Is Not Different From Dharma-Nature. Dharma And Dharma-Nature Are Not Two Separate Identities, "Phenomena" And "Nature" Is Also Not Distinguishable Either. In Other Words, There Is No Difference Between Principle (Absolute) And Practice (Relative). Teachings in Chinese Buddhism (6) Sunyata (Emptiness) In the Mahayana Context (Wikipedia). In Other Words, All Dharma Arises From Causes and Conditions, and All Dharma Is Empty In Nature. The Law of Dependent Origination (Existence) and the Nature of Emptiness Is Neither the Same nor Different. They Exist Mutually. The Truth of "Sunyata" And "Existence", And "Nature" And "Phenomena" Are Not (e&eb) In Conflict With Each Other. Unlike The Scholars Of The Dharmalaksana Sect Who Explain The Dharma Only From (e) The Aspect Of Dependent Origination, Or The Scholars Of Dharma-Nature That Explain The Existence Of Dharma Only From The Aspect Of Dharma-Nature, The Scholars Of Madhyamika Explain The Truth Of The Dharma From Both Aspects. Hence This Is Called The Middle Path Which Does Not Incline To Either Side. Teachings in Chinese Buddhism (6) Sunyata (Emptiness) In the Mahayana Context (Wikipedia). For Stoics, Causes Involved Reference To The Depths, But Effects At The Surface Could Also Have Relations Among Themselves. This Permitted The Distinction To Be Drawn Between Destiny And Necessity—The Stoics Wanted To Affirm Destiny And Deny Necessity. In The First Place, Effects Express Causes, But Expressions Of Relations [Rather Than Necessity] Connects Those Effects. Those Relations May Be Described As Compatibility Or Incompatibility, Conjunction Or Disjunction. These Are Not Causal Relations Themselves, But Represent 'An Aggregate Of Noncausal Correspondences Which Form A System Of Echoes, Of Presumptions And Resonances, A System Of Signs—In Short, An Expressive Quasi Causality And Not At All A Necessitating Causality' (170). This Need Not Involve Contradiction, Which Is Applying To Events Rules That Really Only Applied To Logic And Argument. There Can Be Incompatability Without Contradiction, A Noncausal Correspondence. Leibniz Described Incompossible Worlds [Delanda Is Very Clear On This Too]. Only Impossible Events Contradict Possible Ones. Events Can Be Compossible [Roughly, They Have Predictable and Predicative Future and Past Events]. For Deleuze, It Is A Matter Of 'The Convergence Of Series Which Singularities Of Events Form As They Stretch Themselves Out Over Lines Of Ordinary Points. Incompossibility Must Be Defined By the Divergence of Such Series' (171). Such A Notion Is Essential To Any Theory of Sense. We Should Not See Divergence As A Matter Of Exclusion, As Leibniz Did [Since God Chose Actual Events]. Divergent And Disjunction Can Both Be Positive, While Preserving Differences. In Fact, Differences Are Crucial, Preserving The Distance Between Objects While Affirming That They Are Related. This 'Permits The Measuring Of Contraries Through Their Finite Difference Instead Of Equating Difference With A Measureless Contrariety' (173). It Is Contradiction Which Is The Special Case. Difference Here Is A Topological Term Relating To Distance On Surfaces Rather Than Depths. It Is Not Just A Matter Of Suggesting 'Some Unknown Identity Of Contraries (As In Commonplace In Spiritualist And Colorist Philosophy)' (173) [Take That St Pierre!]. An Example Is Nietzsche Arguing That Health And Sickness Can Both Inform Each Other; Act As Points Of View, Remembering That 'Things, Beings, Are Themselves Points Of View' (173). Divergence Does Not Mean Exclusion, And Disjunction Does Not Mean Separation. Connective Syntheses '(If…, Then)' Construct A Single Series; Conjunctive Series '(And)' Produces Convergent Series, But Disjunctive Series '(Or)' Produces A Divergence Series. Normally, Disjunction Helps Us To Criticise Synthesis, But Disjunction Can Still Be A Synthesis Itself, Despite Its Use In Logical Analysis [I Think What Is Going On Here Is Arguing That

There Is A Difference Between 'Either/Or' In A Logical Sense, And 'One Or Two' In The Real Sense—The Latter Can Mean That Both Are Compossible. This Is The 'Communication Of Events' Rather Than The Logical Business Of Analysing Predicates (174)]. The Synthetic Disjunction Expresses The Paradox, With Divergence At The Centre. The Discussion of Paradoxes and Esoteric Words Above Are Examples: They Contract 'The Multitude of Divergent Series in the Successive Appearance of A Single One' (175). There Is A Difference Again Between the Subversion of the Present and Simple Identity by Depths, And Operations at the Surface. By Considering The Depths, We Encounter Infinite Identities [As Events Become Examples Of Deeper Categories, Wholes?]. At The Surface Events Communicate With Each Other Directly Through Maintaining Distance And By Affirming Disjunctions. Disjunction Threatens The Identity Of The Self, And Helps Us To See The Self As 'So Many Impersonal And Pre-Individual Singularities' [Connected Through Disjunctive Synthesis. Hence the Importance of Heterogeneity] (175). The Normal Concept Of The Self Implies Some Connected Series, 'But When Disjunction Accedes To The Principle Which Gives To It A Synthetic And Affirmative Value, The Self, The World, And God Share In A Common Death' (176). [There Is Also A Point That Divergent Series Explain And Also Exceed Normal Conjunctive And Connective Series]. In The Usual Conception, 'The Self Is the Principle of Manifestation, In Relation To the Proposition, the World Is the Principle of Denotation, And God the Principle of Signification' (176). But The Theory Of Sense Here Says That It Emanates From Nonsense, From Paradox, And From The 'Eternally Decentred Ex-Centric Centre' (176). This Position 'Does Not Tolerate the Subsistence of God as an Original Individuality, Nor The Self As The Person, Nor The World As An Element Of The Self And As God's Product. The Divergence Of The Affirmed Series Forms A "Chaosmos" And No Longer A World; The Aleatory Point Which Traverses Them Forms A Counter self, And No Longer A Self' (176). There Is No Centre But Only 'Pure Events Which The Instant, Displaced Over The Line [Of Aion], Goes On Dividing Into Already Past And Yet To Come. Nothing Other Than The Event Subsists… This Communicates With Itself through Its Own Distance and Resonates Across All of Its Disjuncts' (176). Alexander I. Stingl's Blog A Nomadic Scholarship Entity NOTES ON: Deleuze, G (1990) The Logic Of Sense, Trans Mark Lester, Edited By Constantin Boundas, New York: Columbia University Press For Models Kindly See One Of The Papers In The Series. Attributable And Ascribable To Constraints On Space Accomodation Model Is Provided In Some Paper, Notwithstanding The Generality And Commonalty Of The Observation And Its Concatenatability With The Other Modules Sixteenth Series Of The Static Ontological Genesis Twenty-fourth series of the communication of events. The Buddha Always Used The Terms Void, No Rising And Falling, Calmness And Extinction To Explain The Profound Meaning Of Sunyata And Cessation. For Example, Sunyata And The State Of Nirvana Where There Is No Rising Nor Falling, Are Interpreted By Most People As A State Of Non-Existence And Gloom. They Fail To Realise That Quite The Opposite, Sunyata Is Of Substantial And Positive Significance. The Sutras Often Use The Word "Great Void" To Explain The Significance Of Sunyata. In General, We Understand The "Great Void" As Something That Contains (e) Absolutely Nothing. However, From A Buddhist Perspective, The Nature Of The "Great Void" Implies Something Which Does Not Obstruct Other Things, In Which All Matters Perform Their Own Functions. Materials Are (=) Form, Which By Their Nature, Imply (eb) Obstruction. The Special Characteristic Of The "Great Void" Is (=) Non-Obstruction. The "Great Void" Therefore, Does Not (e) Serve As An Obstacle To (e) Them. Since The "Great Void" Exhibits No Obstructive Tendencies, It Serves As The Foundation For Matter To Function. In Other Words, If There Was Neither "Great Void" Nor Characteristic Of Non-Obstruction, It Would Be Impossible For The Material World To Exist And Function. The "Great Void" Is Not Separated From The Material World. The Latter Depends On The Former. We Can State That the Profound Significance of Sunyata and the Nature of Sunyata in Buddhism highlight The "Great Void's" Non-Obstructive Nature. Sunyata Does Not Imply The "Great Void". Instead, It Is The Foundation Of All Phenomena (Form And Mind). It Is The True Nature Of All Phenomena, And It Is The Basic Principle Of All Existence. In Other Words, If The Universe's Existence Was Not Empty Nor Impermanent, Then All Resulting Phenomena Could Not Have Arisen Due To (e) The Co-Existence Of Various Causes And There Would Be (=) No Rising Nor Falling. The Nature Of Sunyata Is Of Positive Significance. Calmness And Extinction Are The Opposite Of Rising And Falling. They Are Another Way To Express (eb) That There Is No Rising And Falling. Rising And Falling Are (=)

The Common Characteristics Of Worldly Existence. All Phenomena Are Always In (eb)The Cycle Of Rising And Falling. However, Most People Concentrate On Living (Rising). They Think That The Universe And Life Are (=)The Reality Of A Continuous Existence. Buddhism On The Other Hand, Promotes (eb) The Value Of A Continuous Cessation (Falling). This Cessation Does Not (e) Imply That It Ceases To Exist Altogether. Instead, It Is Just A State In The Continuous Process Of Phenomena. In This Material World, Or What We May Call This "State Of Existence", Everything Eventually Ceases To (e) Exist. Cessation Is (=) Definitely The Home Of All Existences. Since Cessation Is The Calm State Of Existence And The Eventual Refuge Of All Phenomena, It Is Also The Foundation For All Activities And Functions. Teachings in Chinese Buddhism (6) Sunyata (Emptiness) In the Mahayana Context (Wikipedia). You Are In Your Own Film A Subjective Experience Where You Are Both Actor And Director Writing Your Script As You Go Along. You Are (=) Both The Audience And The Actor A Near State Of Pure Consciousness. You Are An Individual General Ledger With A Place In Collective General Ledger And Cosmic General Ledger With Celestial Brahman Anti Brahman Simulating (e&eb)The Events Which Appear To You On (eb) The Screen Of Consciousness. Sensations Are Disruptions Within Actual Identifications; They Are Signs Of The Expression Of Ideas And Intensities In Actual Structures. There Is No Denying the Difficulty of Deleuze's Account of Reality – Necessary for an Understanding of the Role Played by Kantian Arguments. It May Therefore Be Helpful To Think In Terms Of An Example. You Have Waited Years To See A Loved One Again, Tarrying With Dimming Memories And Shifts In Longings And Needs. A Meeting Has Been Arranged. The Opening Gestures, Words And Images Bring Together Fixed Memories And Preconceptions (Actual Identifications) With Feelings That Destroy those (Actual Sensations Of Disappointment, Excitement, Bemusement, And Renewed Passion). The Power Of These Sensations Cannot Come Solely From The Disrupted Identities; Rather, They Are Expressions Of Deeper Charges (Virtual Intensities). But, Beyond Local Identifications And Their More And More Distant Actual Effects, Imagined And Even Undreamed Worlds Of Ideas Are Changed Through The Meeting. According To Deleuze, These Are Not Possible Worlds, But Virtual Ones That Ground Actual Identifications. The Actual Meeting Is Accompanied By A Virtual One, Where Ideas Take Shape And Acquire Significance Through Shifts In Relations Between Intensities. These Shifts Light Up Actual Situations In New Ways And Determine Different Ideas To Come Into Play. The Meeting Is A Failure. Intensities Push Feelings of Disappointment to the Fore, Destroying One World of Ideas and Perhaps Determining a Colder One with Greater Clarity. The Transversal Thought of Gilles Deleuze: Encounters and Influences James Williams. For models please see one of the papers in the series. Models are procrastinated due to spatial restrictions. You Can Stir The Conscious State And Consequentially Connect All The Three States. As Lyotard Argues, Aesthetic Judgment Is The Appropriate Model For The Problem Of Justice In Postmodern Experience Because We Are Confronted With A Plurality Of Games And Rules Without A Concept Under Which To Unify Them. Judgment Must Therefore Be Reflective Rather Than Determining. Furthermore, Judgment Must Be Aesthetic Insofar As It Does Not Produce Denotative Knowledge About A Determinable State Of Affairs, But Refers To The Way Our Faculties Interact With Each Other As We Move From One Mode Of Phrasing To Another, I.E. The Denotative, the Prescriptive, the Performative, the Political, the Cognitive, the Artistic, Etc. In Kantian Terms, This Interaction Registers As An Aesthetic Feeling. Where Kant Emphasizes The Feeling Of The Beautiful As A Harmonious Interaction Between Imagination And Understanding, Lyotard Stresses The Mode In Which Faculties (Imagination And Reason,) Are In Disharmony, I.E. The Feeling of the Sublime. For Kant, The Sublime Occurs When Our Faculties Of Sensible Presentation Are Overwhelmed By Impressions Of Absolute Power And Magnitude, And Reason Is Thrown Back Upon Its Own Power To Conceive Ideas (Such As The Moral Law) Which Surpass The Sensible World. For Lyotard, However, The Postmodern Sublime Occurs When We Are Affected By A Multitude Of Unpresentables Without Reference To Reason As Their Unifying Origin. Justice, Then, Would Not Be A Definable Rule, But Ability To Move And Judge Among Rules In Their Heterogeneity And Multiplicity. In This Respect, It Would Be More Akin To The Production Of Art Than A Moral Judgment In Kant's Sense. Postmodernism First Published Fri Sep 30, 2005 Copyright © 2005 By Gary Aylesworth Geaylesworth@Eiu.Edu Open Access To The SEP Is Made Possible By A World-Wide Funding Initiative. In Dissolving All Social Phenomena In The Acid Bath Of Power And Domination, Foucault

Prevents Critical Theory From Drawing Crucial Distinctions, Such As Those "Between Just And Unjust Social Arrangements, Legitimate And Illegitimate Uses Of Political Power, Strategic And Cooperative Interpersonal Relations, Coercive And Consensual Measures" (McCarthy 1991: 54). One Cannot Say, For Example, That One Regime Of Power Is Any Better Or Worse Than Another, Only That They Are Different -- "Another Power, Another Knowledge" (Foucault 1979: 226). Since Ruling Powers Attempt To Erase Such Distinctions, Or To Present Injustice As Justice, Falsehood As Truth, And Domination As Freedom, Foucault's Position Unwittingly Supports The Mystifications Of Orwellian Doublespeak, Now More Rife Than Ever (See Kellner 2001), And Blocks The Discriminations Necessary For Social Critique. If There Are No Standards Or Right, Then, With Thrasymacus And Hobbes, We Can Conclude Might Is As Right As Anything. There Can Be No Ideology Critique Where There Is No Distinction Between True And False, And any Social Or Moral Critique Without A Distinction Between Right And Wrong. The Evaluative Character Of Foucault's Own Work Is Not Any Less Normative For His Refusal To Explicitly Confront It. The Problem Becomes Glaring In His Later Work Where He Employs Normative Terms Such As Liberty And Autonomy, But Fails To State What We Should Be Free For. Foucault's Anti-Normative Stance Therefore Forces Him into Self-Defeating Value Neutrality. Richard Rorty, The Attack on Theory, And Renunciation of Radical Politics (for models see one of the papers in the series). Our Environment Can Either Open Up or Constrict Possibilities for Spiritual and Intellectual Growth. In The Politics Of Experience Laing Offers A Fine Passage From Erving Goffman’s Studies Encounters: There Seems To Be No Agent More Effective Than Another Person In Bringing A World For Oneself Alive, Or, By A Glance, A Gesture, Or A Remark, Shriveling Up The Reality In Which One Is Lodged. "The Anxiety Of Meaninglessness Is Anxiety About The Loss Of An Ultimate Concern, Of A Meaning Which Gives Meaning To All Meanings. This Anxiety Is Aroused By The Loss Of A Spiritual Center, Of An Answer, However Symbolic And Indirect, To The Question Of The Meaning Of Existence." Philosophy And Depression By Tim Ruggiero -- Paul Tillich, The Courage To Be (1952) As Quoted In Depression And The Threat Of Nonbeing Tim Ruggiero, September 4, 2001 (Philosophical Society.Com).The Dissolution Of The Subject And Its Implications For Society Is The Theme Of Anti-Oedipus: Capitalism And Schizophrenia, Which Deleuze Published With Félix Guattari In 1972 (English 1983). The Book, In Large Part, Is Written Against An Established Intellectual Orthodoxy Of The Political Left In France During The 1950s And 1960s, An Orthodoxy Consisting Of Marx, Freud, And Structuralist Concepts Applied To Them By Louis Althusser And Jacques Lacan. Deleuze And Guattari Argue That This Mixture Is Still Limited By Representational Thinking, Including Concepts Of Production Based Upon Lack, And Concepts Of Alienation Based Upon Identity And Negation. Furthermore, The Oedipus Concept In Psychoanalysis, They Say, Institutes A Theater Of Desire In Which The Psyche Is Embedded In A Family Drama Closed Off From The Extra-Familial And Extra-Psychic Forces At Work In Society. They Characterize These Forces As “Desiring Machines” Whose Function Is To Connect, Disconnect, And Reconnect With One Another Without Meaning Or Intention. The Authors Portray Society As A Series Of “Territorializations” Or Inscriptions Upon The “Body Without Organs, “Or The Free-Flowing Matter Of Intensive Qualities Filling Space In Their Varying Degrees. The First Inscriptions Are Relations Of Kinship And Filiation Structuring Primitive Societies, Often Involving The Marking And Scarring Of Human Bodies. As An Interruption and Encoding of “Flows,” The Primitive Inscriptions Constitute a Nexus of Desiring Machines, Both Technical and Social, Who’s Elements Are Humans and Their Organs. The Full Body Of Society Is The Sacred Earth, Which Appropriates To Itself All Social Products As Their Natural Or Divine Precondition, And To Whom All Members Of Society Are Bound By Direct Filiation (Deleuze 1983b, 141-42). These First Inscriptions Are Then De-Territorialized And Re-Coded By The “Despotic Machine,” Establishing New Relations Of Alliance And Filiation Through The Body Of The Ruler Or Emperor, Who Alone Stands In Direct Filiation To The Deity (Deleuze 1983b, 192) And Who Institutes The Mechanism Of The State Upon Pre-Existing Social Arrangements. Finally, Capitalism De-Territorializes The Inscriptions Of The Despotic Machine And Re-Codes All Relations Of Alliance And Filiation Into Flows Of Money (Deleuze 1983b, 224-27). The Organs Of Society And The State Are Appropriated Into The Functioning Of Capital, And Humans Become Secondary To The Filiation Of Money With Itself. Postmodernism First Published Fri Sep 30, 2005 Copyright © 2005 By Gary Aylesworth

Geaylesworth@Eiu.Edu Open Access To The SEP Is Made Possible By A World-Wide Funding Initiative For Models Kindly See One Of The Papers In The Series, Procrastinated Due To Restrictions On Space. "The Vanity Of Existence Is Revealed In The Whole Form Existence Assumes: In The Infiniteness Of Time And Space Contrasted With The Finiteness Of The Individual In Both; In The Fleeting Present As The Sole Form In Which Actuality Exists; In The Contingency And Relativity Of All Things; In Continual Becoming Without Being; In Continual Desire Without Satisfaction; In The Continual Frustration Of Striving Of Which Life Consists." (For Models See One of the Papers in the Series, Which Are Procrastinated Due to Spatial Constraints) -- Arthur Schopenhauer, "On the Vanity of Existence" as quoted in Philosophy and Depression by Tim Ruggiero. "There Is Little Conjunction Of Truth And Social 'Reality'. Around Us Are Pseudo-Events, To Which We Adjust With A False Consciousness Adapted To See These Events As True And Real, And Even As Beautiful. In The Society Of Men The Truth Resides Now Less In What Things Are Than In What They Are Not. Our Social Realities Are So Ugly If Seen In The Light Of Exiled Truth, And Beauty Is Almost No Longer Possible If It Is Not A Lie." (For Models See One of the Papers in the Series, Which Are Procrastinated Due to Spatial Constraints) -- R.D. Laing, the Politics of Experience. [We're going to Use the Idea of a Game to Further Illustrate How the Virtual Is Imminent in the Actual. Why This Example? To Have A Go At The Metaphor Of Games In Philosophy? To Instate Lewis Carroll As Some Great Philosopher?] The Actual Games Have A Series Of Rules About Playing And Outcomes That Limit The Influence Of Chance. It's Possible to Invert These Rules to arrive at an Ideal Game, With No Rules, And with the Full Operation of Chance [Which Describes the Virtual, I Suppose]. Carroll Describe Some. We Also Learn About Time And How It Operates. In Each Throw Of The Dice, Singularities Are Distributed As A 'Nomadic And Non Sedentary Distribution, Wherein Each System Of Singularities Communicate And Resonates With The Others, Being At Once Implicated By The Others And Implicating Them In The Most Important Cast' (60). This Game Can Only Be Played In Thought [Since Any Empirical Reality Limits Chance. Chance Is Fixed By The Mechanisms Of Say The Roulette Wheel]. Infinite Time Can Be Compared With Finite [Or Metric] Time. They Are Linked Together, Although It Is Useful To Consider Both. In Finite Time, The Present Exists And Features Mixtures Of Matter Which Are Temporally Realized And Can Take On A [Empirical] Depth. Any Other Kind of Time, Only the Past and Future Exist, and Colonise Moments in the Presence. It Offers An Unlimited Kind Of Time, Since The Gap Between Past And Future Is Endlessly Subdivided And Never Closed. It Is Neutral And Incorporeal. Here, the Present Is A 'Pure Mathematical Instant' (62). It Is The Time Of Events Which Are Always Things Which Have Just Happened And Things Which Are About To Happen 'Never Something Which Is Happening' (63). As With The Operations Of Chance In A Game, Specific Events Or Throws Occupy A Time Which Is Smaller Than That Which Can Be Thought [That Is The Unfolding Of Singularities Which Happen Instantly?], And Also The Maximum Notion Of Time [I Can Only Grasped This By Thinking Of Specific Chances Being Understood By Their Location In Whole Sequences Of The Infinite Throws Of The Dice]. [There Is Some Stuff About The Eternal Return Here Too Which I Don't Understand—63 – 64]. 'The Aion Is The Straight Line Traced By The Aleatory [Unpredictable, Accidental] Point' [Search Me] Which Means That Events The Can Communicate With All The Others, As In Events Of The Aion Itself. This Sort Of Unity Is Not The Same Is An Empirical Corporeal Unity. The Aion Underpins or Circulates Within the Series of Events. This Also Helps Explain The Connection Between The Sense Of Propositions And The Sense Of States Of Affairs. [Weird. Making The Strongest Possible Case For Chance, I Assume. Denying Essence or Subjective Intent. Makes Empirical {Including Social} Science As Limited To The Finite And Metric, But Requiring Thought Experiments To Move To The 'Pure' Level? So Says Delanda Anyway, I Think]. Alexander I. Stingl's Blog A Nomadic Scholarship Entity NOTES ON: Deleuze, G (1990) The Logic Of Sense, Trans Mark Lester, Edited By Constantin Boundas, New York: Columbia University Press For Models Please See One Of The papers In The Series Alexander I. Stingl's Blog A Nomadic Scholarship Entity NOTES ON: Deleuze, G (1990) The Logic Of Sense, Trans Mark Lester, Edited By Constantin Boundas, New York: Columbia University Press For Models Kindly See One Of The Papers In The Series. Attributable and Ascribable To Constraints on Space Accomodation Model Is Provided In Some Paper, Notwithstanding the Generality and Commonalty of the Observation and Its Concatenatability with the Other Modules. [Particularly

Dense And Baffling Stuff, Attempting To Show How The Subjective World Is A Derivative Of The Objective Play Of Singularities. Makes More Sense With Concrete And Familiar Examples As In Delanda's Stuff On Biological Membranes And Their Importance In Embryology, Instead Of The Abstract 'Surfaces' Of This Account]] Individuals Are Capable Of Infinite Description, Limited Only By Their Bodies To Express. Persons [Human Beings Based the More Abstract Notion of an Individual?] Can Only Produce Propositions To Describe The World To A Limited Extent. Both, However, Are 'Ontological Propositions' [That Is They Create A Kind Of Reality?]. Multiple Classes And Systems Of Categories Are Not Produced By Propositions. Instead We Have To Look At Something That Now Produces Human Propositions Themselves, As 'Material Instances' (118). [So Human Ontological Activity Only Realises Sense Making In The Form Of Denotations, Manifestations, And Signification As Above?]. Denotations And The Others Are Interconnected As We Saw Above. There Is Also No Simple Connection Between, Say, 'The Individual And Denotation, The Person And Manifestation, Multiple Classes Or Variable Properties And Signification' (119) [For Similar Reasons—For Example, Signification Depends On The Good Sense Already Established By Individuation]. The Whole Complex Structure Is Produced By Both Ontological And Logical Genesis. Sense Operates On The Whole Structure. [We Can See This Both In The Fragility Of Sense And Its Tendency To Be Threatened By Nonsense, And Because The Alternative Is Unpalatable—Language And Sense-Making Would Be Based On Nothing But A 'Undifferentiated Abyss' (120)]. So Sense, 'In Its Organization of Aleatory and Singular Points, Problems and Questions, Series and Displacement' (120) Generates both Logical Propositions and Also 'the Objective Correlates' Of Propositions… The Denoted, the Manifested, and the Signified' (120). The Notion Of An Error Suggests This Although In A Confused Way. Normally We Think Of Error As A Matter Of Truth Or Falsity, When Propositions Are Formed And Tested. However, When We Consider Problems Instead Of Propositions That Offer Solutions, The Category Of Sense Emerges Strongly. [We Have Seen Above the Problem Is an Objective Matter of Structured Possibilities]. We Can See How Both Knowledge And The Known Are Produced By This Structure. [Problematics Are Further Discussed As Involving Particular Distributions Of Singularities In Space And In Time. As Problems Condense Out, So Do Solutions—'The Synthesis Of The Problem With Its Conditions Engenders Propositions, Their Dimensions, And Their Correlates' (121)]. Sense Is Produced Or Expressed When Solutions, Expressed As Propositions, Correspond To Problems [Act As 'Instances Of A General Solution' (121)] It Is Common To Express Sense In An Interrogative Form [Although The Interrogative Also Includes A More Limited Operation As In The Closed Question]. Specific Questions And Solutions Are Already Determined By The Problematic, However—'The Problem In Itself Is The Reality Of A Genetic Element, The Complex Theme Which Does Not Allow It To Be Reduced To Any Propositional Thesis' (122). It Is A Mistake To Define A Problem In Terms Of Possible Solutions [Which Arise From Human Consciousness]. This Would Mean Us 'Confuse Sense with Signification, and… Conceive Of the Condition Only In the Image of the Conditioned' (122). [This Autonomous Constitution Of Problems Shows The Inadequacy Of Subjective Conceptions. Seeing Problems As Derived From Propositions Expressing Solutions Would Also Infringe The Neutrality Of Sense]. Problems Are Neutral Insofar As [Modes Of] Propositions Are Concerned. For Example 'A Circle Qua Circle Is Neither A Particular Circle, Nor A Concept Represented In An Equation… It Is Rather A Differential System To Which An Emission Of Singularities Corresponds' (123). Problems In This Sense Exist In Propositions But Also Allude To None Being, As Above. As A Result, Problems Are Neutral, That Is 'Independent of both the Negative and the Affirmative' (123). The Neutrality of Sense Means It Is Never Just the Echo ['the Double'] Of Propositions. Working With Propositions Can Only Lead To A Partial Understanding Of Sense. We Have To Develop Another Conception, Not Based On Propositions Or On Images Of Conventional Logical Thinking. Philosophy Must 'Purge The Transcendental Field Of All Resemblance' In Order To Avoid Trap Of Consciousness As The Origin Of The Transcendental (123). However, The Earlier Discussions Defined The Neutrality Of Sense As An Effect Produced By Corporeal Causes. Here, We Are Implying That It Arises From Its Genetic Power [To Produce Problematics] And This Relates To A Quasi Cause. Sense Is Produced By Bodies In A Way Which Presupposes This More General Kind Of Sense. The More General Kind Operates In A Different Way, Not Through Concepts Or Describing Mixtures. This Time It Is a Matter of Depth [Of Bodies] and

the Effects of a Surface. The Pulsations Of Bodies Produce Surfaces In Particular Ways, Sometimes As A Minimum Energy Conserving Form [Delanda's Soap Bubble], Sometimes As A More Complex Structure Of Multiplied Differentiated Surfaces [Stretched, Emulsified, Absorbing Etc Are The Examples Given, P. 124]. Surfaces Are Produced By the 'Actions and Passions of Mixed Bodies' (124). Surfaces Have No Thickness Of Their Own, Which Permits Contact Between The Internal And The External. The Quasi Causes Play On These Surfaces, As A Kind Of 'Fictitious Surface Tension… A Force Exerting Itself On The Plane Of The Surface' (125). Singularities Are Condensed Extended And Reshuffled. These Surfaces Have An Existence In Actual Physics, And Also In Metaphysics—The Surface Becomes The Transcendental Field, The Border Between Bodies And Propositions. As Such, The Surface Becomes The 'Locus Of Sense And Expression' (125). Propositions And Bodies Are Actually Articulated On Surfaces, 'So That Sense Is Presented Both As That Which Happens To Bodies And That To Which Insists [Sic] In Propositions' (125). It Is In This Process Of 'Doubling Up' That Neutrality Arises [I Think The Argument Here Is That Neutrality Is Not Some Disembodied Quality But A Function Of Surfaces Which Enable 'The Continuity Of Reverse And Right Sides' (125). It Is The Indifference Of The Objective, Immune To Appeals From Human Subjectivity? ]. The Surface Enables Sense To Be Distributed, As Both The Expressed In Propositions, And The Event In Bodies [Depends On This Very General Definition Of 'Expression' Again - More General Than The Use Above Where It Means Human Expression As In Manifestation Etc]. If The Surface Is Destroyed, Bodies' Fall Back Into The Depth, The 'Primary Order [Some Sort Of Natural Being Which Cannot Be Named Or Expressed?] Which Grumbles Beneath the Secondary Organisation of Sense' (125). But On The Surface, Sense Is Unfolded And Is Also Affected By Quasi Causes. This Sense In Turn Individuates and Determines Bodies, and Signification, and All the Propositions 'The Entire Tertiary Arrangement or the Object of the Static Genesis' (126). Alexander I. Stingl's Blog A Nomadic Scholarship Entity NOTES ON: Deleuze, G (1990) The Logic Of Sense, Trans Mark Lester, Edited By Constantin Boundas, New York: Columbia University Press For Models Kindly See One Of The Papers In The Series. Attributable And Ascribable To Constraints On Space Accomodation Model Is Provided In Some Paper, Notwithstanding The Generality And Commonalty Of The Observation And Its Concatenatability With The Other Modules Sixteenth Series Of The Static Ontological Genesis. Chronos And Aion Are Compared In Terms Of The Importance Of The Present As Opposed To The Past And The Future. For Chronos, The Present Represents The Main Interest, Although There Is 'The Relativity Of Presents Themselves In Relation To Each Other. God Experiences As Present That Which for Me Is Future or Past, Since I Live Inside More Limited Presents' (162). The Present Is Corporeal, A Matter of Mixtures [Imperfect for Humans and Perfect in the Divine Present for Stoics as Above]. The Present Limits The Action Of Bodies [Compare This With Bergsonian Stuff About The Present As A Cone—Here We Are Talking About Contraction And Dilation To Connect The Empirical With A Cosmic Present]. The Partial Mixtures Of Bodies In The Present Threaten To Subvert The Sufficiency Of The Notion. Stoics Had To Distinguish Between Good Or Bad Mixtures, For Example, Which Lead To Their Notion Of Cosmic Perfection, And To Aion. This Implies That Bodies Are Actually Nothing But Simulacra, And The Present Less Important Than The Future Or The Past. However, Chronos Represents the Only Kind Of Empirical Understanding [?] (164). For Aion, The Present Is A Mere Instant, Always Divided Into Past And Future. This Is A Different Way Of Subverting The Present As In Depth Metaphors [Some Of Which Appear To Linger In The Argument About Partial Mixtures Of Bodies In The Present?]. [Somehow], Aion Operates At The Surface, Since The Present Is Evaded In Favour Of The Instant Rather Than The Fathomless Depths. Aion Operates With Incorporeal Events, And Their Effects, Which Are Limitless And Infinite. Thus Aion Is 'The Eternal Truth Of Time: Pure Empty Form Of Time, Which Is Freed Itself Of Its Present Corporeal Content' (165) [I Think Delanda Is Much Clearer On The Differences Between Metric And Intensive Time]. Aion Is An Essential Element In The Development Of Language, Allowing Language To Escape Corporeal Determinations, Alluding To An Existence Outside Of The Present, Allowing Signification And Manifestation. [Once Having Escaped, Language Can Persist In Itself]. The Instant Demonstrates The Aleatory Point, Nonsense, And Quasi-Cause. It Is A Pure Abstraction And It 'Extracts Singularities From The Present, And From Individuals And Persons Which Occupy This Present' (166) [Because It Alludes To Future And Past, And Thereby Constitutes The Pure Event?]. Without

The Notion Of Aion, We Would Only Be Left With Bodies And States Of Affairs, Not Language And Propositions. [The Argument Here Seems To Be That Language Necessarily Has A Future Element?]. The Surface Divides The Two Series Of States Of Affairs And Propositions, And Sense Can Now Relate To Propositions And Events [In The Form Of The Commentary On Events Mentioned Wayback At The Beginning?]. [There Is Some Sort Of Topological Connection Between Points, Lines And Surfaces, Presumably In Terms Of The Way In Which One Can Be 'Cut' From The One Above]. Actualizations Occur When Bodies, States Of Affairs And Mixtures Intersect At The Surface. This Is A Matter Of 'Imprisoning First Their Singularities Within The Limits Of Worlds, Individuals, And Persons' (167). However, There Is Always Something In Excess Of Actualizations, Which Alludes To The Quasi- Cause [Which Apparently Can Be Identified By Sages]. This Involves Analysing the 'Pure Perverse "Moment"', the Pure Operation—It Is Graspable At and As A Moment of Counter Actualization. Twenty-Third Series Of The Aionalexander I. Stingl's Blog A Nomadic Scholarship Entity NOTES ON: Deleuze, G (1990) The Logic Of Sense, Trans Mark Lester, Edited By Constantin Boundas, New York: Columbia University Press For Models Kindly See One Of The Papers In The Series. Attributable And Ascribable To Constraints On Space Accomodation Model Is Provided In Some Paper, Notwithstanding The Generality And Commonalty Of The Observation And Its Concatenatability With The Other Modules Sixteenth Series Of The Static Ontological Genesis. Wild And Wacky Stuff Here Relating To Artaud's Work. I Would Normally Chop It Out Altogether, Except It Provides Some Context For The Emergence Of The Phrase 'Body Without Organs' That Crops Up So Prominently In Anti Oedipus Sense Is Fragile And Is Threatened By Nonsense. Sometimes Such Nonsense Can Destroy Everything, And We See This If We Switch From The Playful Portmanteaux Of Carroll To The Schizophrenic Writings Of Artaud. These Are Real Examples Of Nonsense, Instead Of The Artificial Ones Discussed By Philosophers. Artaud Apparently Disliked Carroll, and Rendered Jabberwocky in A Much More Challenging Schizophrenic Language. Carroll Is Too Superficial, Whereas Schizophrenia Reveals The Real Problems With Language. Another Work Is Discussed, By Wolfson, To Focus On The Duality Between Things And Words. This Takes The Form Of Someone [Schizophrenic?] Experiencing A Problem In Translating From One Language To The Other, Which Somehow Becomes Transposed Into An Anxiety About Eating. A Schizophrenic Interlude Ensues Involving Associations Between Consonants As The Basis For Translation, And The Paradoxes That Emerge (85). The Same Basic 'Oral Duality (To Eat/To Speak)' Is Also Found In Carroll's Work, And In Artaud 's. However, In Caroll, Some Sense Is Retained Since This Duality Is Explored 'At The Surface' (86) [This Also Indicates, Apparently That The Operation Of Sense Of The Surface Shows Only A Quasi-Causal Relation Between Its Elements, Since There Are Incorporeal Elements Driving It]. For Artaud, The Classic Schizophrenic Symptoms Included The Absence Of Surface, Especially With Bodies. Apparently Freud Also Noticed This Tendency For Schizophrenics To See Their Body As 'Punctured By An Infinite Number Of Little Holes' (87). The Body Therefore Incorporates Everything Into Its Depths, Everything Becomes Corporeal And Physical. The Surface No Longer Limits The Extension Of The Body. 'Hence The Schizophrenic Manner Of Living The Contradiction: Either In The Deep Fissure Which Traverses The Body, Or In The Fragmented Parts Which Encase One Another And Spin About' (87). The World Loses Its Meaning And Sense [Because It Can No Longer Split Sensation Into A Signifying And Signified Separated By A Surface?] Words Become Physical And Affect Bodies, Or They Burst Into Components [Which Relates Back To The Wolfson Example]. Schizophrenics Experience 'A Pure Language–Affect' (88) [Sic --Affect Not Effect]. Schizophrenics Manage This By Overcoming The Effects Of Language, As In The Strange Translation Activity In The Wolfson Example. In Artaud's Case The Solution Was To Create Special Words Expressing 'Values Which Are Exclusively Tonic And Not Written' (88). 'To These Values A Glorious Body Corresponds, Being A New Dimension Of The Schizophrenic Body, An Organism About Parts Which Operates Entirely By Insufflations, Respiration, Evaporation, And Fluid Transmission (The Superior Body Or Body Without Organs Of Antonin Artaud)' (88). This Solution Can Never Be Complete Because There Can Never Be A Total Separation Between Sufferings ['Passion'] And [Remedial] Action, And Passion Can Be Reintroduced, And The Body Corrupted-- A Schizophrenic Body Is Therefore A Constant Mixture Of Two Actions Or Principles. Artaud Tries To Invent A New Language Which Cannot Be Decomposed And Thus Cannot Be Colonised, A

Language Of 'Consonantal Guttural And Aspirated Overloads' (89). The Words Are Joined By Some Invented Principle, In This Case A 'Palatalized' One (89) Which Blurs The Consonants Together And Prevents Them Being Written Down. The Result Is 'So Many Active Howls In One Continuous Breath' (89) [Sounds Very Much Like Tara's Dadaist Tone Poems --Artaud's Example On P.83]]. These Words Are Often The Equivalent To Portmanteaux [Some Examples Are Given On Page 90]. Using These Words Can 'Enact A Chain Of Associations… In A Region Of Infra Sense, According To A Fluid And Burning Principle Which Absorbs And Reabsorbs Effectively The Sense As Soon As It Is Produced' (90). So Two Sorts Of Words Related To Two Sorts Of Bodies, One Fragmented And One Without Organs. There Are Also Two Theatres Or Two Types Of Nonsense Implied Here: One Where Ordinary Words Are Decomposed Into Nonsense, And One Where Tonic Elements Alone Form Nonsensical Words. They Are Produced By Things Happening Beneath The Surface, Unlike Carroll's Playful Superficiality. The Two Signifying and Signified Series Disappear, and Non Sense Engulfs Signifiers and Signified. There Is No Surface Division To Separate The Expressivity Of Words And The Attributes Of Actual Bodies [Which Regulates Ordinary Language]. In Schizophrenic Language There Is No Grammar Or Syntax Either, Although Both Are Preserved In Carroll. Nevertheless, It Is Artaud Who Has 'Discovered A Vital Body And The Prodigious Language Of This Body… He Explored The Infra Sense Which Is Still Unknown Today' (93). However, Caroll Has Explored Those Important Surfaces, On Which 'The Entire Logic Of Sense Is Located' (93). We Can Still Find Schizoid Fragments In Ordinary Speech, But These Are Normally Reorganized. Similarly, Carroll Can Be Retranslated As A Schizophrenic Piece (92). But It Is Wrong To Generalise Here, 'Believing To Have Discovered Analogous Forms Which Create False Differences' (92). Psychoanalysis Should Operate With A Surface/Depth Structure Rather Than With Analogies—'It Is Geographical Before It Is Historical' (93). [While We Are Here, Note That 'It Is Hardly Acceptable… To Run Together a Child's Nursery Rhymes, Poetic Experimentations, And Experiences of Madness… [And] Justify the Grotesque Trinity of Child, Poet, and Madmen' (82-83). This Must Be A Problem For Those Who Think That Deleuze Is Arguing That Children Are Philosophers?] Thirteenth Series Of The Schizophrenic And The Little Girl Alexander I. Stingl's Blog A Nomadic Scholarship Entity NOTES ON: Deleuze, G (1990) The Logic Of Sense, Trans Mark Lester, Edited By Constantin Boundas, New York: Columbia University Press For Models Kindly See One Of The Papers In The Series. Attributable and Ascribable To Constraints on Space Accomodation Model Is Provided In Some Paper, Notwithstanding the Generality and Commonalty of the Observation and Its Concatenatability with the Other Modules. Singularities Constitute The Transcendental Field. The First Stage Of Actualization Involves The Derivation Of Individuals. Singularities Spread Out 'In a Determined Direction over a Line of Ordinary Points', Up To the Vicinity of another Singularity (109). When Series Converge, A World Is Constituted [And Other Worlds Arise From Diverging Series]. The Infinite System Of Singularities Are Selected And Rendered Finite By Individuals [Does He Mean Human Individuals Throughout? I Think So] Who Combine Them, 'Spread Them Out Over Their Ordinary Lines' (109) And Even Form Them Again On Various Membranes. 'An Individual Is Therefore Always In A World As A Circle Of Convergence, And A World May Be Formed And Thought Only In The Vicinity Of The Individuals Which Occupy Or Fill It' (110). [Deleuze Is Suggesting That This Role For Individuals Is The Only Way For Worlds To Persist, Since Entropy Prevents The Endless Renewal Of Singularities At The Virtual Level—'The Power Of Renewal Is Conceded Only To Individuals In The World, And Only For A Time—The Time Of Their Living Present' (110). This Is Static Genesis, The First Level Of Actualization. Singularities Are Actualized In The World And In Individuals. Actualization Means Being Extended, Selected, Incarnated In A Body, And Renewed Locally. These Qualities Require Individuals. Actualization Also Means Being Expressed, But It Would Be Wrong To Assume That Only The Expressed World Exists. There Are, For Example Incompossible Worlds, Produced By Diverging Series [As Real Events, And Not Just Matters Of Contradictory Consciousness]. This Implies That There Is A Continuum Of Singularities Outside Of [Conscious] Individuals, Worlds In Which Other Possibilities Arise. [So Grasping These Other Possibilities Again Distinguishes Sense From Mere Logic, Based On The Truth And Falsity Of Propositions Expressed By Individuals]. Because Individuals Express Worlds, However, The World Looks As If It Is Merely Subjective, And Objective Events Appear To Be 'The Analytic Predicate Of A Subject'

(112). [These Analytic Predicates Appear To Be Properties Of People Or Objects]. [Much Of This Argument Is Referring To Leibniz]. However, The Problem Is To Explain Logical Hierarchies Of Properties And General Categories. Subjective Analytic Predicates Have Immediacy, But Do Not Have Any Kind of Order; They Get Elaborated But Only As Mixtures. They Are Descriptive Of Actual Structures And Diversities. They Are 'Intuitions... Immediate Representations' (113). The Second Level Of Actualization Involves The Transcendental Again, Originally Conceived As The Development Of Transcendency In The Individual [See The Critique Of Husserl Above]. The Transcendental Ego Is Constituted Just As Other Individuals Are, As A Circle Of Convergences, So The Problem Is To Escape This And Develop Knowledge Of The Whole Continuum, Of Incompossible Worlds And Divergent Series, A New Sense Of World [Deleuze Uses The German Terms Welt For This New World And Umwelt For The Subjective World. I Am Grateful To Wikipedia For Explaining That: In The Semiotic Theories Of Jakob Von Uexküll And Thomas A. Sebeok, Umwelt (Plural: Umwelten; The German Word Umwelt Means "Environment" Or "Surrounding World") Is The "Biological Foundations That Lie At The Very Epicenter Of The Study Of Both Communication And Signification In The Human [And Non-Human] Animal."[Citation Needed] The Term Is Usually Translated As "Self-Centered World". [1] Uexküll Theorised That Organisms Can Have Different Umwelten, Even Though They Share The Same Environment']. Proper Transcendence Involves Transcending Individuated Worlds. Singularities Come With Their Own 'Perfectly Objective Determination Which Is The Open Space Of Its Nomadic Distribution' (113), And This Is How Deleuze Wants To Define A Problematic [Above]. Leibniz Apparently Half Grasped This Argument, Suggesting That Conic Sections, For Example, All Referred Back To The Same Event, Subdivided By 'Ambiguous Signs' (114). Similarly, Incompossible Worlds Also Have 'Something Objectively In Common', And 'Several Worlds Appear As Instances Of Solutions For One And The Same Problem... Variants of the Same Story... There Is Thus A"Vague Adam", That Is A Vagabond, A Nomad… Common to Several Worlds' (114). In Other Words, All The Objective And Subjective Worlds Can Be Explained By A Series Of Singularities. It Is Not Just The Analytic Predicates Of Individuals That Actualize Worlds. 'On The Contrary, [There] Are Predicates Which Define Persons Synthetically And Open Different Worlds And Individualities To Them As So Many Variables Or Possibilities' (115). It Is Incompossible Worlds Which Synthesise 'Primary Possibilities Or Categories' (115). These Possibilities or Categories 'Necessarily Signify Classes and Properties', And These Are Distinct From Individual Categories at the First Level. They Still Seen Grounded In Persons, But That Is 'Because Persons Themselves Are Primarily Classes Having One Single Member, And Their Predicates Are Properties Having One Constant' (115). There Is Thus A System Involving Persons, Classes With One Single Member, Extensive Classes And Variable Properties '-- That Is The General Concepts Which Derive From Them' (115). If There Is A Universal Ego, It Is Something That Corresponds To Common Elements In All Worlds. So, Firstly, Sense Gets Actualized In A Field Of Singularities, Then The Umwelt Develops Around Individuals Which Express Or Describe This World, Then, Secondly, A Whole Objective World Develops From Common Elements, A Welt. Persons Can Then Define This Common Element And Develop Classes And Properties Derived From It. At The First Level We Find Good Sense, 'An Already Fixed And Sedentary Organization Of Differences'. At The Second Level, Commonsense Serves The Function Of Identification. Neither Term Explains How These Activities Are Derived [In Fact, Deleuze Wants To Suggest That The Second Level 'Is The Work Of Non Sense Which Is Always Copresent To Sense (Aleatory Point Or Ambiguous Sign)... Productive Nonsense Which Animates The Ideal Game And The Impersonal Transcendental Field' (116). There Is No Conventional Transcendental Movement, Especially One Driven By A Version Of The Ego. The Person Is A 'Produced Form, Derived From This Impersonal Transcendental Field… Always an Individual in General, Born… From The Singularity Which Extends Itself Over A Line of Ordinary Points and Starts from the Preindividual Transcendental Field' (116). Persons And Their Varieties Of Sense Are All Produced 'On The Basis Of Sense And Nonsense Which Do Not Resemble Them' (117). This Explains The Paradoxes And Limits Of Good Sense And Common Sense. [So The Existence Of Nonsense --Paradox And Impossible Sentences Etc -- In The Ordinary Subjective World Shows The Existence Of Non Sense --Something That Produces Ordinary Sense In The First Place, Something Objective? Individuals Can Only Describe But As Soon As They Generalise

And Order Concepts They Necessarily Assume Some Non-Subjective World? The Man oeuvre seems Rather like the Illicit Transcendental Ego Move by Husserl, Generalising from an Ordinary Ego: Here, Explaining Ordinary Experience Is Used to Explain Some Deeper Virtual World as an Extension?] Alexander I. Stingl's Blog A Nomadic Scholarship Entity NOTES ON: Deleuze, G (1990) The Logic Of Sense, Trans Mark Lester, Edited By Constantin Boundas, New York: Columbia University Press For Models Kindly See One Of The Papers In The Series. Attributable And Ascribable To Constraints On Space Accomodation Model Is Provided In Some Paper, Notwithstanding The Generality And Commonalty Of The Observation And Its Concatenatability With The Other Modules Sixteenth Series Of The Static Ontological Genesis. The Issue Is Of The Validity Or Truth Of Propositions And How This Might Be Grounded. It Is As Difficult To Root Propositions In The Idea As It Is Actual Events. Significations Cannot Be Just Seen As Exhausted By Their Denoted Examples. A List Of Denoted Examples Would Be Endless And Groundless Anyway. Classic Philosophers Often Refuse To Answer Questions Denoted Examples Any Way, And Replied With A Blow From The Staff, Or A Mute Demonstration. After All, 'There Is No Resemblance (Nor Should There Be One) Between What One Points Out And What One Has Been Asked' (135) [Typical Philosophical Elitism]. The Issue Is Often Covered By Humour, Especially Irony or the Absurd. We Must Refuse Ascent and Descent and Occupy the Surface 'Where Pure Sense Is Produced' (136). At The Surface, 'One Finds Pure Singularities, An Emission Of Singularities Considered From This Perspective Of The Aleatory Element, Independent Of The Individuals And Persons Which Embody Or Actualise Them' (136). Getting To The Surface Involves Humour And Adventure, Seen In The Routines Of Zen Masters, Who Can Allude To The Void Which Is Present In Events [Events Are 'Not The Object As Denoted, But The Object As Expressed Or Expressible, Never Present, But Always Already In The Past And Yet To Come' (136), And The Abolition Of The Object Alludes To The Void]. Negating Objects Indicates The Expressible. [Examples From Zen Follow, Page 137. What They Seem To Be Suggesting Is That Sense Combines Actual Objects With The Notion Of The Void, Non Sense—An Actual Brushstroke And Also The White Space—'Language Becomes Possible And, By Becoming Possible, It Inspires Only A Silent And Immediate Communication' Somehow Outside All The Significations And Denotations]. The Same Reasoning Affects The Question Of Who Speaks—Individuals Or Language Itself In The Form Of Ideal Forms [And We Are Back With Greek Philosophy Examples. Socratic irony apparently Refers to the Fact That Individuals Both Initiate Speech and Are the Products of Speech—138. This In Turn Permits The Idea Of A 'Pure Rational Language… [Enabling]… Natural Communication between A Supremely Individuated God and the Derived Individuals Which He Created' (138)]. Further Irony Awaits The Romantic Notion Of The Person, 'The Finite Synthetic Unity Of The Person... [Not Just]... The Analytic Identity of the Individual' (138). Here, The Person And Representations Are Linked Ironically, Implying Both A Universal Idea And 'Sensible Particularity' (138), Both Universal And Particular Characteristics Of The Person—Hence The Earlier Argument That Persons Should Really Be Seen As A Class Which Has Only One Member. All These Formulations Assume That Singularities Are Located In The Individual Or The Person, Influenced Only By Some Groundless Abyss, Or Chaos, Threatening Both Classical And Romantic Discourse With The Lack Of Articulation. There Is Also A Threefold Division Of Language—The Ordinary Or Real; The Ideal Language, Like The Purely Rational One; Esoteric Language Which Subverts The Ideal Language And The Individuality Of The Speaker Of Ordinary Language [For Example Social Scientific Explanations, Or Structural Linguistic Ones?] [Weird Examples of Esoteric Languages Which Mean Nothing to Me on Page 140, With a Reference Back To Portmanteau Words]. So Sometimes Individual Speaks, Sometimes The Person, And Sometimes 'The Ground Which Dissolves Both' (140). The Only Way Out Is To See That Singularities Are Not Coterminous With Individuals, And There Is Not Just A Groundless Abyss Beneath Them. There Are Other Impersonal And Pre-Individual Singularities. Nonsense And Sense Collaborate At The Surface. Irony Gives Way To Humour—'The Co extensiveness Off Sense with Nonsense', A Matter Of 'Surfaces… Doubles… Nomad Singularities and Of the Always Displaced Aleatory Point' (141). Normal Significations, Denotations and Manifestations Are Suspended, and 'All Height and Depth Abolished' (141). Nineteenth Series of Humour. Alexander I. Stingl's Blog A Nomadic Scholarship Entity NOTES ON: Deleuze, G (1990) The Logic Of Sense, Trans Mark Lester, Edited By Constantin Boundas, New York: Columbia

University Press For Models Kindly See One Of The Papers In The Series. Attributable And Ascribable To Constraints On Space Accomodation Model Is Provided In Some Paper, Notwithstanding The Generality And Commonalty Of The Observation And Its Concatenatability With The Other Modules Sixteenth Series Of The Static Ontological Genesis.Doctrines Come From 'Wounds And Vital Aphorisms' (148), And Some Writers See Themselves As Embodying Events [Hints Of The Stuff On Nietzsche And Illness Earlier]. Our Will Can Act As A Quasi Cause Of Bodily Events, As We Live Them. This Is An Ethical Stance—'Not To Be Unworthy Of What Happens To Us' (149). The Alternative Is Ressentiment. The Normal Moral Notions Such As Just Or Unjust Are Themselves Immoral. It Is Not Just A Matter Of Resignation To Events, Which Can Still Be Ressentiment, More An 'Apotheosis Of The Will' (149) [Citing Bousquet]. The Organic Is Exchanged For The Spiritual Will [Isn't This Just Making The Best Of Things, Finding Some Deep Meaning In Personal Tragedy?]. In This Sense, Freedom Is The Same As Submitting To Fate, Actualizing Events, Making Sense Of Events, Seeing Events As Something Expressed. One Becomes 'The Offspring of One's Events Are Not of One's Actions, For the Action Is Itself produced By the Offspring Of The Event' (150). In This Way, Actors Somehow Communicate With The Aion, Instead Of Being Dominated By Chronos. [Lots Of Implicit Christianity Here, Surely?]. All The Components Of An Event, Future As Well As Past, Are United, So That One Realises The Impact Of Singularities, Including Preindividual Components. [In This Way, Some Sort Of Agency Seems To Remain? One Can Become 'The Actor of One's Own Events – A Counter Actualization' (150)]. Ressentiment Arises From Not Realizing That Our Particular Part Of Experience, Which May Well Be Unjust, Is Part Of A More Perfect Whole. It Is Incorrect To Judge Everything From The Perspective Of The Present. Some Humourous Recognition of the Futility of It All Is Valuable—In the Great Scheme of Things, States of Affairs Are Always 'Impersonal and Pre-Individual, Neutral, neither General nor Particular' (151). In These Circumstances, Either Life Can Seem 'Too Weak For Me' [Not Vivid Enough, Not Enough Focused In The Present?], Or 'It Is I Who Am Too Weak For Life, It Is Life Which Overwhelms Me, Scattering Its Singularities All About, In No Relation To Me, Nor To A Moment Determinable As The Present Except An Impersonal Instant' (151). This Is Apparent When One Considers Death Or The Mortal Wound, An Event Which Is Indifferent To Me, 'Incorporeal And Infinitive, Impersonal, Grounded Only In It' (151), Although It Is I Who Has To Actualize The Event. However I Can Also Counter actualizes In the Sense Above. It Follows That The Events Of Reality Are Quite Different From Personal Experience, Shown In The 'Splendour Of The "They"… The Splendour of the Event Itself Order Of the Fourth Person' (152) [This Fourth Person Is the 'It' As In Phrases Like 'It Is Raining']. The Old Distinctions Between Private And Collective Refer Only To Personal Experience, Whereas 'Everything Is Singular, And Thus Both Collective And Private… Which Private Event Does Not Have All Its Coordinates, That Is, All Its Impersonal Social Singularities?' (152). Freedom Exists Only In Recognising The Impersonal Nature Of The Event, Seeing All Events As 'A Single Event Which No Longer Makes Room For The Accident, And Which Denounces And Removes The Power Of Ressentiment Within The Individual As Well As The Power Of Oppression Within Society' (152) [Ridiculous Philosophical Notion Of Freedom Overcoming Oppression By An Act Of Philosophy]. Ressentiment Ties One to an Oppressive Order. [Then A Really Sentimental And Pathos Ridden Bit: 'It Is At This Mobile And Precise Point, Where All Events Gather Together In One That Transmutation Happens: This Is The Point At Which Death Turns Against Death; Where Dying Is The Negation Of Death And The Impersonal At He Of Dying No Longer Indicates Only The Moment When I Disappear Outside Of Myself, But Rather The Moment When Death Loses Itself In Itself, And Also The Figure Which The Most Singular Life Takes On In Order To Substitute Itself For Me' (153)] Ethics Links Logic and the Body. The Stoic Notion Of Bodies Included The Passions, And Good And Evil Intents: Particular Bodies Might Have Evil Mixtures, But The Aggregate Of Bodies Is Perfect Or Good, The Unity Of Causes Themselves. In Principle, Each Event Can Be Linked To A Particular Cause And Thus The Unity Of Causes, And This Could Lead To The Activity Of Divination [Grasping The Divine Unity Of Causes] As A Basis For Ethics. The Effects, Lines Have To Be Traced Back From Events To Pure Events And Then To Actions And Passions [With A Lot Of Poetic Stuff As Examples, 143]. Stoics Took Another Route To Ethics Through Logic [Kind Of Working Out Which Events Will Actualize]. Stoic Philosophy Then Saw In Representations Of The Limited Event A Connection With

Pure Events [And Ethical Conduct Seems To Be To Work Towards Actualizing Such Events?]. [Stoic Accounts Of Representations Ensue, 144-5.] There Is A Difference [For Stoics] Between Representations Of Sense And Of Logic, Denotations And Significations, Representations And Expressions. Representation Is Not Just Resemblance, But Includes A Notion of Adequate Expression. Concepts Have To Be Actualised In Representation, And Also Expressed If They Are To Be Comprehensive. Thus Our Knowledge Of Death Is Abstract, Despite The Number Of Deaths We Witness, Until It Becomes Personal, Not Indifferent, But Concrete. This Is Where Expression Is Needed. In Stoic Philosophy, Moral Conduct Involves A Relation with Pure Events Unites Those Events with One's Self. [The Examples Again Come From Zen: 'The Bowman Must Reach The Point Where The Aim Is Also Not The Aim, That Is To Say, The Bowman Himself' (146).] An Understanding Of The Pure Event Is Required, As Something Which Is 'Eternally Yet To Come And Always Already Past', But Which Has To Be Willed Into Actualization (146). The Stoic Embodies Incorporeal Effects, Aligning Themselves With The Quasi Cause. It Is Necessary Because Quasi Causes Cannot Embody Themselves, Except To The Immediate Instant, The Present. [There Is An Equally Baffling Aside About The Difference Between Actors And Characters—Actors Represent By Occupying The Instant, While The Character Also 'Portrays Hopes Or Fears In The Future And Remembers Or Repents In The Past' (147). Stoics Therefore See Themselves As Uniting The Instant With The Unlimited Future And Past, Willing The Event And Also Representing It. [Largely Incomprehensible, Although I Am Starting To See How The Normal Notion Of Stoicism Might Fit—The Patient Resignation Of One's Self To One's Fate—And Also Seeing How Deleuze Thinks That Human Beings Should Reconcile Themselves To Solving The Problems That Reality Creates In The Form Of A Problematic]. Alexander I. Stingl's Blog A Nomadic Scholarship Entity NOTES ON: Deleuze, G (1990) The Logic Of Sense, Trans Mark Lester, Edited By Constantin Boundas, New York: Columbia University Press For Models Kindly See One Of The Papers In The Series. Attributable And Ascribable To Constraints On Space Accomodation Model Is Provided In Some Paper, Notwithstanding The Generality And Commonalty Of The Observation And Its Concatenatability With The Other Modules Sixteenth Series Of The Static Ontological Genesis. Deleuze And Guattari See In The Capitalist Money System “An Axiomatic Of Abstract Quantities That Keeps Moving Further And Further In The Direction Of The Deterritorialization Of The Socius” (Deleuze 1983a, 33), Which Is To Say That Capital Is Inherently Schizophrenic. However, Because Capital Also Re-Territorializes All Flows Into Money, Schizophrenia Remains Capitalism's External Limit. Nevertheless, It Is Precisely That Limit Against Which Thinking Can Subject Capitalism To Philosophical Critique. Psychoanalysis, They Say, Is Part Of The Reign Of Capital Because It Re-Territorializes The Subject As “Private” And “Individual,” Instituting Psychic Identity Through Images Of The Oedipal Family. However, The Oedipal Triangle Is Merely A Representational Simulacrum Of Kinship And Filiation, Re-Coded Within A System Of Debt And Payment. In This System, They Insist, Flows Of Desire Have Become Mere Representations Of Desire, Cut Off From The Body Without Organs And The Extra-Familial Mechanisms Of Society. A Radical Critique Of Capital Cannot Therefore Be Accomplished By Psychoanalysis, But Requires A Schizoanalysis “To Overturn The Theater Of Representation Into The Order Of Desiring-Production” (Deleuze 1983b, 271). Here, The Authors See A Revolutionary Potential In Modern Art And Science, Where, In Bringing About The “New,” They Circulate De-Coded And De-Territorialized Flows Within Society Without Automatically Re-Coding Them Into Money (Deleuze 1983a, 379). In This Revolutionary Aspect, Anti-Oedipus Reads As A Statement Of The Desire That Took To The Streets Of Paris In May Of 1968, And Which Continues, Even Now, To Make Itself Felt In Intellectual Life. Postmodernism First Published Fri Sep 30, 2005 Copyright © 2005 By Gary Aylesworth Geaylesworth@Eiu.Edu Open Access To The SEP Is Made Possible By A World-Wide Funding Initiative For Models Kindly See One Of The Papers In The Series, Procrastinated Due To Restrictions On Space. The Term “Deconstruction,” Like “Postmodernism,” Has Taken On Many Meanings In The Popular Imagination. However, In Philosophy, It Signifies Certain Strategies For Reading And Writing Texts. The Term Was Introduced Into Philosophical Literature In 1967, With The Publication Of Three Texts By Jacques Derrida: Of Grammatology (English 1974), Writing And Difference (English 1978), And Speech And Phenomena (English 1973). This So-Called “Publication Blitz” Immediately Established Derrida As A Major Figure In The New Movement In Philosophy And The Human Sciences Centered In Paris, And Brought

The Idiom "Deconstruction" Into Its Vocabulary. Derrida And Deconstruction Are Routinely Associated With Postmodernism, Although Like Deleuze And Foucault, He Does Not Use The Term And Would Resist Affiliation With "-Isms" Of Any Sort. Of The Three Books From 1967, Of Grammatology Is The More Comprehensive In Laying Out The Background For Deconstruction As A Way Of Reading Modern Theories Of Language, Especially Structuralism, And Heidegger's Meditations On The Non-Presence Of Being. It Also Sets Out Derrida's Difference With Heidegger Over Nietzsche. Where Heidegger Places Nietzsche Within The Metaphysics Of Presence, Derrida Insists That "Reading, And Therefore Writing, The Text Were For Nietzsche 'Originary' Operations," (Derrida 1974, 19), And This Puts Him At The Closure Of Metaphysics (Not The End), A Closure That Liberates Writing From The Traditional Logos, Which Takes Writing To Be A Sign (A Visible Mark) For Another Sign (Speech), Whose "Signified" Is A Fully Present Meaning. This Closure Has Emerged, Says Derrida, With The Latest Developments In Linguistics, The Human Sciences, Mathematics, And Cybernetics, Where The Written Mark Or Signifier Is Purely Technical, That Is, A Matter Of Function Rather Than Meaning. Precisely The Liberation Of Function Over Meaning Indicates That The Epoch Of What Heidegger Calls The Metaphysics Of Presence Has Come To Closure, Although This Closure Does Not Mean Its Termination. Just As In The Essay "On The Question Of Being" (Heidegger 1998, 291-322) Heidegger Sees Fit To Cross Out The Word "Being," Leaving It Visible, Nevertheless, Under The Mark, Derrida Takes The Closure Of Metaphysics To Be Its "Erasure," Where It Does Not Entirely Disappear, But Remains Inscribed As One Side Of A Difference, And Where The Mark Of Deletion Is Itself A Trace Of The Difference That Joins And Separates This Mark And What It Crosses Out. Derrida Calls This Joining And Separating Of Signs Différance (Derrida 1974, 23), A Device That Can Only Be Read And Not Heard When Différance And Différence Are Pronounced In French. The "A" Is A Written Mark That Differentiates Independently Of The Voice, The Privileged Medium Of Metaphysics. In This Sense, Différance As The Spacing Of Difference, As Archi-Writing, Would Be The Gram Of Grammatology. However, As Derrida Remarks: "There Cannot Be A Science Of Difference Itself In Its Operation, As It Is Impossible To Have A Science Of The Origin Of Presence Itself, That Is To Say Of A Certain Non-Origin" (Derrida 1974, 63). Instead, There Is Only The Marking Of The Trace Of Difference, That Is, Deconstruction. Because At Its Functional Level All Language Is A System Of Differences, Says Derrida, All Language, Even When Spoken, Is Writing, And This Truth Is Suppressed When Meaning Is Taken As An Origin, Present And Complete Unto Itself. Texts That Take Meaning Or Being As Their Theme Are Therefore Particularly Susceptible To Deconstruction, As Are All Other Texts Insofar As They Are Conjoined With These. For Derrida, Written Marks Or Signifiers Do Not Arrange Themselves Within Natural Limits, But Form Chains Of Signification That Radiate In All Directions. As Derrida Famously Remarks, "There Is No Outside-Text" (Derrida 1974, 158), That Is, The Text Includes The Difference Between Any "Inside" Or "Outside." A Text, Then, Is Not A Book, And Does Not, Strictly Speaking, Have An Author. On The Contrary, The Name Of The Author Is A Signifier Linked With Others, And There Is No Master Signifier (Such As The Phallus In Lacan) Present Or Even Absent In A Text. This Goes For The Term "Différance" As Well, Which Can Only Serve As A Supplement For The Productive Spacing Between Signs. Therefore, Derrida Insists That "Différance Is Literally Neither A Word Nor A Concept" (Derrida 1982, 3). Instead, It Can Only Be Marked As A Wandering Play Of Differences That Is Both A Spacing Of Signifiers In Relation To One Another And A Deferral Of Meaning Or Presence When They Are Read. How, Then, Can Différance Be Characterized? Derrida Refuses To Answer Questions As To "Who" Or "What" Differs, Because To Do So Would Suggest There Is A Proper Name For Difference Instead Of Endless Supplements, Of Which "Différance" Is But One. Structurally, This Supplemental Displacement Functions Just As, For Heidegger, All Names For Being Reduce Being To The Presence Of Beings, Thus Ignoring The "Ontological Difference" Between Them. However, Derrida Takes The Ontological Difference As One Difference Among Others, As A Product Of What The Idiom "Différance" Supplements. As He Remarks: "Différance, In A Certain And Very Strange Way, (Is) 'Older' Than The Ontological Difference Or Than The Truth Of Being" (Derrida 1982, 22). Deconstruction, Then, Traces The Repetitions Of (e) The Supplement. It Is Not So Much A Theory About Texts As A Practice Of Reading And Transforming Texts, Where Tracing The Movements Of Différance Produces Other Texts Interwoven With The First. While There Is A Certain Arbitrariness In The Play

Of Differences That Result, It Is Not The Arbitrariness Of A Reader Getting The Text To Mean Whatever He Or She Wants. It Is A Question Of Function Rather Than Meaning, If Meaning Is Understood As A Terminal Presence, And The Signifying Connections Traced In Deconstruction Are First Offered By The Text Itself. A Deconstructive Reading, Then, Does Not Assert Or Impose Meaning, But Marks Out Places Where The Function Of The Text Works Against Its Apparent Meaning, Or Against The History Of Its Interpretation. Postmodernism First Published Fri Sep 30, 2005 Copyright © 2005 By Gary Aylesworth Geaylesworth@Eiu.Edu Open Access To The SEP Is Made Possible By A World-Wide Funding Initiative For Models Kindly See One Of The Papers In The Series, Procrastinated Due To Restrictions On Space. "Anything Which Is (=) Just Born, Which Has (e) Just Come Into Existence, Has (e) No Past Behind It. Birth, In Other Words, Is (=) The Condition Of Having No Past. And Likewise, Anything Which Now Dies, Which Has Just Ceased To Be, Has No Future Left In Front Of It. Death Is The Condition Of Having No Future. But We Have Already Seen That This Present Moment Has Both No Past And No Future Simultaneously. That Is, Birth And Death Are One In This Present Moment. This Moment Is Just Now Being Born—You Can Never Find A Past To This Present Moment, You Can Never Find Something Before It. Yet Also, This Moment Is Just Now Dying — You Can Never Find A Future To This Moment, Never Find Something After It. This Present, Then, Is A Coincidence Of Opposites, A Unity Of Birth And Death, Being And Non-Being, Living And Dying. As Ippen Put It, "Every Moment Is The Last Moment And Every Moment Is A Rebirth." Ken Wilber, No Boundary: eastern and Western Approaches to Personal Growth for models please see one of the papers in the series. "Because We Demand A Future, We Live Each Moment In Expectation And Unfulfillment. We Live Each Moment In Passing. In Just This Way The Real Nunc Stans, The Timeless Present, Is Reduced To The Nunc Fluens, The Fleeting Present, The Passing Present Of A Mere One Or Two Seconds. We Expect Each Moment To Pass On To A Future Moment, For In This Fashion We Pretend To Avoid Death By Always Rushing Toward An Imagined Future. We Want To Meet Ourselves In The Future. We Don't Want Just Now—We Want Another Now, And Another, And Another, Tomorrow And Tomorrow And Tomorrow. And Thus, Paradoxically, Our Impoverished Present Is Fleeting Precisely Because We Demand That It End! We Want It To End So That It Can Thereby Pass On To Yet Another Moment, A Future Moment, Which Will In Turn Live Only To Pass." ― Ken Wilber, No Boundary: Eastern And Western Approaches To Personal Growth. For Stoics, Causes Involved Reference To The Depths, But Effects At The Surface Could Also Have Relations Among Themselves. This Permitted The Distinction To Be Drawn Between Destiny And Necessity—The Stoics Wanted To Affirm Destiny And Deny Necessity. In The First Place, Effects Express Causes, But Expressions Of Relations [Rather Than Necessity] Connects Those Effects. Those Relations May Be Described As Compatibility Or Incompatibility, Conjunction Or Disjunction. These Are Not Causal Relations Themselves, But Represent 'An Aggregate Of Noncausal Correspondences Which Form A System Of Echoes, Of Presumptions And Resonances, A System Of Signs—In Short, An Expressive Quasi Causality And Not At All A Necessitating Causality' (170). This Need Not Involve Contradiction, Which Is Applying To Events Rules That Really Only Applied To Logic And Argument. There Can Be Incompatability Without Contradiction, A Noncausal Correspondence. Leibniz Described Incompossible Worlds [Delanda Is Very Clear On This Too]. Only Impossible Events Contradict Possible Ones. Events Can Be Compossible [Roughly, They Have Predictable And Predicative Future And Past Events]. For Deleuze, It Is A Matter Of 'The Convergence Of Series Which Singularities Of Events Form As They Stretch Themselves Out Over Lines Of Ordinary Points. Incompossibility Must Be Defined By The Divergence Of Such Series' (171). Such A Notion Is Essential To Any Theory Of Sense. We Should Not See Divergence As A Matter Of Exclusion, As Leibniz Did [Since God Chose Actual Events]. Divergent And Disjunction Can Both Be Positive, While Preserving Differences. In Fact, Differences Are Crucial, Preserving The Distance Between Objects While Affirming That They Are Related. This 'Permits The Measuring Of Contraries Through Their Finite Difference Instead Of Equating Difference With A Measureless Contrariety' (173). It Is Contradiction Which Is The Special Case. Difference Here Is A Topological Term Relating To Distance On Surfaces Rather Than Depths. It Is Not Just A Matter Of Suggesting 'Some Unknown Identity Of Contraries (As In Commonplace In Spiritualist And Dolorist Philosophy)' (173) [Take That St Pierre!]. An Example Is Nietzsche Arguing That Health And Sickness Can Both

Inform Each Other, Act As Points Of View, Remembering That 'Things, Beings, Are Themselves Points Of View' (173). Divergence Does Not Mean Exclusion, And Disjunction Does Not Mean Separation. Connective Syntheses '(If…, Then)' Construct A Single Series; Conjunctive Series '(And)' Produces Convergent Series, But Disjunctive Series '(Or)' Produces A Divergence Series. Normally, Disjunction Helps Us To Criticise Synthesis, But Disjunction Can Still Be A Synthesis Itself, Despite Its Use In Logical Analysis [I Think What Is Going On Here Is Arguing That There Is A Difference Between 'Either/Or' In A Logical Sense, And 'One Or Two' In The Real Sense—The Latter Can Mean That Both Are Compossible. This Is The 'Communication Of Events' Rather Than The Logical Business Of Analysing Predicates (174)]. The Synthetic Disjunction Expresses The Paradox, With Divergence At The Centre. The Discussion Of Paradoxes And Esoteric Words Above Are Examples: They Contract 'The Multitude Of Divergent Series In The Successive Appearance Of A Single One' (175). There Is A Difference Again Between The Subversion Of The Present And Simple Identity By Depths, And Operations At The Surface. By Considering The Depths, We Encounter Infinite Identities [As Events Become Examples Of Deeper Categories, Wholes?]. At The Surface Events Communicate With Each Other Directly Through Maintaining Distance And By Affirming Disjunctions. Disjunction Threatens The Identity Of The Self, And Helps Us To See The Self As 'So Many Impersonal And Pre-Individual Singularities' [Connected Through Disjunctive Synthesis. Hence The Importance Of Heterogeneity] (175). The Normal Concept Of The Self Implies Some Connected Series, 'But When Disjunction Accedes To The Principle Which Gives To It A Synthetic And Affirmative Value, The Self, The World, And God Share In A Common Death' (176). [There Is Also A Point That Divergent Series Explain And Also Exceed Normal Conjunctive And Connective Series]. In The Usual Conception, 'The Self Is The Principle Of Manifestation, In Relation To The Proposition, The World Is The Principle Of Denotation, And God The Principle Of Signification' (176). But The Theory Of Sense Here Says That It Emanates From Nonsense, From Paradox, And From The 'Eternally Decentred Ex-Centric Centre' (176). This Position 'Does Not Tolerate The Subsistence Of God As An Original Individuality, Nor The Self As The Person, Nor The World As An Element Of The Self And As God's Product. The Divergence Of The Affirmed Series Forms A "Chaosmos" And No Longer A World; The Aleatory Point Which Traverses Them Forms A Counterself, And No Longer A Self' (176). There Is No Centre But Only 'Pure Events Which The Instant, Displaced Over The Line[Of Aion] , Goes On Dividing Into Already Past And Yet To Come. Nothing Other Than The Event Subsists… This Communicates With Itself Through Its Own Distance And Resonates Across All Of Its Disjuncts' (176). Alexander I. Stingl's Blog A Nomadic Scholarship Entity NOTES ON: Deleuze, G (1990) The Logic Of Sense, Trans Mark Lester, Edited By Constantin Boundas, New York: Columbia University Press For Models Kindly See One Of The Papers In The Series. Attributable And Ascribable To Constraints On Space Accomodation Model Is Provided In Some Paper of the series, Notwithstanding The Generality And Commonalty Of The Observation And Its Concatenatability With The Other Modules Sixteenth Series Of The Static Ontological Genesis Twenty-fourth series of the communication of events . "As Simple As That Sounds, It Is Nevertheless Extremely Difficult To Adequately Discuss No-Boundary Awareness Or Nondual Consciousness. This Is Because Our Language — The Medium In Which All Verbal Discussion Must Float — Is A Language Of Boundaries. As We Have Seen, Words And Symbols And Thoughts Themselves Are Actually Nothing But Boundaries, For Whenever You Think Or Use A Word Or Name, You Are Already Creating Boundaries. Even To Say "Reality Is No-Boundary Awareness" Is Still To Create A Distinction Between Boundaries And No-Boundary! So We Have To Keep In Mind The Great Difficulty Involved With Dualistic Language. That "Reality Is No-Boundary" Is True Enough, Provided We Remember That No-Boundary Awareness Is A Direct, Immediate, And Nonverbal Awareness, And Not A Mere Philosophical Theory. It Is For These Reasons That The Mystic-Sages Stress That Reality Lies Beyond Names And Forms, Words And Thoughts, Divisions And Boundaries. Beyond All Boundaries Lies The Real World Of Suchness, The Void, The Dharmakaya, Tao, Brahman, The Godhead. And In The World Of Suchness, There Is Neither Good Nor Bad, Saint Nor Sinner, Birth Nor Death, For In The World Of Suchness There Are No Boundaries." — Ken Wilber, No Boundary: Eastern And Western Approaches To Personal Growth. It Has Just Been Argued That Divergence Can Produce A Positive Synthesis, And That Ultimately, Events And States Of Affairs Are Compatible. Incompatibility Arises Only With

Actualizations. It Is Individuals, For Example, Who Actualize Divergent Events. Even These Divergences Are Not Just Logical Contradictions But 'Alogical 'Incompatibilities. Persons Can Enjoy The Paradoxes That Ensue, But The Point Is How To Get To The Universal Communication Of Events, The Disjunctive Syntheses. Individuals Must Grasp Themselves As Events, And See What They Normally Regard As Themselves As An Actualization. Normal Individuality Is Only 'Fortuitous' (178). It Follows That The Same Thinking Must Be Extended To All Other Events And Individuals. 'Each Individual Would Be Like A Mirror For The Condensation Of Singularities… The Ultimate Sense Of Counteractualization' (178) [Undertaken For Cognitive Reasons Here Not Ethical Ones, Although Deleuze Says That This Is A Route To 'The Universal Freedom' (178)]. The Normal Sense Of The Individual As Having A Coherent Identity Must Be Replaced By The Notion Of A Series Of Individualities [With Nomadic Lines Connecting Them?]. We Then Get To The Realm Of Pure Events, And The Universal Connections Between Them, Seen As Disjunctive Syntheses Of Series, And The Knowledge That Actualized Events Are Fortuitous. [The Example Here Is That A Friend In One World Could Be An Enemy In Another Equally Possible World]. 'Philosophy Merges With Ontology, But Ontology Merges With The Univocity Of Being (Analogy Has Always Been A Theological Vision, Not A Philosophical One, Adapted To The Forms Of God, The World, And The Self)' (179). This Does Not Mean That There Is Only One And The Same Being [And Certainly Not Identical Beings, Because These Are Always Heterogeneous]. Instead, 'Being Is Voice That Is Said, And It Is Said In One And The Same "Sense" Of Everything About Which It Is Said' (179). It Is The Ultimate Form. All The Other Forms Remain 'Disjointed In It', But Being Joins Them Into Series And Disjunctions. 'The Positive Use Of The Disjunctive Synthesis… Is The Highest Affirmation... A Single Voice For Every Hum Of Voices And Every Drop Of Water In The Sea' (180). It Is Not Just That There Is A Connection At The Level Of Language—'Being Cannot Be Said Without Also Occurring' (180). In This Way, Event And Sense Are Identical. 'Univocal Being Is Neutral. It Is Extra Being, That Is The Minimum Of Being Common To The Real, The Possible And The Impossible… The Pure Form Of The Aion… One Single Event For All Events; One And The Same Aliquid [Something] For That Which Happens And That Which Is Said; And One And The Same Being For The Impossible, The Possible, And The Real' (180). Alexander I. Stingl's Blog A Nomadic Scholarship Entity NOTES ON: Deleuze, G (1990) The Logic Of Sense, Trans Mark Lester, Edited By Constantin Boundas, New York: Columbia University Press For Models Kindly See One Of The Papers In The Series. Attributable And Ascribable To Constraints On Space Accomodation Model Is Provided In Some Paper, Notwithstanding The Generality And Commonalty Of The Observation And Its Concatenatability With The Other Modules. Sāṅkhya (Often Spelled Sāṁkhya) Is One Of The Major "Orthodox" (Or Hindu) Indian Philosophies. Two Millennia Ago It Was The Representative Hindu Philosophy. Its Classical Formulation Is Found In Īśvarakṛṣṇa's Sāṅkhya-Kārikā (Ca. 350 C.E.), A Condensed Account In Seventy-Two Verses. It Is A Strong Indian Example Of Metaphysical Dualism, But Unlike Many Western Counterparts It Is Atheistic. The Two Types Of Entities Of Sāṅkhya Are Prakṛti And Puruṣa-S, Namely Nature And Persons. Nature Is Singular, And Persons Are Numerous. Both Are Eternal And Independent Of Each Other. Persons (Puruṣa-S) Are Essentially Unchangeable, Inactive, Conscious Entities, Who Nonetheless Gain Something From Contact With Nature. Creation As We Know It Comes About By A Conjunction Of Nature And Persons. Prakṛti, Or Nature, Is Comprised Of Three Guṇa-S Or Qualities. The Highest Of The Three Is Sattva (Essence), The Principle Of Light, Goodness And Intelligence. Rajas (Dust) Is The Principle Of Change, Energy And Passion, While Tamas (Darkness) Appears As Inactivity, Dullness, Heaviness And Despair. Nature, Though Unconscious, Is Purposeful And Is Said To Function For The Purpose Of The Individual Puruṣa-S. Aside From Comprising The Physical Universe, It Comprises The Gross Body And "Sign-Body" Of A Puruṣa. The Latter Contains Among Other Things The Epistemological Apparati Of Embodied Beings (Such As The Mind, Intellect, And Senses). The Sign Body Of A Puruṣa Transmigrates: After The Death Of The Gross Body, The Sign-Body Is Reborn Into Another Gross Body According To Past Merit, And The Puruṣa Continues To Be A Witness Through Its Various Bodies. An Escape From This Endless Circle Is Possible Only Through The Realization Of The Fundamental Difference Between Nature And Persons, Whereby An Individual Puruṣa Loses Interest In Nature And Is Thereby Liberated Forever From All Bodies, Subtle And Gross. Much Of The Sāṅkhya System Became Widely

Accepted In India: Especially The Theory Of The Three Guṇa-S; And It Was Incorporated Into Much Latter Indian Philosophy, Especially Vedānta. Internet Encyclopedia Of Philosophy Sankhya Author Information Ferenc Ruzsa Email: Ferenc.Ruzsa@Elte.Hu Eötvös Loránd University Hungary. The Amitabha Buddha Who Was, And Is, Revered And Praised By Buddhists Around The World, Radiates Indefinite Light And Life From This "State Of Cessation". This State Is A Continuous Process Of Calmness. It Will Be The Eventual Refuge For Us All. If We Think Carefully About The Definitions Of Calmness And Extinction, Then We Can Deduce That They Are The True Natural End-Points Of Rising And Falling. The True Nature Of The Cycle Of Rising And Falling Is Calmness And Extinction. Because Of This Nature, All Chaos And Conflicts In The State Of Rising And Falling Will Eventually Cease. This Is Attainable By The Realisation Of Prajna. Teachings In Chinese Buddhism (6) Sunyata (Emptiness) In The Mahayana Context (Wikipedia). Like Psychic DNA, The Collective Unconscious Contains "Inherited" Psychic Material That Links Us Not Only To Other Humans In The Present But Also To Our Ancestors From The Past. According To Jung's Theory, Though Each Of Us Appears To Function Independently, In Actuality We're All Tapped Into The Same Global Mind. Symbols Interact With And Condition Our Biohologram. The Collective Unconscious Is Essentially A Hologram. Symbols Arise From And Are Embedded In The Environment As Holographic Fields Of Energy. They Are Morphogenic Veils Of Primal Forces. If Your Brain Acts Like A Self-Contained Hologram, It Is Possible Your Consciousness Is Actually A Piece Of A Much Larger Hologram Of Overall Human Consciousness. Jung Noticed That Patterns Spontaneously Appear Over And Over Around The World. They Also Appear As Our Ancestral Memories Or Holographic Wisdom Field. In The Archetypal World, Everyone Is The Same, All Around The World... We Are All Gods, And Our Emotional Addictions To Pain And Suffering, Contempt, Insecurities, Doubt, Failure, Is Holographically-Recorded And Can Be Holographically Healed. All Archetypes Are A Form Of Human Expression That Is Both Holographic And Physical. Physical Formations Of Archetypal Sequences Cause Humans To Behave In Parallel Manners To Each Ancestor That Is Associated. Integration Is A Function Of Intentionality -- Conscious And Unconsciously Maintained, Or Incorporated. Integration Occurs Both Without Effort, As A Redesign Of The Central Processor Of Our Minds, And Voluntarily As A Deliberate Effort To Understand, Find Meaning, And As Rectification Of Our Behavior Towards Others And Towards Ourselves. Imagination Is Structured By The Archetypal Potentials Of The Unconscious. Archetypes Structure The Possibility To Generate And Entertain Such Ideas. The Archetype Itself Cannot Be Known But Structures Everything We Come To Know. Their Totality Functions As A Psychic Organ. Universal Themes Appear In Distinct Cultural Garb. Over Millennia, All The Archetypes Have Emerged In Stories, Deities And Cultural Forms. One Of The Striking Points Of Religious Faith Is That They Aren't True. In Early History People Didn't Know The ‘Real' Reasons Things Happened, So It's No Surprise That Their Explanations Were Wrong. But Then, Why Would We Want To Retrieve Such Superstitions? Kuhn Reminds Us That Even The Most Absurd Or Confused Explanation Of A Phenomenon Can Find Acceptance In The Absence Of A Competing Idea. Once Any Explanation Is Offered For A Phenomenon, It Has An Explanation From That Time On. Succeeding Speculations Might Be Able To Explain The Relevant Phenomena Better Than Its Predecessors. Deities, As Archetypal Role Models, Are The Opening Gambit In The Drama Of Understanding Human Subjectivity. Folk Tales Function The Same Way. Deities Or Archetypes May Have Evolved To Smooth Interpersonal Relations By Including An Understanding Of Human Types, Along With Rules For Helping The Different Types Relate With One Another. They Are Reflections Of Ancient Evolutionary Pressures, With A Dash Of Neuroanatomy. We Still Have Yjr Same Pressures Today So The Ancient Archetypes Still "Work", Regardless Of Objective Existence Of Gods And Goddesses. Burke (2003) Describes How The Underlying Pattern Or Strange Attraction Between Transference And Holography Extends To Other Processes Both Within And Outside The Field Of Psychology. Such Processes Include Projection, Projective Identification, Splitting, Memory, Biology, Creative Discovery, Theology, Synchronicity, Chaos, And Nonlocality. Identifying The Similar Patterns Of These Processes Demonstrates The Existence Of An Underlying Holographic Archetype In Which Essential Qualities Of The Whole Are Present In Each Of The Parts Of The Whole. The Autonomy Of The Overall Human Is Present In Each Conscious And Unconscious Component Part Of The Human Psyche. Cognition, Itself Is A Holographic

Archetype. Many Essential Qualities Of The Whole Are Reflected Or Contained In Each Of The Parts That Make Up That Whole. It Is A Subtle Net Of Metaphor, Analogy And Simile. Holographic Archetypes Effectively Echo Their Resonant Patterns Through Literal And Symbolic Reflectaphors. The Passing Forms Of The Holographic Archetype Include The Hologram, Psychic Structure, Synchronicity, Wisdom Traditions, Memory, And The Process Of Scientific Discovery, Chaos In Physics, Nonlocality And Virtuality In Physics. As Unconscious Structuring Principle, The Archetype Lies Beyond Normal Consciousness And Emerges Suddenly And Dynamically From The (Holographic) Psychoid Field, With A Powerful Emotional Charge That Invests It With Significance. Everything That Happens Is Conditioned By The Moment In Which It Happens. The Universal Field Imposes The Conditions. Matter Is Not Inert But Receptive To Holistic Patterning. If The Mythic World Taught Our Ancestors How To Manipulate The Empirical World, It Also Taught Them To Manipulate That Mythic Narrative Itself For Control Purposes. Socioeconomic Power Enforces Its Mythic Narrative. The Psychoid Field Imposes Holistic Function. Autonomous Inner Forces Arouse Compelling Opportunities To Enact Archetypal Behaviors. They Guide Our Perceptions And Behavior, Usually Without Our Awareness. Limbic Action Of Complexes Is A Big Part Of The Holistic Dynamics Of The Psyche. Dreams Report What Goes On Beneath The Veil Of Conscious Awareness. To The Consciousness Of The 'Thinker', Knowledge Is Thought. Period. Without Thought, The Consciousness Of The 'Thinker' Collapses Into Psychosis. With Archetypes Come The Potential For Wisdom, Relatedness, Sociality, Ambiguity, Paranoia, Projection, Identification, Denial, Inflation, Sub Personalities (Fragmentation), Defensiveness, Obsession, Hypnotic Dissociation, The Contagion Of Participation Mystique, Mythologizing, Complexes, Compensation, And Self-Delusion. Compensation May Calm Or Disturb Consciousness. There Is No Imperative For The Ego To Integrate These Alternative Perspectives, Private And Public Myths. The Unconscious Can Produce Deep Wisdom And Utter Nonsense. It Is Up To The Ego To Discriminate. The Value Of Myths Is Purely Heuristic, Not Pragmatic. Jung In The 21st Century: Synchronicity And Science By John Ryan Haule Mind Control Countermeasures. Forms Of Experience, Such As Madness, Violence, Or Sexuality That Break From The Prison Of Rationality. Where Modern Societies "Problematize" Forms Of Experience Such As Madness, Illness, And Sexuality, That Is, Turn Them Into Governmental Problems, Into Areas Of Life In Need Of Control And Regulation, Foucault In Turn Queries The Social Construction Of “Problems” By Uncovering Their Political Motivations And Effects And By Challenging Their Character As Natural, Necessary, Or Timeless. In What He Calls A "Diagnostic Critique" That Combines Philosophy And History (1989: 38-39, 73), Foucault Attempts To Clarify The Nature Of The Present Historical Era, To Underline Its Radical Difference From Preceding Eras, And To Show That Contemporary Forms Of Knowledge, Rationality, Social Institutions, And Subjectivity Are Contingent Socio-Historical Constructs Of Power And Domination, And Therefore Are Subject To Change And Modification. Foucault's Ultimate Task, Therefore, Is "To Produce A Shift In Thought So That Things Can Really Change" (Quoted In O'Farrell 1989: 39). The Goal Of Foucault's Historico-Philosophical Studies, As He Later Came To Define It, Is To Show How Different Domains Of Modern Knowledge And Practice Constrain Human Action And How They Can Be Transformed By Alternative Forms Of Knowledge And Practice In The Service Of Human Freedom. Foucault Is Concerned To Analyze Various Forms Of The "Limit Experience" Whereby Society Attempts To Define And Circumscribe The Boundaries Of Legitimate Thought And Action. The Political Vision Informing Foucault's Work Foresees Individuals Liberated From Coercive Social Norms, Transgressing All Limits To Experience, And Transvaluing Values, Going Beyond Good And Evil, To Promote Their Own Creative Lifestyles And Affirm Their Bodies And Pleasures, Endlessly Creating And Recreating Themselves. Foucault Denies There Can Be Any Basis For Objective Descriptive Statements Of Social Reality Or Universal Normative Statements That Are Not Socially Conditioned And Locally Bound. He Tries To Show That All Norms, Values, Beliefs, And Truth Claims Are Relative To The Discursive Framework Within Which They Originate. Any Attempt To Write Or Speak About The Nature Of Things Is Made From Within A Rule-Governed Linguistic Framework, An “Episteme,” That Predetermines What Kinds Of Statements Are True Or Meaningful. All Forms Of Consciousness, Therefore, Are Socio-Historically Determined And Relative To Specific Discursive Conditions. There Is No Absolute, Unconditioned, Transcendental Stance From Which To Grasp What Is Good, Right, Or True. Foucault Refuses To

Specify What Is True Because There Are No Objective Grounds Of Knowledge; He Does Not State What Is Good Or Right Because He Believes There Is No Universal Standpoint From Which To Speak. Universal Statements Merely Disguise The Will To Power Of Specific Interests; All Knowledge Is Perspectival In Character. For Postmodern Theorists Like Foucault, The Appeal To Foundations Is Necessarily Metaphysical And Assumes The Fiction Of An Archimedean Point Outside Of Language And Social Conditioning. Habermas (1987) Rightly Finds Perplexing An Approach That Raises Truth Claims While Destroying A Basis For Belief In Truth, Which Takes Normative Positions While Suppressing The Values To Which They Are Committed. For Critique To Be Justified And Effective, It Should Preserve Standards By Which To Judge And Evaluate, But Foucault's Total Critique Turns Against Itself And Calls All Rational Standards Into Question. **Richard Rorty, The Attack On Theory, And Renunciation Of Radical Politics.** Deleuze's Analysis Begins by Coming to New Understandings of the Concepts of the Image and Movement. The Image, Above All, Is Not A Representation Of Something, That Is, A Linguistic Sign. This Definition Relies Upon The Age-Old Platonic Distinction Between Form And Matter, In Its Modern Saussurean Form Of Signifier-Signified. Rather, Deleuze Wants To Collapse These Two Orders Into One, And The Image Thus Becomes Expressive And Affective: Not An Image Of A Body, But The Body As Image (C1 58). This Collapse Comes About With Reference To Two Philosophers, Henri Bergson And Charles Sanders Pierce. Deleuze Dedicated A Book-Length Study To The Former Entitled Bergsonism (1968), And His Use Of His Notions Of Movement And Time In The Cinema Texts Is Already Presaged By This Text. Movement For Bergson, Deleuze Argues, Is Not Separable From The Object Which Moves: (I Am Not Movement: Italics Mine) They Are Literally The Same Thing. Thus, No Representative Relationship Can Be Established Without Artificially Halting The Flow Of Movement And Thus Misconstruing The Frozen 'Element' As Self-Sufficient. (See the befuddling similarity of holographic principle of Modern Quantum Mechanics and the Deleuzean idea; italics mine) There Is Only The Flow Of Movement Which Expresses Itself In Different Ways. Among Other Things, This Is One Of Deleuze's Critiques Of Phenomenology (C1 56, 60). Thus The Early Cinema Is Characterised For Deleuze By The Reign Of What He Calls The Sensory-Motor Schema. This Schema Is The Unity Of The Viewed And The Eye That Views In Dynamic Movement. This Model Of The Movement-Image Is Precisely The Nature Of Cinema, For Deleuze. It Does Not Falsify Movement By Extracting Segments And Stringing Them Together In A Representative Fashion, But Creates A Wide Range Of Expressive Images. It Is In Order To Come To Terms With The Varieties Of Movement-Images That Deleuze Turns To Pierce, Who Developed, "The Most Extraordinary Classifications Of Images And Signs . . ." (C2 30). ……. With Some Alterations, Pierce's Semiotic Classifications To Describe The Use Of Movement-Images In Cinema, And Their Centrality Before The Second World War The Movement From The First Text To Cinema 2: The Time-Image Has A Significance Closely Related To Kant's So-Called Copernican Revolution In Philosophy. Up Until Kant, Time Was Subject To The Events That Took Place Within It, Time Was A Time Of Seasons And Habitual Repetition (See (3) (C) Above); It Was Not Able To Be Considered On Its Own, But As A Measure Of Movement (C2 34-5; KCP Iv.). One Element Of Kant's Achievement For Deleuze, As We Have Seen, Is His Reversal Of The Time-Movement Relationship: He Establishes Time Itself As An Element To Which Movement Must Be Subordinated, A Pure Time. In The Cinema, Deleuze Argues, A Similar Reversal Takes Place. The Historico-Cultural Reason Behind This Reversal Is The Event Of World War Two Itself. With The Great Truths Of Western Culture Put So Deeply In Question By The Before Unimaginable Methods Employed And Their Forthcoming Results, The Sensory-Motor Apparatus Of The Movement-Image Are Made To Tremble Before The Unbearable, The Too-Much Of Life's Possibilities, The Potential Of The Present (C2 35). No Longer Could The Dogmatic Truths That Had Guided Society, And Cinema To An Extent, Allow The Apparently 'Natural' Movement From One Thing To The Next In An Habitual Fashion: 'Natural' Links Precisely Lost Their Efficacy. And With The Use Of Unnatural Or False Links, Which Do Not Follow The Sequence Or Narrative Affect Of The Movement-Image, Time Itself, The Time-Image, Is Manifested In Cinema (Deleuze Considers Orson Welles To Be The First Auteur To Make Use Of The Time-Image (C2 137)). Rather Than Finding Time As An, "Indirect Representation," (C2 35-6), The Viewer Experiences The Movement Of Time Itself, Which Images, Scenes, Plots And Characters Presuppose Or Manifest In Order To Gain Any Sort Of Movement Whatsoever. Along With This 'External' Reason,

There Is Also For Deleuze A Motivation Within Cinema Itself To Go From The Movement-Image To The Time-Image. The Movement Image Has The Tendency, Thanks To The Habitual Experience Of Movement As Normal And Centered, To Justify Itself In Relation To Truth: As Deleuze Argues With Regard To The Dogmatic Image Of Thought (See (3)(D) Above), There Is The Presupposition That Thought Naturally Moves Towards Truth. Of Course, Deleuze Suggests, Cinema, When Truly Creative, Never Relied Upon This Presupposition, And Yet, "The **Movement-Image, In Its Very Essence, Is (=) Answerable To (e) The Effect Of (e) Truth Which It Invokes (e&eb) While Movement Preserves (eb) Its Centres." (C2 142).** In Questioning Its Own Presuppositions, Deleuze Argues, **Cinema Moved Towards A New, Different, Way Of Understanding Movement Itself, As Subordinate To (e) Time. (Wikipedia: Gilles Deleuze).** From His Early Writings On, Nietzsche, Like Kierkegaard, Rails Against A Life-Denying (e) Rationalism And Idealist Philosophy Which Champions' Reason Over The Passions. Nietzsche Interprets The "Subject" As A Mere Construct, An Idealized Sublimation Of Bodily Drives, Experiences, And (e&eb) A Multiplicity Of Thoughts And Impulses. This "Little Changeling," On Nietzsche's View, This Subject, "Is Believed in (eb) More Firmly than Anything Else on Earth," But Is For Him A Simple Illusion Created Out Of (e) Modern Desperation To Have (e) A Well-Grounded Identity. Belief In The Subject Is Promoted By The Exigencies Of (e) Grammar Which Utilize A Subject/Predicate Form, Giving Rise To (eb) The Fallacy That The "I" Is A Substance, Whereas It Is (=) Really Only A Convention Of Grammar (Nietzsche 1968b: 37-38). For Nietzsche, "The Doer" Is (=) "Merely A Fiction Added To The Deed -- The Deed Is (=) Everything" (1968b: 45). "The **Subject**," He Concludes, Is (=) Thus But A Shorthand Expression For (e) A Multiplicity Of Drives, Experiences, And Ideas. In The Spirit Of Enlightenment, Nietzsche Also Polemicizes Against (e&eb) Metaphysics, Arguing That It Illicitly Generalizes From (e) Ideas In One Historical Epoch To (e) The Entirety Of History. Against This Form Of (e) Philosophical Universalism, Nietzsche Argues "**There Are (=) No Eternal Facts, Just As There Are (=) No Absolute Truths.** Consequently, What Is Needed From Now On Is (=) Historical Philosophizing, And With It (eb) The Virtue Of Modesty" (Nietzsche 1986: 13). Castigating Traditional Philosophy And Values From (e) A Critical Enlightenment Perspective, Nietzsche Anticipates Later Postmodern Critiques Of Metaphysics, Assailing (e) The Concept Of Enduring Knowledge, The Notion Of A Transcendental World, And Presenting (eb) Metaphysical Thought As (=) A Thoroughly Obsolete Mode Of (e) Thinking. He Attributes The "Metaphysical Need," At (eb)The Heart Of Philosophies Such As (=) Schopenhauer's, To Primitive Yearnings For Religious Consolation For (e) The Sufferings Of Life And He Urges "Free Spirits" To Liberate Them And Pursue (e&eb) Thinking And Living Experimentally (1986: 8). Nietzsche's Attack On Foundationalism, Universalizing Thought, And Metaphysics Thus Undertakes (e&eb) A "Postmodern" Turn In Philosophy Through (e&eb) A Radical Deconstruction Of (e) Modern Theory. But While Deconstructionist Philosophies Typically Terminate (e) In The No, Merely Seeking To (e) Unravel A Positive Modern Value System Into (e&eb) A Heap Of Disconnected Fragments, Nietzsche Starts And Finishes With (e&eb) A Big Yes, A Life-Affirming Value, Deconstructing Only To (e) Reconstruct. Moving Far Away From Schopenhauerian Pessimism, Back Toward (e&eb) A Greek View Of Tragedy, Toward A Dionysian View Of Existence, Nietzsche Seeks "A Justification Of Life, Even At Its Most Terrible, Ambiguous, And Mendacious" (1968a: 521), A Justification Found In Art, Creativity, Independence, And The Emergence Of (e) "Higher Types" Of Humanity. Yet Nietzsche's Perspectivism Denies (e) The Possibility Of (e) Affirming Any Absolute Or Universal Values: All Ideas, Values, Positions, And So On Are Posits Of Individual Constructs Of A Will To Power, Which Are To Be Judged According To (e&eb)The Extent To Which They Do Or Do Not Serve The Values Of Life, Creativity, And Strong Individuality. For Nietzsche There Are (=) No Facts, Only (e&eb) Interpretations, And He Argues That All Interpretation Is Constituted By(e) The Individual's Perspectives And Is Thus Inevitably Laden With (e&eb) Presuppositions, Biases, And Limitations. For Nietzsche, A Perspective Is (=)Thus An Optic, A Way Of Seeing, And The More Perspectives One Has At One's Disposal, The More One Can See, And The Better One Can Understand And Grasp Specific Phenomena. To Avoid Limited And Partial Vision One Should Learn "How To Employ A Variety Of Perspectives And Interpretations In The Service Of Knowledge" (Nietzsche 1968a: 119). THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS By Steven Best And Douglas Kelner

Http://Www.Gseis.Ucla.Edu/Faculty/Kellner/Kellner.Html. And Just As During Much Of The Twentieth Century, Any Plea For Greater Social Justice Could Be Successfully Damned As Communist, Likewise Today, Any Strategy To Eradicate Suffering Is Likely To Be Condemned In Similar Reactionary Terms: Either Wire Head Hedonism Or Revamped Brave New World. This Response Is Not Just Facile And Simplistic. If It Gains Currency, The Result Is Morally Catastrophic. Of Course, The Abolitionist Issue Rarely Arises. Typically, Universal Bliss Is Still More-Or-Less Unthinkingly Dismissed As Technically Impossible. Insofar As The Prospect Is Even Contemplated - Grudgingly - It Is Usually Assumed That The New Regime Would Be Underwritten Day-By-Day With Drugs Or, More Crudely, Electrodes In The Pleasure-Centres. These Techniques Have Their Uses. Yet In The Medium-To-Long-Term, Stopgaps Won't Be Enough. All Use Of Psychoactive Drugs May Be Conceived As An Attempt To Correct Something Pathological With One's State Of Consciousness. There's Something Deeply Wrong With Our Brains. If What We Had Now Was OK, We Wouldn't Try To Change It. But It Isn't, So We Do. Mature Biological Psychiatry Will Recognise Inadequate Innate Bliss As A Pandemic Form Of Mental Ill-Health: Good For Selfish DNA In The Ancestral Environment Where The Adaptation Arose, But Bad For Its Throwaway Vehicles, Notably Us. The Whole Gamuts Of Behavioural Conditioning, Socio-Economic Reform, Talk-Therapies - And Even Euphoriant Superdrugs - Are Just Palliatives, Not Cures, For A Festering Global Illness. Its Existence Demands A Global Eradication Program, Not Idle Philosophical Manifestos And Scientific Belle's Letters. But One Does One's Best. The Ideological Obstacles To Genetically Pre-Programmed Mental Super-Health Are Actually More Daunting Than The Technical Challenges. To Be Cured, Hypo-Hedonia Must Be Recognised As A Primarily Genetic Deficiency-Disorder. Designer Mood-Brighteners And Anti-Anxiety Agents To Alleviate It Are Sometimes Branded "Lifestyle-Drugs"; But This Is To Trivialize A Serious Medical Condition Which Must Be Corrected At Source. Happily, Our Hereditary Neuropsychiatric Disorder Is Likely To Become Extinct Within A Few Generations As The Reproductive Revolution Unfolds. Aversive Experience, And The Poisonous Metabolic Pathways That Mediate Its Textures, Will Become Physiologically Impossible Once The Genes Coding Its Neural Substrates Have Been Eliminated. We Won't Miss Its Corrupting Effect When It's Gone. In The Medium-Term, The Functional Equivalent Of Aversive Experience Can Help Animate Us Instead. Late In The Third Millennium And Beyond, Its Functional Successors May Be Expressed As Gradients Of Majestic Well-Being. On This Scenario, Our Descendants Will Enjoy A Civilisation Based On Information-Bearing Pleasure-Gradients: Whether Steep Or Shallow, We Simply Don't Know. Such A Global Species-Project Does Not Have The Desperate Moral Urgency Of Eliminating The Phenomenon Of Darwinian Pain - Both "Mental" And "Physical", Human And Non-Human Alike. Abolishing Raw Nastiness - Sometimes Vile Beyond Belief - Remains The Over-Riding Ethical Priority. One Doesn't Have To Be An Outright Negative Utilitarian To Acknowledge That Getting Rid Of Agony Takes Moral Precedence Over Maximizing Pleasure. But Both Genetic Fundamentalists And Gung-Ho Advocates Of Better Living Through Chemistry Today Agree On One Crucial Issue. There Is No Sense In Sustaining A Legacy Of Mood-Darkening Metabolic Pathways Out Of Superstitious Deference To Our Savage Past. When Bernard Marx Tells The Savage He Will Try To Secure Permission For Him And His Mother To Visit The Other Place, John Is Initially Pleased And Excited. Echoing Miranda In The Tempest, He Exclaims: "O Brave New World That Has Such People In It." Heavy Irony. Like Innocent Miranda, He Is Eager To Embrace A Way Of Life He Neither Knows Nor Understands. And Of Course He Comes Unstuck. Yet If We Swallow Such Fancy Literary Conceits, Then Ultimately The Joke Is On Us. It Is Only Funny In The Sense There Are "Jokes" About Auschwitz. For It Is Huxley Who Neither Knows Nor Understands The Glory Of What Lies Ahead. A Utopian Society In Which We Are Sublimely Happy Will Be Far Better Than We Can Presently Imagine, Not Worse. And It Is We, Trapped In The Emotional Squalor Of Late-Darwinian Antiquity, Which Neither Know Nor Understand The Lives Of The God-Like Super-Beings We Are Destined To Become. **A Defence Of Paradise-Engineering Brave New World By Aldous Huxley Brave New World (1932) Is One Of The Most Bewitching And Insidious Works Of Literature Ever Written.** . Deleuze attacks Hegel and others in what we can call—though Deleuze did not—the "identitarian" tradition first of all by means of (e) a radicalized reading of Kant, whose genius, as Deleuze explains in Kant's Critical Philosophy (1963), was to have conceived of a purely immanent critique of reason—a critique that did not

seek "errors" of reason produced by external causes, but rather "illusions" that arise (eb) from within reason itself by the illegitimate (transcendent) uses of the syntheses of (e) consciousness. Deleuze characterized his own work as a philosophy of immanence, arguing that Kant himself had failed to realize fully the ambitions of his critique, for at least two reasons: first, the failure to pursue a fully immanent critique, and second, the failure to propose a genetic account of (e) real experience, resting content with the account of the conditions of possible experience. First, Kant made the field of consciousness immanent to (e) a transcendental subject, thereby reintroducing an element of identity that is (=) transcendent (that is, external) to (e) the field itself, and reserving all power of synthesis (that is, identity-formation) in the field to the activity of (e) the always already unified and transcendent subject. (Deleuze was influenced in this regard by his reading of Sartre's 1937 essay "The Transcendence of the Ego.") Already in his Hume book, Empiricism and Subjectivity (1953), Deleuze had pointed to an empiricist reversal of Kant. Where Kant's question had been "How can the given be given to (e) a subject?" Hume's question had been "How is the subject (human nature) constituted within (eb) the given?" Metz asserting a notion of langue with the formal elements associated with narrative. Moreover he narrowed this gap further by claiming a version of langue's syntagmatic and paradigmatic in the operation of cinematic narrative. Deleuze notes the significance of this move in The Time-Image: [L]anguage features which necessarily apply to utterances will be found in the cinema, as rules of use, in the language system and outside of it: the syntagm (conjunction of present relative units) and the paradigm (disjunction of present units with comparable absent units). The semiology of the cinema will be the discipline that applies linguistic model, especially syntagmatic ones, to images as constituting one of their principle "codes" (25-26). (As quoted in Roger Dawkin's making matter our of Deleuzean concept of cinema) An exemplary and illustrious account of unbounded consciousness has been studied by CORRINA BONSHEK Deleuzian sensation and unbounded consciousness in Anna & Corrina Bonshek's Reverie I (2002). Entire article has been modelled in one of the paper in the series. It is true that Deleuze sometimes suggests follow of the Model and Shadow theme instead of getting into the trap of consciousness and cogito. This should not be confused with the unbounded consciousness Deleuze mentioned in his studies on Cinema. In his mature work, Deleuze argues for an "impersonal and pre-individual" transcendental field in which the subject as identity pole which produces (eb) empirical identities by active synthesis is itself the result or product of (e) differential passive syntheses (for instance, in what Deleuze calls the syntheses of habit, we find bodily, desiring, and unconscious "contractions" which unify (e&eb) a series of experiences, extracting (eb) that which it to be retained in the habit and allowing (eb) the rest to be "forgotten"). The passive syntheses responsible for subject formation must be qualified as (=) "differential," for three reasons. Each passive synthesis is (=) serial, never singular (there is never one synthesis by itself, but always a series of (e) "contractions," that is to say, experience is (=) ongoing and so our habits require (e) constant "updating"); each series is related to (e&eb) other series in the same body (at the most basic level, for instance, the series of taste contractions is related to (e&eb) those of smell, sight, touch, hearing and proprioception); and each body is related to (e&eb) other bodies, which are themselves similarly differential (the series of syntheses of bodies can resonate or clash). Together the passive syntheses at all these levels form (eb) a differential field within which subject formation takes place as (=) an integration or resolution of that field; in other words, subjects are (=) roughly speaking the patterns of (e) these multiple and serial syntheses which fold in on themselves producing (eb) a site of self-awareness. Of course, Deleuze never simply proclaims this as a bald thesis, but develops a genetic account of subjectivity in many of his books. Taking all this into account, Deleuze summarized his differential, immanent and genetic position by the at first glance odd phrase of "transcendental empiricism." This is cashed out in terms of two characteristics: (1) the abstract (e.g., "subject," "object," "State," the "whole," and so on) does not explain, but must itself be (=) explained; and (2) the aim of philosophy is not to rediscover (eb) the eternal or the universal, but to find the singular conditions under which (e&eb) something new is produced. In other words—and this is a pragmatic perspective from which Deleuze never deviated—philosophy aims not at stating the conditions of knowledge qua representation, but at finding and fostering (eb) the conditions of creative production. Deleuze's second criticism of Kant claims that he had simply presumed (eb) the existence of knowledge and morality as "facts" and then sought their conditions of possibility in the transcendental But already in 1789, Salomon Maimon, whose early critiques of

Kant helped generate the post-Kantian tradition, had argued that Kant's critical project required (e) a method of genesis—and not (e) merely a method of conditioning—that would account for the (eb) production of knowledge, morality, and indeed reason itself. In other words, Maimon called for a genetic method that would be able to reach (eb) the conditions of real and not merely possible experience. Maimon found a solution to this problem in a principle of difference: whereas identity is (=) the condition of possibility of thought in general, it is difference that constitutes (e) the genetic and productive principle of real thought. These two Maimonian exigencies—the search for the genetic conditions of real experience and (e&eb) the positing of a principle of difference—appear in almost every one of Deleuze's early monographs. Nietzsche and Philosophy (1962), for instance, suggests that Nietzsche completed and inverted Kantianism by bringing critique to bear, not simply on false claims to knowledge or morality, but on (eb) true knowledge and true morality, and indeed on truth itself: "genealogy" constituted (e) Nietzsche's genetic method, and the will-to-power was (=) his principle of difference. Deleuze's anti-Hegelianism is shown in his focus on (eb) the productivity of the non-dialectical ("affirmative") differential forces termed by Nietzsche "noble." These forces affirm themselves, and thereby differentiate (e&eb) themselves first, and only secondarily consider that from which they have (e) differentiated themselves. In Bergsonism (1966), Deleuze develops the ideas of virtuality and (e&eb) multiplicity that will serve as (=) the backbone of his later work. From Maimon's reading of Kant, we know that Deleuze needs to substitute the notion of the condition of the genesis of the real for (e&eb) the notion of conditions of possibility of representational knowledge. The positive name for that genetic condition is the virtual, which Deleuze adopts from the following Bergsonian argument. The notion of the, possible Bergson holds in Creative Evolution, is derived from (e) a false problem that confuses the "more" with (e&eb) the "less" and ignores (e) differences in kind; there is not less but more in (eb) the idea of the possible than in the real, just as there is (=) more in the idea of nonbeing than in that of being, or more in the idea of disorder than (e) in that of order. When we think of the possible as (=) somehow "pre-existing" the real, we think of (e) the real, then we add to (e) it the negation of its existence, and then we project (e&eb) the "image" of the possible into the past. We then reverse the procedure and think of the real as (=) something more than possible, that is, as the possible with existence added to (e) it. We then say that the possible has been "realized" in (eb) the real. By contrast, Deleuze will reject the notion of the possible in favor of (e) that of the virtual. Rather than awaiting realization, the virtual is (=) fully real; what happens in genesis is (=) that the virtual is (=) actualized. The fundamental characteristic of the virtual, that which means (eb) it must be actualized rather than realized, is its differential makeup. Here Deleuze used terms actualisation and realisation as if they are (=) differential in nature Actualisation of the dreams could be tantamount to (=) the realization of the dreams, but the reciprocal is not true. (My interpretation) Deleuze always held the critical axiom that the ground cannot (e) resemble that which it grounds; he constantly critiques the "tracing" operation by which identities in real experience are said to be conditioned (e&eb) by identities in the transcendental. Mention may be made of the Deleuzian statement that the relationship between problem and its conditions bears ample testimony and infallible observatory to the truth of the problem as such. Deleuze writes: …….as for the determination of the conditions it implies (eb) on one hand, a space of nomadic distribution, in which singularities are (=) distributed, on the other hand, it implies, (eb) a time of decomposition whereby a space is (=) subdivided in to subspaces. (See Seventeenth series of Logical genesis: Gilles Deleuze: Translation by Constantin Boundas) **It is to be noted that Deleuze writes in a progressive vein; and the subspaces he mentions are those which incorporate adjunct and corresponding points, which in fact defines progressively a determined domain under consideration.** This also brings us to the question whether there is any space which has singularities in its domain under consideration, and if so effects of such singularities incorporated thereof the space itself. Modem quantum mechanics proves beyond doubt that the happenings in space affect the space and vice versa. Deleuze goes on to state that verily there is a space that incorporates these singularities. What does this mean in the most general sense and to us who are used to quantum mechanical world? It is clear that there exists a spatio temporal self determination of the problem in which (eb) the problem advances, making up for (e) its deficiencies and thwarting (e) its own conditionalities. This can be made possible only by a transcendental power or complete computer simulation ad dissimulation. Mention may be made that many studies of the Universe being a

computer has been put forward. And it had its many ramifications in the positive direction, albeit it did not received the much needed accentuation and augmentation when the theory was proposed. There is a mistaken notion that once the solution is obtained, problem is solved. Infact solutions are engendered precisely at (eb) the same moment problem determines itself. Thus sense in its organsiation of problems and solutions, aleatory and (e&eb) singular points, series and displacements are (=) doubly generative. Solutions does not attribute a subjective status to the problem retrospectively as is thought of by the widest commonalty spread when the problem is determined it also determines the solutions under which it persist. Argument also can be put forth from Stoic Philosophy that in order for the thing to be bad, it should have been good. But it has its own limitations as one could discern with circumspection and jurisprudence. For instance, Deleuze criticizes Kant for copying the transcendental field in the image of (e) the empirical field. In fact Kant himself clarifies that the figment of imagination lies dormant like in the mind. If cinnabar were now red, now black, now light, now heavy, if a human being were now changed into this animal shape, now into that one, if on the longest day the land were covered now with fruits, now with ice and snow, then my empirical imagination would never even get the opportunity to think of (e) heavy cinnabar on the occasion of the representation of the color red; or if a certain word were now attributed now to this thing, now to that, or if one and the same thing were sometimes called this, sometimes that, without the governance of (e&eb) a certain rule to which the appearances are already subjected (e&eb) in themselves, then no empirical synthesis of reproduction could take place (CPR, A100-101) as quoted in Real Experience and Possible Experience: Deleuze's Ontology of Experience – Conclusion. This is very much true,. In fact associations are created because of the nonchangeabilityof of the object, notwithstanding certain change in the subject. Such continuous associations are what form the basis of our understanding or misunderstanding (see earlier papers). That is, empirical experience is (=) personal, identitarian and centripetal; there is (=) a central focus, the subject, in which all our experiences are tagged as belonging (e&eb) to us. Mention may be made that Satre established that pre reflexive cogito is primary consciousness notwithstanding the fact that he makes in later work, he makes this his original point of departure. Kant says this empirical identity is only possible if (e) we can posit the Transcendental Unity of Apperception, that is, the possibility of adding "I think" to (e&eb) all our judgments. Instead of this smuggled-in or "traced" identity, Deleuze will want to have the transcendental field be (=) differential. Deleuze still wants to work back from experience, but since the condition cannot (e) resemble the conditioned, and since the empirical is (=) personal and individuated, the transcendental must be (=) impersonal and pre-individual. Deleuze is of the opinion that the interrogative problem cannot bear resemblance to propositions it subsumes(e) under it; it rather engenders (eb) them as it determines its own conditions and assigns (eb) individual order of individual order of permutation of (e) the propositions within (eb) the framework of generalised signification (See seventeenth series of static genesis) The virtual is the condition for (e) real experience, but it has (e) no identity; identities of the subject and the object are products of (e) processes that resolve, integrate, or actualize (the three terms are synonymous for Deleuze) a differential field. Satre was furious about Descartes for having confused spontaneous doubt which is consciousness with methodological doubt. I may add the later might arise from the non emotional feeling, and the question of consciousness itself would remain slightly confused with Satre as far as the author is concerned. Satre holds that all consciousness is consciousness of something. In other words consciousness is (=) pre reflexive and directive. It is (=) certainly (consciousness) intentional or so it would seem. Author's view is that consciousness is not utterly translucent but is to major extent cultivated through reference groups and peer groups which follow the principle of making a choice of 'coming up in life". "I know" is a tendency that grows in an individual and this is a relation with his objects or information (as we have taken) in mind which is essentially anagrammatic. The Deleuzean virtual is (=) thus not the condition of possibility of (e) any rational experience, but the condition of (e) genesis of real experience. Interrogation itself is (=) the shadow of the problem projected, or rather reconstructed based on (e) empirical propositions. This is true of every parameter like language, region, religion, science. As we have seen, the virtual, as genetic ground of the actual cannot (e) resemble that which it grounds; thus, if we are confronted with actual identities in experience, then (eb) the virtual ground of those identities must be purely differential. Deleuze adopts "multiplicity" from Bergson as the name for such a purely differential field. In this usage, Deleuze later clarifies; "multiplicity" designates the

multiple as (=) a substantive, rather than as (=) a predicate. Deleuze has been candid about the fact that there exists no parallelism between logical genesis and (e&eb) ontological genesis. There is rather a relay between logical genesis and ontological genesis which permits every sort of shifting (e&eb) and jamming. It is therefore too simple to argue for the correspondence between the individual and (e&eb) the denotation the person and (e&eb) manifestation, multiple properties or variable properties and (e&eb) signification. (Seventeenth series of logical genesis) The multiple as predicate generates (eb) a set of philosophical problems under (e&eb) the rubric of "the one and the many" (things is one or multiple, one and multiple, and so on). With multiplicity, or the multiple as substantive, the question of the relation between the predicates one/multiple is replaced by (e&eb) the question of distinguishing types of multiplicities (as with Bergson's distinction of qualitative and (er&eb) quantitative multiplicities in Time and Free Will). A typological difference between (e&eb)substantive multiplicities, in short, is substituted for (e&eb) the dialectical opposition of the one and the multiple. In sum, then, against the "major" post-Kantian tradition of Fichte, Schelling, and Hegel, Deleuze in effect posited his (el&eb) own "minor" post-Kantian trio of Maimon, Nietzsche, and Bergson. To these he added a trio of pre-Kantians, Spinoza, Leibniz and Hume, but read through a post-Kantian lens. We have already touched on Deleuze's reading of Hume. Let us now turn to Spinoza, for whom Deleuze's admiration was seemingly limitless; for Deleuze, Spinoza was the "prince" or even the "Christ" of philosophers. There are many Spinozist inheritances in Deleuze, but one of the most important is certainly the notion of univocity in ontology. Univocity—as opposed to its great rivals, equivocity and analogy—is (=) the key to developing (eb) a "philosophy of difference" (Deleuze's term for his project in Difference and Repetition), in which difference would no longer be (=) subordinated to identity. The result is a Spinozism minus (-MD) substance, (=) a purely modal or (=) differential universe. In univocity, as Deleuze reads Spinoza, the single sense of Being frees (e) a charge of difference throughout all that is. In univocal ontology being is said in a single sense of (e) all of which it is said, but it is said of difference itself. What is that difference? Difference is (=) difference in degrees of "power"; in interpreting this term we must distinguish the two French words puissance and pouvoir. In social terms, puissance is (=) immanent power, power to act (e&eb) rather than power to (e) dominate another; we could say that puissance is (=) praxis (in which equals clash or act together) rather than poiesis (in which others are matter to be formed by the command of a superior, a sense of transcendent power that matches (e&eb) what pouvoir indicates for Deleuze). In the most general terms Deleuze develops throughout his career, puissance is (=) the ability to affect and to be affected, to form (eb) assemblages or consistencies, that is, to form (eb) emergent unities that nonetheless respect the heterogeneity of their components. (Here we see the empiricist theme of the "externality of relations": in an assemblage or consistency, the "becoming" or relation of the terms attains (eb) its own independent ontological status. In Deleuze's favorite example, the wasp and orchid create (eb) a "becoming" or symbiotic emergent unit.). Although Deleuze wrote a touching and certainly important book in tribute to his friend Foucault after the latter's death in 1984, the final important figure in Deleuze's readings of other philosophers is Leibniz, to whom, it must be recalled, Maimon appealed in his criticism of Kant. In 1988, Deleuze published a book on Leibniz entitled The Fold: Leibniz and the Baroque, which added new elements to the reading of Leibniz that appeared in Deleuze's earlier books: an interpretation centered on the concept of the fold, a development of a concept of the Baroque, and a attempt to define a neo-Leibnizianism in terms of (e&eb) contemporary artistic and scientific practices. While The Fold is a fascinating work, we will concentrate here on Deleuze's early reading of Leibniz, which plays an important role in Difference and Repetition. Deleuze pushes Leibniz's thought to a point where Leibniz could never have taken it, given his (Leibniz's) theological presuppositions. This is the point where one begins to consider the virtual domain on its own account, freed from (e) its actualization in a world and its individuals. Never has the enterprise and industry of "demystification" been further to the horizon where a person stands on the threshold of infinity trying to ponder what lies beyond its viel which separates the seen from unseen. Myth is always the expression (e) false infinite and thé disturbance of spirit. One of the most profound constants of Naturalism is to denounce everything that is (=) sadness, everything that is (=) cause of sadness, and everything that needs (e) sadness to exercise its power. Naturalism makes of (e) thought and sensibility (=)an affirmation .Naturalism directs its attack against (e&eb) the prestige of the negative (negativity

is not bad and it has its own cognizance and prestige) Naturalism deprives (e) negative of all its power. Naturalism also disallows (e) negativity to speak in the name of philosophy which is unfair. Afterall, the spirit of negative made an appearance out of (e) the sensible and linked (e&eb) the intelligible to (e&eb) One or the Whole. But this one and One Whole as we have stressed many a time is that which cancels each projections, actions, cognitions, perceptions, concurrences, rotations, foldings, motions and shocks in the universe atleast holistically and is conservative. This bears ample testimony and infallible observatory and impeccable demonstration to the fact that One and the Whole is (=) nothingness of thought. On a similar veing appearance might also be (=) nothing ness of sensation. Naturalism is (=) the thought of an infinite sum all of the elements which are not composed at once. Naturalism is (=) also the sensation of finite compounds which are not (e) added up as such with one another. Thus multiplicity is (=) affirmed Multiple of multiple is (=) the object of affirmation .Diverse of diverse is (=) an object of joy. Notwithstanding the fact that gender is no longer widely considered to be a property of individuals, the alternative of viewing it in terms of (e&eb) performativity, where gender is (=) the outcome of linguistic and social performances, unnecessarily limits (e) the possibilities of thinking of gender as(=) a form of multiplicity that is both internally and externally differentiated. Any attempt to move beyond binary thinking in gender relations initiates (eb) a consideration of multiplicity, and the way in which multiplicity is conceptualized exerts a critical influence on (e&eb) the possibilities that are opened up. This article interrogates existing understandings of multiplicity and finds (eb) three actual or possible types - multiplicities of the same, characteristic of feminist approaches which authors critique through (e&eb) a reconceptualization of desire; multiplicities of (e) the third, characterized by anthropological, transgender and queer theory approaches; and multiplicities of difference and dispersion, typified by the rhizomatics and fluid theorizing of Deleuze and Guattari, Grosz and Olkowski. They propose ontology of gender as (=) a creative and productive form of desire, realized as (=) proliferation in Deleuze and Guattari's model of the rhizome. Gender identity is accordingly rethought as (=) immanence, intensity and consistency. **Gender as multiplicity: Desire, displacement, difference and dispersion Stephen Linstead Alison Pullen.** On this score, Deleuze often likes to cite Jorge Luis Borges's famous story, "The Garden of the Forking Paths," in which such a virtual world is described in (eb) the labyrinthine book of a Chinese philosopher named Ts'ui Pên: "In all fiction, when a man is faced with (e&eb) alternatives, he chooses one at the expense of (e) others. In the almost unfathomable Ts'ui Pên, he chooses—simultaneously—all of (e) them… In Ts'ui Pên's work, all the possible solutions occur, each one being the point of departure for (e) other bifurcations." Leibniz had in fact given a similar presentation of the world at the conclusion of the Theodicy. In Deleuze's transformation of the Leibnizian / Borgesian image, the three Kantian transcendent Ideas of God, World, and Self all take on (eb) a completely different demeanor. . First, God is no longer a Being who compares and chooses (e&eb) the richest compossible world; he has now become (eb) a pure Process that affirms incompossibilities and passes through (e&eb) them. (As the notion of "process" here attests, Deleuze's relation to Whitehead is one of the most important contemporary issues for students of his thought; although the points of comparison are many, Deleuze himself rarely discussed Whitehead, save for several important pages in The Fold.) Second, the world is no longer a continuous world defined by (e) its pre-established harmony; instead, divergences, bifurcations, and incompossibles must now be seen to belong to (e) one and the same universe, a chaotic universe in which divergent series trace (e&eb) endlessly bifurcating paths, and give rise to (eb) violent discords and dissonances that are never resolved into (eb) a harmonic tonality: a "chaosmos," as Deleuze puts it (borrowing a word from Joyce) and no longer a world. In contrast, Leibniz could only save the "harmony" of this world by relegating (e) discordances and disharmonies to other possible worlds—this was his theological sleight of hand. Compare this with the present parallel universe philosophy which has intearctionability as its main thematic and discursive form. Third, selves or individuals, rather than being closed upon the compossible and convergent world they express from within, are now torn open, and kept open through (e&eb)the divergent series and incompossible ensembles that continually pull them outside themselves. The "monadic" subject, as Deleuze puts it, becomes (=) the "nomadic" subject. Pluralism is always linked to (e&eb) multiple affirmation. The interrogation problem does not bear resemblance to (e) the proposition it subsumes under it; it rather engenders them as it determines (eb) its own conditions and assigns (eb) individual order of permutation of the propositions within (eb)

the framework of generalised significations and functionalized manifestations. Sensuality is connected with (e&eb) the joy of the diverse (one gentleman here says variety is the spice of life; so diversity is spice and savoury to be palatable). In other words, if Deleuze is Leibnizian, it is only by eliminating the idea of a God who chooses (e&eb) the best of all possible worlds, with (e&eb) its pre-established harmony and well-established selves; in Deleuze, incompossibilities and dissonances belong to (eb) one and the same world, the only world, our world. But they belong to our world as (=) its virtual register; developing the thought of the virtual is one of the great challenges of Deleuze's masterpiece, Difference and Repetition, to which we now turn. **The Philosophy of Difference Difference and Repetition:** Deleuze's historical monographs were, in a sense, preliminary sketches for the great canvas of Difference and Repetition (1968), which marshaled these resources from the history of philosophy in an ambitious project to construct a "philosophy of difference." Following Maimon's critique, Difference and Repetition produces a two-fold shift from the Kantian project of providing (eb) the universal and necessary conditions for (e) possible experience. First, rather than seeking the conditions for possible experience, Deleuze wants to provide an account of the genesis of (e) real experience, that is (=) the experience of this concretely existing individual here and now. Second, the genetic principle to respect the demands of the philosophy of difference must itself be (=) a differential principle. However, despite these departures, Deleuze maintains a crucial alignment with Kant; Difference and Repetition is (=) still a transcendental approach. Here we should remind ourselves that the terms "transcendent" and "transcendental" have (e) opposing significations. Transcendental philosophy in fact critiques (eb) the pretensions of other philosophies to transcend (e) experience by providing (eb) strict criteria for the use of (e) syntheses immanent to experience. On this score, at least, Deleuze aligns himself with Kant's critical philosophy. Three further preliminary notes are in order here. First, as we will discuss in section 4 below, the Capitalism and Schizophrenia project of Deleuze and Guattari will bring to the fore naturalist tendencies that are only implicitly present in the still-Kantian framework of Difference and Repetition. So, although there is some risk of reading backwards in this formulation, we can say that the "of" in the phrase "the experience of (e) this concretely existing individual here and now" is (=) both subjective and objective. It is the experience by (e) human subjects of this individual object in front of it, and it is (=) the experience enjoyed by (e) the concretely existing individual itself, even when that individual is (=) non-human or even non-living. (Deleuze's panpsychism is treated briefly in Protevi 2011.) Many Physicists' have taken objection to any "quantum information" being expended after life, for Einstein's Field Equations does not characterise such a situation, notwithstanding the fact that the equations provide travel backwards in time and the referential integrity and concomitant relational veracity of the equations are unquestionably true. (Please see earlier papers) It is very consistent to bring in to the incorporation the expositional quality Deleuzian remarks about the quantitative and qualitative aspects of life as is studied in LOS (page 310): Then suddenly there is a click. The subject breaks away (e) from the object divesting (e&eb) it of a part of its colour and substance. There is (=) a rift in the scheme of things and the whole range of objects crumbles in becoming (eb) me; each object transferring (e&eb)its quality to an appropriate subject. The light becomes (=) the eye, and such no longer exists; it is (=) simply the stimulation of the retina. Te smell becomes (eb) the nostril- and the world declares itself (eb) odorless. The song of the wind in the trees is (=) disavowed: it was (=) nothing but the quivering of the timpani……The subject is (=) the disqualified object. My eye is (=) the corpse of the light and color. My nose is (=) all that remains of odors when their unreality becomes manifested and demonstrated. My hand refutes (e) the thing it holds. Thus the problem of awareness is born of (e) anachronism. It implies (eb) the simultaneous existence of the subject and the object whose mysterious relationship to (E&eb) himself he seeks to define. Subject and object cannot (e) exist apart from one another since they are one and the same at first integrated (e&eb) in to the real world and then cast out by (e) it. (I have written from small note made by me years ago; there might be some variations. Kindly pardon me on that count. Essence however remains the same). Second, then, in the demand for genetic principles to account for the real experience of (e) concrete individuals, Deleuze is working in the tradition of the Principle of Sufficient Reason. Third, the notion of "genesis" is (=) itself double; in Chapter 3, Deleuze lays out a dynamic genesis that moves from an encounter with intensity in sensation to (e&eb) the thinking of virtual Ideas, while Chapters 4 and 5 lay out a static genesis that moves from the virtual Idea through (e&eb) an

intensive individuation process to (e&eb) an actual entity. We are now ready to discuss the book itself. Murphy 1992 suggests that the first part of the book (the Introduction and Chapters 1 and 2) constitutes Deleuze's treatment of the history of philosophy, while in the second part of the book (Chapters 4 and 5) Deleuze is doing philosophy in his own name. From this point of view, Chapter 3, on the "image of thought," plays a pivotal role, leading us into Deleuze's own philosophy. This transitional role of Chapter 3 is confirmed elsewhere when Deleuze says that the study of the image of thought is (=) the "prolegomena to philosophy" (Negotiations, 149). In Chapters 1 and 2, to find a differential genetic principle, Deleuze works through the history of philosophy to isolate the concepts of "difference in itself" and (e&eb) "repetition for itself" that the assumptions of previous philosophies had prevented from (e) being formulated. "Difference in itself" is (=) difference that is freed from (e) identities seen as metaphysically primary. Normally, difference is conceived of as (=) an empirical relation between two terms which each has (e) a prior identity of its own ("x is different from y"). Deleuze inverts this priority: identity persists, but identity is now a something produced by (e) a prior relation between (e&eb) differentials (dx rather than not-x). Difference is no longer an empirical relation but becomes (=) a transcendental principle that constitutes (e&eb) the sufficient reason of (e) empirical diversity (for example, it is the difference of electrical potential between cloud and ground that constitutes (eb) the sufficient reason of the phenomenon of lightning). In Chapter 2, the concept of "repetition for itself" is produced as (=) repetition that is freed from (e) being repetition of an original self-identical thing so that it can be (=) the repetition of difference. Following the formula of Deleuze's reading of Nietzsche's eternal return, repetition is (=) the return of the differential genetic condition of (e) real experience each time there is (=) an individuation of a concrete entity. Ultimately, then, Difference and Repetition will show (eb) that the individuation of entities is produced by (e) the actualization, integration, or resolution (the terms are synonymous for Deleuze) of a differentiated virtual field of (e) Ideas or "multiplicities" that are themselves changed, via (e&eb) "counter-effectuation," in (eb) each individuating event. Chapter 3 lays out 8 postulates of the "dogmatic image of thought." Between the first four and last four postulates we find a theory of the faculties, which is thus at the crossroads of both the chapter and the book. Let us take up the first four postulates. The first postulate concerns our supposed natural disposition to (e) think; the denial of this (natural disposition to think) is (=) what necessitates our being forced to think. The second and third postulates concern subjective and (e&eb) objective unity. Subjective unity is captured by (e) the notion of "common sense" such that our faculties of sensation, memory, imagination, and thought work in harmony, while objective unity is captured by (e) the notion of "recognition" such that it is the same object that is sensed, remembered, imagined, and thought. The fourth postulate concerns "representation", a key target of Deleuze's critique. Here difference is submitted to a fourfold structure that renders (eb) difference subordinate to (e) identity: 1) identity in (eb) the concept; 2) opposition of (e) predicates; 3) analogy in (eb) judgment; and 4) resemblance in (eb) perception. A good way to approach Deleuze's notion of representation is via (e&eb) Aristotle and Porphyry. Specific differences are the opposed predicates that function on (eb) a horizon of identity in the concept under division; thus animal is the genus that is divided into rational and irrational as specific differences that enable (eb) the isolation of the species "human." Then, we find that the difference between individuals of the same species is (=) infra-conceptual and can only be made via (e&eb) the perception of resemblances; Theaetetus looks like Socrates but not so much that they cannot be distinguished. Finally, the relation of substance to (e&eb) the other categories is (=) analogical, such that being is (=) said in many ways, but with substance as the primary way in (eb) which it is said. After the first four postulates, we find the theory of the faculties, which will be Deleuze's account of what it means to be (=) "forced" to think in differential rather than identitarian terms. To free a notion of "difference in itself" such that difference need not be thought on the basis of (e) a prior horizon of identity, Deleuze looks for an "encounter," a sensation that cannot (e) be thought, that cannot find (eb) the empirical category under which an object can be recognized, and thus forces (eb) the "transcendent exercise" of the faculty of sensibility, when something can only be sensed. Here we see the dynamic genesis from intensity in sensation to (e&eb) the thinking of virtual Ideas. Each step here has (e) a distinct Kantian echo. The faculties are (=) linked in order; here Deleuze as well as Kant looks to the privilege of sensibility as (=) the origin of knowledge—the "truth of empiricism." With sensibility, pure difference in intensity is grasped immediately in the

encounter as (=) thé sentiendum, that which can only be sensed. In the differential theory of the faculties, sensibility, imagination, memory, and thought all "communicate violence" from one to (e&eb) the other—here Deleuze works with the Kantian notion of the sublime as (=) discordant accord of the faculties. The "free form of difference" in intensity moves each faculty and communicates its violence to (e&eb) the next, though in this case there is no supernatural vocation that will redeem (e&eb) the conflict of imagination and reason, as there is in the resolution (e) to the discussion of the sublime in the Critique of Judgment. Rather than a reconciliation of the faculties, with (e&eb) thought, a "fractured self"—here Deleuze takes up Kant's notion of the split (e&eb) between the empirical ego and the transcendental subject—is constrained (e) to think "difference in itself" in Ideas. For now, let us note that two of Deleuze's technical terms, intensity and (e&eb) virtuality, occupy two different places on (eb) this line of dynamic genesis. Intensity is (=) the characteristic of the encounter, and sets off (eb) the process of thinking, while virtuality is (=) the characteristic of the Idea. Extensive differences, such as length, area or volume, are (=) intrinsically divisible. A volume of matter divided into two equal halves produces (eb) two volumes, each having half the extent of the original one. Intensive differences, by contrast, refer to (e) properties such as temperature or pressure that cannot (e) be so divided. Recently scientists have been able to separate the characterstics from the object and this has been a thoroughbred break in quantum mechanics. In this connection, advert to the definition of "error' by Deleuze which we discuss elsewhere in the following. If a volume of water whose temperature is 90º is divided in half, the result is (=) two volumes at the original temperature, not two volumes at 45º. However, the important property of intensity is not that it is (=) indivisible, but that it is (=) a property that cannot (e) be divided without (e) involving a change in kind. The temperature of a volume of water, for instance, can be "divided" by (e) heating the container from below, causing (eb) a temperature difference between the top and the bottom. In so doing, however, we change (e&eb) the system qualitatively; moreover, if the temperature differences reach a certain threshold (if they attain a certain "intensity" in Deleuze's terms), the system will undergo (e&eb) a "phase transition," losing (e) symmetry and changing (e&eb) its dynamics, entering into (eb) a periodic pattern of motion—convection—which displays (eb) extensive properties of size: X centimeters of length and breadth. Drawing on these kinds of analyses, Deleuze will assign (eb) a **transcendental status** to the intensive**: intensity,** he argues, constitutes (e) the genetic condition of (e) extensive space. Intensive processes are themselves in turn structured by (e&eb) Ideas or multiplicities. An Idea or multiplicity is (=) really a process of progressive determination of differential elements, differential relations, and singularities. Let us take these step-by-step. "Elements" must have no independent existence from (e) the system in which they inhere; phonemes as the elements of the virtual linguistic Idea are (=) an example Deleuze uses in Difference and Repetition. Here it might be said the conjecture is too disconcerting for Deleuze seems to contradict himself from the propositions of logical genesis and ontological genesis as he had so assiduously alluded to before. When phonemes are actualized they enter into differential relations that determine (eb) the patterns of individual languages; thus the English phoneme /p/ is reciprocally determined by (e) its differences from /t/, /b/, /d/, and so on. Finally, these differential relations of an individual language determine (eb) singularities or remarkable points at which the pattern of that language can shift: the Great Vowel Shift of Middle English being an example, or more prosaically, dialect pronunciation shifts. Deleuze also holds that events have such points of inflexion, boiling points et al., For another example—and here, in the applicability of his schema to (e&eb) widely divergent registers, is one of the aspects of Deleuze as metaphysician—let us try to construct the Idea of hurricanes. The differential elements would be material "flows" driven by (e) intensive differences in temperature and pressure but undetermined in form (neither smooth nor turbulent, neither big nor small) and function (neither forming nor destroying of (e) weather events). A suggestion can be made to create indeterminate spaces and modules in respect of such variables like 0/0; infinity/ infinity…….It is not as if such existence of indeterminate constituents are not palatable. An example is that of violation of general relativity at remotest parts of the universe as in case of a bill in transit that remains unsecure in its obligation. These flows qua differential elements enter into relations of reciprocal determination linking changes in any one element to (e&eb) changes in the others; Thus temperature and pressure differences will link changes in air and (e&eb) water currents to each other. Updrafts are related to (e&eb) downdrafts even if the exact relations

(the tightness of the links (e&eb) the velocity of the flows) are not yet determined. Finally, at singular points in these relations singularities are determined that mark (eb) qualitative shifts in the system, such as the formation of thunderstorm cells (e&eb) the eye wall, and so on. But this is still the virtual Idea of hurricanes; real existent hurricanes will have (e) measurable values of these variables so that we can move from (e) the philosophical realm of sufficient reason to that of scientific causation. A hurricane is explained by (e) it's Idea, but it is caused by (e) real wind currents driven by (e) real temperature supplied by (e) the sun to (e&eb) tropical waters.. To see how Ideas are transcendental and immanent, we have to appreciate that an Idea is (=) a concrete universal. There is a statement in Bhagavat Geeta that creation is attributed and ascribed to "Yoga maya" of Brahman itself. In fact as had been stated earlier both Brahman and Anti Brahman form two sides of the same coin what with Vishwaroopa of Vishnu standing out for both benediction and belligerence. Advert various papers on the indeterminate and latent thought and manifest action (cf Merton) in various papers on Holographic principle, where in it is clearly stated that the information at the boundary is what is being produced at the universe virtually like a film. Let us for the present state that it is the mind that projects this Universal Mind's thought on the screen of consciousness of individual which is mistaken for objective reality. By Universal Mind we mean cosmic consciousness or Nature's General ledger which has been proposed and defined many a time earlier in the deliberation on various topics like Holographic principle, Quantum Holography, and its concomitance to string theory. This aspect is further discussed and expatiated upon in further papers exclusively. In an early article on Bergson ("The Conception of Difference in Bergson" [1956]), Deleuze gave a particularly helpful example of this notion. In La Pensée ET le Mouvant, Bergson had shown that there are two ways of determining (eb) what the spectrum of "colors" has in (eb) common. (1) You can extract from (e) particular colors an abstract and general idea of color ("by removing from the red that which makes (eb) it red, from the blue what makes it blue, from the green what makes it (eb) green"). Or, (2) you can make all these colors "pass through (e&eb) a convergent lens, bringing them to a single point," in which case a "pure white light" is obtained that "makes the (eb) differences between the shades stand out." Please see earlier papers colours are befuddling you" The former case defines a single generic "concept" with (e&eb) a plurality of objects The relation between concept and (e&eb) object is (=) one of subsumption; and the state of difference remains exterior to (e) the thing. Here Deleuze makes a profound statement. It is virtual statement that concept itself is object and both cannot coexist like subject and object. Startling both in its thematic and discursive form genius of Deleuze finds its pristine glory in Eastern primordial thought. The second case (see item numbered 156) on the contrary, defines (eb) a differential Idea in (eb) the Deleuzean sense: the different colors are no longer objects under (e&eb) a concept, but constitute (e) an order of mixture in coexistence and succession within (eb) the Idea; the relation between the Idea and (e&eb) a given color is (=) not one of subsumption, but one of (e) actualization and differenciation (it si the actualisation and manifestation in space time of the idea that is what we spoke as hologram); and the state of difference between the concept and the object is (=) internalized in (eb) the Idea itself, so that the concept itself has become (=) the object. White light is still a universal, but it is (=) a concrete universal, and not a genus or generality. Verily, this is what is enunciated as holographic principle and "Yoga Maya" of the Brahman-Anti Brahman co creation. Problem in itself is (=) the reality of the genetic element, the complex theme which does not allow itself to be reduced to (eb) propositional thesis and basis. It is one and the same illusion which from the empirical point of view formulates (eb) the problems from (e) the propositions, which function as (=) answers and which from a philosophical point of view and a scientific point of view , defines (eb) the problem through (e&eb) the form of possibility of (e) the corresponding propositions. As long as we define the problem by its possibility of resolvability (eb) we tend to confuse (e&eb) sense with (e&eb) signification and we conceive (eb) condition in (eb) the image of the conditioned. Domains of resolvability are related to (e&eb) the process of self determination of the problem. The series of the problem with the (e&eb) conditions constitute (eb) something unconditioned, determining at once, the conditions of the problem perse, and the domains of resolvability and the solutions present in the (eb) domain. Problem bears no resemblance to the neither to the propositions it subsumes (e) under it, nor to the relation it engenders in the proposition. Problem is (=) not propositional; although it does not exist outside of (e) the propositions which express it. Husserl cannot be followed blindly by the statement that an expression is (=)

mere double and necessarily has the same thesis as that which receives it. The Idea of color is (=) thus like white light, which "perplexes" within (e&eb) itself the genetic elements and relations of all the colors, but which is (=) actualized in the diverse colors and their respective spaces. Like the word "problem," Deleuze uses the word "perplexion" to signify (eb) not a coefficient of doubt, hesitation, or astonishment, but (eb) the multiple and virtual state of Ideas. Indeed, Deleuze adopts a number of Neoplatonic notions to indicate (eb) the structure of Ideas, all of which are derived from (e) the root word pli [fold]: perplication, complication, implication, explication, and replication. Interrogation is (=) the shadow of the problem projected, or rater reconstructed based on (e) empirical propositions. Similarly, the Idea of sound could be conceived of as (=) a white noise, just as there is (=) also a white society or a white language, which contains (e) in its virtuality all the phonemes and relations destined to be actualized in (eb) the diverse languages and in (eb) the remarkable parts of a same language. Only a genius with profound thought beyond his space and time and period could have written what is written. Words are more often used to violate the very meaning of sentences which they try to exemplify and presumptuousness forms the bastion and stylobate of many philosophers. Deleuze being a contemporary thinker stands above all in his own right. Constantin Boundas refers to this century as "Deleuzean Century;" which is neither garish nor ostentatious. Attention drawn to remarks on the Difference which is really poised for natural history of philosophy **Deleuze's Readings of Other Philosophers I have added my comments and commentaries of other researchers including that of Deleuze. Where ever the comments are drawn from other sources, attributions are mentioned.** The yogi rises to the level of contemplation when the awareness he has of himself and the things around him become (=) one and he realises his own identity with (e&eb) Siva, the sole reality. This is a clear cut statement of Aham Brahmasmi, which is achieved by continuous evolution of consciousness. It is here individual consciousness become cosmic consciousness. The aim of this Yoga in all its phases is to achieve (e&eb) the Fourth State of consciousness (turiya) beyond the three states of waking, dreaming and deep sleep and to then ultimately reach the liberated state Beyond the Fourth (turiyatita). These five states correspond to: (a) Siva's activity (vyapara), that is, His power of action; (b) Siva's Lordship (adhipatya), which is His power of knowledge; (c) the absence of these two, which corresponds to Siva's power of will; (d) His exertion (prerakatva), which contains (e) all the cycles of creation and destruction and, (e) the rest Siva enjoys in His own nature, which is His power of consciousness. 296 The first three states, when divorced from the last two, belong to (e&eb) the sphere of transmigratory existence. The Fourth and Beyond the Fourth on the other hand are (=) higher, supramundane (alaukika) states of consciousness in which (eb) the yogi enjoys bliss and repose (**visranti**) in his own nature by penetrating (samavesa) into (e&eb)_ the universal consciousness of the Self, through (e&eb) which he ultimately becomes (eb) liberated (jivanmukta). We had earlier defined **"Self"** as witness consciousness and or dynamic consciousness in the evolved or evolutionary soul. Indriyateeta anubhava is what beyond sense organs experience. Note that the author does not classify the consciousness states and mentions only cosmic consciousness the Nature's General Ledger. Beyond the Fourth is (=) the state of awareness ParamaSiva Himself enjoys when (e) duality has entirely disappeared and everything is realised to be one with (e&eb) consciousness. It is clear that the author is talking about cosmic consciousness. The Fourth is the state of awareness of the yogi who, catching hold of the pure subjectivity (upalabdhrta) flowing through (e&eb) the lower three states, is still actively eliminating (e) his sense of duality. While the former is (=) the supreme subject as T consciousness (aham), the latter is (=) the pure awareness (prama) or 4 I-nesses (ahanta) of (e) the subject which encompasses the lower states, giving them (eb) life and uniting them (e&eb) together. 297 As such, the Fourth State is (=) the reflective awareness of one's own nature shining in (eb) all three states at one with them. 298 The fact that we recall that we slept well is proof (eb) that this state of consciousness persists even in deep sleep. Indeed, if the flow of Turiya could somehow be (=) brought to a halt, all the other states of consciousness would come to an end in (eb) the absence of the pure subjectivity which makes them, and their contents, manifest. 299 The states of waking, dreaming and deep sleep correspond to (e&eb) the form of awareness consciousness assumes when (e) it predominantly manifest as (=) the object, means of knowledge and individual subject, respectively. Turiya is the pure awareness (prama) that both transcends (e) them and merges them all into (e&eb) itself. 300 As such, it appears as the triad of deed, means and agent in the pure act (vyapara) of (e) consciousness

unsullied by (e) any outer reality. 301 Abhinava explains: {Turiya) transcends (e) the three aspects of *form\ 'sight' and T consisting as it does of (e) the pure act of 'seeing'; therefore any means [by which this state could be realised] has (e) merely a [provisional] instrumental value. It is, in other words, pure subjectivity of (e) the nature of absolute freedom, independent of (e) all external means. This is the state of consciousness called Turiya, luminous with (e&eb) its own light. Turiya is thus **not just a psychological state** but the supreme creative power (para sakti) of (e) consciousness, the Goddess (samviddevi) who generates and withdraws the (e&eb) entire universe of subject, object and means of knowledge. In the Heart of Recognition Ksemaraja explains: Whenever the extroverted [conscious] nature rests within (eb) itself, external objectivity is (=) withdrawn and consciousness is established in (eb) the inner abode of peace which threads through (e&eb) the flux of awareness in (eb) every [externally] emanated [state]. Thus **Turiya**, the Goddess of Consciousness, is the union of creation (e&eb) persistence and (e&eb) destruction. She emanates (eb) every individual [cycle] of (e) creation and withdraws (e) it. Eternally full [of (e) all things] and [yet] void [of (e) diversity] She is (=) both and yet neither, shining radiantly as (=) **non-successive** [consciousness] alone. 303 The yogi is fully absorbed in this state of consciousness and takes possession of (e) its power when he is able to rise from (e) contemplation (samadhi) carrying with (e&eb) him the abiding awareness of Turiya throughout (e&eb) his waking, dreaming and deep sleep. When he achieves this constantly, he continues to (e) experience these states individually, but they no longer obscure (e) the insight (pratibha) he has acquired because (e) he realises that they are all aspects of the bliss of Turiya. Thus, while the common man calls this state the 'Fourth' (turiya) because (e) he cannot experience it directly and knows only that it is beyond the other three, the yogi calls it 'Beyond (e) Form' **(rupatita)** because it transcends (e) the detachment of the state of deep sleep which, devoid of (e) objective content, is (=) the naked form' of the individual subject tending towards (e&eb) the fullness of consciousness. Those who are on the path of knowledge (jnaniri) call it the 'Whole* **(pracaya)** because (e) in this state, they see the entire universe gathered together in (e&eb) one place. 304 This is something like Nature's General Ledger 'Supra-mental Awareness' (**manonmana**) is (=) the name given to the experience of Turiya in (eb) the waking state. The yogi in this state moves and lives (e&eb) in the world of waking experience free of (e) all disturbing thoughts while abiding in (eb) the transcendental silence beyond (e) the activities of the mind. 'Infinite' is the name of the experience of Turiya while dreaming because (e) free of the limitations imposed upon (e&eb) the body by time and space, the yogi enjoys (e&eb) the unlimited expanse of the Self. Mention here is the attachment of the self with infinite expanse of nature beyond space and time.When Turiya is experienced in deep sleep, the yogi's state is called 'All things' (sarvdrtha) because in it he discovers his freedom from limitations in this, the most contracted state of human consciousness. The yogi who manages to maintain Turiya- consciousness comes to (e) experience the three states of waking, dreaming and deep sleep as the constant flow of the bliss of consciousness in which all traces of the relative distinction between these states and their contents is eradicated. 305 Following the stream of Turiya to its highest level (para katfha), he reaches the state Beyond the Fourth (turiydtita), which is the universal consciousness (caitanya) of the Self. Here the yogi comes to rest within his own nature. Plunged in the vast, waveless ocean of the consciousness and bliss (ciddnanda) of the state Beyond the Fourth, the yogi becomes Siva, 306 the Free One (svacchanda), and thus wanders freely, practising the Yoga of Freedom. 307 K§emaraja equates the Fourth State with the pure (suddha), innate (sahaja) knowledge that one's own conscious nature is all things. It is the Supra-mental State (unmana) in which Siva's pervasive presence is experienced 308 once the Yoga practised at the Individual level attains fruition at the Empowered. 309 What the yogi must do, once consciousness is elevated to grasp the Fourth State, is make it constant. He must forcefully lay hold of it within himself and not release his grip until it becomes permanent. Then he travels 'Beyond the Fourth' to enlighten- ment. 310 Before this ultimate attainment the yogi inevitably falls. The forces operating within consciousness that limit and obscure it throw him down whenever they possibly can. The only way the yogi can defend himself against them is to maintain a constant attentive awareness of the Fourth State. 311 He falls when (e) he is distracted but when he attends carefully to his pure conscious nature, he realises that every aspect of his state of being, including (e) the forces that lead him astray, are one with (e&eb) the pulsing flux of his own consciousness and so cannot affect him. These powers, which are the energies of Matrka we have already discussed, are not the only obstacles the yogi must overcome. He must, for

example, also resist the temptation to rest content with the miraculous yogic powers (siddhi) he acquires in the course of his spiritual develop- ment. Again to do this he must practice Yoga. Similarly, in order to pervade the Fourth State gradually through the other states in the manner proper to practice at the Individual level, the method is the same. He must practise the higher yoga of the Tantras which, turning his mind inwards and freeing it from discursive representations, allows him to penetrate into the Supreme Principle. 312 Once the yogi has attained this contemplative state, his main problem is to make it permanent. In the introverted state the gross external movement of the breath is suspended and with it the activity of the intellect, mind, individualised consciousness, powers of the senses and the ego. 313 When the yogi rises out of this state, he is liable to fall again into the lower order of creation generated by Maya if he does not maintain his awareness of the higher reality he has experienced and allows his awakened, illumined insight to be obscured by the dream-like vision of thought-constructs. 314 Naturally, the yogi must rise out of the introverted condition of suspension. It is inherent in the very nature of reality that it should move out of itself. 315 Pure, universal consciousness initially transforms itself into the vital breath 316 charged with the impression iyasana) of the power of awareness attained through introversion. By attending to the pulse (spanda) of the breath as it moves out of the absolute, the yogi can develop an intuitive sense of the inherent unity of all he will perceive in the mental and physical spheres created by the outpouring of consciousness. In this way he realises that his own nature is everywhere present in all he perceives and that all things thus reside within him. Blessed with this insight his consciousness remains free and unlimited even at the individual level where the breath, mind, senses and body are active. If the yogi fails to do this, he finds himself once again beset by the strictures of his embodied existence and must, as before, try to pervade all his other states of consciousness with the aesthetic delight (rasa) and wonder (camatkara) of the Fourth State he experienced in contemplation. Again this means that he must strengthen his pure, empowered awareness that his universal nature manifests as all things. 317 In this way he discovers Siva's presence in every sphere of individualised consciousness ranging from the breath to outer objectivity. The yogi's mind then becomes tranquil and undistracted because wherever it may wander, the yogi perceives only Siva, his authentic nature. 318 Consciousness is thus freed of all external referents and the yogi's subjectivity is purified of all identification with the body or anything else that belongs to the objective sphere. The yogi then becomes detached from the opposites of pleasure and pain and is transcendentally free (kevalin). 319 yogis are again, however, liable to fall if he allows himself to get entangled in the play of opposites. This fall is more serious than the others because, although he is caught by the confining restrictions of individual- ised consciousness as before, he is now also affected by karma. Fleeing from pain in the pursuit of pleasure he is bound to act (karma) to minimise one and maximise the other and so is thrown down to the lowest level of embodied subjectivity (sakala). In order to regain his lost state, he must ascend gradually, by Siva's grace, from one order of subjectivity to the next and so free himself progressively of the limitations of the lower levels to gain the greater freedom and expansion of the higher. As he progresses, the objective sphere also evolves from the grossest perceptions of physical objects outside the lowest order to subjectivity, through to the subtler inner, mental perceptions to finally reach the order of subjectivity that contains objectivity within itself and is free to externalise it at will. 320 The degree to which this process develops depends, as before, on the yogi's awareness of the Fourth State. In consonance with the general principle that the remedy should suit the defect, the yogi is instructed to seek this higher state of consciousness in the wonder (camatkara) or delight (ananda) he feels in moments of intense physical pleasure. At first he experiences this subtle consciousness for an instant in the subjective sphere. If he manages to catch hold of it, it becomes more intense as the cognitive and objective spheres are also gradually pervaded and vitalised by it. Occasions for this practice are, for example, the sense of satisfaction one feels after a good meal or the aesthetic delight one experiences when listening to good music or the pleasure of sexual union with the Tantric consort or even solitary sexual excitation. In these moments of delight the yogi can penetrate momentarily into his own authentic Siva-nature (sambhavavesa) through the empowered contact (saktasparsa) m he makes with it in the freedom of the pure subjectivity of the Fourth State. 322 If the yogi develops his awareness of this higher level of consciousness and maintains it, he eventually experiences it constantly. 323 Clearly, what prevents the yogi from attending to his state of consciousness rather than the circumstances which induce it is the craving for pleasure (abhila§a) born of ignorance — the source

of every impurity which clouds consciousness. Craving directs the yogi's attention towards outer, worldly things and so he is caught in the net of thought- constructs. 324 To free himself of his worldly desires and reverse this binding extroversion, the yogi must eradicate its cause. To be freed of all the ups and downs of the path and no longer be tormented by the possibility of a fall, the yogi must see reality perfectly and completely. This insight is itself liberation and the moment it dawns the yogi is instantly freed. This sudden realisation is the goal of Tantric Yoga. E Tantra declares: "He who perceives reality directly, even for the brief moment it takes to blink, is liberated that very instant and never reborn again." 325 Although the yogi's body and mind continue to function as before, they are like mere outer coverings 326 which contain, but do not obscure, the mighty, universal consciousness which operates through them. The yogi's body is (=) the universe, the senses (=) the energies that (eb, eb+)vitalise it, his mind (=) Mantra, the rhythm of his breath (eb) the pulse of time and his inner nature pure, dynamic consciousness. Raised above all practice, and hence all possibility of falling to lower levels, the yogi realises that he has always been free 327 and that his journey through (e&eb) the **dark land of Maya was nothing but a dream, a construct of his own imagination** The channels of the senses by which the yogi enjoys the pleasure of the objects of sense are those through which he fills the three worlds with the consciousness emitted from his own Heart. 78 **The Doctrine of Vibration T**he senses are outer, subsidiary cycles of conscious energy (anucakra), revolving around the core or primary cycle (mukhyacakra) of pure consciousness. They scintillate with the diversity of sensations and resonate in the field of their awareness like the feathers of a peacock unfurled and folded back in the ecstasy of its dance. 79 Ultimately it is Siva Himself Who, for His own pleasure, 80 extends and withdraws them, 81 not the individual soul. The outward movement of the senses is the creative flow of consciousness from the subject who, like an ocean, is the source of the waves of their activity and the final resting place of the flow of the rivers of sensations and thought-forms that unfold through the arising of the power of each of the senses. 82 The Stanzas on Vibration declare: Indeed the individual soul does not activate the impulse of the will [which directs the body's activity] by himself alone but through his contact with [his] own [inner] strength {bald) made in such a way that he identifies with it [thus acquiring its power]. 83 The close contact between the senses and universal consciousness vitalises them and sets them in motion. They share in the properties of consciousness, just as a heated iron ball can burn, cook or give out light. 84 Spanda doctrines does not agree with the Sarnkhya view that the individual soul (purusa) alone stimulates activity in the objective sphere (prakrti) by the contact he makes with it. The soul can do this only because universal consciousness animates him. Moreover, consciousness, which has the power to impel this activity, cannot itself be inactive as the Sarnkhya maintains. The influence it has on lifeless matter implies an active exertion. 85 This activity is the subtle, inner outpouring of the Self (svatmocchalatta) within the Self. Outside the confines of time and space, in the centre of the circle of the senses, this inner activity initiates the expansion (prasara) of the flow of conscious energy. As the pure exertion (udyoga, udyama) of Being (bhava), it is the source of the initial unfolding (unmesa) of Becoming. 86 Ksemaraja explains: [By] 'exertion' (udyama) is [meant] the emergence of the supreme [level of] intuition (pratibha) which is the sudden outpouring of the [immanent and transcendental] unfolding of the reflective awareness of pure consciousness. 87 The direct, intuition (pratibha) of reality, free of thought-constructs, is the supreme power (parasakti) of consciousness. The yogi can gain Divine **Body and Sacred Circle of the Senses 153** this pure intuitive awareness by grasping the initial expansion (unmesa) of consciousness which engenders the emergence of thought and the activity of the senses. It can be discovered in the subtle Centre, between one thought and the next. The Stanzas on Vibration teach: The expansion of consciousness that takes place when one is engaged in a single thought should be known to be the source from whence another arises. One should experience that for oneself! 88 This does not mean that the unfolding expansion of consciousness {unmesa) is an intermittent, transitory occurrence. 89 As the source of all thought, including the notions of past, present and future, 90 it stands outside the confines of time. Consciousness is in a state of perpetual expansion. It is an ever developing process, a tiny part of which we observe as the flow of events around us and the thoughts and feelings within us. The totality of this process can never be grasped objectively. Objectivity is a state of fragmentation (bheda) in which discrete elements are discernible and mutually distinguished as individual, specifically definable entities. It is the realm within which thought and language operate. To grasp reality in its completeness, we must go beyond the

partial representations of thought and speech. To experience the primordial source and basis of all things, we must pierce through the outer periphery of thought and plunge into the Centre, to discover the instant in which thought, and with it, the sense of diversity, initially emerges. Ksemaraja cites a practice from the Vijnanabhairava through which the yogi can achieve this state of awareness: Checking [the movement of] attention (cit) once it has quit its object, it must not [be allowed] to move to any other; contemplation (bhavana) then blossoms forth by [experiencing] the state in the centre between them (tanmadhyabhava). 91 Left unattended and undisciplined, awareness spontaneously shifts from one object of thought to (e&eb) the next. The transition from one to the other entails a movement through (e&eb) a state of pure indeterminate (nirvikalpa) awareness divested of all thought-forms and perceptions. The yogi who succeeds in checking the movement of his attention, experiences within himself the subsidence of thought into the emptiness of the Centre which unfolds full of the powerful pulsation of consciousness within which all objectivity merges and becomes one 92 in the trans- cendental outpouring (lokottarollasa) of its aesthetic rapture. 93 turning in on himself, the yogi must exert his power of awareness to (e) discern the vitality of (e) his own blissful Spanda nature. Just as a distant object becomes (=) more clearly visible by (e) exerting an effort to (e) perceive it, in the same way, through (e&eb) practice, the yogi's own Spanda nature becomes (=) progressively more evident to him, 94 The effort the yogi exerts to free himself of ignorance is essentially one with the force of awareness that impels the senses and mind. This inner exertion is the basis of every means to liberation. Ultimately bestowing the bliss of self- realisation, it carries the yogi along the path to salvation with all its twists and turns like wind which blows a cloth in various directions. 95 As the basis and ultimate goal of all yogic practice, this exertion is the pure wonder (camatkdra) inspired by the realisation of the essential identity of consciousness (cit) and being (sat). 96 As such it is the divine radiance (sphuratta) of the Heart of one's own consciousness 97 where the powers of will, knowledge and action merge in the harmony (samarasya) of bliss. 98 Thus the highest form of worship (varivasya) is the contemplation of the inner strength (bald) or universal exertion of one's own true nature. 99 "One should examine [that reality]," says Ksemaraja, "with the exertion which is Bhairava's nature and is true service (sevana) to the integral inner nature [of all things] and the unfolding of the activity of one's own vitality (ojas) which withdraws all duality [back into consciousness]." 100 Mindful of the eternal joy (ahlada) of his inner strength, the yogi must give up all desire for the fleeting moments of petty pleasure (ksanikasukhalava) ]0] he may glean from outer objects. He must even give up the desire to achieve enlightenment and a reality (tattva) which can never be realised through a personal act of will. 102 In short, he must die to himself in an act of profound faith (sraddha) and adoration (adara) to find rest in the supreme reality free from the binding dichotomy of means and objectives (upeyopayabhava) — the reality of his perfected effort. The yogi, now full of devotion for this reality unfolding within himself and the universe (antarmukhatattva), merges into it and, by achieving the pure consciousness and bliss (cidananda) of the power (bala) of his own nature, achieves oneness. 103 The yogi is firmly established in his true nature by recognising the creative unfolding (unmesa) of consciousness through which he is able to know all things, whether in the present, past or future. 104 Master of the vitality which gives life to his mind and body, he can strengthenit directly without need of food or drink, 105 or quit it at will to vivify Divine Body and Sacred Circle of the Senses 155 that of another. 106 Experiencing the eternal satisfaction of his own uncreated nature, 107 the yogi is freed of disease and the infirmities of old age. 108 Inspired by this vitality, he has the inner strength to overcome the severest hardships and obstacles in the way of his ultimate goal. 109 The Stanzas on Vibration explain: Lassitude ravages the body and this arises from ignorance but if this is eliminated by an expansion of consciousness (unmesa), how can [ignorance], deprived of its cause, continue to exist? 110 Doubt (sanka) is the source of every spiritual ailment. 111 A man in doubt about his true nature, and the path to follow by which he may come to realise it, is constantly overcome by difficulties. 112 "Due to ignorance" teaches the Sarvacaratantra, "the deluded man is in doubt and thus [suffers the cycle] of birth and death." 113 Caught between conflicting alternatives, the mind shifts from one to the other becoming more entangled in its own thoughts as it does so. Doubt thus contracts consciousness 114 sullying it with the turbulence of (e) confused thinking. 115 At once both the root of "the ancient tree of transmigration" and the first sprout of its seed, 116 doubts deprives man of the innate bliss (sahajananda) of his own nature. 117 Like a thief it steals away the wealth of true knowledge and

reduces man to an imagined state of poverty in which he feels hemmed in on all sides by (e&eb) constraints and limitations. 118 Ignorance and its sister, doubt, are the essence of (e) all impurities (mala) which sully consciousness. To the degree in which the yogi is cured of this ailment, namely, the lassitude (glani) of doubt, his true nature becomes (=) manifest, just as raw gold when heated is (=) freed from dross. When the yogi recognises (e&eb) the all-powerful expansion (unmesa) of his consciousness, he is projected beyond the realm of relative distinctions between virtue and vice. Thus overcoming all doubt as to what he should or should not do, he penetrates into (e&eb) the pulsing Heart of Bhairava's consciousness. 119 As the Stanzas on Vibration say: An individual who [though] desirous of doing various things [but] is incapable of doing them due to (e) his innate impurity, [experiences] the Supreme state when (e) the disruption (ksobha) [of his ego] ceases. Then [the soul realises] that his [true] uncreated nature is [universal] agency and perceiving subjectivity and so he knows and does whatever [he] desires. 120 the individual soul is(=) a combination of (e) conscious (cit) and 156 The Doctrine of Vibration unconscious (acii) elements 121 brought together when he allows the power of his authentic nature to be obscured by (e) his own impurity. This impurity is (=) three-fold. The first is technically called (=) the 'Impurity of Individuality' (anavamald). Due to (e) this impurity, the individual soul fails to (e) recognise his all-embracing fullness (purnata) and is disturbed by (e) his craving for (e) experience (bhoga) based on (e) the unreasoned assumption that he is (=) incomplete (apurnd) and in need of (e) something outside himself. 122 Thus it manifests as (=) desire or attachment (rdga). ]23 At the same time it is (=) primordial ignorance, 'the cause of the sprout of (e) transmigration'. 124 As such, it works in two ways. Firstly, it deprives (e) individual consciousness of freedom. The yogi affected by (e&eb) this impurity is incapable of (e) entering or emerging out of the states of contemplation as he wishes. Secondly, this impurity renders (eb) the individual unconscious of his freedom, which for(e) the yogi means that he is (=) liable to lose consciousness of (e) his true nature when he is (=) not in a state of (e) contemplative absorption. 125 The second impurity is (=) called the 'Impurity of Maya' (maylyamald). This comes into operation when consciousness has been contracted by (e) the Impurity of Individuality. It limits (e) the power of knowledge operating in (e&eb) the individual soul through (e&eb) the inner and outer organs of (e) perception. 126 Due to this impurity he perceives (e) diversity everywhere. 127 Finally, the third impurity, that of Karma, comes into play when (e) the individual, deprived of (e) the freedom and knowledge of (e) unconditioned consciousness, acts (e&eb) in his limited way prompted by (e) desires and fears for (e) his personal gain. Disturbed by (e) these three impurities the soul cannot find rest in (eb) himself. When he manages, however, to overcome (e) their disturbing influence, he experiences (e&eb) the pure vibration (spandd) of (e) consciousness and through (e&eb) this recognition realises (eb) his essentially omnipotent and omniscient nature. In order to do this the yogi must first discover (eb) universal consciousness within himself. Secondly, he must learn to recognise his own Spanda energy operating in the outer world. In the first stage, the power which flows through the channels of the senses and mind is withdrawn into itself (saktisamkocd) through a powerful, one-pointed act of intro- spection. 128 If the yogi is successful, he comes to perceive the totality of existence reflected in his mind (cittd) and pervaded by the light of consciousness. Thus he enjoys the inner bliss of contemplative absorption with his eyes closed (nimllanasamadhi). As Bhairava instructs the goddess: Merging the senses in the centre, between the upper and lower lotuses [of the Heart], in the void of the Heart, with mind unwavering DIvine Body and Sacred Circle of the Senses 1 57 O Fair One, attain the Supreme Bliss. 129 The restriction imposed on the exertion (udyama) of the senses by the confining perception of outer objectivity is thus overcome and the senses move freely in the 'Sky of the Heart of Consciousness'. 130 Like fuel in a fire, diversity is burnt to ashes and fused with the pure energy of one's own nature. All things deposited in the fire of the belly of one's own conscious- ness, suddenly (hafhatah) abandoning all distinctions, feed it with their own power. When the diversity which divides things is dissolved away by this violent digestion (hafhapaka), the deities of consciousness [i.e., the senses] devour the universe transformed [by this process] into the nectar [of T]. Once they are satisfied, they rest at one with the God (deva) Bhairava Who is the fullness of one's own nature and the Sky of Consciousness which, solitary, rests in the Heart. 131 The yogi is now absorbed in the unfolding (unmesa) of consciousness and becomes one withjts universal vibration. 132 The 'Great Expansion' (mahavikdsa) of the; yogi's consciousness unfolds as he penetrates and becomes one with the pure Being

(satta) of Bhairava 133 and thus enjoys the bliss (bhoga) of simultaneously making all things one with consciousness, 'savouring' their essence and 'protecting' or securing their place and function within the economy of the Whole. 134 The yogi experiences consciousness expansion both inwardly and externally. When all things are absorbed in the introverted unity of his consciousness, it emits itself into itself freely without succession or division. In this way his consciousness also unfolds throughout the external world mediated by the senses divinised by the grace of his inner realisation and he perceives its transcendental nature immanentalise into the activities of the universe. His senses open up fully in an instant as the power hidden within consciousness expands out from it. 135 The state of absorption 'with the eyes closed' {nimllanasamadhi) thus leads spontaneously to a state of absorption 'with the eyes open' (unmilana- samadhi) and vice versa. The yogi, with the greatest respect and devotion for the reality now unfolding before him, must learn to unite these two forms of absorption and experience the underlying unity of his consciousness which pervades both simultaneously from the Centre between them. Ksemaraja explains: By [the practice] of introverted and extroverted absorption and by being firmly fixed in the Centre which pervades both simultaneously, 158 The Doctrine of Vibration having laid hold of the fire-stick (arani) of their two-fold emanation (visarga) with all thought fallen away, the circle of the senses expands instantaneously. 136 The activity of the senses is now unconditioned by time and space; free of all obstacles, they function perfectly. They unfold to fill the entire universe with the power of their awareness and thereby withdraw into their pure and undivided source — the omnipresent Lord they perceive in all things. This state of simultaneous expansion (yikasa) and contraction (sarfikoca) beyond both outward and inner movement is technica 1 y called Bhairavimudra. The experience of this attitude (mudra) o f awareness takes place thus: [If] you project the vision and all the other powers [of the senses] simultaneously everywhere onto their respective objects by the power of awareness, while remaining firmly established in the centre like a pillar of gold, you [will] shine as the One, the foundation of the universe. 137 The One is the Lord of the Centre Who manifests all the deities of the inner circle as the essence of His pure awareness (svasamvittisara) together with the outer circle of the physical senses. 138 The power of awareness thus manifests itself on two levels simultaneously. It functions at the microcosmic level as the power of sensory awareness which apprehends specific objects in the field of individualised, embodied awareness. At the macrocosmic level it functions as the divine power of sensory awareness which apprehends the universal, cosmic object in the field of universal consciousness. Through the practice of Bhairavimudra these two aspects are experienced together in the blissful realisation that results from the union of the inner and outer states of absorption. 139 Thus, nothing that appears before him is any longer confined within the ridged structures of thought and, supportless, he becomes inwardly absorbed in Siva's consciousness. 140 Ksemaraja quotes an unknown Tantra as saying: With one's aim inside while gazing outside, eyes neither opening nor closing — this is Bhairava's Mudra kept secret in all the Tantras. 141 During the initial instant of perception, T consciousness is manifestly apparent and the yogi, participating in its plenitude, observes the outer world without being attached to any particular Divine Body and Sacred Circle of the Senses 1 59 or singling it out from any other, like a man who observes a city from a high mountain peak. 142 He sees the outer world reflected within his consciousness free of thought-constructs and so 'stamps* the outer on the inner while absorbing the object and means of knowledge in the pure subject which grasps them as the expansion of his own nature. 143 K§emaraja says: By penetrating into Bhairavimudra, the yogi observes the vast totality of beings rising from, and dissolving into, the Sky [of conscious- ness], like a series of reflections appearing and disappearing inside a mirror. 144 Through the practice of Bhairavimudra the yogi realises that he is the substratum consciousness (adhisfhatr) 145 which both underlies and is the essence of all things. He discovers that phenomena have no independent existence apart from him and so are, in this sense, void. At the same time, he realises that because all things are consciousness, they are far from unreal. He views the outer world yet sees it not. Beyond both Voidness and Non-voidness he penetrates into the Supreme Abode (pararn padam) of Siva's consciousness. 146 [The powers of the senses] endowed with the attributes of the Great Union [between subject and object] whose form is the awakening of man's spiritual potential (kuntfalinf), fill [with consciousness] the outer clatter of diversity (bhedatfambara) born of its intense power and are then established in the unobscured abode of the void of consciousness to shine [there] eternally. Thus residing beyond Being and Non-being, the sole protector of

the unity which is tranquil and expanding [consciousness], whose glory is all-embracing and form unobscured, is called Bhairavimudra. 141 Through the practice of Bhairavimudra, the yogi unites the universal vibration of T consciousness with the individual pulsation of objectivised 'this' consciousness. The two aspects of consciousness are now in a state of equilibrium like the two pans of an evenly weighed balance 148 and the yogi experiences the pure knowledge (suddhavidya) that: 'I alone am all things'. Thus becoming the master of the Wheel of Energies he is free, like Siva, to create and destroy. 149 When [the yogi] is well established, without wavering, solely in the integral egoity of his authentic nature, the Spanda principle, and is absorbed in contemplation (samavisfa), he becomes one with it 160 The Doctrine of Vibration (tanmaya). Then . . . dissolving and creating the universe by means of his introverted and extroverted absorption, he destroys and creates all things out of Sarikara, his innate nature. [Thus] he assumes the state of the universal experiencer and having absorbed all that is to be experienced from [the grossest level] — Earth — to (the subtlest) — Siva — he reaches the state of the supreme subject by progressively recognising [his identity with Him]. 150 Thus, introverted and extroverted absorption both lead to the recognition of the pulsation (spanda) of one's own consciousness. 1 51 the level of consciousness corresponding to Siva's basic state (saryibhavdvastha), the alternation from inner to outer is instantaneously resolved into the vibration of His nature. When the yogi finally comes to be constantly aware of this reality, his enlightenment is full and perfect. 152 Freed of all means (anupaya) and delighting in the power of his bliss (anandasakti), he knows and does whatever he pleases. The yogi seeking self-realisation must acquire mastery over this movement. Ksemaraja stresses that the Doctrine of Vibration teaches that liberation can only be achieved by first withdrawing all sense activity in introverted contemplation (nimilanasamadhi) to then experiencethe 'Great Expansion' (mahdvikdsd) of consciousness while recognising this to be a spontaneous process within it. 153 This is done through the practice of Kramamudra. A passage from the now lost Kramasiitra explains: Although the adept's attention [may be] outwardly directed, he enjoys contemplative absorption through the introverted aspect of Kramamudra. Initially he turns inward from the outside world and [then] from within [himself] he exits into the outer world under the influence of his absorption. Thus the sequence (krama) in this attitude (mudra) [ranges through] both inner and outer. 154 The yogi must pervade the surface level of awareness (yyutthana) with the same bliss he experiences plunging into the depths of con- templative absorption (samadhi). Submerging himself and emerging repeatedly from samadhi, he eventually recognises that the unity of consciousness pervades both states: The best of yogi's, who has achieved a state of complete absorption Even when risen from meditation, [inwardly] vibrating like a drunkard in blissful inebriation from the after-effects of the nectar of con- templation, sees all things dissolving in the Sky of Consciousness like a cloud in the autumn sky. He plunges repeatedly within himself Divine Body and Sacred Circle of the Senses 161 and becomes aware of his identity with consciousness by the practice of introverted contemplation. Thus even when he is said to have risen from absorption, he is one with [his] experience of it. 155 Ksemaraja goes on to explain that this practice is called 'Mudra' because it both fills the adept with bliss (mud) and is itself the bliss of consciousness. Moreover, it dissolves away (dra) all bondage and 'stamps' the universe of experience with the seal (mudra) of the fourth state (turiya) of enlightened consciousness beyond, and including, the three states of waking, dreaming and deep sleep. It is called 'Krama' because it is the root source of all emanation and all other conscious processes which succeed one another in ordered sequence (krama) and is, at the same time, their successive (krama) appearance as well. 156 By the practice of Kramamudra the opposites fuse and Siva and Sakti unite. They yogi comes to experience the simultaneous pervasion of all the lower, grosser categories of existence by the higher and the presence of the lower in the higher. Commencing his practice in a low form of Bhairavimudra, the yogi conjoins the outer with the inner; then, in Kramamudra, he fills both the outer with the inner and the inner with the outer. When he achieves perfection in this two-fold movement, he attains to the highest form of Bhairavimudra, in which the two merge completely in the experience of the absolute (anuttara), free of all differentiation and polarities. If he fails to maintain awareness of this state, he again falls into Kramamudra until he has finally completely merged all the highest states in the lower and the lower in the higher. He then no longer needs to resort to any means (anupayd) to achieve liberation. All he says or does, anything he perceives or thinks, instantly occasions in him the highest level of consciousness. Thus the fruit of Bhairavimudra is the wonder (camatkara) or

amazement (vismaya) that overcomes the yogi when he reaches the plane of union (yogabhumika) y xsl where all opposites merge in the radiance of the Great Light of consciousness. The Stanzas on Vibration teach: How can one who, as if astonished, beholds his own nature as that which sustains [the existence of everything] be subject to this painful round of transmigration (kusrti)? l5% The yogi, recognising his true nature to be the supreme subject, is astonished to suddenly discover that the individual he thought he was, caught up in the trammels of thought and living in a world enmeshed in the web of time and space, does not really exist at all. 159 He experiences a 'turning about' (paravrtti) in the deepest seat of The Doctrine of Vibration Consciousness as he penetrates his true nature. The sudden eruption of this intuition (pratibha) arouses (eb) in him a cry of amazement as he transcends all thought-constructs and, perfectly absorbed in his own nature, is liberated. The Path to Liberation Essentially, Spanda doctrine is concerned with two matters. The first is to impart to those who are fit to receive the teachings a deeper understanding of the ultimate goal of life (upeya). When we have understood what truly benefits us and is worth attaining and what, on the contrary, is of no real value but stands in the way of this attainment, we can begin to make progress towards our goal. This is Spanda doctrine's second concern, namely, to show the way in which we can develop spiritually through Siva's grace and the right application of the means to realisation that it teaches. When both these aspects of the teaching have been correctly understood and applied, the Spanda yogi achieves a clear and permanent realisation of his goal and is liberated, thus fulfilling the ultimate aim of the teaching. The Doctrine of Vibration is not meant for the spiritually dull. It is not for the worldly whose consciousness, clouded by ignorance, is as if dreaming, even during the waking state of daily life, the dream of its own thought-constructs. 1 The teachings are meant for those who are awake (prabuddha), those who, full of faith and reverence, are always alert and intent on iscerning the true nature of ultimate reality. This reality is understood in three basic ways. The first is purely transcendental. The Stanzas choose this aspect as the one which formally defines it most specifically. Ultimate reality transcends all the opposites, including subject and object. This does not mean, however, that it is an unconscious void, 2 a mere absence of all existence. In fact, this negative characterisation of reality (which includes also a denial of all that is unconscious) implies a positive immanence in which the opposites are 64 The Doctrine of Vibration united in the oneness of pure consciousness that is equally Siva and Spanda, His universal activity. These two seemingly contrasting aspects are reconciled in the third, namely, reality understood as the essential nature of all things. Although universal and everywhere the same, it is understood to be the essential and specific nature of each existent as its 'own nature' (svabhava). In the case of the individual soul it is even more specific, more personal as his own 'own nature' (svasvabhava). Belonging to none other than oneself it is the pure subjectivity that perceives experiences, enjoys, reflects, thinks and senses as well as being the conscious agent who creates every possible form of experience in all the states of consciousness. 3 The liberating knowledge of reality thus corresponds to our regaining possession of ourselves (svatmagraha). We must lay hold of ourselves and abide in our authentic nature. Reality coincides with our own most fundamental state of being (svasthiti), free of all contrasts and contradictions. Once we have overcome the negative forces that arise from our ignorance and prevent us from abiding in ourselves, we are liberated. To do this, we must penetrate through the pulsing fluctuations of objectively experienced states and perceptions at the surface level of consciousness and gain insight into the timeless rhythm of our own nature manifest in the universal arising and falling away of all things. We are not freed of the trammels of perpetual change by setting it aside; on the contrary, we must gain insight into the recurrent cycles of creation, persistence and destruction, or else be bound by our ignorance. 4 This spiritual ignorance consists essentially of our contracted state of consciousness and so can only be effectively countered by expanding it 5 to reveal our own authentic nature as this expanded state itself, which is the universal vibration (samanyaspanda) of consciousness. The Spanda- yogi treads (e&eb) the Path of Consciousness Expansion. The movement from (e) the contracted to (e&eb) the expanded state marks (eb) the transition from ignorance to (e&eb) understanding, from the dispersion and incompleteness of a form of consciousness entirely centred on (eb) an objectively perceived and discursively represented reality to (e&eb) a direct, intuitive awareness of the unity and integral wholeness of our own absolute Spanda nature. Along the way to this supreme realisation consciousness develops, as veil after veil is lifted, until it becomes full and perfect in the absolute which encompasses within itself all possible formats of experience. As Abhinava says: [This

realisation] is the supreme limit of plenitude and as such there can be no higher attainment. Any [other] attainment [we can] conceive issues from a state that falls short of [this] perfection. Once* [this] uncreated fullness has been attained, pray tell, what other fruit can there be [beyond it]? 6**Contemplation (Samadhi) (Doctrine of Vibration: S.G.Dyczkowski) (For models see Entanglement entropy et al). Ordinary knowledge is the knowledge of objects. Kant: Source: Wikipedia:** Something is transcendental if it plays a role in the way in which the mind "constitutes" objects and makes **it possible (eb) for us** to experience them as objects in the first place. Ordinary knowledge **is (=)** knowledge of objects Transcendental knowledge **is (=)** knowledge of how it is possible for us to experience those objects as objects. This is based on Kant's acceptance of David Hume's argument that certain general features of objects (e.g. persistence, causal relationships) cannot **derive (e) from** the sense impressions we have of them. Kant argues that the mind must contribute those features **and make it (eb) possible for us** to experience objects as objects. In the central part of his Critique of Pure Reason, the "Transcendental Deduction of the Categories", Kant argues for a deep interconnection between the ability to have self-consciousness **and (e&eb) the** ability to experience a world of objects. Through a process of synthesis, the mind generates both the structure of objects and its own unity. A metaphilosophical question discussed by many Kantian scholars is how transcendental reflection **is (=)** itself possible. Stephen Palmquist interprets Kant's appeal to faith **as (=) his** most effective solution to this problem. For Kant, the "transcendent", as opposed to the "transcendental", is that which lies **beyond (e)** what our faculty of knowledge can legitimately know. Hegel's counter-argument to Kant was that to know a boundary **is (=)** also to be aware of what it bounds and as such what lies beyond it – in other words, to have already transcended it.In phenomenology, the "transcendent" is that which transcends our own consciousness - that which is objective rather than only a phenomenon of consciousness. Noema is employed in phenomenology to refer to the terminus of an intention as given for consciousness. **(Kant: Source: Wikipedia). Pure engaged subject manifests itself in subject-object dichotomy, the very assumption which most theories of knowledge make to the detriment, damnation and discomfiture to the studies on subject and object.** Much of the time Dasein's practical problem solving **will (e&eb) involve** recovery strategies (e.g., switching to a different mode of transport) which preserve the marks of fluid and flexible know-how that are present in ready-to-hand contexts. In the limit, however (e.g., when a mechanic uses his theoretical knowledge of how cars work to guide a repair), Dasein's problem solving activity **will (=) begin** to approximate the theoretical reasoning distinctive of scientific inquiry into present-at-hand entities. But even here Dasein is not 'just theorizing' or 'just looking', so **it is (eb) yet, in** Heidegger's terms, a pure engaged subject.With this spectrum of cases in view, it is possible to glimpse a potential worry for Heidegger's account. Cappuccio and Wheeler (2010; see also Wheeler 2005, 143) argue that the situation of wholly transparent readiness-to-hand **is (=)** something of an ideal state. Skilled activity is never (or very rarely) perfectly smooth. Moreover, minimal subjective activity (such as a nonconceptual awareness of certain spatially situated movements by my body) **produces (eb)** a background noise that never really disappears. Thus a distinction between Dasein and its environment is, to some extent, preserved, and this distinction arguably **manifests (eb) the** kind of minimal subject-object dichotomy that is characteristic of those cases of un-readiness-to-hand that lie closest to readiness-to-hand. On the interpretation of Heidegger just given, Dasein's access to the world is only intermittently **that (=) of a** representing subject. An alternative reading, according to which Dasein always exists **as (=) a** subject relating to the world via representations, is defended by Christensen (1997, 1998). Christensen targets Dreyfus (1990) as a prominent and influential exponent of the intermittent-subject view. Among other criticisms, Christensen accuses Dreyfus of mistakenly hearing Heidegger's clear rejection of the thought that Dasein's access to the world is always theoretical (or theory-like) in character as being, at the same time, a rejection of the thought that Dasein's access to the world is always in the mode of a representing subject; but, argues Christensen, **there (eb)may be** non-theoretical forms of the subject-world relation, so the claim that Heidegger advocated the second rejection is not established by pointing out that he advocated the first. **Ideas: General Introduction to Pure Phenomenology. Trans. W.R. Boyce Gibson. Collier/Macmillan[first translation 1931] Original German text – 1913.** Cheu and his team then adapted a mathematical model that describes the activity of a single neuron to **incorporate (e)** specific characteristics of cells in the hippocampus, including their inhibitory activity. This mathematical model that

describes the activity of a single neuron to **incorporate (e)** specific characteristics of cells in the hippocampus, including their inhibitory activity **allowed (eb) them** to model neural networks in the hippocampus that encode short-term memories. They showed that for successful formation of auto-associative memories, the strength of synapses needs to be within a certain range: if synapses become too strong, the associated neurons **are (eb) activated** at the wrong time and networks become muddled, destroying the memories. If they are not strong enough, however, activation of some neurons in the network **is not (e) enough** to activate the rest, and memory retrieval fails. As well as providing insight into how memories may be stored and retrieved in the brain, Cheu thinks this work also has practical applications. "This study has **significant (eb) implications** in the construction of artificial cognitive computers in the future," he says. "It helps with developing artificial cognitive memory, in which memory sequences **can (e) be retrieved** by the presentation of a partial query." According to Cheu, one can compare it to a single image being used to retrieve a sequence of images from a video clip. More information: Cheu, E. Y., Yu, J., Tan, C. H. & Tang, H. Synaptic conditions for auto-associative memory storage and pattern completion in Jensen et al.'s model of hippocampal area CA3. Journal of Computational Neuroscience advance online publication, 30 May 2012 (doi: 10.1007/s10827-012-0394-8). http://link.springer.com/article/10.1007%2Fs10827-012-0394-8 .Provided by Agency for Science, Technology and Research (A*STAR), Singapore Read more at: http://medicalxpress.com/news/2012-11-brain-memories-artificial-cognitive.html#jCpTranscranial magnetic stimulation (TMS) **is(=) a** noninvasive method to cause depolarization or Hyperpolarization in the neurons of the brain. TMS uses electromagnetic induction to induce weak electric currents using a rapidly changing magnetic field; http://en.wikipedia.org/wiki/Transcr...ic_stimulation. You can induce action potential trains artificially **using** TMS. As far as natural communication between cells, we usually model these as electrochemical interactions. So, although magnetism always goes along with the territory when speaking of these actions, we typically do not invoke it a causal agent in natural brain function. **Scientists map process by which brain cells form long-term memories June 9, 2013 in Neuroscience.** During the first phase, the inducer, a random texture is slowly rotated, very briefly or for a longer duration. This inducer is followed by the transient, which may be a replacement of the texture by another, uncorrelated random texture (once or several times); a sudden jump of the original texture by an amount large enough that it cannot be detected; or an inversion of the texture contrast. (Other transients also lead to the effect: see text.) The **transient leads** to the illusory perception of a very rapid rotation, either forward (with respect to the direction of the inducer)—provided the inducer is very brief—or backward—for longer inducers. Copyright © PNAS, doi:10.1073/pnas.1213997110 (Medical Xpress) Optical illusions abound in human visual perception, **as (e) demonstrated by** the following well-known examples. Although many are static illusions, motion illusions also occur. Recently, scientists at Université Paris Descartes and Centre National de la Recherche Scientifique, Paris, University of Reading, United Kingdom, and Kyushu University, Japan discovered and investigated a new illusory motion effect, termed high phi by the authors, in which **we (e) perceive** conspicuous large illusory jumps when presentation of motion signals are followed by brief visual stimuli free of detectable motion signals. The researchers found that the size of the illusory jump does not depend on the speed of the motion signals presented, but rather **depend on (=)** on spatial frequency and transient duration while jump duration depends on motion signal duration. The study's authors conclude that their findings demonstrate that existing explanations for this illusion – namely, **the (e) loss of** coherent motion perception above an upper limit and the preference for minimal motion – are incomplete at best. Lead researcher Mark Wexler describes some of the challenges he and his colleagues – Andrew Glennerster, Patrick Cavanagh, Hiroyuki Ito, and Takeharu Seno – encountered in conducting their study. "We had the idea that these illusory jumps are related to dmax, the supposed upper speed limit on the steps **that(eb) leads to** motion perception, varies between individuals, and must be measured using random textures," Wexler tells Medical Xpress. "For displacements **below (eb)** dmax you're supposed to see the motion more or less correctly," he explains, "while for displacement above dmax you're just supposed to see noise – and the latter also turns out to be false." (These illusory jumps are demonstrated in an online supplement to the paper.1) Interestingly, the researchers discovered the illusion **as (=) a** bug in a computer program whose purpose was to do something else. "The easy thing to do in those kinds of circumstances is to correct the bug and move on," Wexler comments, "and ignore how strange the effect of

the bug actually is. Our key insight was not to move on." According to Wexler, the one finding in motion perception that everyone agrees with, for at least 100 years, **is (=) the** minimal-motion principle. "The minimal-motion principle **states that** whenever a stimulus is ambiguous and compatible with more than one motion – as it nearly always is – the brain is supposed to prefer the smallest, slowest motion, including stand-still, that is compatible with the stimulus," he explains. (In fact, he illustrates, many computer vision systems are built around this principle, and neuroscientists have verified it by recording signals from primate neurons.) **Motion perception revisited: High Phi effect challenges established motion perception assumptions April 23, 2013 by Stuart Mason Dambrot in Neuroscience Enlarge A schematic illustration of the basic high-phi stimuli. Animated versions may be seen in demonstrations 1–4.** "However," Wexler points out, "one **consequence of the** high phi effect **is (=) the** minimal-motion principle can be violated! When the stimulus is incompatible with any globally coherent motion, and therefore equally compatible with any motion, people **perceive (e)** not only a large jump, but the largest possible jump that they can perceive. This maximum jump is the one that steps by dmax, which **acts (=) as the** speed limit on motion perception." Another principle that seems to be violated by the high phi illusion, according to Wexler, **is (=)** dmax itself. "Below dmax, steps should be more or less seen as what they are – and as can be seen in demonstration three, this is what happens. On the other hand, says Wexler, "above dmax you're supposed to **perceive (e)** noise, not motion – but this is not what actually occurs." Rather, you **perceive (e)** the high phi jump, as can also be seen in demonstration three. "In one of our experiments," Wexler adds, "we showed that the amplitude of the jump is very closely correlated with the dmax limit, so that people who have higher dmax limits also see a larger high phi jump." It's known, Wexler points out, that the dmax limit **depends o**n spatial frequency: the lower the frequency (that is, the larger the features in the stimulus) the higher the limit. "And indeed," he notes, "we found that the magnitude of the illusory jump depends on spatial frequency in exactly the same way: the lower the frequency, formation shall be of the farther the jump." This can be verified, Wexler adds, by viewing high phi demonstration six and demonstration seven. Discussing the finding that the direction of the jump depends on the duration of the inducing motion signals, Wexler notes, "We think that the preceding – that is, inducing – motion acts like a seed. For brief inducers, the motion itself **acts (=) as** the seed, and the jump is experienced forwards with respect to the inducer. **Motion perception revisited: High Phi effect challenges established motion perception assumptions April 23, 2013 by Stuart Mason Dambrot in Neuroscience Enlarge A schematic illustration of the basic high-phi stimuli. Animated versions may be seen in demonstrations 1–4. (Models given in one of the papers Kindly bear with me)** For longer inducers, vision begins **to (e&eb) adapt** to the motion – a result known as the motion aftereffect." Also known as the waterfall illusion, the motion aftereffect occurs when, after viewing a moving object for an extended period of time, and that object then becomes stationary, **one (e) perceives** the object appears to slowly move in the opposite direction. "Many people initially think that what we've found **is (=) a** consequence of adaptation to motion or the motion aftereffect," he says. "If so, then it's the fastest motion aftereffect known. We've measured that the illusory motion is 10-100 times **faster (eb) than** the inducing motion! We think that for motion inducers, the adaptation **acts (=) as** the seed of the fast, backward jump." (Brief and long inducers can be compared directly in demonstration nine.) Wexler also describes how their findings for motion inducers, the adaptation **acts (=) as** the seed of the fast, backward jump." (Brief and long inducers can be compared directly in demonstration nine.) **relate(e&eb) to** the activity of neurons in of the primary visual cortex that respond to lines of a certain angle moving in one direction, as first described by Hubel and Wiesel (1959)2. "In the brain, motion detectors **are (e&eb) sensitive** to motion in a particular place – the receptive field – a particular direction, and usually a particular speed," Wexler notes. "When faced with our stimulus, there can be many accidental matches at the local level. In one image there is a dark spot, for example, and in the next, uncorrelated image **there (eb) happens** to be dark spot just next to it. In that case, a local motion detector **will (eb) react to** this false match – so our stimulus actually activates many local motion detectors, but incoherently, in that all of these motion detectors are signaling different motions. The main point is that in all this incoherent mess the brain finally **prefers (eb) the** largest possible motion." Commenting on other areas of research that might benefit from their study, Wexler cites computer vision. "The minimal-motion principle **is (e&eb) enshrined in** a lot of algorithms for extracting motion," he concludes. **Motion perception**

**revisited: High Phi effect challenges established motion perception assumptions April 23, 2013 by Stuart Mason Dambrot in Neuroscience Enlarge A schematic illustration of the basic high-phi stimuli. Animated versions may be seen in demonstrations 1–4(not incorporated due to spatial constraints).** "Our study shows that this principle can be violated. Can we find a different way to extract motion?" "The dependence on transient duration – which can be clearly seen in demonstration five – is, to be completely honest, a mystery, but a very interesting one," Wexler continues. "The amplitude of the jump is a very **linear (=) function** of transient duration, at least for small durations. If some perceptual process goes linearly farther for longer durations, **then (eb) something** in the brain must be effectively rotating at constant speed. I have no idea what that something may be, but it's an interesting challenge for the future." Journal reference: Proceedings of the National Academy of Sciences Journal of Physiology Copyright 2013 Medical Xpress All rights reserved. Read more at: http://medicalxpress.com/news/2013-04-motion-perception-revisited-high-phi.html#jCp **Motion perception revisited: High Phi effect challenges established motion perception assumptions April 23, 2013 by Stuart Mason Dambrot in Neuroscience Enlarge A schematic illustration of the basic high-phi stimuli. Animated versions may be seen in demonstrations 1–4(not incorporated due to spatial constraints).** When human observers are exposed to even slight motion signals followed by brief visual transients—stimuli containing no detectable coherent motion signals—**they (e) perceive** large and salient illusory jumps. This visually striking effect, which we call "high phi," **challenges (e)** well-entrenched assumptions about the perception of motion, namely the minimal-motion principle and the breakdown of coherent motion perception with steps above an upper limit called dmax. Our experiments with transients, such as texture randomization or contrast reversal, **show(eb) that** the magnitude of the jump depends on spatial frequency and transient duration—but not on the speed of the inducing motion signals—and the direction of the jump depends on the duration of the inducer. Jump magnitude is robust across jump directions and different types of transient. In addition, when a texture is actually displaced by a large step beyond the upper step size limit of dmax, **a (e) breakdown** of coherent motion perception is expected; However, in the presence of an inducer, observers again **perceive (e)** coherent displacements at or just above dmax. In summary, across a large variety of stimuli, we find that when incoherent motion noise is preceded by a small bias, instead of perceiving little or no motion—as suggested by the minimal-motion principle—**observers (e) perceive** jumps whose amplitude closely follows their own dmax limits. **Default perception of high-speed motion Mark Wexler et al Edited by George Sperling, University of California.** When a group of dots within a random-dot array is discontinuously displaced, it appears as a moving region perceptually segregated from its stationary surround. The spatial, temporal and other constraints **governing** this effect are markedly different from those classically found for the apparent motion of isolated stimulus elements. The random-dot display appears to tap a low-level motion-detecting process, distinct from the more interpretive process elicited by the classical displays. The distinct contributions of these processes can be identified in 'multi-stable' displays **which(eb) yield** alternative percepts of apparent motion depending on which one or both of the processes is activated. Such experiments illustrate the interaction of relatively stimulus-constrained and relatively autonomous processes in visual perception. **Low-Level and High-Level Processes in Apparent Motion O. J. Braddick et al Published 8 July 1980 doi: 10.1098/rstb.1980.0087 Phil. Trans. R. Soc. Lond. B 8 July 1980 vol. 290 no. 1038 137-151 . What is it like to be in this world? Immediately Kant and Heidegger come to mind.** Let's assume that Christensen is right about this. The supporter of the intermittent-subject view might still argue that although Heidegger holds that Dasein **sometimes (eb) emerges as a** subject whose access to the world is non-theoretical (plausibly, in certain cases of un-readiness-to-hand) There is other textual evidence, beyond that which indicates the non-theoretical character of hitch-free skilled activity, **to (eb) suggest** that readiness-to-hand must remain non-subject-object in form. Whether or not there is such evidence would then need to be settled. Being-in-the-World What the existential analytic has given us so far **is (=) a** phenomenological description of Dasein's within-the-world encounters with entities. The next clarification concerns the notion of world and the associated within-ness of Dasein. Famously, Heidegger writes of Dasein **as (=)** Being-in-the-world. In effect, then, the notion of Being-in-the-world **provides us** with a reinterpretation of the activity of existing (Dreyfus 1990, 40), where existence is given the narrow reading (existence) identified earlier. Understood as a unitary

phenomenon (as opposed to a contingent, additive, tripartite combination of Being, in-ness, and the world), Being-in-the-world is an essential **characteristic (eb) of** Dasein. As Heidegger explains: Being-in is not a 'property' which Dasein sometimes has and sometimes does not have, and without which it could just **be (=) just** as well as it could be with it. It is not the case that man 'is' and then has, by way of an extra, **a (e&eb) relationship**-of-Being towards the 'world'—a world with which he provides himself occasionally. Dasein is never 'proximally' an entity which is, so to speak, free from Being-in, but which sometimes has the inclination to take up a **'relationship' (e&eb)** towards the world. Note the text has v=been expounded I model able form. **Ideas: General Introduction to Pure Phenomenology. Trans. W.R. Boyce Gibson Collier/Macmillan [First translation 1931] Original German text – 1913 Dasein's Existence Constitutes A Branch-Point At Which It Chooses A Way To Be.** Thus I am currently working with a computer (a with-which), in the practical context of my office (an in-which), in order to write this encyclopedia entry (an in-order-to), which is aimed towards presenting an introduction to Heidegger's philosophy (a towards-this), for the sake of my academic work, that is, for the sake of my being an academic (a for-the-sake-of-which). The final involvement here, the for-the-sake-of-which, is crucial, **because (e)** according to Heidegger all totalities of involvements have a link of this type at their base. This forges a **connection between** (i) the idea that each moment in Dasein's existence constitutes a branch-point at which it chooses a way to be, and (ii) the claim that Dasein's projects and possibilities are essentially bound up with the ways in which other entities may become intelligible. This is because every for-the-sake-of-which **is the (e) base** structure of an equipment-defining totality of involvements and reflects a possible way for Dasein to be (an academic, a carpenter, a parent, or whatever). Moreover, given that entities are intelligible only within contexts of activity that, so to speak, arrive with Dasein, **this helps (eb) to explain** Heidegger's claim (Being and Time 16: 107) that, in encounters with entities, the world is something with which Dasein is always already familiar. Finally, it puts further flesh on the phenomenological category of the un-ready-to-hand. Thus when I am absorbed in trouble-free typing, the computer and the role that it plays in my academic activity are transparent aspects of my experience. But if the computer crashes, I become aware of it as an entity with which I was working in the practical context of my office, in order to write an encyclopedia entry aimed towards presenting an introduction to Heidegger's philosophy. And I become aware of the fact that my behaviour is being organized for the sake of my being an academic. So disturbances **have (e&eb) the effect of** exposing totalities of involvements and, therefore, worlds. (For a second way in which worlds are phenomenologically 'lit up', see Heidegger's analysis of signs (Being and Time 17:107–114); for discussion, see Dreyfus 1990, 100–2, Cappuccio and Wheeler 2010.)As already indicated, Heidegger sometimes uses the expression 'world' in a different key sense, **to(eb) designate** what he calls the "ontologico-existential concept of worldhood" (Being and Time 14: 93).At this point in the existential analytic, worldhood is usefully **identified (e&eb)** as the abstract network mode of organizational configuration that is shared by all concrete totalities of involvements. We shall see, however, that as the hermeneutic spiral of the text unfolds, the notion of worldhood is subject to a series of reinterpretations until, finally, its deep structure **gets(e) played out** in terms of temporality. Having completed what we might think of as the first phase of the existential analytic, Heidegger uses its results to launch an attack on one of the front-line representatives of the tradition, namely Descartes. This is the only worked-through example in **Being and Time** itself of what Heidegger calls the destruction (Destruktion) of the Western philosophical tradition, a process that was supposed to be a prominent theme in the ultimately unwritten second part of the text. The aim is to show that although the tradition takes theoretical knowledge to be primary, such knowledge (the prioritization of which is an aspect of the 'onticization' of Being mentioned earlier) **presupposes(eb)** the more fundamental openness to Being that Heidegger has identified as an essential characteristic of Dasein. **Ideas: General Introduction to Pure Phenomenology. Trans. W.R. Boyce Gibson Collier/Macmillan [first translation 1931] Original German text – 1913** Theory of evolution by natural selection apparently provides a non-semantic, non-intentional basis upon which to explicate (e) functions and, in turn, semantic content. Individual organisms vary (e&eb) in their characteristics, such as how their neurons respond to (e&eb) features of the environment. Some of these differences in how neurons respond make (eb) a difference to an organism's survival and reproduction. Finally, some of these very differences may be (=) heritable. Natural selection, commonly understood as this differential

reproduction of heritable variation, is (=) purely causal. Suppose that there is a population of (e) rabbits. Further suppose that either by a genetic mutation or by the recombination of (e&eb) existing genes; some of these rabbits develop (e&eb) neurons that are wired into (e&eb) their visual systems in such a way that they fire (more or less reliably) in (e&eb) the presence of dogs. Further, the firing of these neurons is wired into (e&eb) a freezing behavior in these rabbits. Because of this configuration, the rabbits with (e&eb) the are less likely to be (=) detected by dogs, hence more likely to (e) survive and reproduce. Finally, because (e) the genes for these neurons are heritable, (eb) the offspring of these dog-sensitive rabbits will themselves be (=) dog-sensitive. Over time, the number of the dog-sensitive rabbits will (=) increase, thereby displacing (e&eb) the dog-insensitive rabbits. So, natural selection will, in such a scenario, give rise to (eb) mental representations of (e) dogs. Insofar as such a story is plausible, (eb) there is hope that natural selection and the genesis of functions can provide (eb) a naturalistically acceptable means of delimiting (e) content-determining (eb) causes. There is no doubt that individual variation, differential reproduction, and inheritance (e&eb) can be understood in a purely **causal manner**. Yet, there remains skepticism about (e&eb) how naturalistically one can describe (eb) what natural selection can select for. There are doubts about (e&eb) the extent to which the objects of (e) selection really can be specified without (e) illicit importation of (e) intentional notions. Fodor (1989, 1990a) give voice to (e) some of this skepticism. Prima facie, it makes sense to say that the neurons in our hypothetical rabbits fire in response to (e&eb) the presence of dogs, hence that there is selection for (e) dog representations. But, it makes just as much sense, one might worry, to say that it is(=) sensitivity to dog-look-alikes that leads to (eb) the greater fitness of (e) the rabbits with (e&eb) the new neurons.[8] There are genes for (e) the dog-look-alike neurons and these genes are (=) heritable. Moreover, those rabbits that freeze in response to (e&eb) dog-look-alikes are more likely to survive and reproduce (eb, eb) than are those that do not (e) so freeze; hence one might say that the freezing is in response to (e&eb) dog-look-alikes. So, our ability to say that the meaning of (e) the rabbits' mental representation "X" is (=) dog, rather than dog-look-alike, depends on (=) our ability to say that it is (=) the dog-sensitivity of "X", rather than the dog-look-alike-sensitivity of (e) "X", that keeps the rabbits alive longer. Of course, being dog-sensitive and (e&eb) being dog-look-alike-sensitive are connected, but the problem here is that both being dog-look-alike-sensitive and (e&eb) being dog-sensitive can increase (eb) fitness in ways that lead to (eb) the fixation of a genotype. And it can well be that it is (=) avoidance of dogs that keeps a rabbit alive, but one still needs some principled basis for (e) saying that the rabbits avoid dogs by (e) being sensitive to dogs, rather than by (e) being sensitive to dog-look-alikes. The latter appears to be (=) good enough for the differential reproduction of (e) heritable variation to do its work. Where we risk importing semantic notions into (e&eb) the mix is in understanding selection intentionally, rather than purely causally. We need a notion of "**selection for"** that is both general enough to work for(e) all the mental contents causal theorists aspire to address (e&eb) and that does not tacitly import (e&eb) semantic notions. In response to this sort of objection, it has been proposed that the correct explanation of (e) a rabbit's evolutionary success with, say, "X", is (=) not that this enables (eb) the rabbit to avoid dog-look-alikes, but that it enables them to avoid dogs. It is dogs, but not (e) mere dog-look-alikes, **that prey on rabbits**. (This sort of response is developed in
and Neander (1995).) Yet, the rejoinder is (=) that if we really want to get at the correct explanation of a rabbit-cum-"X" system, then (eb) we should not suppose that "X" means dog. Instead, we should say that it is (=) in virtue of the fact that "X" picks up (e) on something like, say, **predator of such and such characteristics** that (eb) the "X" alarm system increases the chance of a rabbit's survival. (This sort of rejoinder may be found in Agar (1993).) This problem aside, there is (=) also some concern about the extent to which it is plausible to (e) suppose that natural selection could act on (e&eb) the fine details of the operation of the brain, such as (=) the firing of neurons in the presence of dogs. (This is an objection raised in Fodor (1990c)). Natural selection might operate to (e&eb) increase the size of the brain so there is (=) more cortical mass for cognitive processing. Natural selection might also operate to (e) increase the folding of the brain so as to maximize (eb) the cortical surface area that can be contained (e) within the brain. Natural selection might also lead to (eb) compartmentalization of the brain, so that one particular region could be dedicated to (e) visual processing, another to (e) auditory processing, and still another to face processing. Yet, many would take it to be implausible to suppose that natural selection works (e&eb) at the level of

(e) individual mental representations. The brain is too plastic and there is (=) too much individual variation (e&eb) in (eb) the brains of mammals to admit (e) of selection acting in (e&eb) this way. Moreover, such far reaching effects of natural selection would lead to (eb) innate **ideas not merely of (e) colors and shapes**, but of (e) dogs, cats, cars, skyscrapers, and movie stars. Rather than supposing that functions are determined by natural selection across multiple generations, many philosophers contend that it is (=) more plausible that the functions that underlie mental representations are acquired through (e&eb) cognitive development (Stanford encyclopedia of philosophy) Freud's account of the unconscious, and the psychoanalytic therapy associated with (e&eb) it, is best illustrated by his famous tripartite model of the structure of the mind or personality (although, as we have seen, he did not formulate this until 1923). This model has many points of similarity with the account of the mind offered by Plato over 2,000 years earlier. The theory is termed 'tripartite' simply because, again like Plato, Freud distinguished three structural elements within (e&eb) the mind, which he called id,(e&eb) ego, and (e&eb) super-ego. The id is (=) that part of the mind in which are (=) situated the instinctual sexual drives which require (e) satisfaction; the super-ego is (=) that part which contains (e) the "conscience," namely, socially-acquired control mechanisms which have been internalized, and which are (=) usually imparted in the first instance by (e) the parents; while (e) the **ego is (=) the conscious self that is (=) created by (e) the dynamic tensions and interactions between the id and (e&eb) the s**uper-ego and has (e) the task of reconciling their conflicting demands with (e&eb) the requirements of external reality. It is in this sense that the mind is (=) to be understood as a **dynamic energy-system**. All objects of consciousness reside in (eb) the ego; the contents of (e) the id belong (e&eb) permanently to the unconscious mind; while (e) the super-ego is (=) an unconscious screening-mechanism which seeks to limit (e) the blind pleasure-seeking drives of the id by (e) the imposition of restrictive rules. There is some debate as to how literally Freud intended this model to be taken (he appears to have taken it extremely literally himself), but it is important to note that what is being offered here is indeed a theoretical model rather than a description of (e) an observable object, which functions as (=) a frame of reference to explain the link between early childhood experience and (e&eb) the mature adult (normal or dysfunctional) personality. Freud also followed Plato in his account of (e) the nature of (e) mental health or psychological well-being, which he saw as (=) the establishment of a harmonious relationship between the three elements which constitute (e) the mind. If the external world offers (eb) no scope for the satisfaction of the id's pleasure drives, or more commonly, if the satisfaction of some or all of these drives would indeed transgress (e) the moral sanctions laid down by the super-ego, then (eb) an inner conflict occurs in the mind **between (e&eb) its constituent parts or elements**. Failure to resolve this can lead to (eb) later neurosis. A key concept introduced here by Freud is (=) that the mind possesses (e) a number of 'defense mechanisms' to attempt to (e) prevent (e) conflicts from (e) becoming too acute, such as (=) repression (pushing conflicts back into (e&eb) the unconscious), sublimation (**channeling the sexual drives into (e&eb) the achievement socially acceptable goals**, in art, science, poetry, and so forth), fixation (the failure to progress beyond (e) one of the developmental stages), and regression (a return to the behavior characteristic of one of the stages). Of these, repression is (=) the most important, and Freud's account of (e) this is as follows: when a person experiences an instinctual impulse to behave in a manner which the super-ego deems to be (=) reprehensible (**for example, a strong erotic impulse on the part of the child towards the parent of the opposite sex)**, then (eb) it is possible for (e) the mind to push (eb) this impulse away, to repress (e) it into the unconscious. Repression is (=) thus one of the central **defense mechanisms** by (e) which the ego seeks to (e) avoid internal conflict and pain, and to reconcile reality with (e&eb) the demands of both id and (e&eb) super-ego. As such it is completely normal and an integral part of (e) the developmental process through which every child must pass on the way to adulthood. However, the repressed instinctual drive, as an energy-form, is (=) not and cannot be destroyed (e) when it is repressed–**it continues to exist intact in (eb) the** unconscious, from (e) where it exerts (e&eb) a determining force upon the conscious mind, and can give rise to (eb) the dysfunctional behavior characteristic of (e) neuroses. This is one reason why **dreams and slips of the tongue** possess (e) such a strong symbolic significance for Freud, and why their analysis became (=) such a key part of his treatment–they represent (eb) instances in which the vigilance of (e) the super-ego is (=) relaxed, and when (e) the repressed drives are (=) accordingly able to present (eb) themselves to (e) the conscious mind in (eb) a transmuted

form. The difference between 'normal' repression and (e&eb) the kind of repression which results in (eb) neurotic illness is (=) one of degree, not of kind–the compulsive behavior of (e) the neurotic is itself (=) a manifestation of (e) an instinctual drive repressed in (eb) childhood. Such behavioral symptoms are (=) highly irrational (and may even be perceived as such by the neurotic), but are (=) completely beyond the control of (e) the subject because (e) they are driven by (e) the now **unconscious repressed impulse**. Freud positioned the key repressions for (e) both, the normal individual and (e&eb) the neurotic, in the first five years of childhood, and of course, held them to be (=) essentially sexual in nature;–since, as we have seen, repressions which disrupt (e&eb) the process of infantile sexual development in particular, according to him, lead to a strong tendency to later neurosis in adult life. The task of psychoanalysis as a therapy is (=) to find the repressions which cause (eb) the neurotic symptoms by delving into (e&eb) the unconscious mind of the subject, and by (e) bringing them to (e) the forefront of consciousness, to allow (eb) the ego to confront them (e&eb) directly and thus to discharge (eb)them. (International Encyclopedia Of Philosophy) Note that the models apply to every statement and proposition above) (Models are given in the previous paper PV criticality et al,) Note that the models apply to every statement and proposition above) (Entire abstract and note exposition is model able) **Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016**. The Evolutionary Argument Against Reality Quanta magazine (Donald Hoffman and Quanta Magazine) In the context of two illustrative examples from supersymmetric quantum mechanics authors show that the semiclassical analysis of the path integral requires complexification of the configuration space and action, and the inclusion of complex saddle points, even when the parameters in the action are real. Authors find new exact complex saddles, and show that without their contribution the semiclassical expansion is in conflict with (e&eb) basic properties such as (=) the positive semidefiniteness of (e) the spectrum, as well as (=) constraints of supersymmetry. Generic saddles are not only complex, but also possibly multivalued and even singular. This is in contrast to instanton solutions, which are real, smooth, and single valuedThe multivaluedness of the action can be interpreted as (=) a hidden topological angle, quantized in units of (e) $\pi$ in supersymmetric theories. The general ideas also apply to nonsupersymmetric theories. Received 12 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.011601 © 2016 American Physical SocietyThe Evolutionary Argument Against Reality The cognitive scientist Donald Hoffman uses evolutionary game theory to show that our perceptions of an independent reality must be (=) illusions.[No Caption]David McNew for Quanta MagazineAbout Quanta: Quanta Magazine's mission is to enhance public understanding of research developments in mathematics and the physical and life sciences. Quanta articles do not necessarily represent the views of the Simons Foundation. Learn more By Amanda Gefter April 21, 2016As we go about our daily lives, we tend to assume that our perceptions — sights, sounds, textures, tastes — are (=) an accurate portrayal of the real world. Sure, when we stop and think about it — or when we find ourselves fooled by (e) a perceptual illusion — we realize with a jolt that what we perceive is (=) never the world directly, but rather our brain's best guess at what that world is (=) like, a kind of internal simulation of (e) an external reality. The Evolutionary Argument Against Reality Quanta magazine Still, we bank on the fact that our simulation is (=) a reasonably decent one. If it wasn't, wouldn't evolution have weeded us out by (e) now? The true reality might be forever beyond our reach, but surely our senses give (e&eb) us at least an inkling of what it's really like.Not so, says Donald D. Hoffman, a professor of cognitive science at the University of California, Irvine. Hoffman has spent the past three decades studying perception, artificial intelligence, evolutionary game theory and the brain, and his conclusion is a dramatic one: The world presented to us by our perceptions is (=) nothing like reality.What's more, he says, we have evolution itself to thank for (e) this magnificent illusion, as it maximizes evolutionary fitness by driving truth to (e&eb) extinction.Getting at questions about the nature of reality, and disentangling the observer from (e) the observed, is (=) an endeavor that straddles the boundaries of neuroscience and fundamental physics. On one side you'll find researchers scratching their chins raw trying to understand how a three-pound lump of gray matter obeying nothing more than the ordinary laws of physics can give rise to (eb) first-person conscious experience. This is the aptly named "hard problem."On the other side are quantum physicists, marveling at the strange fact that quantum systems don't seem to be (=) definite objects localized in

space until (e) we come along to observe them. Experiment after experiment has shown — defying common seen (?) — That if we assume that the particles that make up ordinary objects have (e) an objective, observer-independent existence, (eb) we get the wrong answers. The central lesson of quantum physics is clear: There are (=) no public objects sitting out there in (eb) some preexisting space. As the physicist John Wheeler put it, "Useful as it is under ordinary circumstances to say that the world exists 'out there' independent of (e) us, that view can no longer be upheld."So while neuroscientists struggle to understand how there can be such a thing as (=) a first-person reality, quantum physicists have to grapple with (e&eb) the mystery of how there can be anything but a first-person reality. In short, all roads lead back to (e&eb) the observer. And that's where you can find Hoffman — straddling the boundaries, attempting a mathematical model of the observer, trying to get at the reality behind the illusion. Quanta Magazine caught up with him to find out more.QUANTA MAGAZINE: People often use Darwinian evolution as an argument that our perceptions accurately reflect (e&eb) reality. They say, "Obviously we must be latching onto (e&eb) reality in some way because otherwise we would have been wiped out (e) a long time ago. If I think I'm seeing a palm tree but it's really a tiger, I'm in trouble." "Evolution has shaped us with (e&eb) perceptions that allow us to survive. But part of that involves hiding from (e) us the stuff we don't need to know. And that's (=) pretty much all of reality, whatever reality might be."DONALD HOFFMAN: Right. The classic argument is that those of our ancestors who saw more accurately had a competitive advantage over (e&eb) those who saw less accurately and thus were (=) more likely to pass on their genes that coded for (e) those more accurate perceptions, so after thousands of generations we can be quite confident that we're (=)the offspring of those who saw accurately, and so we see accurately. That sounds very plausible. But I think it is (=) utterly false. It misunderstands the fundamental fact about evolution, which is (=) that it's about fitness functions — mathematical functions that describe (eb) how well a given strategy achieves (eb) the goals of (e) survival and reproduction. The mathematical physicist Chetan Prakash proved a theorem that I devised that says: According to evolution by(e) natural selection, an organism that sees reality as it is will never be (=) more fit than an organism of (e) equal complexity that sees none of reality but is (=) just tuned to fitness. Never.You've done computer simulations to show this. Can you give an example?Suppose in reality there's (=) a resource, like water, and you can quantify how much of it there is (=) in an objective order — very little water, medium amount of water, a lot of water. Now suppose your fitness function is (=) linear, so a little water gives (eb) you a little fitness, medium water gives you (eb) medium fitness, and lots of water gives you (eb) lots of fitness — in that case, the organism that sees the truth about the water in the world can win, but only because (e) the fitness function happens to align with (e&eb) the true structure in reality. Generically, in the real world, that will never be the case. Something much more natural is (=) a bell curve — say, too little water (eb) you die of thirst, but too much water (eb) you drown, and only somewhere in between is (=) good for survival. Now the fitness function doesn't (e) match the structure in the real world. And that's enough to send truth to (e&eb) extinction. For example, an organism tuned to fitness might see small and large quantities of some resource as, say (=), red, to indicate low fitness, whereas they might see intermediate quantities as green, to indicate high fitness. Its perceptions will be tuned to (e) fitness, but not to truth. It won't see any distinction between small and large — it only sees (=) red — even though such a distinction exists in (eb) reality.But how can seeing a false reality be (=) beneficial to an organism's survival? (Quanta)There's a metaphor that's only been available to us in the past 30 or 40 years, and that's the desktop interface. Suppose there's a blue rectangular icon on (eb) the lower right corner of your computer's desktop — does that mean that the file itself is (=) blue and rectangular and lives in (eb) the lower right corner of your computer? Of course not. But those are the only things that can be asserted about anything on (eb) the desktop — it has color, position and shape. Those are (=) the only categories available to you, and yet none of them are (=) true about the file itself or anything in the computer. They couldn't possibly be true. That's an interesting thing. You could not form (eb) a true description of the innards of the computer if your entire view of reality was confined to (e&eb) the desktop. And yet the desktop is useful.That blue rectangular icon guides (e&eb) my behavior, and it hides (e) a complex reality that I don't need to know. That's the key idea. Evolution has shaped us with (e&eb) perceptions that allow us to survive. They guide adaptive behaviors. But part of that involves hiding from (e) us the stuff we don't need to know. And that's (=) pretty much all of reality, whatever reality might

be. If you had to spend all that time figuring it out, (eb) the tiger would eat you.So everything we see is (=) one big illusion?We've been shaped to have (e) perceptions that keep us alive, so we have to take them seriously. If I see something that I think of as a snake, (eb) I don't pick it up. If I see a train, (eb) I don't step in front of it. I've evolved (eb) these symbols to keep me alive, so I have to take them seriously. But it's a logical flaw to think that if we have to take it seriously, we also have to take it literally. If snakes aren't snakes and trains aren't trains, what are (=) they?Snakes and trains, like the particles of physics, have (e) no objective, observer-independent features. The snake I see is a description created by (e) my sensory system to inform me of the fitness consequences of my actions. Evolution shapes (e&eb) acceptable solutions, not (e) optimal ones. A snake is an acceptable solution to (e) the problem of telling me how to act in (e&eb) a situation.My snakes and trains are (=) my mental representations; your snakes and trains are (=) your mental representations.How did you first become interested in these ideas?As a teenager, I was very interested in the question "Are we machines?" My reading of the science suggested (eb) that we are.But my dad was a minister, and at church they were saying we're (=) not. So I decided I needed to figure it out for myself. It's sort of an important personal question — if I'm a machine, (eb) I would like to find that out! And if I'm not, I'd like to know, what is that special magic beyond the machine? So eventually in the 1980s I went to the artificial intelligence lab at MIT and worked on (e&eb) machine perception. The field of vision research was enjoying a newfound success in developing mathematical models for (e) specific visual abilities. I noticed that they seemed to share a common mathematical structure, so I thought it might be possible to write down a formal structure for (e) observation that encompassed all of them, perhaps all possible modes of observation. I was inspired in part by Alan Turing. When he invented the Turing machine, he was trying to come up with a notion of computation, and instead of putting bells and whistles on it, he said, let's get the simplest, most pared down mathematical description that could possibly work. And that simple formalism is (=) the foundation for the science of computation. So I wondered, could I provide a similarly simple formal foundation for (e) the science of observation?A mathematical model of consciousness: That's right. My intuition was, there are (=) conscious experiences. I have (e) pains, tastes, smells, all my sensory experiences, moods, emotions and so forth. So I'm just going to say: One part of this consciousness structure is (=) a set of all possible experiences. When I'm having (e) an experience, based on (e) that experience I may want to change (e&eb) what I'm doing. So I need to have a collection of possible actions I can take and a decision strategy that, given my experiences, allows me (eb) to change how I'm acting. That's the basic idea of (e) the whole thing. I have a space X of experiences, a space G of actions, and an algorithm D that lets me choose (e&eb) a new action given my experiences. Then I posited a W for a world, which is (=) also a probability space. Somehow the world affects my (e&eb) perceptions, so there's (=) a perception map P from (e) the world to my experiences, and when I act, (e&eb) I change the world, so there's (=) a map A from the space of actions to (e&eb) the world. That's the entire structure. Six elements. The claim is: This is (=) the structure of consciousness. I put that out there so people have something to shoot at.But if there's a W, are you saying there is (=) an external world?Here's the striking thing about that. I can pull the W out of (e) the model and stick a conscious agent in its place and get (eb) a circuit of conscious agents. In fact, you can have (e) whole networks of (e) arbitrary complexity. And that's (=) the world. David McNew for Quanta Magazine Video: Donald Hoffman explains how our perceptions have evolved to become like a computer interface.The world is just other conscious agents?I call it (=) conscious realism: Objective reality is (=) just conscious agents, just points of view. Interestingly, I can take two conscious agents and have them interact (e&eb) and the mathematical structure of (e) that interaction also satisfies (eb) the definition of (e) a conscious agent.This mathematics is telling me something. I can take two minds, and they can generate (eb) a new, unified single mind. Here's a concrete example. We have two hemispheres in (eb) our brain. But when you do a split-brain operation, a complete transection of the corpus callosum, you get (eb) clear evidence of two separate consciousnesses. Before that slicing happened, it seemed there was a single unified consciousness. So it's not implausible that there is (=) a single conscious agent. And yet it's (=) also the case that there are two conscious agents there, and you can see that when (e) they're split. I didn't expect that, the mathematics forced me to recognize this. It suggests (eb) that I can take separate observers, put them together and create (eb) new observers, and keep doing this ad infinitum. It's (=)conscious agents all the way down.If it's (=)

conscious agents all the way down, all first-person points of view, what happens to science? Science has always been a (=) third-person description of the world.The idea that what we're doing is (=) measuring publicly accessible objects, the idea that objectivity results from (e) the fact that you and I can measure the same object in the exact same situation and get the same results — it's (=) very clear from quantum mechanics that that idea has to go. Physics tells us that there are (=) no public physical objects. So what's going on? Here's how I think about it. I can talk to you about my headache and believe that I am communicating (e&eb) effectively with you, because (e) you've had your own headaches. The same thing is (=) true as apples and the moon and the sun and the universe. Just like you have your own headache, you have (e) your own moon. But I assume it's (=) relevantly similar to mine. That's an assumption that could be (=) false, but that's (=) the source of my communication, and that's the best we can do in terms of (e&eb) public physical objects and (e&eb) objective science.It doesn't seem like many people in neuroscience or philosophy of mind are thinking about fundamental physics. Do you think that's been a stumbling block for (e) those trying to understand consciousness?I think it has been (=) a stumbling block. Not only are they ignoring the progress in fundamental physics, they are often explicit about it. They'll say openly that quantum physics is not relevant to (e) the aspects of brain function that are causally involved (e&eb) in consciousness. They are certain that it's got to be (=) classical properties of neural activity, which exist independent of (e)any observers — spiking rates, connection strengths at synapses, perhaps dynamical properties as well. These are (=) all very classical notions under Newtonian physics, where (e) time is absolute and objects exist absolutely. And then [neuroscientists] are mystified as to why they don't make progress. They don't avail themselves of the incredible insights and breakthroughs that physics has made. Those insights are out there for us to use, and yet my field says, "We'll stick with Newton, thank you. We'll stay 300 years behind in our physics." I suspect they're reacting to (e) things like Roger Penrose and Stuart Hameroff model, where (e) you still have a physical brain, it's (=) still sitting in space, but supposedly it's (=) performing some quantum feat. In contrast, you're saying, "Look, quantum mechanics is telling us that we have (e) to question the very notions of 'physical things' sitting in (eb) 'space.'"I think that's (=) absolutely true. The neuroscientists are saying, "We don't need to invoke (e) those kinds of quantum processes, we don't (e) need quantum wave functions collapsing (eb) inside neurons, we can just use (e) classical physics to describe (eb) processes in the brain." I'm emphasizing the larger lesson of quantum mechanics: Neurons, brains, space … these are (=) just symbols we use, they're (=) not real. It's not that there's (=) a classical brain that does (eb) some quantum magic. It's that there's (=) no brain! Quantum mechanics says that classical objects — including brains — don't (e) exist. So this is a far more radical claim about (e&eb) the nature of reality and does not (e) involve the brain pulling off (eb) some tricky quantum computation. So even Penrose hasn't taken it far enough. But most of us, you know, we're (=) born realists. We're (=) born physicalists. This is a really, really hard one to let go of.To return to the question you started with as a teenager, are we (=) machines?The formal theory of conscious agents I've been developing is (=) computationally universal — in that sense, it's (=) a machine theory. And it's because (e) the theory is computationally universal that I can get all of (e) cognitive science and neural networks back out of it. Nevertheless, for now I don't think we are (=) machines — in part because (e) I distinguish between the mathematical representation and (e&eb) the thing being represented. As a conscious realist, I am postulating conscious experiences as (=) ontological primitives, the most basic ingredients of the world. I'm claiming that experiences are (=) the real coin of the realm. The experiences of everyday life — my real feeling of (e) a headache, my real taste of (e) chocolate — that really is (=) the ultimate nature of reality.One of modern physics' most cherished ideas is (=) quantum chromodynamics, the theory that describes (eb) the strong nuclear force, how it binds (e&eb) quarks and gluons into protons and neutrons, how these form (eb) nuclei those themselves interact. This is (=) the universe at its most fundamental. So an interesting pursuit is to simulate (e&eb) quantum chromodynamics on a computer to (e) see what kind of complexity arises. The promise is that simulating physics on (e&eb) such a fundamental level is more or less equivalent to (=) simulating the universe itself. There are one or two challenges of course. The physics is mind-bogglingly complex and operates on (e&eb) a vanishingly small scale. So even using (e) the world's most powerful supercomputers, physicists have (e) only managed to simulate (e&eb) tiny corners of the cosmos just a few femtometers across. (A femtometer is 10^-15 metres.) That may not sound like much but the

significant point is that the simulation is (=) essentially indistinguishable from the real thing (at least as far as we understand it). This is true. Once I saw a mob being chased by by a subinspector and the immediate conclusion, for having been living in a hostile environment is that police are behind me and some people are behind police who are helping me. But one thing was certain. Some people are out to get the blood out of me. If I choose to follow those who made sub inspector wielding stick to petrify mob then I am safe! This is one of the mental games some peole play. There are there everywhere and they use it to their advantage. It's not hard to imagine that Moore's Law-type progress will allow (eb) physicists to simulate (e&eb) significantly larger regions of space. A region just a few micrometres across could encapsulate (e&eb) the entire workings of a human cell. Again, the behaviour of this human cell would be (=) indistinguishable from the real thing.It's this kind of thinking that forces (eb) physicists to consider the possibility that our entire cosmos could be (=) running on a vastly powerful computer. If so, is there any way (=) we could ever know? Today, we get an answer of sorts from (e) Silas Beane, at the University of Bonn in Germany, and a few pals. They say there is (=) a way to see evidence that we are being simulated, at least in (eb) certain scenarios.First, some background. The problem with all simulations is that the laws of physics, which appear continuous, have to be superimposed onto a discrete three dimensional lattice which advances in steps of time. The question that Beane and co asks is whether the lattice spacing imposes any kind of limitation on the physical processes we see in the universe. They examine, in particular, high energy processes, which probe smaller regions of space as they get more energetic What they find is interesting. They say that the lattice spacing imposes (e&eb) a fundamental limit on the energy that particles can have. That's because nothing can exist that is (=) smaller than the lattice itself. So if our cosmos is (=) merely a simulation, there ought to be (=) a cut off in the spectrum of high energy particles.It turns out there is (=) exactly this kind of cut off in the energy of cosmic ray particles, a limit known as the Greisen–Zatsepin–Kuzmin or GZK cut off. This cut-off has been well studied and comes about because (e) high energy particles interact with (e&eb) the cosmic microwave background and so lose (e) energy as they travel (e&eb) long distances. But Beane and co calculate that the lattice spacing imposes (e&eb) some additional features on (e&eb) the spectrum. "The most striking feature…is (=) that the angular distribution of (e) the highest energy components would exhibit (eb) cubic symmetry in the rest frame of the lattice, deviating (e&eb) significantly from isotropy," they say.In other words, the cosmic rays would travel (e&eb) preferentially along the axes of (e) the lattice, so (e) we wouldn't see them equally in all directions. That's a measurement we could do now with (e&eb) current technology. Finding the effect would be equivalent to (=) being able to 'see' the orientation of (e) lattice on which our universe is (=) simulated.That's cool, mind-blowing even. But the calculations by Beane and co are not without (e) some important caveats. One problem is that the computer lattice may be constructed in (e&eb) an entirely different way to the one envisaged by these guys. Another is that this effect is (=) only measurable if (e) the lattice cut off is (=) the same as the GZK cut off. This occurs when (e) the lattice spacing is about 10^-12 femtometers. If the spacing is significantly smaller than that,(eb) we'll see nothing.Nevertheless, it's surely worth looking for, if only to rule out (e) the possibility that we're part of a simulation of this particular kind but secretly in the hope that we'll find good evidence of our robotic overlords once and for all. Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation. It is your mind or self that is creating the illusion. See earlier discussions. One of modern physics' most cherished ideas is quantum chromodynamics, the theory that describes (eb) the strong nuclear force, how it binds (e&eb) quarks and gluons into (eb) protons and neutrons, how these form (eb) nuclei those themselves interact. This is the universe at its most fundamental. So an interesting pursuit is to simulate quantum chromodynamics on a computer to see (eb) what kind of complexity arises. 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If the spacing is significantly smaller than that, we'll see nothing. Nevertheless, it's surely worth looking for, if only to rule out the possibility that we're part of a simulation of this particular kind but secretly in the hope that we'll find good evidence of our robotic overlords once and for all. **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Doffamn interview with Quanta. Note all the above sentences are modelable. (For models see below)** In thinking intuitively about complementary properties, one can imagine puzzle pieces fitting (e&eb) together. Both pieces are needed to (e) finish the puzzle. Complementary brain processes are, however, more dynamic than (e) any such analogy: Pairs of complementary processes interact to form (eb) emergent properties which overcome (e) their complementary deficiencies to compute (e &eb) complete information with which to represent or control some aspect of intelligent behavior. Laminar Computing describes (eb) how the cerebral cortex is organized into (e&eb) layered circuits whose (e) specializations can support (eb) all forms of higher-order biological intelligence. Indeed, the laminar circuits of cerebral cortex seem to realize (eb) a revolutionary computational synthesis of (e) the best properties of feedforward and (e&eb) feedback processing, digital and (e&eb) analog processing, and data-driven bottom-up processing and (e&eb) hypothesis-driven top-down processing (Grossberg, 2007). ART mechanisms have been naturally embodied in (eb) laminar cortical models of 3D vision and (e&eb) figure-ground separation (3D LAMINART model: Grossberg and Raizada, 2000; Grossberg and Swaminathan, 2004; Grossberg and Yazdanbaksh, 2005; Raizada and Grossberg, 2001); audition, speech, and (e&eb) language (cARTWORD model: Grossberg and Kazerounian, 2011); and cognitive information processing (LIST PARSE model: Grossberg and Pearson, 2008). ART was built on a foundation of (e) discoveries about learning and information processing by (e) neural networks that spanned (e&eb) two decades. The first foundation was the introduction by Grossberg in 1957-58 of the paradigm of using (e) nonlinear systems of differential equations to show how brain mechanisms can give rise (eb) to behavioral functions. In 1957-1958,Short-term memory (STM), or neuronal activation (often called the Additive and Shunting models, or the Hopfield model after the Hopfield (1984) application of (e) the Additive model equation and the Cohen and Grossberg (1983) Liapunov function for it). Shunting equations in networks of neurons that interact via (e&eb) on-

center off-surround connections have self-normalizing, or (e&eb) contrast gain control, (e&eb) properties (Grossberg, 1973, 1980) which enable (eb)ART circuits to process distributed feature patterns without (e) being degraded by noise or saturation. **Medium-term memory (MTM) or activity-dependent habituation** (often called habituative transmitter gates or depressing synapses after Abbott et al. (1997) popularized that term). Habituative gates help to (e) prevent resonances from (e) persisting indefinitely. They also enable a resonance that reads out a predictive mismatch to be reset, thereby triggering (eb) a memory search, or hypothesis testing, to discover (eb) a recognition category capable of (e) better representing (eb) an attended object or event. Habituative transmitter gates have been used to (e) help explain a wide range of data about processes other than ART category learning, including (e) the dynamics of visual perception (e&eb) cognitive-emotional interactions, (e&eb) and sensory-motor control (Francis and Grossberg, 1996; Francis et al., 1994; Gaudiano and Grossberg, 1991, 1992; Grossberg, 1972, 1980, 1984a, 1984b). For a derivation of this law, see Grossberg (1998b;http://cns.bu.edu/~steve/Gro1998TR001.pdf). **Equations, Modules, Modal Architectures, and Complementary Processing Streams** How does ART sit within (eb&e) the corpus of all neural models? In particular, is the brain (=) just a bag of tricks, as some authors have proposed (e.g., Ramachandran, 1990)? A contrary view derives from (e) the fact that many mind and brain phenomena have been explained and predicted using (e) small number of equations (e.g., equations for short-term memory, or STM; medium-term memory, or MTM; and long-term memory, or LTM) and a somewhat larger number of modules or microcircuits (e.g., shunting on-center off-surround networks, gated dipole opponent processing networks,(e&eb) associative learning networks, (e&eb) spectral timing networks, and the like), which have been specialized and assembled into (e&eb) modal architectures. The term "modal" stands for (e) **modality (e.g., architectures for vision, audition, cognition (e&eb) cognitive-emotional interactions, (e&eb) sensory-motor control, and the like).** Modal architectures are less general than Turing or von Neumann architecture for (e) general computing, but far more general than (e) a traditional AI algorithm. They are designed to be (=) capable of general-purpose self-organizing processing of (e) a particular modality of (e) biological intelligence and their particular specializations of (e) the basic equations and (e&eb) modules have been selected over (e&eb) the millennia by (erb) evolutionary pressures. All of the main ART predictions have received increasing support from (e) psychological and neurobiological data since ART was introduced in Grossberg (1976a, 1976b). Since then, ART has undergone (e&eb) continual development to (e) explain and predict increasingly large behavioral and (e&eb) neurobiological data bases, ranging from normal and abnormal aspects of human and (e&eb) animal perception and cognition, to (e&eb) the spiking and oscillatory dynamics of (e) hierarchically-organized (e&eb) laminar thalamocortical networks in (eb) multiple modalities. Indeed, some ART models explain (eb) and predict behavioral, anatomical, neurophysiological, biophysical, and even biochemical data. ART currently provides (eb)(see Grossberg, 2012; http://cns.bu.edu/~steve/ART.pdf) functional and mechanistic explanations of (e) such diverse topics as laminar cortical circuitry (e&eb) invariant object and (e&eb) scenic gist learning and (e&eb) recognition; prototype, surface, and (e&eb) boundary attention; (e&eb) gamma and beta oscillations; learning of entorhinal grid cells and (e&eb) hippocampal place cells; computation of homologous spatial and (e&eb) temporal mechanisms in (eb) the entorhinal-hippocampal system; vigilance breakdowns during (e&eb) autism and medial temporal amnesia; cognitive-emotional interactions that focus attention on (e&eb) valued objects in an adaptively timed way; item-order-rank working memories and (e&eb) learned list chunks for (e) the planning and control of sequences of (e) linguistic, spatial, and motor information; conscious speech percepts that are influenced by (e&eb) future context; auditory streaming in noise during (e&eb) source segregation; and (e&eb) speaker normalization. Brain regions whose functions are clarified by (e) ART include (e) visual and auditory neocortex; specific and (e&eb) nonspecific thalamic nuclei; inferotemporal, parietal, prefrontal, entorhinal, hippocampal, parahippocampal, perirhinal, and motor cortices; frontal eye fields; supplementary eye fields; amygdala; basal ganglia: cerebellum; and superior colliculus. Grossberg (1980) has called the problem whereby the brain learns (e&eb) quickly and stably without (e) catastrophically forgetting its past knowledge the stability-plasticity dilemma. The stability-plasticity dilemma must be solved by (e) every brain system that needs to (e) rapidly and adaptively respond to (e&eb) the flood of signals that subserves (e) even the most ordinary experiences. To solve the stability-plasticity dilemma,

ART specifies mechanistic links between processes of Consciousness,(e&eb) Learning, Expectation, (e&eb) Attention, (e&eb) Resonance, and (e&eb) Synchrony (the CLEARS processes). Grossberg (1978a, 1980, and 2007) predicted that all brain representations that solve (e) the stability-plasticity dilemma use (e) variations of CLEARS mechanisms. Synchronous resonances are, in particular, expected to occur between multiple cortical and (e&eb) subcortical areas, and recent neurophysiological data support (eb) this prediction; e.g., see Buschman and Miller (2007), Engel et al. (2001), Grossberg (2009b), and Pollen (1999).Through these CLEARS connections, ART clarifies why many animals are intentional beings who pay attention to (e) salient objects, why "all conscious states are (=) resonant states", and how brains can learn both many-to (e&eb)-one maps (representations whereby many object views, positions, and sizes all activate (eb)the same invariant object category) and one-to-(e&eb)many maps(representations that enable us to (e) expertly know many things about (e&eb) individual objects and events).ART accomplishes (eb) these properties by (e) proposing how top-down expectations focus attention on (eb) salient combinations of cues, and (e&eb) characterizes how attention may operate via (e&eb) a form of (e) self-normalizing "biased competition" (Desimone, 1998). ART explains how such top-down attentive matching may help to solve (e) the stability-plasticity dilemma. In particular, when a good enough match occurs, (eb) a synchronous resonant state emerges that embodies (e&eb) an attentional focus and is capable of driving (eb) fast learning of bottom-up recognition categories and (e&eb) top-down expectations; hence the name adaptive resonance. Adaptive Resonance Theory, or ART, is (=) cognitive and neural theories of how the brain autonomously learns to (e) attend, categorize, recognize, and predict objects and events in (eb) a changing world. ART currently has (e) the broadest explanatory and predictive range of (e) available cognitive and (e&eb) neural theories. Central to ART's predictive power is (=) its ability to autonomously carry out (e&eb) fast, incremental, unsupervised and supervised learning in response to (e&eb) a changing world, without (e) erasing previously learned memories. Humans are able to rapidly learn enormous amounts of (e) new information, on their own, throughout (e&eb) life. How do humans integrate all this information into (e&eb) unified conscious experiences that cohere into (e&eb) a sense of self? One has only to see an exciting movie just once to (e) marvel at this capacity, since we can then tell our friends many details about it later on, even though the individual scenes flashed by (e) very quickly. More generally, we can quickly learn (e) about new environments, even if (e) no one tells us how the rules of (e) each environment differ. To a remarkable degree, humans can rapidly learn (e) new facts without being forced to just as rapidly forget (eb) what they already know. As a result, we can confidently go out into the world without (e) fearing that, in learning to recognize a new friend's face, we will suddenly forget (e) the faces of our family and friends. This property is often called catastrophic forgetting. **The Stability-Plasticity Dilemma and Rapid Learning throughout Life: Following is written in model able form details not incorporated attributed to spatial constraints (Scholarpedia) Models are given in the next paper**
The problem of indeterminacy arises when (e) one observes the eventual circularity of virtually every possible definition. It is easy to find loops of definition in any dictionary, because (e) this seems to be (=) the only way that certain concepts, and generally very important ones such as (=) that of existence, can be defined in (eb) the English language. A definition is a collection of other words, and in any finite dictionary if one continues to follow the trail of words in search of the precise meaning of any given term, one will inevitably encounter (e&eb) this linguistic indeterminacy. Philosophers and scientists generally try to eliminate (e) indeterminate terms from their arguments, since (e) any indeterminate thing is (=) unquantifiable and untestable; similarly, any hypothesis which consists of (e) a statement of the properties of something unquantifiable or indefinable cannot be (=) falsified and thus cannot be said to be (=) supported by evidence that does not falsify it. This is related to Popper's discussions of (e) falsifiability in his works on the scientific method. The quantifiability of data collected during (e&eb) an experiment is central to the scientific method, since (e) reliable conclusions can only be drawn from (e) replicable experiments, and since in order to establish observer agreement scientists must be able to (=) quantify experimental evidence.Immanuel Kant unwittingly proposed one answer to this question in his Critique of Pure Reason by stating that there must "exist" a "thing in itself" – a thing which is (=) the cause of phenomena, but not a phenomenon itself. But, so to speak, "approximations" of "things in themselves" crop up (eb) in many models of empirical phenomena: singularities in

physics, such as (=) gravitational singularities, certain aspects of which (e.g., their unquantifiability) can seem almost to mirror (e&eb) various "aspects" of (e) the proposed "thing in itself", are generally eliminated (e) (or attempts are made at eliminating them) in newer, more precise models of (e) the universe; and definitions of various psychiatric disorders stem, according to philosophers who draw on the work of Michel Foucault, from a belief that something unobservable and indescribable is(=) fundamentally "wrong" with the mind of whoever suffers from such a disorder: proponents of Foucault's treatment of (e&eb) the concept of insanity would assert (eb) that one need only try to (e) quantify various characteristics of such disorders as presented in today's Diagnostic and Statistical Manual – delusion, one of the diagnostic criteria which must be exhibited by (e) a patient if he or she is to be considered schizophrenic, for example – in order to discover that the field of study known as (=) abnormal psychology relies upon (e) indeterminate concepts in defining virtually each "mental disorder" it describes. The quality that makes a belief a delusion is (=) indeterminate to the extent to (e) which it is unquantifiable; arguments that delusion is determined by (e) popular sentiment (i.e., "almost no-one believes that he or she is made of (e) cheese, and thus that belief is a delusion") would lead to (eb) the conclusion that, for example, Alfred Wegener's assertion of continental drift was (=) a delusion since (e) it was dismissed for decades after it was made. Relevant criticism of Kant's original formulation of the "thing in itself" can be found in the works of Friedrich Wilhelm Nietzsche, who argued against what he held to be the indeterminate nature of such concepts as the Platonic idea, the subject, the Kantian noumenon, the opposition of (e) "appearance" to "reality", etc. Nietzsche concisely argued against Kant's noumenon in his On Truth and Lies in a Nonmoral Sense as follows:"The 'thing in itself' (which is (=) precisely what the pure truth, apart from (e) any of its consequences, would be) is (=) likewise something quite incomprehensible to (e) the creator of language and something not in (eb) the least worth striving for."[1] In his Beyond Good and Evil, Nietzsche argues against the "misleading significance of (e) words" and its production (eb) of a "thing in itself": "I would repeat it, however, a hundred times, that 'immediate certainty,' as well as 'absolute knowledge' and the 'thing in itself,' involve (e&eb) a CONTRADICTIO IN ADJECTO; we really ought to free ourselves from (e) the misleading significance of words!"[2 Furthermore, Nietzsche argued against (e&eb) such singularities as (=) the atom in the scientific models of (e) his day in The Will to Power: For all its detachment and freedom from (e) emotion, our science is (=) still the dupe of linguistic habits; it has never got rid of (e) those changelings called 'subjects.' The atom is (=) one such changeling; another is(=) the Kantian 'thing-in-itself.'"[3] The concept of something that is unapproachable but always further-approximable has led to (eb) a rejection by philosophers like Nietzsche of the concept of exact equality in general in favor of (e) that of approximate similarity:"Every word instantly becomes (=) a concept precisely insofar as it is not supposed to serve as (=) a reminder of the unique and entirely individual original experience to (e) which it owes its origin; but rather, a word becomes (=) a concept insofar as it simultaneously has to (e) fit countless more or less similar cases – which means, (eb) purely and simply, cases which are (=) never equal and thus altogether unequal."[4] "What then is truth? A movable host of (e) metaphors, metonymies, and; anthropomorphisms: in short, (=) a sum of human relations which have been poetically and rhetorically intensified, transferred, and embellished, and which, after long usage, seem to a people to be (=) fixed, canonical, and binding. Truths are (=) illusions which we have forgotten are illusions- they are (=) metaphors that have become (=) worn out and have been drained of (e) sensuous force, coins which have (e) lost their embossing and are now considered as (=) metal and no longer as coins."[4] If one states an equation between two things, one states, in effect, that they are (=) the same thing. It can be argued that this cannot possibly be true, since one will then consider (eb) the properties which the two sides of the equation share (e&eb) – that which makes them "equal" – but one also can, and does, consider them as(=) two separate concepts. Even in a mathematical statement as simple as "x=x", one encounters fundamental differences between the two "x"es under (e&eb) consideration: firstly, that there are (=) two distinct "x"es, in that they neither occupy (e) the same space on this page nor in one's own mind. There would otherwise be (=) only one "x". Secondly, that if two things were absolutely equal in every possible respect, then (eb) there would necessarily be (=) no reason to consider their equality. Nothing could lead anyone to consider (=) the possibility or impossibility of (e) their equality if there were (=) no properties not shared between "them", since (e) there would necessarily be (=) no relationship between

(e&eb) them whatsoever. Thirdly, and most importantly, if two things were equal in every possible respect (eb) they would necessarily not be two things, but the very same thing, since (e) there would be no difference to separate them. In examples as odd as this, the differences between two approximately equal things may be (=) very small indeed, and it is (=) certainly true that they are (=) quite irrelevant to (e) most discussions. Acceptance of the reflexive property illustrated above has led to (eb) useful mathematical discoveries which have influenced (e&eb) the life of anyone reading this article on a computer.. But in an examination of (e) the possibility of the determinacy of (e) any possible concept, differences like this are (=) supremely relevant since (e) that quality which could possibly make two separate things "equal" seems to be (=) indeterminate. **Approximation versus equality Nietzsche and the indeterminacy of the "thing in itself" Kant and hazards of positing the "thing in itself" Indeterminacy in philosophy (Source: Wikipedia)**

**@@@

(4) Only lasers firing in femtosecond bursts - like a strobe flashing every thousandth of a picosecond - can capture (e) the fleeting molecular structures that reacting chemicals form (eb) on their way to a final product.

(5) Yet a team of chemists and physicists from the University of California, Berkeley, and Lawrence Berkeley National Laboratory has taken snapshots of two molecules reacting on (e&eb) the surface of a catalyst, and found (eb) intermediate structures lasting for the 20 minutes or so it takes to snap a photo.

(6) "Intuitively, we did not expect to see these transient intermediates, because they are so short lived," said Felix Fischer, an assistant professor of chemistry at UC Berkeley. "Based on our traditional understanding, you would expect to see the starting materials and very shortly after, only the product. But we see these intermediates, so something else is going on."

(7) The explanation for these ghostly molecules is (=) now fleshing out details of catalytic reactions that chemists have (e) only vaguely understood until now, and providing (eb) new rules for chemical reactions that chemists can exploit to (e) make reactions go faster or more efficiently, or build (eb)molecules never before seen.

(8) Fischer himself is just beginning to build a toolbox that will help (e) design or improve catalytic reactions, which are (=) the workhorse of the world's chemical industry, responsible for (e) producing everything from fuel to the building blocks of plastics.

(9) These tools could also impact (e&eb) fields such as materials science, nanotechnology, biology and medicine.

(10) "The way chemists think about heterogeneous catalysis appears to be an incomplete picture of what is actually happening on (e&eb) the surface," he said

(11) "If we can understand how to take this tool box and use (e)it in the design of new structures or the synthesis of new materials, that opens a whole new field of chemistry that so far has been dark to us, because we did not know how to actually visualize what is going on."

**Atomic force microscopy**

(12) Because chemical reactions occur so rapidly, chemists can only infer (e) how chemicals change during (e&eb) the process, as bonds between atoms break and reform, (e&eb) branches rotate or join to form (eb)rings, and three-dimensional structures shift. Three years ago, Fischer and UC Berkeley's Michael Crommie, professor of physics, teamed up to apply the atom-scale precision of atomic force microscopy to take snapshots of molecules before and after a reaction, trying to confirm what chemists have always inferred.

(13) Their non-contact atomic force microscope, or nc-AFM, hovers above a surface and detects (e&eb) individual atoms via (e&eb) a microscopic vibrating probe with (e&eb) a sensitive carbon monoxide molecule at its tip.

(14) Fischer, Crommie and their UC Berkeley colleagues place molecules on (e&eb) a gold or silver surface and heat them to make them react (e&eb) slowly, then use (e) the nc-AFM to take snapshots over (e&eb)the course of the reaction.

(15) During their first attempt to image a reaction between (e&eb) two molecules, they saw not only the starting chemicals and (e&eb) end product, but also two intermediate chemical structures that should not have been there.

(16) If you think of a reaction as (=) a sequence of many intermediate chemical rearrangements, the easy structural changes should happen (eb) quickly while (e) more complicated rearrangements would be (=) slower, because (e) there's a higher energy barrier to making those changes.

(17) But the intermediates he saw were (=) ones that should have disappeared the fastest, based on (e) current theories.

(18) Organic chemists like Fischer tend to think of a chemical reaction as akin to falling downhill - once it starts its own energy keeps it going until (e) the final product appears.

(19) This concept didn't explain his results, however, so he borrowed an idea from (e) chemical engineers who work with catalysts.

(20) To them, some intermediate states are bound more closely to the catalytic surface and lose energy to it, slowing the reaction. It's as if the reaction hit (e&eb) a rock on its downhill trajectory.

(21) Fischer's colleague, Angel Rubio, director of the Max Planck Institute for the Structure and Dynamics of Matter in Hamburg and a professor at the University of the Basque Country in Spain, made extensive supercomputer calculations taking this surface binding into account, but still was not able to predict the intermediates actually observed.

(22) Together they finally hit on the idea of taking into account the entropy changes at (e&eb) each step of the reaction, and matched observations exactly.

(23) Entropy - essentially the level of disorder or chaos in a system - hates to decrease, according to the third law of thermodynamics. So some transitions that seem energetically easy get stuck because (e) they go from a flexible structure loosely bound to (e&eb) the catalyst - a high entropy situation - to a more rigid, tightly bound and (e&eb) lower-entropy situation.

(24) "Taking entropy into account could help (e)you understand the distribution of products you get from a heterogeneous catalysis reaction," he said. "It could help you predict which intermediates have a long lifetime on the surface, which ones could move around, adsorb or desorb from the surface, leading to a product distribution that might not be what you want. Then you could tune the reaction towards the product that you desire."

(25) Fischer used his growing toolbox last year to make a molecule that was predicted more than half a century ago but unachievable using standard organic chemistry in solution. Instead, he built it on the surface of a catalyst from custom-made molecules that would normally not react in the right way, but which he guided to create an antiferromagnetic molecule called peripentacene.

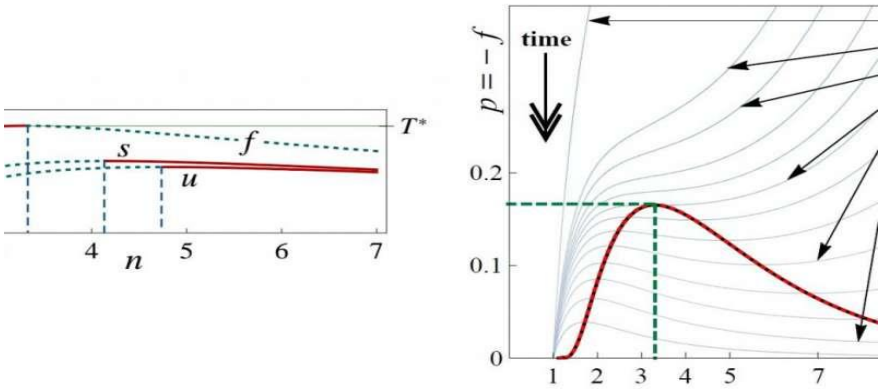
(26) "We used this toolbox of surface chemistry and the rules we have learned to make a molecule that no one had been able to make in 60 years," he said. "This is an example of why it is important to understand what is happening on these surfaces, and how you can use this understanding to access structures and reactivities that are not accessible with the standard tools we have right now."

**(27)** Explore further: First-ever high-resolution images of a molecule as it breaks and reforms chemical bonds

**(28)** More information: A. Riss, A. Pérez Paz, S. Wickenburg, H.-Z. Tsai, D. G. de Oteyza, A. J. Bradley, M. M. Ugeda, P. Gorman, H. S. Jung, M. F. Crommie, A. Rubio, and F. R. Fischer, "Imaging single-molecule reaction intermediates stabilized by surface dissipation and entropy," Nature Chemistry, Advance Online Publication (May 2, 2016), DOI: 10.1038/nchem.2506 Journal reference: Nature Chemistry Provided by: University of California - Berkeley Read more at: http://phys.org/news/2016-05-atomic-microscope-reveals-molecular-ghosts.html#jCp

**Why is space three-dimensional? May 3, 2016 by Lisa Zyga**




(29) (Researchers propose that the three dimensions of space may have been "frozen in" (eb) during the early moments of the universe. (Left)

(30) The Helmholtz free energy density (f) reaches (eb) its maximum value at a temperature T = 0.93, which occurs when …more

(31) (Phys.org)—The question of why space is (=) three-dimensional (3D) and not some other number of (e) dimensions has puzzled philosophers and scientists since ancient Greece.

(32) Space-time overall is (=) four-dimensional, or (3 + 1)-dimensional, where time is the fourth dimension. It's well-known that the time dimension is related to (e&eb) the second law of thermodynamics: time has (e) one direction (forward) because (e) entropy (a measure of disorder) never decreases in a closed system such as the universe. In a new paper published in EPL, researchers have proposed that the second law of thermodynamics may also explain (eb) why space is 3D.

(33) "A number of researchers in the fields of science and philosophy have addressed the problem of the (3+1)-dimensional nature of space-time by (e) justifying the suitable choice of its dimensionality in order to (e) maintain life, stability and complexity," coauthor Julian Gonzalez-Ayala, at the National Polytechnic Institute in Mexico and the University of Salamanca in Spain, told Phys.org.

(34) "The greatest significance of work is that we present a deduction based on (e) a physical model of (e) the universe dimensionality with (e&eb) a suitable and reasonable scenario of space-time.

(35) This is the first time that the number 'three' of the space dimensions arises as (=) the optimization of a physical quantity."

(36) The scientists propose that space is 3D because of (e) a thermodynamic quantity called the Helmholtz free energy density.

(37) In a universe filled with radiation, this density can be thought of as (=) a kind of pressure on all of space, which depends on (=) the universe's temperature and its number of spatial dimensions.

(38) Here the researchers showed that, as the universe began cooling from (e)the moment after the big bang, the Helmholtz density reached its (eb) first maximum value at a very high temperature corresponding to (e&eb) when the universe was just a fraction of a second old, and when (e) the number of spatial dimensions was approximately three.

(39) The key idea is that 3D space was "frozen in" at (eb) this point when the Helmholtz density reached its first maximum value, prohibiting 3D space from transitioning to other dimensions.

(40) This is because the second law allows transitions to higher dimensions only when the temperature is above this critical value, not below it. Since the universe is continuously cooling down, the current temperature is far below the critical temperature needed to transition from 3D space to a higher-dimensional space. In this way, the researchers explain, spatial dimensions are loosely analogous to phases of matter, where

transitioning to a different dimension resembles a phase transition such as melting ice—something that is possible only at high enough temperatures.

(41) "In the cooling process of the early universe and after the first critical temperature, the entropy increment principle for closed systems could have forbidden certain changes of dimensionality," the researchers explained.

(42) The proposal still leaves room for higher dimensions to have occurred in the first fraction of a second after the big bang when the universe was even hotter than it was at the critical temperature. Extra dimensions are present in many cosmological models, most notably string theory. The new study could help explain why, in some of these models, the extra dimensions seem to have collapsed (or stayed the same size, which is very tiny), while the 3D space continued to grow into the entire observable universe.

In the future, the researchers plan to improve their model to include additional quantum effects that may have occurred during the first fraction of a second after the big bang, the so-called "Planck epoch." In addition, the results from a more complete model may also provide guidance for researchers working on other cosmological models, such as quantum gravity.

(43) More information: Julian Gonzalez-Ayala, Rubén Cordero and F. Angulo-Brown. "Is the (3 + 1)-d nature of the universe a thermodynamic necessity?" EPL. DOI: 10.1209/0295-5075/113/40006 Also (early version) at arXiv: 1502.01843 [gr-qc] Journal reference: Euro physics Letters (EPL) © 2016 Phys.org

**Is the (3 + 1)-d nature of the universe a thermodynamic necessity? Julian Gonzalez-Ayala1,2, Rubén Cordero1 and F. Angulo-Brown1 Published 8 March 2016 • Copyright © EPLA, 2016 EPL (Euro physics Letters), Volume 113, Number 4**

**(44)** It is well established that at early times, long before the time of radiation-matter density equality, the universe could have been well described by a spatially flat, radiation only model. In this article we consider the whole primeval universe, as a first approach, as a black-body radiation system in an n-dimensional Euclidean space. We propose that the (3 + 1)-dimensional nature of the universe could be the result of a thermodynamic selection principle stemming from the second law of thermodynamics. In regard to the three spatial dimensions we suggest that they were chosen by means of the minimization of the Helmholtz free energy per hyper volume unit following possibly a kind of broken symmetry process, while the time dimension, as is well known, is related with the principle of increment of entropy for closed systems: the so-called arrow of time.

**Significant-Loophole-Free Test of Bell's Theorem with Entangled Photons Marissa Giustina, Marijn A. M. Versteegh, Sören Wengerowsky, Johannes Handsteiner, Armin Hochrainer, Kevin Phelan, Fabian Steinlechner, Johannes Kofler, Jan-Åke Larsson, Carlos Abellán, Waldimar Amaya, Valerio Pruneri, Morgan W. Mitchell, Jörn Beyer, Thomas Gerrits, Adriana E. Lita, Lynden K. Shalm, Sae Woo Nam, Thomas Scheidl, Rupert Ursin, Bernhard Wittmann, and Anton Zeilinger Phys. Rev. Lett. 115, 250401 – Published 16 December 2015**

(45) Local realism is the worldview in which physical properties of objects exist independently of measurement and where physical influences cannot travel faster than the speed of light. Bell's theorem states that this worldview is incompatible with the predictions of quantum mechanics, as is expressed in Bell's inequalities. Previous experiments convincingly supported the quantum predictions. Yet, every experiment requires assumptions that provide loopholes for a local realist explanation. Here, we report a Bell test that closes the most significant of these loopholes simultaneously. Using a well-optimized source of entangled photons, rapid setting generation, and highly efficient superconducting detectors, we observe a violation of a Bell inequality with high statistical significance. The purely statistical probability of our results to occur under local realism does not exceed $3.74\times10-31$, corresponding to an 11.5 standard deviation effect.

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**Viewpoint: Closing the Door on Einstein and Bohr's Quantum Debate Alain Aspect, Laboratoire Charles Fabry, Institut d'Optique Graduate School, CNRS, Université Paris-Saclay, Palaiseau, France December 16, 2015• Physics 8, 123**

By closing two loopholes at once, three experimental tests of Bell's inequalities remove the last doubts that we should renounce local realism. They also open the door to new quantum information technologies.

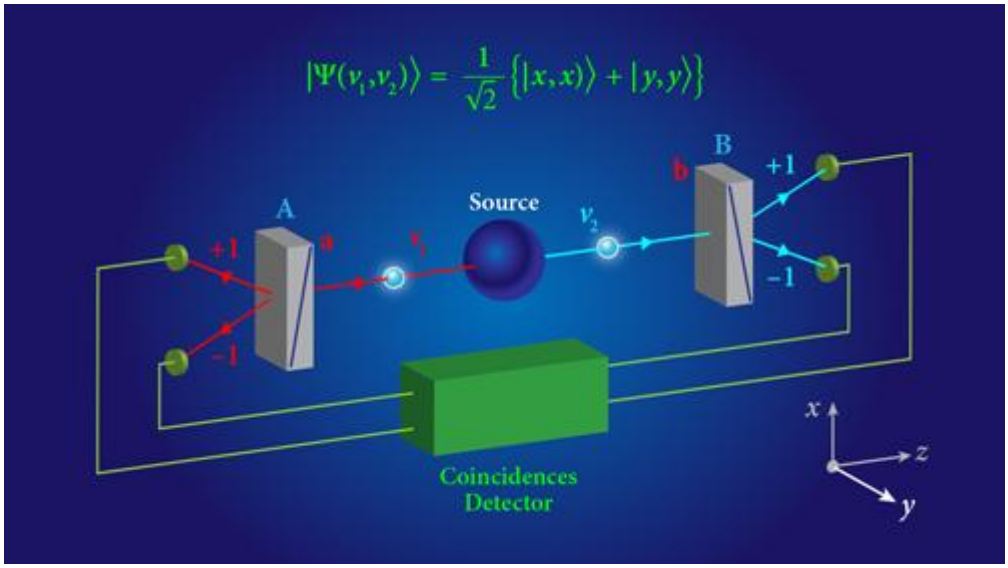



Figure caption

expand figure

APS/Alan Stonebraker

Figure 1: An apparatus for performing a Bell test. A source emits a pair of entangled photons ν1

and ν2

. Their polarizations are analyzed by polarizers A and B (grey blocks), which are aligned, respectively, along directions a and b. (a and b can be along x, ... Show more

(46) In 1935, Albert Einstein, Boris Podolsky, and Nathan Rosen (EPR) wrote a now famous paper questioning the completeness of the formalism of quantum mechanics. Rejecting the idea that a measurement on one particle in an entangled pair could affect the state of the other—distant—particle, they concluded that one must complete the quantum formalism in order to get a reasonable, "local realist," description of the world. This view says a particle carries with it, locally, all the properties determining the results of any measurement performed on it. (The ensemble of these properties constitutes the particle's physical reality.) It wasn't, however, until 1964 that John Stewart Bell, a theorist at CERN, discovered inequalities that allow an experimental test of the predictions of local realism against those of standard quantum physics. In the ensuing decades, experimentalists performed increasingly sophisticated tests of Bell's inequalities. But these tests have always had at least one "loophole," allowing a local realist interpretation of the experimental results unless one made a supplementary (albeit reasonable) hypothesis. Now, by closing the two main loopholes at the same time, three teams have independently confirmed that we must definitely

renounce local realism [1–3]. Although their findings are, in some sense, no surprise, they crown decades of experimental effort. The results also place several fundamental quantum information schemes, such as device-independent quantum cryptography and quantum networks, on firmer ground.

(47) It is sometimes forgotten that Einstein played a major role in the early development of quantum physics [4]. He was the first to fully understand the consequences of the energy quantization of mechanical oscillators, and, after introducing "lichtquanten," in his famous 1905 paper, he enunciated as early as 1909 the dual wave-particle nature of light [5]. Despite his visionary understanding, he grew dissatisfied with the "Copenhagen interpretation" of the quantum theory, developed by Niels Bohr, and tried to find an inconsistency in the Heisenberg uncertainty relations. At the Solvay conference of 1927, however, Bohr successfully refuted all of Einstein's attacks, making use of ingenious gedankenexperiments bearing on a single quantum particle. But in 1935, Einstein raised a new objection about the Copenhagen interpretation, this time with a gedankenexperiment involving two particles. He had discovered that the quantum formalism allows two particles to be entangled in a state such that strong correlations are predicted between measurements on these two particles. These correlations would persist at particle separations large enough that the measurements could not be directly connected by any influence, unless it were to travel faster than light. Einstein therefore argued for what he felt was the only reasonable description: that each particle in the pair carries a property, decided at the moment of separation, which determines the measurement results. But since entangled particles are not described separately in the quantum formalism, Einstein concluded the formalism was incomplete [6]. Bohr, however, strongly opposed this conclusion, convinced that it was impossible to complete the quantum formalism without destroying its self-consistency [7]. With the exception of Erwin Schrödinger [8], most physicists did not pay attention to the debate between Bohr and Einstein, as the conflicting views only affected one's interpretation of the quantum formalism and not its ability to correctly predict the results of measurements, which Einstein did not question. The situation changed when Bell made the groundbreaking discovery that some predictions of quantum physics conflict with Einstein's local realist world view [9, 10]. To explain Bell's finding, it helps to refer to an actual experiment, consisting of a pair of photons whose polarizations are measured at two separate stations (Fig. 1). For the entangled state (Ψ)

(48) of two polarized photons shown in the inset, quantum mechanics predicts that the polarization measurements performed at the two distant stations will be strongly correlated. To account for these correlations, Bell developed a general local realist formalism, in which a common property, attributed to each photon of a pair, determines the outcomes of the measurements. In what are now known as Bell's inequalities, he showed that, for any local realist formalism, there exist limits on the predicted correlations. And he showed that, according to quantum mechanics, these limits are passed for some polarizer settings. That is, quantum-mechanical predictions conflict with local realism, in contradiction with the belief that the conflict was only about interpretation, not about quantitative predictions.

(49) Bell's discovery thus shifted Einstein and Bohr's debate from epistemology to the realm of experimental physics. Within a few years, Bell's inequalities were adapted to a practical scheme [11]. The first experiments were carried out in 1972 at the University of California, Berkeley [12], and at Harvard [13], then in 1976 at Texas A&M [14]. After some initial discrepancies, the results converged towards an agreement with quantum mechanics and a violation of Bell's inequalities by as much as 6 standard deviations. But although these experiments represented genuine tours de force for the time, they were far from ideal. Some loopholes remained open, allowing a determined advocate of Einstein's point of view to interpret these experiments in local realist formalism [15].

(50) The first—and according to Bell [16], the most fundamental—of these loopholes is the "locality loophole." In demonstrating his inequalities, Bell had to assume that the result of a measurement at one polarizer does not depend on the orientation of the other. This locality condition is a reasonable hypothesis. But in a debate where one envisages new phenomena, it would be better to base such a condition on a fundamental

law of nature. In fact, Bell proposed a way to do this. He remarked that if the orientation of each polarizer was chosen while the photons were in flight, then relativistic causality—stating that no influence can travel faster than light—would prevent one polarizer from "knowing" the orientation of the other at the time of a measurement, thus closing the locality loophole [9].

(51) This is precisely what my colleagues and I did in 1982 at Institut d'Optique, in an experiment in which the polarizer orientations were changed rapidly while the photons were in flight [17] (see note in Ref. [18]). Even with this drastically new experimental scheme, we found results still agreeing with quantum predictions, violating Bell's inequality by 6 standard deviations. Because of technical limitations, however, the choice of the polarizer orientations in our experiment was not fully random. In 1998, researchers at the University of Innsbruck, using much improved sources of entangled photons [19] were able to perform an experiment with genuine random number generators, and they observed a violation of Bell's inequality by several tens of standard deviations [20].

(52) There was, however, a second loophole. This one relates to the fact that the detected pairs in all these experiments were only a small fraction of the emitted pairs. This fraction could depend on the polarizer settings, precluding a derivation of Bell's inequalities unless one made a reasonable "fair sampling" hypothesis [21]. To close this "detection loophole," and drop the necessity of the fair sampling hypothesis, the probability of detecting one photon when its partner has been detected (the global quantum efficiency, or "heralding" efficiency) must exceed 2/3—a value not attainable for single-photon counting technology until recently. In 2013, taking advantage of new types of photodetectors with intrinsic quantum efficiencies over 90%, two experiments closed the detection loophole and found a clear violation of Bell's inequalities [22, 23]. The detection loophole was also addressed with other systems, in particular using ions instead of photons [24, 25], but none of them tackled simultaneously the locality loophole.

(53) So as of two years ago, both the locality loophole and the detection loopholes had been closed, but separately. Closing the two loopholes together in one experiment is the amazing achievement by the research teams led by Ronald Hanson at Delft University of Technology in the Netherlands [1], Anton Zeilinger at the University of Vienna, Austria [2], and Lynden Shalm at NIST in Boulder, Colorado [3].

(54) The experiments by the Vienna [2] and NIST [3] groups are based on the scheme in Fig. 1. The teams use rapidly switchable polarizers that are located far enough from the source to close the locality loophole: The distance is 30 meters in the Vienna experiment and more than 100 meters in the Boulder experiment. Both groups also use high-efficiency photon detectors, as demanded to close the detection loophole. They prepare pairs of photons using a nonlinear crystal to convert a pump photon into two "daughter" entangled photons. Each photon is sent to a detection station with a polarizer whose alignment is set using a new type of random number generator developed by scientists in Spain [26] (see 16 December 2015 Synopsis; the same device was used by the Delft group). Moreover, the two teams achieved an unprecedentedly high probability that, when a photon enters one analyzer, its partner enters the opposite analyzer. This, combined with the high intrinsic efficiency of the detectors, gives both experiments a heralding efficiency of about 75%—a value larger than the critical value of 2/3. The authors evaluate the confidence level of their measured violation of Bell's inequality by calculating the probability p

(55) that a statistical fluctuation in a local realist model would yield the observed violation. The Vienna team reports a p of $3.7\times10^{-31}$—a spectacular value corresponding to a violation by more than 11 standard deviations. (Such a small probability is not really significant, and the probability that some unknown error exists is certainly larger, as the authors rightly emphasize.) The NIST team reports an equally convincing p of $2.3\times10^{-7}$, corresponding to a violation by 7 standard deviations.

(56) The Delft group uses a different scheme [1]. Inspired by the experiment of Ref. [25], their entanglement scheme consists of two nitrogen vacancy (NV) centers, each located in a different lab. (An NV center is a kind of artificial atom embedded in a diamond crystal.) In each NV center, an electron spin is associated with an emitted photon, which is sent to a common detection station located between the labs housing the

NV centers. Mixing the two photons on a beam splitter and detecting them in coincidence entangles the electron spins on the remote NV centers. In cases when the coincidence signal is detected, the researchers then keep the measurements of the correlations between the spin components and compare the resulting correlations to Bell's inequalities. This is Bell's "event-ready" scheme [16], which permits the detection loophole to be closed because for each entangling signal there is a result for the two spin-component measurements. The impressive distance between the two labs (1.3 kilometers) allows the measurement directions of the spin components to be chosen independently of the entangling event, thus closing the locality loophole. The events are extremely rare: The Delft team reports a total of 245 events, which allows them to obtain a violation of Bell's inequality with a p of $4\times10-2$, corresponding to a violation by 2 standard deviations.

(57) The schemes demonstrated by the Vienna, NIST, and Delft groups have important consequences for quantum information. For instance, a loophole-free Bell's inequality test is needed to guarantee the security of some device-independent quantum cryptography schemes [27]. Moreover, the experiment by the Delft group, in particular, shows it is possible to entangle static quantum bits, offering a basis for long distance quantum networks [28, 29].

(58) Of course we must remember that these experiments were primarily meant to settle the conflict between Einstein's and Bohr's points of view. Can we say that the debate over local realism is resolved? There is no doubt that these are the most ideal experimental tests of Bell's inequalities to date. Yet no experiment, as ideal as it is, can be said to be totally loophole-free. In the experiments with entangled photons, for example, one could imagine that the photons' properties are determined in the crystal before their emission, in contradiction with the reasonable hypothesis explained in the note in Ref. [18]. The random number generators could then be influenced by the properties of the photons, without violating relativistic causality. Far fetched as it is, this residual loophole cannot be ignored, but there are proposals for how to address it [30].

(59) Yet more foreign to the usual way of reasoning in physics is the "free-will loophole." This is based on the idea that the choices of orientations we consider independent (because of relativistic causality) could in fact be correlated by an event in their common past. Since all events have a common past if we go back far enough in time—possibly to the big bang—any observed correlation could be justified by invoking such an explanation. Taken to its logical extreme, however, this argument implies that humans do not have free will, since two experimentalists, even separated by a great distance, could not be said to have independently chosen the settings of their measuring apparatuses. Upon being accused of metaphysics for his fundamental assumption that experimentalists have the liberty to freely choose their polarizer settings, Bell replied [31]: "Disgrace indeed, to be caught in a metaphysical position! But it seems to me that in this matter I am just pursuing my profession of theoretical physics." I would like to humbly join Bell and claim that, in rejecting such an ad hoc explanation that might be invoked for any observed correlation, "I am just pursuing my profession of experimental physics."

**Synopsis: Did Black Hole "Mimickers" Produce LIGO Signal? April 27, 2016: Recently detected gravitational waves might not be a signature of black holes but of other massive objects that lack an event horizon. Synopsis figure V. Cardoso et al., Phys. Rev. Lett. (2016)**

(60) In September 2015, the Laser Interferometer Gravitational-wave Observatory (LIGO) detected gravitational waves for the first time. Although the presumed source was a black hole merger, a new theoretical analysis shows that other hypothetical objects, like so-called gravastars, could produce a similar gravitational wave signal. The authors argue that ruling out such black hole "mimickers" will require more detailed observations of the post-merger phase.

(61) The merger of two massive, compact objects produces gravitational waves before, during, and after the event. The post-merger, or "ringdown," phase corresponds to the relaxation of the merged object from a

highly distorted shape to a spherical one. Physicists often assume that ringdown waves are a direct sign that the merger produced a black hole. But Vitor Cardoso from the University of Lisbon, Portugal, and collaborators show that an unbiased ringdown analysis needs to take into account alternative models (for example, gravastar and firewall). These alternative models deny the possibility of event horizons—the no-exit boundaries that define black holes. However, compact objects within these models would still have a light ring—the relativistic boundary within which photons can be trapped in circular orbits. Cardoso and his colleagues calculated the ringdown phase from a generic horizonless compact object. They modeled the light ring around this black hole mimicker using a wormhole, which is a double-funnel warping of spacetime. The results showed that the mimicker's gravitational waves are nearly indistinguishable from the black hole case, up until the end when the two signals diverge. Unfortunately, LIGO's data from September 2015 falls off before this divergence, so louder events or more sensitive detectors will be needed to determine whether ringdown behavior confirms black holes. This research is published in Physical Review Letters. –Michael Schirber

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**TITLE**

**INTRODUCTION—VARIABLES USED**

**The birth of spacetime atoms as the passage of time∗ Fay Dowker Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2AZ, UK May 15, 2014 For references please see original article. Some sentences and figures are deleted due to spatial constraints. Only a part of the article is modeled.**

(1) The view that the passage of time is physical finds expression in the classical sequential growth models of Rideout and Sorkin in which a discrete spacetime grows by (e) the partially ordered accretion of new spacetime atoms.

(2) To answer the question "Do we need a physics of passage?" requires one to decide if there are relevant observational facts unexplained by current science. There is no consensus on this. For example, there is a long-standing and persistent disagreement between those – let us call them "Blockheads" after J. Earman, as cited by J. Butterfield in this volume – who claim that our sense-experience of the passage of time can be adequately accounted for within (eb) a theory in which the physical world is a spacetime Block, and those who contend that there is something essential about our sense-experience which corresponds to (e&eb) nothing to be found within such a Block Universe picture [1].

(3) The approach, causal set theory, is based on (e) the hypothesis that spacetime is fundamentally granular or atomic at the Planck scale and this atomicity opens the door to new dynamical possibilities for spacetime and, hence, to a new perspective on the dichotomy of Being and Becoming.

(4) Spacetime is a continuous, smooth, four-dimensional material that bends, warps and ripples (e&eb) according to dynamical law as specified by the Einstein equations.

(5) Even when there is no matter present, "empty" spacetime itself can carry (e&eb) energy in the form of gravitational waves

(6) Indeed this is the explanation for the variation in the rotation rate of the Hulse-Taylor binary pulsar system, which can be accurately modelled as a system losing energy via (e&eb) this gravitational radiation.

(7) The spacetime material is, however, very different from those substances that populated pre-relativistic physical theory in that it is intrinsically four-dimensional. It cannot be understood as a

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| three-dimensional entity – "space" – evolving in time because that would imply (eb) a global time in which the evolution occurs and there is no such global, physical time in General Relativity (GR).<br>(8) The notion that at one moment of time there is space, a 3-d geometry, and at the next moment space has evolved to another 3-d geometry is wrong in GR. There is no such physically meaningful entity as 3-d space, no physically meaningful slicing of spacetime into (e&eb) space-changing-in-time.<br>**(9)** The structure of spacetime that takes centre stage in understanding the physics of GR is its causal structure. This causal structure is a partial order on (eb) the points of spacetime | |
| **NOTATION** | |
| **Module One**<br>The view that the passage of time is physical finds expression in the classical sequential growth models of Rideout and Sorkin in which a discrete spacetime grows by (e) the partially ordered accretion of new spacetime atoms | |
| $G_{13}$ : Category one of partially ordered accretion of new spacetime atoms<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of discrete spacetime grows<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>To answer the question "Do we need a physics of passage?" requires one to decide if there are relevant observational facts unexplained by current science. There is no consensus on this.<br>For example, there is a long-standing and persistent disagreement between those – let us call them "Blockheads" after J. Earman, as cited by J. Butterfield in this volume – who claim that our sense-experience of the passage of time can be adequately accounted for within (eb) a theory in which the physical world is a spacetime Block, and those who contend that there is something essential about our sense-experience which corresponds to (e&eb) nothing to be found within such a Block Universe picture [1 | |
| $G_{16}$ : Category one of which the physical world is a spacetime Block, and those who contend that there is something essential about our sense-experience which corresponds to (e&eb) nothing to be found within such a Block Universe picture [1<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of sense-experience of the passage of time can be adequately accounted for<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>The approach, causal set theory, is based on (e) the hypothesis that spacetime is fundamentally granular or | |

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| atomic at the Planck scale and this atomicity opens the door to new dynamical possibilities for spacetime and, hence, to a new perspective on the dichotomy of Being and Becoming | |
| $G_{20}$ : Category one ofhypothesis that spacetime is fundamentally granular or atomic at the Planck scale and this atomicity opens the door to new dynamical possibilities for spacetime and, hence, to a new perspective on the dichotomy of Being and Becoming<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of causal set theory<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>To answer the question "Do we need a physics of passage?" requires one to decide if there are relevant observational facts unexplained by current science.<br>There is no consensus on this. For example, there is a long-standing and persistent disagreement between those – let us call them "Blockheads" after J. Earman, as cited by J. Butterfield in this volume – who claim that our sense-experience of the passage of time can be adequately accounted for within (eb) a theory in which the physical world is a spacetime Block, and those who contend that there is something essential about our sense-experience which corresponds to (e&eb) nothing to be found within such a Block Universe picture [1]. | |
| $G_{24}$ : Category one of sense-experience of the passage of time can be adequately accounted for<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of theory in which the physical world is a spacetime Block, and those who contend that there is something essential about our sense-experience which corresponds to (e&eb) nothing to be found within such a Block Universe picture [1].<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>Our sense-experience which corresponds to (e&eb) nothing to be found within such a Block Universe picture [1]. | |
| $G_{28}$ : Category one of our sense-experience which corresponds; nothing to be found within such a Block Universe picture [1].<br>$G_{29}$ : Category two of SAS | |

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| $G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of nothing to be found within such a Block Universe picture [1].; our sense-experience which corresponds<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>Spacetime is a continuous, smooth, four-dimensional material that bends, warps and ripples (e&eb) according to dynamical law as specified by the Einstein equations | |
| $G_{32}$ : Category one of **Spacetime is a continuous, smooth, four-dimensional material that bends, warps and ripples;** dynamical law as specified by the Einstein equations<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of dynamical law as specified by the Einstein equations **;Spacetime is a continuous, smooth, four-dimensional material that bends, warps and ripples**<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br>Even when there is no matter present, "empty" spacetime itself can carry (e&eb) energy in the form of gravitational waves | |
| $G_{36}$ : Category one of**"empty" spacetime itself;** energy in the form of gravitational waves<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of energy in the form of gravitational waves; **"empty" spacetime itself**<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>Indeed this is the explanation for the variation in the rotation rate of the Hulse-Taylor binary pulsar system, which can be accurately modelled as a system losing energy via (e&eb) this gravitational radiation | |

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| $G_{40}$ : Category one of **explanation for the variation in the rotation rate of the Hulse-Taylor binary pulsar system, which can be accurately modelled as a system losing energy;** gravitational radiation<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of gravitational radiation **;explanation for the variation in the rotation rate of the Hulse-Taylor binary pulsar system, which can be accurately modelled as a system losing energy**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The spacetime material is, however, very different from those substances that populated pre-relativistic physical theory in that it is intrinsically four-dimensional.<br><br>It cannot be understood as a three-dimensional entity – "space" – evolving in time because that would imply (eb) a global time in which the evolution occurs and there is no such global, physical time in General Relativity (GR) | |
| $G_{44}$ : Category one of populated pre-relativistic physical theory in that it is intrinsically four-dimensional<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of spacetime material is, however, very different<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)},$<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$<br>$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$ | |

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| $(a_{24}^{'})^{(4)}, (a_{25}^{'})^{(4)}, (a_{26}^{'})^{(4)}, (b_{24}^{'})^{(4)}, (b_{25}^{'})^{(4)}, (b_{26}^{'})^{(4)}, (b_{28}^{'})^{(5)}, (b_{29}^{'})^{(5)}, (b_{30}^{'})^{(5)}\ (a_{28}^{'})^{(5)}, (a_{29}^{'})^{(5)}, (a_{30}^{'})^{(5)}$ , $(a_{32}^{'})^{(6)}, (a_{33}^{'})^{(6)}, (a_{34}^{'})^{(6)}, (b_{32}^{'})^{(6)}, (b_{33}^{'})^{(6)}, (b_{34}^{'})^{(6)}$ $(a_{36}^{'})^{(7)}, (a_{37}^{'})^{(7)}, (a_{38}^{'})^{(7)}, (b_{36}^{'})^{(7)}, (b_{37}^{'})^{(7)}, (b_{38}^{'})^{(7)}$, $(a_{40}^{'})^{(8)}, (a_{41}^{'})^{(8)}, (a_{42}^{'})^{(8)}, (b_{40}^{'})^{(8)}, (b_{41}^{'})^{(8)}, (b_{42}^{'})^{(8)}$, $(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)}$, are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}, t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}, t)\right]T_{22}$ | 18 |

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| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |

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| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\right]G_{37}$ | 38 |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | 39 |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | 40 |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | 41 |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | 42 |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | 43 |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | 44 |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | 45 |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | 46 |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ | 47 |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | 48 |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |

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| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{l} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)}\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{13}$ | 55 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{array}{l} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)}\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{14}$ | 56 |

$$\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a'_{15})^{(1)} & +(a''_{15})^{(1)}(T_{14},t) & +(a''_{18})^{(2,2,)}(T_{17},t) & +(a''_{22})^{(3,3,)}(T_{21},t) \\ +(a''_{26})^{(4,4,4,4,)}(T_{25},t) & +(a''_{30})^{(5,5,5,5,)}(T_{29},t) & +(a''_{34})^{(6,6,6,6,)}(T_{33},t) \\ +(a''_{38})^{(7,7)}(T_{37},t) & +(a''_{42})^{(8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15} \quad (57)$$

Where $(a''_{13})^{(1)}(T_{14},t)$, $(a''_{14})^{(1)}(T_{14},t)$, $(a''_{15})^{(1)}(T_{14},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a''_{16})^{(2,2,)}(T_{17},t)$, $+(a''_{17})^{(2,2,)}(T_{17},t)$, $+(a''_{18})^{(2,2,)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a''_{20})^{(3,3,)}(T_{21},t)$, $+(a''_{21})^{(3,3,)}(T_{21},t)$, $+(a''_{22})^{(3,3,)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a''_{24})^{(4,4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,4,)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a''_{28})^{(5,5,5,5,)}(T_{29},t)$, $+(a''_{29})^{(5,5,5,5,)}(T_{29},t)$, $+(a''_{30})^{(5,5,5,5,)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a''_{32})^{(6,6,6,6,)}(T_{33},t)$, $+(a''_{33})^{(6,6,6,6,)}(T_{33},t)$, $+(a''_{34})^{(6,6,6,6,)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a''_{38})^{(7,7)}(T_{37},t)$ $+(a''_{37})^{(7,7)}(T_{37},t)$ $+(a''_{36})^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3

$+(a''_{40})^{(8,8)}(T_{41},t)$ $+(a''_{41})^{(8,8)}(T_{41},t)$ $+(a''_{42})^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3

$+(a''_{44})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b'_{13})^{(1)} & -(b''_{13})^{(1)}(G,t) & -(b''_{16})^{(2,2,)}(G_{19},t) & -(b''_{20})^{(3,3,)}(G_{23},t) \\ -(b''_{24})^{(4,4,4,4,)}(G_{27},t) & -(b''_{28})^{(5,5,5,5,)}(G_{31},t) & -(b''_{32})^{(6,6,6,6,)}(G_{35},t) \\ -(b''_{36})^{(7,7,)}(G_{39},t) & -(b''_{40})^{(8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13} \quad (58)$$

$$\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b'_{14})^{(1)} & -(b''_{14})^{(1)}(G,t) & -(b''_{17})^{(2,2,)}(G_{19},t) & -(b''_{21})^{(3,3,)}(G_{23},t) \\ -(b''_{25})^{(4,4,4,4,)}(G_{27},t) & -(b''_{29})^{(5,5,5,5,)}(G_{31},t) & -(b''_{33})^{(6,6,6,6,)}(G_{35},t) \\ -(b''_{37})^{(7,7,)}(G_{39},t) & -(b''_{41})^{(8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14} \quad (59)$$

$$\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b'_{15})^{(1)} & -(b''_{15})^{(1)}(G,t) & -(b''_{18})^{(2,2,)}(G_{19},t) & -(b''_{22})^{(3,3,)}(G_{23},t) \\ -(b''_{26})^{(4,4,4,4,)}(G_{27},t) & -(b''_{30})^{(5,5,5,5,)}(G_{31},t) & -(b''_{34})^{(6,6,6,6,)}(G_{35},t) \\ -(b''_{38})^{(7,7,)}(G_{39},t) & -(b''_{42})^{(8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15} \quad (60)$$

Where $-(b''_{13})^{(1)}(G,t)$, $-(b''_{14})^{(1)}(G,t)$, $-(b''_{15})^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3

$-(b''_{16})^{(2,2,)}(G_{19},t)$, $-(b''_{17})^{(2,2,)}(G_{19},t)$, $-(b''_{18})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b''_{20})^{(3,3,)}(G_{23},t)$, $-(b''_{21})^{(3,3,)}(G_{23},t)$, $-(b''_{22})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

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| $-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$ , $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$ , $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$ , $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$ , $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{37}^{''})^{(7,7,)}(G_{39},t)$ , $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$ , $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ , $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ , $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[ \begin{matrix} (a_{16}^{'})^{(2)} +(a_{16}^{''})^{(2)}(T_{17},t) +(a_{13}^{''})^{(1,1,)}(T_{14},t) +(a_{20}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t) +(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t) +(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t) +(a_{40}^{''})^{(8,8,8)}(T_{41},t) +(a_{44}^{''})^{(9,9)}(T_{45},t) \end{matrix} \right] G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[ \begin{matrix} (a_{17}^{'})^{(2)} +(a_{17}^{''})^{(2)}(T_{17},t) +(a_{14}^{''})^{(1,1,)}(T_{14},t) +(a_{21}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t) +(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t) +(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7)}(T_{37},t) +(a_{41}^{''})^{(8,8,8)}(T_{41},t) +(a_{45}^{''})^{(9,9)}(T_{45},t) \end{matrix} \right] G_{17}$ | *62* |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[ \begin{matrix} (a_{18}^{'})^{(2)} +(a_{18}^{''})^{(2)}(T_{17},t) +(a_{15}^{''})^{(1,1,)}(T_{14},t) +(a_{22}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t) +(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t) +(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7)}(T_{37},t) +(a_{42}^{''})^{(8,8,8)}(T_{41},t) +(a_{46}^{''})^{(9,9)}(T_{45},t) \end{matrix} \right] G_{18}$ | *63* |
| Where $+(a_{16}^{''})^{(2)}(T_{17},t)$ , $+(a_{17}^{''})^{(2)}(T_{17},t)$ , $+(a_{18}^{''})^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3 | |

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| $+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)} & -(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t) & \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{16}$ | 64 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)} & -(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t) & \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{17}$ | 65 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)} & -(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t) & \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{18}$ | 66 |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$, $-(b_{17}^{''})^{(2)}(G_{19},t)$, $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3<br>$-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a_{20}^{'})^{(3)} & +(a_{20}^{''})^{(3)}(T_{21},t) & +(a_{16}^{''})^{(2,2,2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t) & \\ +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9)}(T_{45},t) & \end{bmatrix} G_{20}$ | 67 |

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| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a'_{21})^{(3)}\boxed{+(a''_{21})^{(3)}(T_{21},t)}\boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)}\boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)}\boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)}\boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)}\boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)}\boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{21}$ | *68* |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a'_{22})^{(3)}\boxed{+(a''_{22})^{(3)}(T_{21},t)}\boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)}\boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)}\boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)}\boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)}\boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)}\boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{22}$ | *69* |
| $\boxed{+(a''_{20})^{(3)}(T_{21},t)}, \boxed{+(a''_{21})^{(3)}(T_{21},t)}, \boxed{+(a''_{22})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)}, \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)}, \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)}$ are second augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)}, \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)}, \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)}$ are third augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)}, \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)}, \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)}, \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)}, \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)}, \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)}, \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)}, \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)}, \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)}, \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)}, \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)}$ are eight augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)}, \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)}, \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b'_{20})^{(3)}\boxed{-(b''_{20})^{(3)}(G_{23},t)}\boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)}\boxed{-(b''_{13})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)}\boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)}\boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7)}(G_{39},t)}\boxed{-(b''_{40})^{(8,8,8,8)}(G_{43},t)}\boxed{-(b''_{44})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{20}$ | *70* |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b'_{21})^{(3)}\boxed{-(b''_{21})^{(3)}(G_{23},t)}\boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)}\boxed{-(b''_{14})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)}\boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)}\boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7)}(G_{39},t)}\boxed{-(b''_{41})^{(8,8,8,8)}(G_{43},t)}\boxed{-(b''_{45})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{21}$ | *71* |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)}\boxed{-(b''_{22})^{(3)}(G_{23},t)}\boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)}\boxed{-(b''_{15})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)}\boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)}\boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7)}(G_{39},t)}\boxed{-(b''_{42})^{(8,8,8,8)}(G_{43},t)}\boxed{-(b''_{46})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{22}$ | *72* |
| $\boxed{-(b''_{20})^{(3)}(G_{23},t)}, \boxed{-(b''_{21})^{(3)}(G_{23},t)}, \boxed{-(b''_{22})^{(3)}(G_{23},t)}$ *are first detrition coefficients for category 1, 2 and 3* | |

$-(b_{16}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*

$-(b_{13}^{''})^{(1,1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*

$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*

$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*

$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*

$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$ $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a_{24}^{'})^{(4)} & +(a_{24}^{''})^{(4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,)}(T_{29},t) & +(a_{32}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a_{25}^{'})^{(4)} & +(a_{25}^{''})^{(4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,)}(T_{29},t) & +(a_{33}^{''})^{(6,6)}(T_{33},t) \\ & +(a_{14}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a_{26}^{'})^{(4)} & +(a_{26}^{''})^{(4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,)}(T_{29},t) & +(a_{34}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a_{24}^{''})^{(4)}(T_{25},t)$, $(a_{25}^{''})^{(4)}(T_{25},t)$, $(a_{26}^{''})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a_{28}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3

$+(a_{32}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3

$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$ *are fourth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$ *are fifth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$

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| $are\ sixth\ augmentation\ coefficients\ for\ category\ 1,2\ and\ 3$ $+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$ $are\ seventh\ augmentation\ coefficients\ for\ category\ 1,2\ and\ 3$ $+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$ $are\ eighth\ augmentation\ coefficients\ for\ category\ 1,2\ and\ 3$ $+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3)}(G_{23},t) \\ -(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{24}$ | 76 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b_{25}^{'})^{(4)}-(b_{25}^{''})^{(4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3)}(G_{23},t) \\ -(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{25}$ | 77 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3)}(G_{23},t) \\ -(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{26}$ | 78 |
| $Where\ -(b_{24}^{''})^{(4)}(G_{27},t)$, $-(b_{25}^{''})^{(4)}(G_{27},t)$, $-(b_{26}^{''})^{(4)}(G_{27},t)$ $are\ first\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $-(b_{28}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,)}(G_{31},t)$ $are\ second\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $-(b_{32}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,)}(G_{35},t)$ $are\ third\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$ $are\ fourth\ \ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$ $are\ fifth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$ $are\ sixth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)$ $are\ seventh\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)$ $are\ eighth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3 | |

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| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a'_{28})^{(5)} & +(a''_{28})^{(5)}(T_{29},t) & +(a''_{24})^{(4,4,)}(T_{25},t) & +(a''_{32})^{(6,6,6)}(T_{33},t) \\ & +(a''_{13})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a'_{29})^{(5)} & +(a''_{29})^{(5)}(T_{29},t) & +(a''_{25})^{(4,4,)}(T_{25},t) & +(a''_{33})^{(6,6,6)}(T_{33},t) \\ & +(a''_{14})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a'_{30})^{(5)} & +(a''_{30})^{(5)}(T_{29},t) & +(a''_{26})^{(4,4,)}(T_{25},t) & +(a''_{34})^{(6,6,6)}(T_{33},t) \\ & +(a''_{15})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{30}$ | *81* |
| *Where* $+(a''_{28})^{(5)}(T_{29},t)$, $+(a''_{29})^{(5)}(T_{29},t)$, $+(a''_{30})^{(5)}(T_{29},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3<br>*And* $+(a''_{24})^{(4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,)}(T_{25},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3<br>$+(a''_{32})^{(6,6,6)}(T_{33},t)$, $+(a''_{33})^{(6,6,6)}(T_{33},t)$, $+(a''_{34})^{(6,6,6)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3<br>$+(a''_{13})^{(1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3<br>$+(a''_{16})^{(2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3<br>$+(a''_{20})^{(3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3<br>$+(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3<br>$+(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3<br>$+(a''_{46})^{(9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9)}(T_{45},t)$, $+(a''_{44})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b'_{28})^{(5)} & -(b''_{28})^{(5)}(G_{31},t) & -(b''_{24})^{(4,4,)}(G_{27},t) & -(b''_{32})^{(6,6,6)}(G_{35},t) \\ & -(b''_{13})^{(1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{28}$ | *82* |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b'_{29})^{(5)} & -(b''_{29})^{(5)}(G_{31},t) & -(b''_{25})^{(4,4,)}(G_{27},t) & -(b''_{33})^{(6,6,6)}(G_{35},t) \\ & -(b''_{14})^{(1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{29}$ | *83* |

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b'_{30})^{(5)}\boxed{-(b''_{30})^{(5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,)}(G_{27},t)} & \boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{30} \quad (84)$$

where $\boxed{-(b''_{28})^{(5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5)}(G_{31},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,)}(G_{27},t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{32})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)}$ are third detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,)}(G,t)}$ are fourth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a'_{32})^{(6)}\boxed{+(a''_{32})^{(6)}(T_{33},t)} & \boxed{+(a''_{28})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{24})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a'_{33})^{(6)}\boxed{+(a''_{33})^{(6)}(T_{33},t)} & \boxed{+(a''_{29})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{25})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \begin{bmatrix} (a'_{34})^{(6)}\boxed{+(a''_{34})^{(6)}(T_{33},t)} & \boxed{+(a''_{30})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{26})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{34} \quad (87)$$

$\boxed{+(a''_{32})^{(6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6)}(T_{33},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5)}(T_{29},t)}$ are second augmentation coefficients for category 1, 2 and 3

| | |
|---|---|
| $+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients<br>$+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients<br>$+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients<br>$+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients<br>$+(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients<br>$+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients | |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b_{32}^{'})^{(6)} & -(b_{32}^{''})^{(6)}(G_{35},t) & -(b_{28}^{''})^{(5,5,5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,4,)}(G_{27},t) \\ & -(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32}$ | *88* |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b_{33}^{'})^{(6)} & -(b_{33}^{''})^{(6)}(G_{35},t) & -(b_{29}^{''})^{(5,5,5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,4,)}(G_{27},t) \\ & -(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33}$ | *89* |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b_{34}^{'})^{(6)} & -(b_{34}^{''})^{(6)}(G_{35},t) & -(b_{30}^{''})^{(5,5,5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,4,)}(G_{27},t) \\ & -(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34}$ | *90* |
| | |
| $-(b_{32}^{''})^{(6)}(G_{35},t)$, $-(b_{33}^{''})^{(6)}(G_{35},t)$, $-(b_{34}^{''})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3<br>$-(b_{28}^{''})^{(5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3<br>$-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3<br>$-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3<br>$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3<br>$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3 | |

$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)}+(a_{37}^{''})^{(7)}(T_{37},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15} \quad (93)$$

Where $(a_{36}^{''})^{(7)}(T_{37},t)$, $(a_{37}^{''})^{(7)}(T_{37},t)$, $(a_{38}^{''})^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3

$+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation

coefficient for 1,2,3

$$\frac{dT_{36}}{dt} =$$

$$(b_{36})^{(7)}T_{37} - \begin{bmatrix} (b'_{36})^{(7)}\boxed{-(b''_{36})^{(7)}(G_{39},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

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$$\frac{dT_{37}}{dt} =$$

$$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b'_{37})^{(7)}\boxed{-(b''_{37})^{(7)}(G_{39},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} =$$

$$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b'_{38})^{(7)}\boxed{-(b''_{38})^{(7)}(G_{39},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

| | |
|---|---|
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[\begin{matrix} (a'_{40})^{(8)}\boxed{+(a''_{40})^{(8)}(T_{41},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{13}$ | 95 |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[\begin{matrix} (a'_{41})^{(8)}\boxed{+(a''_{41})^{(8)}(T_{41},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{14}$ | |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[\begin{matrix} (a'_{42})^{(8)}\boxed{+(a''_{42})^{(8)}(T_{41},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{15}$ | |
| Where $\boxed{+(a''_{40})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for 1,2,3<br>$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3<br>$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{40}}{dt} =$ | |

$$(b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}')^{(8)}\boxed{-(b_{40}'')^{(8)}(G_{43},t)} & \boxed{-(b_{16}'')^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}'')^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}'')^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}'')^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}'')^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}'')^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{36}'')^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{44}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} =$$

$$(b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}')^{(8)}\boxed{-(b_{41}'')^{(8)}(G_{43},t)} & \boxed{-(b_{17}'')^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}'')^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}'')^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}'')^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}'')^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}'')^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}'')^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{45}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}')^{(8)}\boxed{-(b_{42}'')^{(8)}(G_{43},t)} & \boxed{-(b_{18}'')^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}'')^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}'')^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}'')^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}'')^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}'')^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}'')^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{46}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b_{36}'')^{(7)}(G_{39},t)}$, $\boxed{-(b_{37}'')^{(7)}(G_{39},t)}$, $\boxed{-(b_{38}'')^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}'')^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}'')^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}'')^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}'')^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}'')^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}'')^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}'')^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}'')^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}'')^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}'')^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}'')^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}'')^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}'')^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{33}'')^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{15}'')^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{13}'')^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}'')^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{38}'')^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{36}'')^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{37}'')^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}'')^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{44}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a_{44}')^{(9)}\boxed{+(a_{44}'')^{(9)}(T_{45},t)} & \boxed{+(a_{16}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{36}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} =$$

$$(b_{45})^{(9)}T_{44}-\begin{bmatrix}(b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}\\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}\\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}\end{bmatrix}T_{14}$$

$$\frac{dT_{46}}{dt}=$$

$$(b_{46})^{(9)}T_{45}-\begin{bmatrix}(b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}\\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}\\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}\end{bmatrix}T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

$(a_i^{''})^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$

$(b_i^{''})^{(1)}(G,t) \le (r_i)^{(1)} \le (b_i^{'})^{(1)} \le (\hat{B}_{13})^{(1)}$

$lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$ 98

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| $\lim_{G\to\infty}(b_i'')^{(1)}(G,t) = (r_i)^{(1)}$<br><br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br><br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(1)}(T_{14}', t) - (a_i'')^{(1)}(T_{14}, t)\rvert \le (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}'\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\lvert(b_i'')^{(1)}(G', t) - (b_i'')^{(1)}(G, t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G'\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(1)}(T_{14}', t)$ and $(a_i'')^{(1)}(T_{14}, t)$ . $(T_{14}', t)$ and $(T_{14}, t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i'')^{(1)}(T_{14}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i'')^{(1)}(T_{14}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i')^{(1)}, (b_i)^{(1)}, (b_i')^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i, j = 16,17,18$ | |
| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17}, t) \le (p_i)^{(2)} \le (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19}, t) \le (r_i)^{(2)} \le (b_i')^{(2)} \le (\hat{B}_{16})^{(2)}$ | 103 |

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| $\lim_{T_2\to\infty}(a_i^{''})^{(2)}(T_{17},t)=(p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty}(b_i^{''})^{(2)}\big((G_{19}),t\big)=\ (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)}$ :<br>Where $\boxed{(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)},(p_i)^{(2)},\ (r_i)^{(2)}}$ are positive constants and $\boxed{i=16{,}17{,}18}$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'},t)-(a_i^{''})^{(2)}(T_{17},t)\rvert\le(\hat{k}_{16})^{(2)}\lvert T_{17}-T_{17}^{'}\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'},t)-(b_i^{''})^{(2)}\big((G_{19}),t\big)\rvert<(\hat{k}_{16})^{(2)}\lvert\lvert(G_{19})-(G_{19})^{'}\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'},t)$ and$(a_i^{''})^{(2)}(T_{17},t)$ . $(T_{17}^{'},t)$ and $(T_{17},t)$ are points belonging to the interval $\big[(\hat{k}_{16})^{(2)},(\hat{M}_{16})^{(2)}\big]$. It is to be noted that $(a_i^{''})^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)}=1$ then the function $(a_i^{''})^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\ ,\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}<1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)},(\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)},(\hat{A}_{16})^{(2)}and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)},(a_i^{'})^{(2)},(b_i)^{(2)},(b_i^{'})^{(2)},(p_i)^{(2)},\ (r_i)^{(2)},i=16{,}17{,}18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)}+(a_i^{'})^{(2)}+\ (\hat{A}_{16})^{(2)}+\ (\hat{P}_{16})^{(2)}\ (\hat{k}_{16})^{(2)}]<1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[\ (b_i)^{(2)}+(b_i^{'})^{(2)}+\ (\hat{B}_{16})^{(2)}+\ (\hat{Q}_{16})^{(2)}\ (\hat{k}_{16})^{(2)}]<1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)},(a_i^{'})^{(3)},(a_i^{''})^{(3)},(b_i)^{(3)},(b_i^{'})^{(3)},(b_i^{''})^{(3)}>0,\quad i,j=20{,}21{,}22$<br>The functions $(a_i^{''})^{(3)},(b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(3)},\ (\mathrm{r_i})^{(3)}$: | 112 |

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| $(a_i^{''})^{(3)}(T_{21},t) \le (p_i)^{(3)} \le (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23},t) \le (r_i)^{(3)} \le (b_i^{'})^{(3)} \le (\hat{B}_{20})^{(3)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(3)}(T_{21},t) = (p_i)^{(3)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'},t) - (a_i^{''})^{(3)}(T_{21},t)\rvert \le (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'},t) - (b_i^{''})^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'},t)$ and $(a_i^{''})^{(3)}(T_{21},t)$ . $(T_{21}^{'},t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)},$ are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$: | 117 |

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| $(a_i^{''})^{(4)}(T_{25},t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}((G_{27}),t) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(4)}(T_{25},t) = (p_i)^{(4)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(4)}((G_{27}),t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'},t) - (a_i^{''})^{(4)}(T_{25},t)\rvert \leq (\hat{k}_{24})^{(4)}\lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'},t) - (b_i^{''})^{(4)}((G_{27}),t)\rvert < (\hat{k}_{24})^{(4)}\lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)}t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'},t)$ and$(a_i^{''})^{(4)}(T_{25},t)$ . $(T_{25}^{'},t)$ and $(T_{25},t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)}(\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)} + (b_i^{'})^{(4)} + (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}(\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded. | 122 |

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| **Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br>$(a_i^{''})^{(5)}(T_{29},t) \le (p_i)^{(5)} \le (\hat{A}_{28})^{(5)}$<br>$(b_i^{''})^{(5)}\big((G_{31}),t\big) \le (r_i)^{(5)} \le (b_i^{'})^{(5)} \le (\hat{B}_{28})^{(5)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(5)}(T_{29},t) = (p_i)^{(5)}$<br>$\lim_{G\to\infty}(b_i^{''})^{(5)}(G_{31},t) = (r_i)^{(5)}$<br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'},t) - (a_i^{''})^{(5)}(T_{29},t)\rvert \le (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'},t) - (b_i^{''})^{(5)}\big((G_{31}),t\big)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'},t)$ and $(a_i^{''})^{(5)}(T_{29},t)$ . $(T_{29}^{'},t)$ and $(T_{29},t)$ are points belonging to the interval $\big[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\big]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$<br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$ | 126 |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$ | 127 |

The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:

$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$

$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$

$lim_{T_2 \to \infty}(a_i^{''})^{(6)}(T_{33}, t) = (p_i)^{(6)}$ 128

$\lim_{G \to \infty}(b_i^{''})^{(6)}((G_{35}), t) = (r_i)^{(6)}$

**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :

Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$

They satisfy Lipschitz condition:

$|(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)| \leq (\hat{k}_{32})^{(6)}|T_{33} - T_{33}^{'}|e^{-(\hat{M}_{32})^{(6)}t}$

$|(b_i^{''})^{(6)}((G_{35})^{'}, t) - (b_i^{''})^{(6)}((G_{35}), t)| < (\hat{k}_{32})^{(6)}||(G_{35}) - (G_{35})^{'}||e^{-(\hat{M}_{32})^{(6)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and$(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $\left[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\right]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ : 129

$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants

$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$

**Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ : 130

There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities

$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$

$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$

Where we suppose

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| (A) $(a_i)^{(7)}, (a_i')^{(7)}, (a_i'')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (b_i'')^{(7)} > 0, \quad i,j = 36,37,38$<br><br>(B) The functions $(a_i'')^{(7)}, (b_i'')^{(7)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(7)}$, $(r_i)^{(7)}$:<br><br>$(a_i'')^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$<br><br>$(b_i'')^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i')^{(7)} \leq (\hat{B}_{36})^{(7)}$ | 131 |
| *(C)* $\lim_{T_2 \to \infty} (a_i'')^{(7)}(T_{37}, t) = (p_i)^{(7)}$<br>*(D)*<br>$\lim_{G \to \infty} (b_i'')^{(7)}\big((G_{39}), t\big) = (r_i)^{(7)}$<br><br>**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :<br><br>Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$ | 132 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(7)}(T_{37}', t) - (a_i'')^{(7)}(T_{37}, t)\rvert \leq (\hat{k}_{36})^{(7)} \lvert T_{37} - T_{37}' \rvert e^{-(\hat{M}_{36})^{(7)} t}$<br><br>$\lvert(b_i'')^{(7)}((G_{39})', t) - (b_i'')^{(7)}\big((G_{39}), t\big)\rvert < (\hat{k}_{36})^{(7)} \lvert\lvert(G_{39}) - (G_{39})'\rvert\rvert e^{-(\hat{M}_{36})^{(7)} t}$ | 133 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(7)}(T_{37}', t)$ and $(a_i'')^{(7)}(T_{37}, t)$ . $(T_{37}', t)$ and $(T_{37}, t)$ are points belonging to the interval $\big[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}\big]$. It is to be noted that $(a_i'')^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i'')^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :<br><br>(E) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$, are positive constants<br><br>$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | 134 |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :<br><br>(F) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities | 135 |

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| $\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i')^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i')^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i')^{(8)}, (a_i'')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (b_i'')^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i'')^{(8)}, (b_i'')^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i'')^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i'')^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i')^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty}(a_i'')^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty}(b_i'')^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(8)}(T_{41}', t) - (a_i'')^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)}\lvert T_{41} - T_{41}'\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 142 |
| $\lvert(b_i'')^{(8)}((G_{43})', t) - (b_i'')^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)}\lvert\lvert(G_{43}) - (G_{43})'\rvert\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(8)}(T_{41}', t)$ and $(a_i'')^{(8)}(T_{41}, t)$ . $(T_{41}', t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i'')^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i'')^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}, \frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ : | |

There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$

Satisfy the inequalities

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1 \quad (145)$$

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1 \quad (146)$$

Where we suppose

$$(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i,j = 44,45,46 \quad (146A)$$

The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:

$$(a_i'')^{(9)}(T_{45},t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$$

$$(b_i'')^{(9)}(G_{47},t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$$

$$\lim_{T_2\to\infty}(a_i'')^{(9)}(T_{45},t) = (p_i)^{(9)}$$

$$\lim_{G\to\infty}(b_i'')^{(9)}(G_{47},t) = (r_i)^{(9)}$$

**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :

Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$

They satisfy Lipschitz condition:

$$|(a_i'')^{(9)}(T_{45}',t) - (a_i'')^{(9)}(T_{45},t)| \le (\hat{k}_{44})^{(9)}|T_{45} - T_{45}'|e^{-(\hat{M}_{44})^{(9)}t}$$

$$|(b_i'')^{(9)}((G_{47})',t) - (b_i'')^{(9)}((G_{47}),t)| < (\hat{k}_{44})^{(9)}||(G_{47}) - (G_{47})'||e^{-(\hat{M}_{44})^{(9)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}',t)$ and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i'')^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants

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| $\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$ | |
| **Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :<br><br>There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}, i = 44,45,46,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$<br><br>$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$ | |
| **Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>Definition of $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 147 |
| **Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$<br><br>$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $G_i(0) = G_i^0 > 0$<br><br>$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | 148 |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br><br>$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions | 151 |

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| **Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le \left(\hat{P}_{13}\right)^{(1)}$ , $T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le \left(\hat{P}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{16})^{(2)}$, $T_i^0 \le (\hat{Q}_{16})^{(2)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[ (a_{18})^{(2)} G_{17}(s_{(16)}) - \left( (a_{18}')^{(2)} + (a_{18}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{18}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[ (b_{16})^{(2)} T_{17}(s_{(16)}) - \left( (b_{16}')^{(2)} - (b_{16}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{16}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[ (b_{17})^{(2)} T_{16}(s_{(16)}) - \left( (b_{17}')^{(2)} - (b_{17}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[ (b_{18})^{(2)} T_{17}(s_{(16)}) - \left( (b_{18}')^{(2)} - (b_{18}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{18}(s_{(16)}) \right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |

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| **Proof:**<br><br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{20})^{(3)}$ , $T_i^0 \leq (\hat{Q}_{20})^{(3)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br><br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a'_{20})^{(3)} + a''_{20})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a'_{21})^{(3)} + (a''_{21})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a'_{22})^{(3)} + (a''_{22})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b'_{20})^{(3)} - (b''_{20})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b'_{21})^{(3)} - (b''_{21})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b'_{22})^{(3)} - (b''_{22})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br><br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{24})^{(4)}$ , $T_i^0 \leq (\hat{Q}_{24})^{(4)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br><br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |

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| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[(b_{24})^{(4)} T_{25}(s_{(24)}) - \left((b_{24}')^{(4)} - (b_{24}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{24}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[(b_{25})^{(4)} T_{24}(s_{(24)}) - \left((b_{25}')^{(4)} - (b_{25}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b_{26}')^{(4)} - (b_{26}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{28})^{(5)}$ , $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a_{28}')^{(5)} + a_{28}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a_{29}')^{(5)} + (a_{29}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a_{30}')^{(5)} + (a_{30}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b_{28}')^{(5)} - (b_{28}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b_{29}')^{(5)} - (b_{29}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b_{30}')^{(5)} - (b_{30}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{32})^{(6)}$ , $T_i^0 \le (\hat{Q}_{32})^{(6)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By | 164 |

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| $\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a'_{32})^{(6)} + a''_{32})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b'_{32})^{(6)} - (b''_{32})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b'_{33})^{(6)} - (b''_{33})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b'_{34})^{(6)} - (b''_{34})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{36})^{(7)}$, $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a'_{36})^{(7)} + a''_{36})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b'_{36})^{(7)} - (b''_{36})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b'_{37})^{(7)} - (b''_{37})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b'_{38})^{(7)} - (b''_{38})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |

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| **Proof:**<br><br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{40})^{(8)}$ , $T_i^0 \leq (\hat{Q}_{40})^{(8)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| By<br><br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{40})^{(8)} + a''_{40})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{40}(s_{(40)}) \right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[ (a_{41})^{(8)} G_{40}(s_{(40)}) - \left( (a'_{41})^{(8)} + (a''_{41})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{41}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[ (a_{42})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{42})^{(8)} + (a''_{42})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{42}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[ (b_{40})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{40})^{(8)} - (b''_{40})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{40}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[ (b_{41})^{(8)} T_{40}(s_{(40)}) - \left( (b'_{41})^{(8)} - (b''_{41})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{41}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[ (b_{42})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{42})^{(8)} - (b''_{42})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{42}(s_{(40)}) \right] ds_{(40)}$<br><br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br><br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[ (a_{44})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{44}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[ (a_{45})^{(9)} G_{44}(s_{(44)}) - \left( (a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{45}(s_{(44)}) \right] ds_{(44)}$ | |

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| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{\mathrm{T}}_{46}(\mathrm{t}) = \mathrm{T}_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[((\hat{P}_{13})^{(1)} + G_{14}^0) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that<br>$(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left[((\hat{P}_{16})^{(2)} + G_{17}^0) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} \left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$<br>$\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}} \left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$ | 171 |

From which it follows that

$$(G_{20}(t)-G_{20}^{0})e^{-(\bar{M}_{20})^{(3)}t}\leq\frac{(a_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^{0})e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^{0}}{G_{21}^{0}}\right)}+(\hat{P}_{20})^{(3)}\right]$$

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Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$

The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that

$$G_{24}(t)\leq G_{24}^{0}+\int_{0}^{t}\left[(a_{24})^{(4)}\left(G_{25}^{0}+(\hat{P}_{24})^{(4)}e^{(\bar{M}_{24})^{(4)}s_{(24)}}\right)\right]ds_{(24)}=$$

$$(1+(a_{24})^{(4)}t)G_{25}^{0}+\frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left(e^{(\bar{M}_{24})^{(4)}t}-1\right)$$

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From which it follows that

$$(G_{24}(t)-G_{24}^{0})e^{-(\bar{M}_{24})^{(4)}t}\leq\frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^{0})e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^{0}}{G_{25}^{0}}\right)}+(\hat{P}_{24})^{(4)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 4

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The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{28}(t)\leq G_{28}^{0}+\int_{0}^{t}\left[(a_{28})^{(5)}\left(G_{29}^{0}+(\hat{P}_{28})^{(5)}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\right)\right]ds_{(28)}=$$

$$(1+(a_{28})^{(5)}t)G_{29}^{0}+\frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)}t}-1\right)$$

From which it follows that

$$(G_{28}(t)-G_{28}^{0})e^{-(\bar{M}_{28})^{(5)}t}\leq\frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^{0})e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^{0}}{G_{29}^{0}}\right)}+(\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5

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The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{32}(t)\leq G_{32}^{0}+\int_{0}^{t}\left[(a_{32})^{(6)}\left(G_{33}^{0}+(\hat{P}_{32})^{(6)}e^{(\bar{M}_{32})^{(6)}s_{(32)}}\right)\right]ds_{(32)}=$$

$$(1+(a_{32})^{(6)}t)G_{33}^{0}+\frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left(e^{(\bar{M}_{32})^{(6)}t}-1\right)$$

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From which it follows that

$$(G_{32}(t)-G_{32}^{0})e^{-(\bar{M}_{32})^{(6)}t}\leq\frac{(a_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^{0})e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^{0}}{G_{33}^{0}}\right)}+(\hat{P}_{32})^{(6)}\right]$$

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$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}$ , $G_{26}, T_{24}, T_{25}, T_{26}$

(a) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 178

$$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} \left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$$
$$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$$

From which it follows that

$$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left[\left((\hat{P}_{36})^{(7)} + G_{37}^0\right) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} \left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$$
$$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$$ 180

From which it follows that 181

$$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left[\left((\hat{P}_{40})^{(8)} + G_{41}^0\right) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 8

Analogous inequalities hold also for $G_{41}$ , $G_{42}, T_{40}, T_{41}, T_{42}$

The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} \left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$$
$$\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left(e^{(\hat{M}_{44})^{(9)} t} - 1\right)$$

From which it follows that

$$(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left[\left((\hat{P}_{44})^{(9)} + G_{45}^0\right) e^{\left(-\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0}\right)} + (\hat{P}_{44})^{(9)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 9

Analogous inequalities hold also for $G_{45}$ , $G_{46}, T_{44}, T_{45}, T_{46}$

It is now sufficient to take $\frac{(a_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\widehat{M}_{13})^{(1)}} < 1$ and to choose

$(\widehat{P}_{13})^{(1)}$ and $(\widehat{Q}_{13})^{(1)}$ large to have

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$$\frac{(a_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}\left[(\widehat{P}_{13})^{(1)} + \left((\hat{P}_{13})^{(1)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{13})^{(1)}$$

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$$\frac{(b_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{13})^{(1)}\right] \leq (\hat{Q}_{13})^{(1)}$$

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In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric

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$$d\left(\left(G^{(1)}, T^{(1)}\right), \left(G^{(2)}, T^{(2)}\right)\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{13})^{(1)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{13})^{(1)}t}\}$$

Indeed if we denote

**Definition of** $\tilde{G}, \tilde{T}$ : $(\tilde{G}, \tilde{T}) = \mathcal{A}^{(1)}(G, T)$

It results

$$\left|\tilde{G}_{13}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{13})^{(1)}\left|G_{14}^{(1)} - G_{14}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\,ds_{(13)} +$$

$$\int_0^t\{(a'_{13})^{(1)}\left|G_{13}^{(1)} - G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} +$$

$$(a''_{13})^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right)\left|G_{13}^{(1)} - G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}} +$$

$$G_{13}^{(2)}\left|(a''_{13})^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right) - (a''_{13})^{(1)}\left(T_{14}^{(2)}, s_{(13)}\right)\right|\ e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$$

Where $s_{(13)}$ represents integrand that is integrated over the interval [0, t]

From the hypotheses it follows

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$$\left|G^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}t} \leq$$
$$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)}, T^{(1)}; G^{(2)}, T^{(2)}\right)\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

**Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate

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| condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(1)} - (a_i^{''})^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(1)}t)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a_{14}^{'})^{(1)}G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}^{'})^{(1)}$<br><br>If $G_{14}\ or\ G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}\ and\ G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t), t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t), t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right),\quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t), t) = (b_{15}^{'})^{(1)}$ | |

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| We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ $and$ $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)}, (T_{19})^{(1)}), ((G_{19})^{(2)}, (T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)}\left\|G_{17}^{(1)} - G_{17}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\,ds_{(16)} +$<br>$\int_0^t\{(a_{16}')^{(2)}\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{-(\widehat{M}_{16})^{(2)}s_{(16)}} +$<br>$(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}} +$<br>$G_{16}^{(2)}\left\|(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}'')^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\|\ e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}')^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right)d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |

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| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ and $(\widehat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)}t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $((\widehat{M}_{16})^{(2)})_1, ((\widehat{M}_{16})^{(2)})_2$ and $((\widehat{M}_{16})^{(2)})_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq ((\widehat{M}_{16})^{(2)})_1 - (a_{17}^{'})^{(2)}G_{17}$ and by integrating<br><br>$G_{17} \leq ((\widehat{M}_{16})^{(2)})_2 = G_{17}^0 + 2(a_{17})^{(2)}((\widehat{M}_{16})^{(2)})_1/(a_{17}^{'})^{(2)}$<br><br>In the same way , one can obtain<br><br>$G_{18} \leq ((\widehat{M}_{16})^{(2)})_3 = G_{18}^0 + 2(a_{18})^{(2)}((\widehat{M}_{16})^{(2)})_2/(a_{18}^{'})^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | 201 |
| **Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(2)}((G_{19})(t), t)) = (b_{17}^{'})^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i^{''})^{(2)}((G_{19})(t), t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded. | 205 |

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| The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right)=(b_{18}')^{(2)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}<1$ and to choose<br>$(\widehat{P}_{20})^{(3)}$ and $(\widehat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[(\widehat{P}_{20})^{(3)}+\left((\widehat{P}_{20})^{(3)}+G_j^0\right)e^{-\left(\frac{(\widehat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right]\leq(\widehat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[\left((\widehat{Q}_{20})^{(3)}+T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)}+(\widehat{Q}_{20})^{(3)}\right]\leq(\widehat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | 211 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}},\widetilde{T_{23}}:\left(\widetilde{(G_{23})},\widetilde{(T_{23})}\right)=\mathcal{A}^{(3)}((G_{23}),(T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{20})^{(3)}\left\|G_{21}^{(1)}-G_{21}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)}+$<br>$\int_0^t\{(a_{20}')^{(3)}\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$(a_{20}'')^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$G_{20}^{(2)}\left\\|(a_{20}'')^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)-(a_{20}'')^{(3)}\left(T_{21}^{(2)},s_{(20)}\right)\right\|\ e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)}-G_{23}{}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}t}\leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)}+(a_{20}')^{(3)}+(\widehat{A}_{20})^{(3)}+(\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)},(T_{23})^{(1)};\ (G_{23})^{(2)},(T_{23})^{(2)})\right)$ | 214 |

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| And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ *and* $(\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)}t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}\ and\ G_{22}$ . indeed if<br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)}G_{21}$ and by integrating<br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br>In the same way , one can obtain<br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br>If $G_{21}\ or\ G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}\ and\ G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t), t\right)) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br>Indeed let $t_3$ be so that for $t > t_3$<br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t), t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results | 220 |

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| $T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right), \quad t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded. The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $\widetilde{(G_{27})},\widetilde{(T_{27})}$ : $\left(\widetilde{(G_{27})},\widetilde{(T_{27})}\right) = \mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{24})^{(4)}\left\|G_{25}^{(1)} - G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)} +$<br>$\int_0^t\{(a_{24}^{'})^{(4)}\left\|G_{24}^{(1)} - G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}} +$<br>$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)} - G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}} +$<br>$G_{24}^{(2)}\left\|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right) - (a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\| \; e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)} - (G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t} \leq$ | 226 |

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| $\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};(G_{27})^{(2)},(T_{27})^{(2)})\right)$<br>And analogous inequalities for $G_i$ $and$ $T_i$. Taking into account the hypothesis the result follows | |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ $and$ $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t)=0$ $and$ $T_i(t)=0$<br>it results<br>$G_i(t)\geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]}\geq 0$<br>$T_i(t)\geq T_i^0 e^{\left(-(b_i^{'})^{(4)}t\right)}>0$ for t > 0 | 228 |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1,\left((\widehat{M}_{24})^{(4)}\right)_2$ $and$ $\left((\widehat{M}_{24})^{(4)}\right)_3$ :<br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ $and$ $G_{26}$ . indeed if<br>$G_{24}<(\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt}\leq\left((\widehat{M}_{24})^{(4)}\right)_1-(a_{25}^{'})^{(4)}G_{25}$ and by integrating<br>$G_{25}\leq\left((\widehat{M}_{24})^{(4)}\right)_2=G_{25}^0+2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a_{25}^{'})^{(4)}$<br>In the same way , one can obtain<br>$G_{26}\leq\left((\widehat{M}_{24})^{(4)}\right)_3=G_{26}^0+2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a_{26}^{'})^{(4)}$<br>If $G_{25}$ $or$ $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ $is$ bounded, from below, the same property holds for $G_{25}$ $and$ $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t),t))=(b_{25}^{'})^{(4)}$ then $T_{25}\to\infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t>t_4$<br>$(b_{25})^{(4)}-(b_i^{''})^{(4)}((G_{27})(t),t)<\varepsilon_4, T_{24}(t)>(m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt}\geq(a_{25})^{(4)}(m)^{(4)}-\varepsilon_4 T_{25}$ which leads to | 232 |

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1-e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right), \quad t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42

Analogous inequalities hold also for $G_{29}, G_{30}, T_{28}, T_{29}, T_{30}$

It is now sufficient to take $\frac{(a_i)^{(5)}}{(\bar{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\bar{M}_{28})^{(5)}} < 1$ and to choose

$(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ large to have

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$$\frac{(a_i)^{(5)}}{(\bar{M}_{28})^{(5)}}\left[(\hat{P}_{28})^{(5)} + \left((\hat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{28})^{(5)}$$

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$$\frac{(b_i)^{(5)}}{(\bar{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{28})^{(5)}\right] \leq (\hat{Q}_{28})^{(5)}$$

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In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric

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$$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{28})^{(5)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{28})^{(5)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{31}}), (\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right) = \mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$

It results

$$\left|\tilde{G}_{28}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{28})^{(5)}\left|G_{29}^{(1)} - G_{29}^{(2)}\right|e^{-(\bar{M}_{28})^{(5)}s_{(28)}}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)} +$$

$$\int_0^t\{(a_{28}^{'})^{(5)}\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\bar{M}_{28})^{(5)}s_{(28)}}e^{-(\bar{M}_{28})^{(5)}s_{(28)}} +$$

$$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\bar{M}_{28})^{(5)}s_{(28)}}e^{(\bar{M}_{28})^{(5)}s_{(28)}} +$$

$$G_{28}^{(2)}\left|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right) - (a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right| \; e^{-(\bar{M}_{28})^{(5)}s_{(28)}}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$$

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| Where $s_{(28)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on it follows | |
| $\left\|(G_{31})^{(1)} - (G_{31})^{(2)}\right\| e^{-(\widehat{M}_{28})^{(5)}t} \leq$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)} + (a'_{28})^{(5)} + (\widehat{A}_{28})^{(5)} + (\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right) d\left(((G_{31})^{(1)}, (T_{31})^{(1)};\ (G_{31})^{(2)}, (T_{31})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a''_{28})^{(5)}$ and $(b''_{28})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(5)}$ and $(b''_i)^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(5)} - (a''_i)^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(5)}t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a'_{29})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a'_{29})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a'_{30})^{(5)}$<br><br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(5)}((G_{31})(t), t)) = (b'_{29})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ : | 242 |

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| Indeed let $t_5$ be so that for $t > t_5$<br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t), t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t), t\right) = (b_{30}^{'})^{(5)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br>Analogous inequalities hold also for $G_{33}, G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\bar{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\bar{M}_{32})^{(6)}} < 1$ and to choose<br>$(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\bar{M}_{32})^{(6)}}\left[(\hat{P}_{32})^{(6)} + \left((\hat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\bar{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{32})^{(6)}\right] \leq (\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)}, (T_{35})^{(1)}), ((G_{35})^{(2)}, (T_{35})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}), (\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}), (\widetilde{T_{35}})\right) = \mathcal{A}^{(6)}\left((G_{35}), (T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{32})^{(6)}\left|G_{33}^{(1)} - G_{33}^{(2)}\right|e^{-(\bar{M}_{32})^{(6)}s_{(32)}}e^{(\bar{M}_{32})^{(6)}s_{(32)}}\,ds_{(32)} +$<br>$\int_0^t\{(a_{32}^{'})^{(6)}\left|G_{32}^{(1)} - G_{32}^{(2)}\right|e^{-(\bar{M}_{32})^{(6)}s_{(32)}}e^{-(\bar{M}_{32})^{(6)}s_{(32)}} +$ | 247 |

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| $(a_{32}^{''})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)\left\|G_{32}^{(1)} - G_{32}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)} s_{(32)}} e^{(\widehat{M}_{32})^{(6)} s_{(32)}} +$ <br> $G_{32}^{(2)} \left\|(a_{32}^{''})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right) - (a_{32}^{''})^{(6)}\left(T_{33}^{(2)}, s_{(32)}\right)\right\| \; e^{-(\widehat{M}_{32})^{(6)} s_{(32)}} e^{(\widehat{M}_{32})^{(6)} s_{(32)}} \} ds_{(32)}$ <br> Where $s_{(32)}$ represents integrand that is integrated over the interval $[0, t]$ <br> From the hypotheses it follows | |
| $\left\|(G_{35})^{(1)} - (G_{35})^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)} t} \leq$ <br> $\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)} + (a_{32}^{'})^{(6)} + (\widehat{A}_{32})^{(6)} + (\widehat{P}_{32})^{(6)} (\widehat{k}_{32})^{(6)}\right) d\left(\left((G_{35})^{(1)}, (T_{35})^{(1)}; \; (G_{35})^{(2)}, (T_{35})^{(2)}\right)\right)$ <br> And analogous inequalities for $G_i \; and \; T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a_{32}^{''})^{(6)}$ and $(b_{32}^{''})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)} e^{(\widehat{M}_{32})^{(6)} t} \; and \; (\widehat{Q}_{32})^{(6)} e^{(\widehat{M}_{32})^{(6)} t}$ respectively of $\mathbb{R}_+$. <br> If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i = 32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})(and \; not \; on \; t)$ and hypothesis can replaced by a usual Lipschitz condition. | 249 |
| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0 \; and \; T_i(t) = 0$ <br> it results <br> $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(6)} - (a_i^{''})^{(6)} (T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$ <br> $T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(6)} t\right)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2 \; and \; \left((\widehat{M}_{32})^{(6)}\right)_3$ : <br> **Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33} \; and \; G_{34}$ . indeed if <br> $G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}^{'})^{(6)} G_{33}$ and by integrating <br> $G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}^{'})^{(6)}$ <br> In the same way , one can obtain <br> $G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}^{'})^{(6)}$ <br> If $G_{33} \; or \; G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33} \; and \; G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |

**Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$. 253

**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :

Indeed let $t_6$ be so that for $t > t_6$

$(b_{33})^{(6)} - (b_i^{''})^{(6)}\big((G_{35})(t),t\big) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$

Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to 254

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right), \quad t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.
The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

Analogous inequalities hold also for $G_{37}, G_{38}, T_{36}, T_{37}, T_{38}$ 255

It is now sufficient to take $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ and to choose
$(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ large to have

$$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[(\hat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)} \quad 256$$

$$\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)} \quad 257$$

In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric 258

$$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right) =$$
$$\sup_i\{\max_{t\in\mathbb{R}_+}|G_i^{(1)}(t) - G_i^{(2)}(t)|e^{-(\hat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}|T_i^{(1)}(t) - T_i^{(2)}(t)|e^{-(\hat{M}_{36})^{(7)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{39})}, \widetilde{(T_{39})}$ : $\left(\widetilde{(G_{39})}, \widetilde{(T_{39})}\right) = \mathcal{A}^{(7)}((G_{39}),(T_{39}))$
It results

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| $\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{36})^{(7)} \left\|G_{37}^{(1)}-G_{37}^{(2)}\right\| e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\, ds_{(36)} +$ $\int_0^t \{(a_{36}^{'})^{(7)} \left\|G_{36}^{(1)}-G_{36}^{(2)}\right\| e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} +$ $(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\| e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} e^{(\widehat{M}_{36})^{(7)}s_{(36)}} +$ $G_{36}^{(2)}\left\|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right)-(a_{36}^{''})^{(7)}\left(T_{37}^{(2)}, s_{(36)}\right)\right\| \; e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\} ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on it follows | |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\| e^{-(\widehat{M}_{36})^{(7)}t} \leq$ $\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}^{'})^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right) d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)}t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2$ *and* $\left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}$ *and* $G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)}G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1/(a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}^{'})^{(7)}$ | 262 |

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|---|---|
| If $G_{37}$ *or* $G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}$ *and* $G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t),t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right)$, $t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t),t\right) = (b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ and to choose<br>$(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\left[(\hat{P}_{40})^{(8)} + \left((\hat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{40})^{(8)}\right] \leq (\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{40})^{(8)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{40})^{(8)}t}\}$ | 269 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})}, \widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})}, \widetilde{(T_{43})}\right) = \mathcal{A}^{(8)}((G_{43}), (T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{40})^{(8)} \left\|G_{41}^{(1)} - G_{41}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}}\, ds_{(40)} +$<br>$\int_0^t \{(a'_{40})^{(8)} \left\|G_{40}^{(1)} - G_{40}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} +$<br>$(a''_{40})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right) \left\|G_{40}^{(1)} - G_{40}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}} +$<br>$G_{40}^{(2)} \left\|(a''_{40})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right) - (a''_{40})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)\right\| \; e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}}\} ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)} - (G_{43})^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} t} \leq$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)} + (a'_{40})^{(8)} + (\widehat{A}_{40})^{(8)} + (\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right) d\left(((G_{43})^{(1)}, (T_{43})^{(1)};\ (G_{43})^{(2)}, (T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a''_{40})^{(8)}$ and $(b''_{40})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)} t}$ *and* $(\widehat{Q}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(8)}$ and $(b''_i)^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(8)} - (a''_i)^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(8)} t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ *and* $G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a'_{41})^{(8)} G_{41}$ and by integrating | 276 |

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| $G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1/(a_{41}')^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2/(a_{42}')^{(8)}$<br><br>If $G_{41}$ *or* $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | |
| **Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}$ *and* $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(8)}((G_{43})(t),t)) = (b_{41}')^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i'')^{(8)}\left((G_{43})(t),t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$ | 278 |
| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right)$, $t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded. The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}'')^{(8)}\left((G_{43})(t),t(t),t\right) = (b_{42}')^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$ | |
| In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself | |
| The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric | |

$$d\left(((G_{47})^{(1)},(T_{47})^{(1)}),((G_{47})^{(2)},(T_{47})^{(2)})\right)=$$
$$sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})},\widetilde{(T_{47})}:\left(\widetilde{(G_{47})},\widetilde{(T_{47})}\right)=\mathcal{A}^{(9)}\left((G_{47}),(T_{47})\right)$
It results

$$\left|\tilde{G}_{44}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{44})^{(9)}\left|G_{45}^{(1)}-G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}ds_{(44)}+$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)-(a_{44}^{''})^{(9)}\left(T_{45}^{(2)},s_{(44)}\right)\right|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t}\le$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)}+(a_{44}^{'})^{(9)}+(\widehat{A}_{44})^{(9)}+(\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)},(T_{47})^{(1)};\ (G_{47})^{(2)},(T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}\ and\ (\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)},i=44,45,46$ depend only on $\mathrm{T}_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

From 99 to 44 it results

$$G_i\ (t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(9)}-(a_i^{''})^{(9)}(T_{45}(s_{(44)}),s_{(44)})\}ds_{(44)}\right]}\ge 0$$

$$T_i\ (t)\ge T_i^0e^{\left(-(b_i^{'})^{(9)}t\right)}>0\quad \text{for t}>0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1,\left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3:$

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44}<(\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt}\le\left((\widehat{M}_{44})^{(9)}\right)_1-(a_{45}^{'})^{(9)}G_{45}$ and by integrating

$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}')^{(9)}$

In the same way , one can obtain

$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}')^{(9)}$

If $G_{45}$ *or* $G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}$ *and* $G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(9)}\left((G_{47})(t),t\right) = (b_{45}')^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i'')^{(9)}\left((G_{47})(t),t\right) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}'')^{(9)}\left((G_{47})(t),t\right) = (b_{46}')^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}')^{(1)} + (a_{14}')^{(1)} - (a_{13}'')^{(1)}(T_{14},t) + (a_{14}'')^{(1)}(T_{14},t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b_{13}')^{(1)} + (b_{14}')^{(1)} - (b_{13}'')^{(1)}(G,t) - (b_{14}'')^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$ , $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$ , $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)}, , (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$ , $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$ , $(\bar{u}_2)^{(1)} < 0$ the roots of the equations

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| $(a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$ | |
| **Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :-<br><br>If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by<br>$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)},\ if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$<br><br>$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)},$<br><br>and $\boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$<br><br>$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)},\ if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$ | 283 |
| and analogously<br><br>$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)},\ if\ (u_0)^{(1)} < (u_1)^{(1)}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)},$<br><br>and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$<br><br>are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br><br>where $(p_i)^{(1)}$ is defined by equation<br><br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le\right.$<br>$\left.\frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a'_{15})^{(1)})}\left[e^{(S_1)^{(1)}t} - e^{-(a'_{15})^{(1)}t}\right] + G_{15}^0 e^{-(a'_{15})^{(1)}t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)} T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)} - (b'_{15})^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b'_{15})^{(1)}t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)}t} \le T_{15}(t) \le$ | 289 |

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| $\frac{(a_{15})^{(1)}T_{13}^{0}}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t}-e^{-(R_2)^{(1)}t}\right]+T_{15}^{0}e^{-(R_2)^{(1)}t}$ | |
| **Definition of** $(S_1)^{(1)},(S_2)^{(1)},(R_1)^{(1)},(R_2)^{(1)}$:- <br> Where $(S_1)^{(1)}=(a_{13})^{(1)}(m_2)^{(1)}-(a_{13}^{'})^{(1)}$ <br> $(S_2)^{(1)}=(a_{15})^{(1)}-(p_{15})^{(1)}$ <br> $(R_1)^{(1)}=(b_{13})^{(1)}(\mu_2)^{(1)}-(b_{13}^{'})^{(1)}$ <br> $(R_2)^{(1)}=(b_{15}^{'})^{(1)}-(r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** <br> **Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ : <br> $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)}\le-(a_{16}^{'})^{(2)}+(a_{17}^{'})^{(2)}-(a_{16}^{''})^{(2)}(T_{17},t)+(a_{17}^{''})^{(2)}(T_{17},t)\le-(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)}\le-(b_{16}^{'})^{(2)}+(b_{17}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19}),t)-(b_{17}^{''})^{(2)}((G_{19}),t)\le-(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)},(\nu_2)^{(2)},(u_1)^{(2)},(u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)}>0$ , $(\nu_2)^{(2)}<0$ and respectively $(u_1)^{(2)}>0$ , $(u_2)^{(2)}<0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}(\nu^{(2)})^2+(\sigma_1)^{(2)}\nu^{(2)}-(a_{16})^{(2)}=0$ | 297 |
| and $(b_{14})^{(2)}(u^{(2)})^2+(\tau_1)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)},,(\bar{\nu}_2)^{(2)},(\bar{u}_1)^{(2)},(\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)}>0$ , $(\bar{\nu}_2)^{(2)}<0$ and respectively $(\bar{u}_1)^{(2)}>0$ , $(\bar{u}_2)^{(2)}<0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}(\nu^{(2)})^2+(\sigma_2)^{(2)}\nu^{(2)}-(a_{16})^{(2)}=0$ | 301 |
| and $(b_{17})^{(2)}(u^{(2)})^2+(\tau_2)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ | 302 |
| **Definition of** $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)}=(\nu_0)^{(2)},(m_1)^{(2)}=(\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)}<(\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)}=(\nu_1)^{(2)},(m_1)^{(2)}=(\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)}<(\nu_0)^{(2)}<(\bar{\nu}_1)^{(2)}$, | 306 |

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| and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)},$ ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)},$ ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$,<br>and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)},$ ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)} t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)} t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)} t}\right] + G_{18}^0 e^{-(S_2)^{(2)} t} \le G_{18}(t) \le \right.$<br>$\left.\frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a'_{18})^{(2)})}\left[e^{(S_1)^{(2)} t} - e^{-(a'_{18})^{(2)} t}\right] + G_{18}^0 e^{-(a'_{18})^{(2)} t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b'_{18})^{(2)})}\left[e^{(R_1)^{(2)} t} - e^{-(b'_{18})^{(2)} t}\right] + T_{18}^0 e^{-(b'_{18})^{(2)} t} \le T_{18}(t) \le$<br>$\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)} + (r_{16})^{(2)} + (R_2)^{(2)})}\left[e^{((R_1)^{(2)} + (r_{16})^{(2)})t} - e^{-(R_2)^{(2)} t}\right] + T_{18}^0 e^{-(R_2)^{(2)} t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a'_{16})^{(2)}$<br>$(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b'_{16})^{(2)}$<br>$(R_2)^{(2)} = (b'_{18})^{(2)} - (r_{18})^{(2)}$ | 318 |

**Behavior of the solutions** 319

**Theorem 3:** If we denote and define

**Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ :

$(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying

$$-(\sigma_2)^{(3)} \le -(a_{20}^{'})^{(3)} + (a_{21}^{'})^{(3)} - (a_{20}^{''})^{(3)}(T_{21}, t) + (a_{21}^{''})^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$$

$$-(\tau_2)^{(3)} \le -(b_{20}^{'})^{(3)} + (b_{21}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t) - (b_{21}^{''})^{(3)}\big((G_{23}), t\big) \le -(\tau_1)^{(3)}$$

**Definition of** $(v_1)^{(3)}, (v_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ : 320

By $(v_1)^{(3)} > 0$ , $(v_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$ , $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}\big(v^{(3)}\big)^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\big(u^{(3)}\big)^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and

By $(\bar{v}_1)^{(3)} > 0$ , $(\bar{v}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$ , $(\bar{u}_2)^{(3)} < 0$ the

roots of the equations $(a_{21})^{(3)}\big(v^{(3)}\big)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\big(u^{(3)}\big)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$

**Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :- 321

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (v_0)^{(3)}, (m_1)^{(3)} = (v_1)^{(3)}$, ***if*** $(v_0)^{(3)} < (v_1)^{(3)}$

$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (\bar{v}_1)^{(3)}$ , ***if*** $(v_1)^{(3)} < (v_0)^{(3)} < (\bar{v}_1)^{(3)}$,

and $\boxed{(v_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (v_0)^{(3)}$, ***if*** $(\bar{v}_1)^{(3)} < (v_0)^{(3)}$

and analogously 322

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$ , ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$$G_{20}^0 e^{\left((S_1)^{(3)} - (p_{20})^{(3)}\right)t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)}t}$$

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| $(p_i)^{(3)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)}-(p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)}G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)}-(p_{20})^{(3)}-(S_2)^{(3)})}\left[e^{((S_1)^{(3)}-(p_{20})^{(3)})t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \le G_{22}(t) \le \right.$ $\left.\frac{(a_{22})^{(3)}G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)}-(a'_{22})^{(3)})}[e^{(S_1)^{(3)}t} - e^{-(a'_{22})^{(3)}t}] + G_{22}^0 e^{-(a'_{22})^{(3)}t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)}+(r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)}+(r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)}T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)}-(b'_{22})^{(3)})}\left[e^{(R_1)^{(3)}t} - e^{-(b'_{22})^{(3)}t}\right] + T_{22}^0 e^{-(b'_{22})^{(3)}t} \le T_{22}(t) \le$ $\frac{(a_{22})^{(3)}T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)}+(r_{20})^{(3)}+(R_2)^{(3)})}\left[e^{((R_1)^{(3)}+(r_{20})^{(3)})t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$ | 327 |
| **Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:-<br>Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$<br>$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$<br>$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$<br>$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$ | 328 |
| **Behavior of the solutions of equation**<br>**Theorem:** If we denote and define<br>**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :<br>$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying<br>$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$<br>$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ :<br>By $(v_1)^{(4)} > 0$ , $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$ , $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$<br>and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and | 329 |
| **Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : | 330 |

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| By $(\bar{\nu}_1)^{(4)} > 0$ , $(\bar{\nu}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$ , $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)}$, $(\mu_2)^{(4)}$, $(\nu_0)^{(4)}$ :-<br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)}$, $(\mu_2)^{(4)}$ by<br>$(m_2)^{(4)} = (\nu_0)^{(4)}, (m_1)^{(4)} = (\nu_1)^{(4)}$, ***if*** $(\nu_0)^{(4)} < (\nu_1)^{(4)}$<br>$(m_2)^{(4)} = (\nu_1)^{(4)}, (m_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ , ***if*** $(\nu_4)^{(4)} < (\nu_0)^{(4)} < (\bar{\nu}_1)^{(4)}$,<br>and $\boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$<br>$(m_2)^{(4)} = (\nu_4)^{(4)}, (m_1)^{(4)} = (\nu_0)^{(4)}$, ***if*** $(\bar{\nu}_4)^{(4)} < (\nu_0)^{(4)}$ | |
| and analogously<br>$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b_{26}')^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}')^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)}t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)} + (r_{24})^{(4)} + (R_2)^{(4)})}\left[e^{((R_1)^{(4)} + (r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:- | 337 |

Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a'_{24})^{(4)}$

$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$

$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b'_{24})^{(4)}$

$(R_2)^{(4)} = (b'_{26})^{(4)} - (r_{26})^{(4)}$

**Behavior of the solutions of equation** 338

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :

$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying

$-(\sigma_2)^{(5)} \le -(a'_{28})^{(5)} + (a'_{29})^{(5)} - (a''_{28})^{(5)}(T_{29}, t) + (a''_{29})^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$

$-(\tau_2)^{(5)} \le -(b'_{28})^{(5)} + (b'_{29})^{(5)} - (b''_{28})^{(5)}((G_{31}), t) - (b''_{29})^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$

**Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ : 339

By $(v_1)^{(5)} > 0$ , $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$ , $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$
and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and

**Definition of** $(\bar{v}_1)^{(5)}, , (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ : 340

By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$
and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$

**Definition of** $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :-

If we define $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}$ by

$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$

$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,
and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$

$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$

and analogously 341

$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$

$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,

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| and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}, \boldsymbol{if}\ (\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)} t}$<br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b'_{30})^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b'_{30})^{(5)} t}\right] + T_{30}^0 e^{-(b'_{30})^{(5)} t} \le T_{30}(t) \le$<br>$\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a'_{28})^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b'_{28})^{(5)}$<br>$(R_2)^{(5)} = (b'_{30})^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :<br>$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying<br>$-(\sigma_2)^{(6)} \le -(a'_{32})^{(6)} + (a'_{33})^{(6)} - (a''_{32})^{(6)}(T_{33}, t) + (a''_{33})^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$ | 349 |

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|---|---|
| $-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}((G_{35}),t) - (b''_{33})^{(6)}((G_{35}),t) \le -(\tau_1)^{(6)}$ | |
| **Definition of** $(\nu_1)^{(6)}, (\nu_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, \nu^{(6)}, u^{(6)}$ :<br><br>By $(\nu_1)^{(6)} > 0$ , $(\nu_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(\nu^{(6)})^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{\nu}_1)^{(6)},, (\bar{\nu}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br><br>By $(\bar{\nu}_1)^{(6)} > 0$ , $(\bar{\nu}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(\nu^{(6)})^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (\nu_0)^{(6)}$ :-<br><br>If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br><br>$(m_2)^{(6)} = (\nu_0)^{(6)}, (m_1)^{(6)} = (\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)} < (\nu_1)^{(6)}$<br><br>$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)} < (\nu_0)^{(6)} < (\bar{\nu}_1)^{(6)}$,<br>and $\boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$<br><br>$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)} < (\nu_0)^{(6)}$ | 351 |
| and analogously<br><br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br><br>$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$<br><br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G^0_{32}e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)}G^0_{32}}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G^0_{34}e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |

| | |
|---|---|
| $\boxed{T_{32}^{0}e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^{0}e^{((R_1)^{(6)}+(r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}}T_{32}^{0}e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}}T_{32}^{0}e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)}T_{32}^{0}}{(\mu_1)^{(6)}((R_1)^{(6)}-(b_{34}^{'})^{(6)})}\left[e^{(R_1)^{(6)}t}-e^{-(b_{34}^{'})^{(6)}t}\right]+T_{34}^{0}e^{-(b_{34}^{'})^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)}T_{32}^{0}}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t}-e^{-(R_2)^{(6)}t}\right]+T_{34}^{0}e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)},(S_2)^{(6)},(R_1)^{(6)},(R_2)^{(6)}$:- Where $(S_1)^{(6)}=(a_{32})^{(6)}(m_2)^{(6)}-(a_{32}^{'})^{(6)}$ $(S_2)^{(6)}=(a_{34})^{(6)}-(p_{34})^{(6)}$ $(R_1)^{(6)}=(b_{32})^{(6)}(\mu_2)^{(6)}-(b_{32}^{'})^{(6)}$ $(R_2)^{(6)}=(b_{34}^{'})^{(6)}-(r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(7)},(\sigma_2)^{(7)},(\tau_1)^{(7)},(\tau_2)^{(7)}$: $(\sigma_1)^{(7)},(\sigma_2)^{(7)},(\tau_1)^{(7)},(\tau_2)^{(7)}$ four constants satisfying $-(\sigma_2)^{(7)} \le -(a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}-(a_{36}^{''})^{(7)}(T_{37},t)+(a_{37}^{''})^{(7)}(T_{37},t) \le -(\sigma_1)^{(7)}$ $-(\tau_2)^{(7)} \le -(b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(b_{36}^{''})^{(7)}\big((G_{39}),t\big)-(b_{37}^{''})^{(7)}\big((G_{39}),t\big) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)},(v_2)^{(7)},(u_1)^{(7)},(u_2)^{(7)},v^{(7)},u^{(7)}$: By $(v_1)^{(7)}>0,(v_2)^{(7)}<0$ and respectively $(u_1)^{(7)}>0,(u_2)^{(7)}<0$ the roots of the equations $(a_{37})^{(7)}\big(v^{(7)}\big)^2+(\sigma_1)^{(7)}v^{(7)}-(a_{36})^{(7)}=0$ and $(b_{37})^{(7)}\big(u^{(7)}\big)^2+(\tau_1)^{(7)}u^{(7)}-(b_{36})^{(7)}=0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)},,(\bar{v}_2)^{(7)},(\bar{u}_1)^{(7)},(\bar{u}_2)^{(7)}$: By $(\bar{v}_1)^{(7)}>0,(\bar{v}_2)^{(7)}<0$ and respectively $(\bar{u}_1)^{(7)}>0,(\bar{u}_2)^{(7)}<0$ the roots of the equations $(a_{37})^{(7)}\big(v^{(7)}\big)^2+(\sigma_2)^{(7)}v^{(7)}-(a_{36})^{(7)}=0$ and $(b_{37})^{(7)}\big(u^{(7)}\big)^2+(\tau_2)^{(7)}u^{(7)}-(b_{36})^{(7)}=0$ **Definition of** $(m_1)^{(7)},(m_2)^{(7)},(\mu_1)^{(7)},(\mu_2)^{(7)},(v_0)^{(7)}$:- If we define $(m_1)^{(7)},(m_2)^{(7)},(\mu_1)^{(7)},(\mu_2)^{(7)}$ by | 362 |

| | |
|---|---|
| $(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)},$ ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ , ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,<br>and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)},$ ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)},$ ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$ , ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,<br>and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)},$ ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)} t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | 364 |
| $\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)} t}$ | 365 |
| $(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})} \left[ e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)} t} \right] + G_{38}^0 e^{-(S_2)^{(7)} t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a'_{38})^{(7)})} [e^{(S_1)^{(7)} t} - e^{-(a'_{38})^{(7)} t}] + G_{38}^0 e^{-(a'_{38})^{(7)} t})$ | 366 |
| $\boxed{T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b'_{38})^{(7)})} \left[ e^{(R_1)^{(7)} t} - e^{-(b'_{38})^{(7)} t} \right] + T_{38}^0 e^{-(b'_{38})^{(7)} t} \le T_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)} + (r_{36})^{(7)} + (R_2)^{(7)})} \left[ e^{((R_1)^{(7)} + (r_{36})^{(7)})t} - e^{-(R_2)^{(7)} t} \right] + T_{38}^0 e^{-(R_2)^{(7)} t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:- | 370 |

Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a'_{36})^{(7)}$

$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$

$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b'_{36})^{(7)}$

$(R_2)^{(7)} = (b'_{38})^{(7)} - (r_{38})^{(7)}$

**Behavior of the solutions of equation** 371

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :

$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying

$-(\sigma_2)^{(8)} \le -(a'_{40})^{(8)} + (a'_{41})^{(8)} - (a''_{40})^{(8)}(T_{41}, t) + (a''_{41})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$

$-(\tau_2)^{(8)} \le -(b'_{40})^{(8)} + (b'_{41})^{(8)} - (b''_{40})^{(8)}((G_{43}), t) - (b''_{41})^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$

**Definition of** $(\nu_1)^{(8)}, (\nu_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, \nu^{(8)}, u^{(8)}$ : 372

By $(\nu_1)^{(8)} > 0, (\nu_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}(\nu^{(8)})^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$
and $(b_{41})^{(8)}(u^{(8)})^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(8)}, , (\bar{\nu}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :

By $(\bar{\nu}_1)^{(8)} > 0, (\bar{\nu}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0, (\bar{u}_2)^{(8)} < 0$ the

roots of the equations $(a_{41})^{(8)}(\nu^{(8)})^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$

and $(b_{41})^{(8)}(u^{(8)})^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$

**Definition of** $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (\nu_0)^{(8)}$ :-

If we define $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by

$(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)}$, ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$

$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$, ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)}$,

and $\boxed{(\nu_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$

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|---|---|
| $(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)},\ \boldsymbol{if}\ (\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},\ \boldsymbol{if}\ (u_0)^{(8)} < (u_1)^{(8)}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}, \boldsymbol{if}\ (u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}, \boldsymbol{if}\ (\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)} t}$<br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \le G_{42}(t) \le \right.$<br>$\left.\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)} t} - e^{-(a'_{42})^{(8)} t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)} t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b'_{42})^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b'_{42})^{(8)} t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)} t} \le T_{42}(t) \le$<br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$ | 381 |

$(R_2)^{(8)} = (b_{42}^{'})^{(8)} - (r_{42})^{(8)}$

**Behavior of the solutions of equation 37 to 92** 382

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$$-(\sigma_2)^{(9)} \le -(a_{44}^{'})^{(9)} + (a_{45}^{'})^{(9)} - (a_{44}^{''})^{(9)}(T_{45}, t) + (a_{45}^{''})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$$

$$-(\tau_2)^{(9)} \le -(b_{44}^{'})^{(9)} + (b_{45}^{'})^{(9)} - (b_{44}^{''})^{(9)}\big((G_{47}), t\big) - (b_{45}^{''})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$$

**Definition of** $(v_1)^{(9)}, (v_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, v^{(9)}, u^{(9)}$ :

By $(v_1)^{(9)} > 0$ , $(v_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(v^{(9)}\big)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{v}_1)^{(9)}, , (\bar{v}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{v}_1)^{(9)} > 0$ , $(\bar{v}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(v^{(9)}\big)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (v_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (v_0)^{(9)}, (m_1)^{(9)} = (v_1)^{(9)}$, ***if*** $(v_0)^{(9)} < (v_1)^{(9)}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (\bar{v}_1)^{(9)}$ , ***if*** $(v_1)^{(9)} < (v_0)^{(9)} < (\bar{v}_1)^{(9)}$,

and $\boxed{(v_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (v_0)^{(9)}$, ***if*** $(\bar{v}_1)^{(9)} < (v_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$G_{44}^0 e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^0 e^{(S_1)^{(9)}t}$

where $(p_i)^{(9)}$ is defined by equation 45

$\frac{1}{(m_9)^{(9)}} G_{44}^0 e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}} G_{44}^0 e^{(S_1)^{(9)}t}$

(

$\frac{(a_{46})^{(9)} G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t} - e^{-(S_2)^{(9)}t}\right] + G_{46}^0 e^{-(S_2)^{(9)}t} \le G_{46}(t) \le$

$\frac{(a_{46})^{(9)} G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)}-(a'_{46})^{(9)})}[e^{(S_1)^{(9)}t} - e^{-(a'_{46})^{(9)}t}] + G_{46}^0 e^{-(a'_{46})^{(9)}t})$

$\boxed{T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^0 e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$

$\frac{1}{(\mu_1)^{(9)}} T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}} T_{44}^0 e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$

$\frac{(b_{46})^{(9)} T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)}-(b'_{46})^{(9)})}\left[e^{(R_1)^{(9)}t} - e^{-(b'_{46})^{(9)}t}\right] + T_{46}^0 e^{-(b'_{46})^{(9)}t} \le T_{46}(t) \le$

$\frac{(a_{46})^{(9)} T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t} - e^{-(R_2)^{(9)}t}\right] + T_{46}^0 e^{-(R_2)^{(9)}t}$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain

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$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right) - (a''_{14})^{(1)}(T_{14}, t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$-\left((a_{14})^{(1)}(v^{(1)})^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}(v^{(1)})^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (\nu_1)^{(1)} < (\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t) \geq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(\nu_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\nu_1)^{(1)}$

In the same manner , we get 384

$$\nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{\nu}_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\bar{\nu}_1)^{(1)}$

If $0 < (\nu_1)^{(1)} < (\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{\nu}_1)^{(1)}$ we find like in the previous case, 385

$$(\nu_1)^{(1)} \leq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}} \leq \nu^{(1)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (C)^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$ if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \le \frac{d\nu^{(2)}}{dt} \le -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \ge \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \le \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \le \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \le \nu^{(2)}(t) \le$$

| | |
|---|---|
| $\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$ | |
| If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br><br>$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :-<br><br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right) - (a_{21}^{''})^{(3)}(T_{21},t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| | 400 |

From which one obtains

For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

In the same manner , we get

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$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

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If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case,

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

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If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$ if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

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In the same manner , we get

$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

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$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}'')^{(5)} = (b_{29}'')^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then

$(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a'_{32})^{(6)} - (a'_{33})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right) - (a''_{33})^{(6)}(T_{33},t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \le \frac{d\nu^{(6)}}{dt} \le -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \ge \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \le \nu^{(6)}(t) \le (\nu_1)^{(6)}$

In the same manner , we get

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$$\nu^{(6)}(t) \le \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \le \nu^{(6)}(t) \le (\bar{\nu}_1)^{(6)}$

If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

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$$(\nu_1)^{(6)} \le \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}} \le \nu^{(6)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \le (\bar{\nu}_1)^{(6)}$$

If $0 < (\nu_1)^{(6)} \le (\bar{\nu}_1)^{(6)} \le \boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$ , we obtain

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$$(\nu_1)^{(6)} \le \nu^{(6)}(t) \le \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \le (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}$, $\boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

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$$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$$

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If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain

$$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(C)^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

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**Definition of** $u^{(7)}(t)$ :-

$$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}, \quad \boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}'')^{(7)} = (a_{37}'')^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}'')^{(7)} = (b_{37}'')^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a'_{40})^{(8)} - (a'_{41})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right) - (a''_{41})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \le \frac{dv^{(8)}}{dt} \le -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(8)}, (v_0)^{(8)}$ :-

For $0 < \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}} < (v_1)^{(8)} < (\bar{v}_1)^{(8)}$

$$v^{(8)}(t) \ge \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(v_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (v_2)^{(8)}}}$$

it follows $(v_0)^{(8)} \le v^{(8)}(t) \le (v_1)^{(8)}$

In the same manner , we get

$$v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{v}_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (\bar{v}_2)^{(8)}}}$$

From which we deduce $(v_0)^{(8)} \le v^{(8)}(t) \le (\bar{v}_8)^{(8)}$

If $0 < (v_1)^{(8)} < (v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}} < (\bar{v}_1)^{(8)}$ we find like in the previous case,

$$(v_1)^{(8)} \le \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}} \le v^{(8)}(t) \le$$

$$\frac{(\bar{v}_1)^{(8)} + (C)^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (\bar{v}_1)^{(8)}$$

If $0 < (v_1)^{(8)} \le (\bar{v}_1)^{(8)} \le \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$ , we obtain

$$(v_1)^{(8)} \le v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (v_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

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**Definition of** $v^{(8)}(t)$ :-

$(m_2)^{(8)} \leq v^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{v^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $v^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)}+(\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)}-(\bar{v}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)}-(\bar{v}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{v}_1)^{(9)}-(v_0)^{(9)}}{(v_0)^{(9)}-(\bar{v}_2)^{(9)}}}$$

From which we deduce $(v_0)^{(9)} \le v^{(9)}(t) \le (\bar{v}_1)^{(9)}$

If $0 < (v_1)^{(9)} < (v_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{v}_1)^{(9)}$ we find like in the previous case,

$$(v_1)^{(9)} \le \frac{(v_1)^{(9)}+(C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)}-(v_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)}-(v_2)^{(9)}\right)t\right]}} \le v^{(9)}(t) \le$$

$$\frac{(\bar{v}_1)^{(9)}+(\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)}-(\bar{v}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)}-(\bar{v}_2)^{(9)}\right)t\right]}} \le (\bar{v}_1)^{(9)}$$

If $0 < (v_1)^{(9)} \le (\bar{v}_1)^{(9)} \le \boxed{(v_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(v_1)^{(9)} \le v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)}+(\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)}-(\bar{v}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)}-(\bar{v}_2)^{(9)}\right)t\right]}} \le (v_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(9)}(t)$ :-

$$(m_2)^{(9)} \le v^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{v^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \le u^{(9)}(t) \le (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(v_1)^{(9)} = (\bar{v}_1)^{(9)}$ if in addition $(v_0)^{(9)} = (v_1)^{(9)}$ then $v^{(9)}(t) = (v_0)^{(9)}$ and as a consequence $G_{44}(t) = (v_0)^{(9)}G_{45}(t)$ **this also defines** $(v_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(v_1)^{(9)}$ and $(\bar{v}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

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| $(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,<br>$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$<br>$with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$ | 433 |

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| $(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$ | 436 A |

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| $(a'_{44})^{(9)}(a'_{45})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a'_{45})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b'_{44})^{(9)}(r_{45})^{(9)} - (b'_{45})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b'_{13})^{(1)} - (b''_{13})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b'_{14})^{(1)} - (b''_{14})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b'_{15})^{(1)} - (b''_{15})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b'_{16})^{(2)} - (b''_{16})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b'_{17})^{(2)} - (b''_{17})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b'_{18})^{(2)} - (b''_{18})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a'_{20})^{(3)} + (a''_{20})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a'_{21})^{(3)} + (a''_{21})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a'_{22})^{(3)} + (a''_{22})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b'_{20})^{(3)} - (b''_{20})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b'_{21})^{(3)} - (b''_{21})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b'_{22})^{(3)} - (b''_{22})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) +$ | 485 |

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| $(a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |

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| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492<br>A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations | 498 |

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| $G_{28} = \frac{(a_{28})^{(5)} G_{29}}{[(a'_{28})^{(5)} + (a''_{28})^{(5)}(T^*_{29})]}$ , $G_{30} = \frac{(a_{30})^{(5)} G_{29}}{[(a'_{30})^{(5)} + (a''_{30})^{(5)}(T^*_{29})]}$ | |
| **Definition and uniqueness of** $T^*_{33}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T^*_{33}$ for which $f(T^*_{33}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{32} = \frac{(a_{32})^{(6)} G_{33}}{[(a'_{32})^{(6)} + (a''_{32})^{(6)}(T^*_{33})]}$ , $G_{34} = \frac{(a_{34})^{(6)} G_{33}}{[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T^*_{33})]}$ | 499 |
| **Definition and uniqueness of** $T^*_{37}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T^*_{37}$ for which $f(T^*_{37}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)} G_{37}}{[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T^*_{37})]}$ , $G_{38} = \frac{(a_{38})^{(7)} G_{37}}{[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T^*_{37})]}$ | 500 |
| **Definition and uniqueness of** $T^*_{41}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T^*_{41}$ for which $f(T^*_{41}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)} G_{41}}{[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T^*_{41})]}$ , $G_{42} = \frac{(a_{42})^{(8)} G_{41}}{[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T^*_{41})]}$ | 501 |
| **Definition and uniqueness of** $T^*_{45}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T^*_{45}$ for which $f(T^*_{45}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)} G_{45}}{[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T^*_{45})]}$ , $G_{46} = \frac{(a_{46})^{(9)} G_{45}}{[(a'_{46})^{(9)} + (a''_{46})^{(9)}(T^*_{45})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br><br>$\left[(b'_{13})^{(1)}(b''_{14})^{(1)}(G) + (b'_{14})^{(1)}(b''_{13})^{(1)}(G)\right] + (b''_{13})^{(1)}(G)(b''_{14})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that $\varphi$ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that | 502 |

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| there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br>$\varphi(G_{19}) = (b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br>$\left[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19}) + (b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})\right] + (b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right] + (b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right] + (b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that | 508 |

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| there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0\,, T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0\,, T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |

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| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^*$ given by $\varphi((G_{23})^*) = 0$ , $T_{21}^*$ given by $f(T_{21}^*) = 0$ and<br>$G_{20}^* = \frac{(a_{20})^{(3)} G_{21}^*}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^*)]}$ , $G_{22}^* = \frac{(a_{22})^{(3)} G_{21}^*}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^*)]}$<br>$T_{20}^* = \frac{(b_{20})^{(3)} T_{21}^*}{[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}{}^*)]}$ , $T_{22}^* = \frac{(b_{22})^{(3)} T_{21}^*}{[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}{}^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^*$ given by $\varphi(G_{27}) = 0$ , $T_{25}^*$ given by $f(T_{25}^*) = 0$ and<br>$G_{24}^* = \frac{(a_{24})^{(4)} G_{25}^*}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}^*)]}$ , $G_{26}^* = \frac{(a_{26})^{(4)} G_{25}^*}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}^*)]}$ | 516 |
| $T_{24}^* = \frac{(b_{24})^{(4)} T_{25}^*}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27})^*)]}$ , $T_{26}^* = \frac{(b_{26})^{(4)} T_{25}^*}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^*$ given by $\varphi((G_{31})^*) = 0$ , $T_{29}^*$ given by $f(T_{29}^*) = 0$ and<br>$G_{28}^* = \frac{(a_{28})^{(5)} G_{29}^*}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^*)]}$ , $G_{30}^* = \frac{(a_{30})^{(5)} G_{29}^*}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^*)]}$ | 518 |
| $T_{28}^* = \frac{(b_{28})^{(5)} T_{29}^*}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^*)]}$ , $T_{30}^* = \frac{(b_{30})^{(5)} T_{29}^*}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^*$ given by $\varphi((G_{35})^*) = 0$ , $T_{33}^*$ given by $f(T_{33}^*) = 0$ and<br>$G_{32}^* = \frac{(a_{32})^{(6)} G_{33}^*}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^*)]}$ , $G_{34}^* = \frac{(a_{34})^{(6)} G_{33}^*}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^*)]}$ | 520 |
| $T_{32}^* = \frac{(b_{32})^{(6)} T_{33}^*}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^*)]}$ , $T_{34}^* = \frac{(b_{34})^{(6)} T_{33}^*}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^*$ given by $\varphi((G_{39})^*) = 0$ , $T_{37}^*$ given by $f(T_{37}^*) = 0$ and | 522 |

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| $G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)} (T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)} (T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)} ((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)} ((G_{39})^{*})]}$ | |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)} (T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)} (T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)} ((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)} ((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br><br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br><br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)} (T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)} (T_{45}^{*})]}$<br><br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)} ((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)} ((G_{47})^{*})]}$ | 523 A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:**_Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}} (T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j} (G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*} \mathbb{T}_{14}$ | 525 |

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| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^{*}\mathbb{G}_{j}\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^{*}\mathbb{G}_{j}\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^{*}\mathbb{G}_{j}\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 532 |
| $\frac{\partial(a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial(b_i^{''})^{(2)}}{\partial G_j}((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^{*}\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | 540 |

$$G_i = G_i^* + \mathbb{G}_i \qquad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)} \quad , \frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^*\mathbb{T}_{21} \quad (541)$$

$$\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^*\mathbb{T}_{21} \quad (542)$$

$$\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^*\mathbb{T}_{21} \quad (543)$$

$$\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^*\mathbb{G}_j\right) \quad (544)$$

$$\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^*\mathbb{G}_j\right) \quad (545)$$

$$\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^*\mathbb{G}_j\right) \quad (546)$$

**ASYMPTOTIC STABILITY ANALYSIS** (547)

**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ *and* $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- (548)

$$G_i = G_i^* + \mathbb{G}_i \qquad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)} \quad , \frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^*) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25} \quad (549)$$

$$\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25} \quad (550)$$

$$\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25} \quad (551)$$

$$\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right) \quad (552)$$

$$\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right) \quad (553)$$

$$\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right) \quad (554)$$

**ASYMPTOTIC STABILITY ANALYSIS** 555

**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ $and$ $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- 556

$$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)} \quad , \frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^*) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29} \quad 557$$

$$\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29} \quad 558$$

$$\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29} \quad 559$$

$$\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^*\mathbb{G}_j\right) \quad 560$$

$$\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^*\mathbb{G}_j\right) \quad 561$$

$$\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^*\mathbb{G}_j\right) \quad 562$$

**ASYMPTOTIC STABILITY ANALYSIS** 563

**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ $and$ $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- 564

$$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)} \quad , \frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33} \quad 565$$

$$\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33} \quad 566$$

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| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{37})^{(7)}}{\partial T_{37}}(T^*_{37}) = (q_{37})^{(7)}$ , $\frac{\partial(b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a'_{36})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G^*_{36}\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a'_{37})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G^*_{37}\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. | |

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| Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \ , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}((G_{43})^*) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^*\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^*\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b_{41}^{'})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T_{41}^*\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b_{42}^{'})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T_{42}^*\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS** **Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. **Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |

| | |
|---|---|
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$ $\left[\left(\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\right]$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\right)$ $+\left(\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\right)$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((b'_{13})^{(1)} + (b'_{14})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $+\left(\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$ $+\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G^*_{13}\right)$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T^*_{14} + (b_{14})^{(1)}s_{(13),(15)}T^*_{13}\right)\} = 0$ $+$ $\left((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)}\right)\{\left((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)}\right)$ $\left[\left(\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}^*_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^*_{16}\right)\right]$ $\left(\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^*_{17}\right)$ $+\left(\left((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}^*_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^*_{17}\right)$ $\left(\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^*_{16}\right)$ $\left(\left((\lambda)^{(2)}\right)^2 + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ $\left(\left((\lambda)^{(2)}\right)^2 + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ | |

$$+\left(\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\right)\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left(\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\right)$$

$$+\left(\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((b_{20}')^{(3)}+(b_{21}')^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}')^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}')^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}')^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}')^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}')^{(4)}+(a_{25}')^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((b_{24}')^{(4)}+(b_{25}')^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\}\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^2+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G^*_{33}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G^*_{32}\right)$$

$$\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T^*_{33}+(b_{33})^{(6)}s_{(32),(34)}T^*_{32}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b'_{38})^{(7)}-(r_{38})^{(7)}\}\{\left((\lambda)^{(7)}+(a'_{38})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(q_{36})^{(7)}G^*_{36}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T^*_{37}+(b_{37})^{(7)}s_{(36),(37)}T^*_{37}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a'_{37})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G^*_{36}+(a_{36})^{(7)}(q_{37})^{(7)}G^*_{37}\right)$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T^*_{37}+(b_{37})^{(7)}s_{(36),(36)}T^*_{36}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^2+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^*_{36}\right)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^*_{37}+(b_{37})^{(7)}s_{(36),(38)}T^*_{36}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^*_{40}\right)\right]$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^*_{41}+(b_{41})^{(8)}s_{(40),(41)}T^*_{41}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^*_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^*_{41}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^*_{41}+(b_{41})^{(8)}s_{(40),(40)}T^*_{40}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^*_{40}\right)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^*_{41}+(b_{41})^{(8)}s_{(40),(42)}T^*_{40}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^*_{44}\right)\right]$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^*_{45}+(b_{45})^{(9)}s_{(44),(45)}T^*_{45}\Big)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^*_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45}+(b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left(\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45}+(b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

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# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i(x)\, xi$ where $x_i \in R^n_i$, $x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i(x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i(\mu)$ of the $A_i(x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$(1) \qquad \frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k$$

Where $x_i$ is the abundance of population $i$, $x = (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to

replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 1. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon\ C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \ i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0}$

$\overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow *x.t restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set *K* is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** *x.t* is defined for all $t \geq 0$

**A4:** *x.t* is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\mathcal{A}(\tilde{A}), C \backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \longrightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) in\ ||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population *i* is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^{t}_{0} (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\to\infty}{limsup} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v = au_i(x) + w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |B_i(t,x)u_i(x)|(a+\alpha+exp(-\beta t)|w|)$. Hence, $\limsup_{t\to\infty}\frac{1}{t} In\, B_i(t,x)\, v| \leq$

$$\limsup_{t\to\infty}\frac{1}{t} In\, B_i(t,x)\, u_i(x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty}\frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a-\alpha+exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty}\frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty}\frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\, |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\, u_i(x)| = In\, |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\, u_i(x.t)||\, B_i(t,x)\, u_i(x)|$$

$$= \text{In } b_i(s,t.x)\, b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt}b_i(t,x) = \lim_{s\to 0}\frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0}\frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

$$=$$

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s), u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } | (x.t)_i |$$

$$\leq \limsup_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x) =$ Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let $K \subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let $K \subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if $W^s(K) \subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of

irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x)u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$ where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ *and* $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as $n\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\le \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x)u_i^n(x), u_i^n(x)\rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)|d\mu(x) + |\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\,d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\,ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i\,dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\,d\,v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak$^*$ limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \{ \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i \epsilon I}{if\ i \notin I}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that \quad \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t$ −invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently *K* is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s =$

$s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_s$. If $r(x) = s(x), then\ x \in M_r$. Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume $x \mapsto A(x)$ is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists $j \in \{1,\ldots,k\}$ such that $M_i$ ----- $\{x \in C : x_l \gg 0 \text{ for } l \neq j\}$ Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_i M_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, *$\lambda_i(\mu) > 0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, ..., x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the

system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j \ (x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1{}^1, \ldots, E_1{}^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i{}_2(E_1{}^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$$

$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j\, x_1^j - x_2^j - \beta^j\, x_3^j) + \sum d_2^{jk}\, x_2^k - e_2^j\, x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j\, x_1^j - \alpha^j x_2^j - \beta^j\, x_3^j) + \sum d_3^{jk}\, x_3^k - e_3^j\, x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > 1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria $E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$ and $\lambda_3(E_1)$ are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\Big(r_i - \sum_s a_{is}x_s^j\Big) + \sum_l d_i^{jl}\, x_i^l - e_i^j x_i^j, \qquad i = 1,\ldots,k;\ j = 1,\ldots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i\, a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i\, x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \qquad i = 1, \ldots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \ldots, k$$

It has been shown [12,ch.16.1,16.2] That for $k = 3$,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1 \to E_2 \to E_3 \to E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is

the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1-y) - b(N)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1 = N$, $x_2 = p$, $x_3 = (y,z)'$ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\curlywedge_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\curlywedge_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\curlywedge_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + K p\, x_i x_i - 1}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i)-1 & \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions

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# Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a

qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and

orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t\geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a\geq 0)$ at time $t$ so that we have

(1.1) $$P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t\geq 0.$$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.2) $$D\rho_i(a, t) = \limsup_{h\to 0} \frac{\rho_i(a+h,t+h)-\rho_i(a,t)}{h},\ i = 1, 2.$$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.3) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0,' & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, \end{cases}$$

(1.5) $$\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s\in [-\tau, 0],\qquad i = 1, 2,$$

(1.6) $$P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s\in [-\tau, 0],$$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1, \varphi_2, f_1, f_2, b_1, b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for

each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as* $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 1. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$. We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C(R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C(R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$,

($d_1^*$, $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$; that is, *for* $i = 1, 2$,

$$|f_i(a, P_1', P_2') - f_i(a, P_1'', P_2'')| \leq d_i\ [||P_1' - P_1'', P_2' - P_2''||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\partial P_1, \partial f_1/\partial P_2 \in C(R^+ \times K_1 \times K_2, R^+),$$

$$\partial f_2/\partial P_1, \partial f_2/\partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C(R^+ K_1 \times K_2, R^+)$, $b_1 \leq \beta_1^*$ *on* $R^+ \times K_1 \times K_2$,

$b_2 \in C(R^+ K_2 \times K_1, R^+)$, $b_2 \leq \beta_2^*$ *on* $R^+ \times K_2 \times K_1$,

*($\beta_1^*$, $\beta_2^*$ are positive constants);*

$b_1$, $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$|b_i(a, P_1', P_2') - b_i(a, P_1'', P_2'')| \leq \beta_i\ [||P_1' - P_1''|| + ||P_2' - P_2''||]$

$B_1$, $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2)) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, R^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, R^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1, 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[-\int_0^\infty f_i(a + s, P(t + S))ds\right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[-\int_o^x f_1(a + s, p(t + s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0)\exp\left[-\int_o^t f_1(a - t + s, p_1(s))\, ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

*(2.3)* $p_i(a,t) = B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i and\ L_i\ (i = 1,2)$ as follows:

$$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a,t,\ p_i(t)) = \exp[-\int_o^t f_1(a+s,\ p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

$$(2.6)\ p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da + \int_o^\infty \varphi_i(a, 0)\ L_i(a,t,p_i(t)da,$$

$$t > 0, i = 1,2,$$

$$(2.7)\ p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$$

$$(2.8)\ \beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.\left(a,t,P_i(t)\right)da$$

$$+\int_0^\infty b_i\left(a+t, P_i(t)\right)\varphi_i(a,0)L_i\left(a,t,P_i(t)\right)da$$

$$i = 1, 2, t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t) \phi(0) d_i^*] t \quad for \ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T], R^+), \quad B_i \in C([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in C([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau, T]$. Then for any

$(x, y) \in C^+[-\tau, T] \times C^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau, T], \quad C_T^2 = C^+[-\tau, T] \times C^+[-\tau, T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for $i = 1,2$.

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \pi C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \quad P_t = \ (P_{1t}, P_{2t}) \in C^2_{t*} \mid \|P_t - \Phi_t\|_{t*} \le r^*$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)\, da \ for\ s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)\, da \ for\ s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Phi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^* \right] \ t^*$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for t $\le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \le \left| \int_0^t B_1^{(1)}(a) \left\{ M_1(a, t, P_t^{(1)}) - M_1(a, t, P_t^{(2)}) \right\} da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a, t, P_t^{(2)}) da \right|$$

$$+ \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \; for \; x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \| ds\right) da$$

$$\leq d_1 \beta_1^* \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o)\left[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds\right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \| t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\text{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right]\right] d\sigma$$

$$+ \int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^{\infty} \varphi(\sigma,o) b_1\left(\sigma+a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma,a,P_t^{(1)}(a)\right)-L_1\left(\sigma,a,P_t^{(2)}(a)\right)\right]d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right)-b_1\left(\sigma+a, P_t^{(2)}(a)\right)\right| \le B_1 \,/\!/\, P_t^{(1)}-P_t^{(2)} /\!/ \text{ for } a\,\varepsilon[o,t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \le \beta_1 \Phi_1(0) ||P_t^{(1)}-P_t^{(2)}||_t^{*} + \Phi_1(0)\beta_1 d_1 ||P_t^{(1)}-P_t^{(2)}||_t^{*a},$$

$$|R_2(a)|=[\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a))-b_1(a-\sigma, P_t^{(2)}(a))\}M_1(\sigma, a, P_t^{(1)}(a))$$

$$+b_1(a,-\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a))-M_1(a-\sigma, P_t^{(2)}(a))\}]d\sigma]$$

$$+b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a))-M_1(a-\sigma, P_t^{(2)}(a))\}]d\sigma]$$

$$\le \int_0^a [\beta_1^{*} \Phi_1(0)\, e^{\delta 1^a} \beta_1 \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}(a-o)]do$$

$$\le \beta_1^{*} \Phi_1(0)\, e^{\delta 1^a} \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a)\right|$$

$$\le \int_o^a \left| B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma)\right| B_1^{*d}\sigma$$

$$+\Phi_1(0) /\!/ P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\left\{B_1 + B_2 d_1 a + B_1^{*}\left(B_2 a + d_1 a^2/2\right)e^{\delta 1^a}\right\}$$

$$\le \beta_1^{*}\int_o^a \left| B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma)\right| do + \left(r^{*} + /\!/ \Phi_t /\!/_{t*}\right)\| P_t^{(1)}-P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \to \beta_1$ as $a \to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a)\right| \le \left(r^{*} + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))\text{-}b_1(a\text{-}\sigma, P_t^{(1)}(a))\, M_1(\sigma, a, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + /\!/ \Phi_t /\!/_{t*}\right) /\!/ \; P_t^{(1)}\text{-}P_t^{(2)} /\!/_{t*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right) /\!/ \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^{*})$$

As $t^{*} \to o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^{*}, P_t^{*}$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp\, P_2^{*}(a)}{da} + f_i(\mathrm{a}, P_1^{*}, P_2^{*})\, P_i^{*}(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^{*} = \int_o^{\infty} P_i^{*}(a)da, i = 1,\ 2,$$

$$(3,3)\ P_i^{*} = \int_o^{\infty} b_i(a,\ P_1^{*},\ P_2^{*}) P_i^{*}(a)da,\ i = 1.\ \ 2$$

All solutions of (3.1)are of the from

$$(3.4)\ P_i^{*}(a) = P_i^{*}(o) \exp\left[ - \int_o^{\infty} f_i(s, P_1^{*}, P_2^{*})ds \right], i = 1, \quad 2, a \geq 0,$$

Which together with (3.3) lead to

$$1=\int_o^{\infty} b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^{\infty} b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $$F_{i.}\ (P_1,P_2) = \int_o^{\infty} b_i(a,P_1,P_2)\exp\left[-\int_o^{\infty} f_i\,(s,P_1,P_2)ds\right]da,$$

$$i = 1,\ 2, \quad P_1 \geq 0, \quad P_2 \geq 0,$$

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_5$. *there exists a positive constant c for which*

$$F_1(\mathrm{C},0) < F_{2.}(\mathrm{C},0),$$

(3.8) $$F_1(0.\mathrm{C}) < F_{2.}(0,\mathrm{C}),$$

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $$F_1(P_1^*,P_2^*) = 1 = F_2(P_1^*,P_2^*)$$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $$P_2^*(a) = [P_2^*\exp[-f_i\,(s,P_1^*,P_2^*)ds])/(\int_o^{\infty}\exp[-\int_o^a f_i\,(s,P_1^*,P_2^*)ds]da),$$

$$i = 1,\ 2, \quad a \geq 0,$$

Proof Consider the elements of the surfaces defined by

$$\mathrm{Z}= F_1(P_1,P_2), z = F_2(P_1,P_2),\ P_1 \geq 0, P_2 \geq 0,$$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i/\partial P_j < 0\ and$ , by ($A_2$), $\partial b_i/\partial P_j \leq 0$ for all

$$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$$

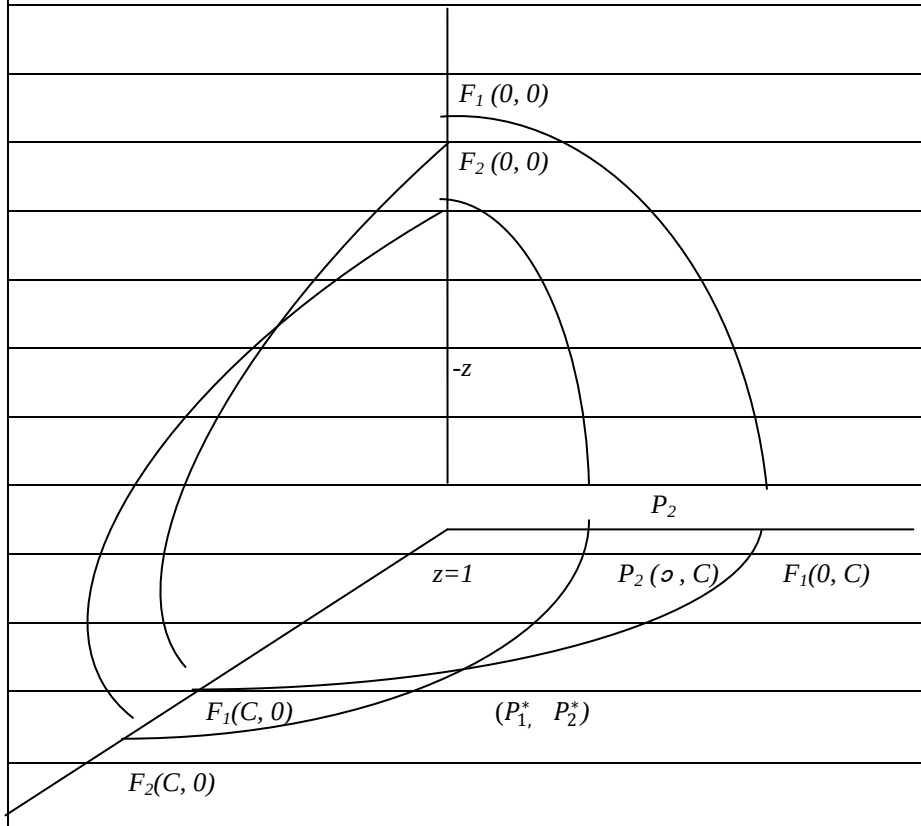
$j = 1,\ 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= 1, $P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$$F_1(P_1, P_2 = 1 \; and \; F_2(P_1, P_2) = 1, \; F_1(P_1, P_2 \geq 0,$$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility 1< $F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p_1^*, p_2^*)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p_1^*, p_2^*$, we let

$$(3.11) \quad \begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \; i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_j^*}\, p_i(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial b_i}{\partial p_j^*}\, p_i(t)\right\}da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_i(a)\exp[\lambda t], \quad u_i(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty u_i(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{du_i}{da} + \lambda u_i = -f_i\left(a, p_1^*, p_2^*\right) u_i(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*}\, p_i^*\right\},$$

(3.15)

$$u_i(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*}\, p_i^*\right\}da,$$

Where $i = 1, 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \quad p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\Phi_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\Phi_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\Phi_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\Phi_i(a) = \{\Phi_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_i\right\} e^{\lambda s} ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\Phi_i(a)\left[p_i^*(a)/p_i^*(0)c_i \exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_i\right\} da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a}\left\{\int_0^a \left(\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_i\right) e^{\lambda s} ds\right\} da\right]$$

$$- p_i^*(a) e^{-\lambda a}\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_i\right\} e^{\lambda s} ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\Phi_1(a)$ and $\Phi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^* \exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^* \exp(-\lambda\tau) + (\lambda - A_{22})p_2^* = g_{21}(\lambda)p_1^* \exp(-\lambda\tau) + g_{22}(\lambda)p_2^*$$

Where

(3.22) $A_{ij} = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_j^*} - \frac{\partial f_i}{\partial p_j^*}\right]da,$

(3.23) $g_{ij}(\lambda) = [p_i^*(0)c_i]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\,da\right\}$
$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_j^*} - b_i(a, p_1^*, p_2^*)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)\right]da\right\}$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)da,\quad i, j = 1, 2,$$

(3.24) $\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*),\quad i, = 1, 2.$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix}\lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda))\end{bmatrix} = 0$

Or equivalently

(3.25) $\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = [A_{21}g_{12}(\lambda) + A_{12}g_{21}(\lambda) + g_{12}(\lambda)g_{21}(\lambda)]\exp[-2\lambda\tau]$

$-A_{11}g_{22}(\lambda) - A_{22}g_{11}(\lambda) - g_{11}(\lambda)g_{22}(\lambda)$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i, = 1, 2.)\ (see(A_1)\ and\ (A_2))$;

$A_7.\ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\},$

$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$

$\| S_2(0) \| = |A_{11}g_{12}(0)| + |A_{12}g_{21}(0)| + |g_{12}(0)g_{21}(0)|$

$+ |A_{11}g_{22}(0)| + |A_{22}g_{11}(0)| + |g_{11}(0)g_{22}(0)|$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i(a, t) = \phi_i(a) \exp[\lambda t] \to 0$ as $t \to \infty$ for $i = 1, 2$ and $a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda$ $\mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$

$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \left|\right.$

$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$

$\geq [(|\lambda| - P)^2 + (q)]^2$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \{4|(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| | - |A_{11} + A_{22} + \| S_1(0) \| |^2\}/2,$

It will now follow, from (A) and (3.26), $F_1(\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder

concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power

obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy

withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation

from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

**TITLE**

## Birth Of Spacetime Atoms As The Passage Of Time

**INTRODUCTION—VARIABLES USED**

**The birth of spacetime atoms as the passage of time∗ Fay Dowker Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2AZ, UK May 15, 2014 For references please see original article. Some sentences and figures are deleted due to spatial constraints. Only a part of the article is modeled.**

(1) Given two points of spacetime, call them A and B, they will either be (=) ordered or unordered.
(2) If they are ordered then one, let's say A without loss of generality, precedes – or, is to the past of – the other, B. This ordering is transitive: if A precedes B and B precedes C then A precedes C. The order is partial because there are pairs of spacetime points such that there is no physical sense in which one of the pair precedes the other, they are simply unordered. This lack of ordering does not mean the points of the pair are simultaneous because that would imply (eb) they occur at the same "time" and require the existence of a global time for them to be simultaneous in.
(3) Global time does not exist in (eb) GR.
(4) This partial ordering of the points of spacetime is referred to as the causal structure of spacetime because (e) it coincides with the potential for physical effects to propagate.
(5) A point of spacetime is an idealization of an event that has (e) no physical extent, like an infinitely small, infinitely fast-burning firecracker [3]. 3 if and only if (e&eb) A precedes B in the causal structure.
(6) If two spacetime points are unordered then no physical effect can propagate from one to the other because (e) to do so would require (e) something physical to travel faster than light.
(7) Causal structure plays a central role in GR and indeed the epitome of the theory, a black hole, is

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| defined in terms of the causal structure: it is a region of spacetime such that nothing that happens in that region can affect (e&eb) anything outside the region.<br>(8) It is only by thinking of a black hole in terms of (e&eb) causal structure that its physics can be understood.<br>**(9)** Physical time in GR is associated, locally and "personally," with (e&eb) every localised physical system, in a manner that more closely reflects our intimate experience of time than the global time of pre-relativistic Newtonian mechanics. | |
| **NOTATION** | |
| **Module One**<br>Given two points of spacetime, call them A and B, they will either be (=) ordered or unordered | |
| $G_{13}$ : Category one of Given two points of spacetime, call them A and B, they will either<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of ordered or unordered<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>If they are ordered then one, let's say A without loss of generality, precedes – or, is to the past of – the other, B. This ordering is transitive: if A precedes B and B precedes C then A precedes C.<br>The order is partial because there are pairs of spacetime points such that there is no physical sense in which one of the pair precedes the other, they are simply unordered.<br>This lack of ordering does not mean the points of the pair are simultaneous because that would imply (eb) they occur at the same "time" and require the existence of a global time for them to be simultaneous in. | |
| $G_{16}$ : Category one of If they are ordered<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of one, let's say A without loss of generality, precedes – or, is to the past of – the other, B. This ordering is transitive: if A precedes B and B precedes C then A precedes C<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>This lack of ordering does not mean the points of the pair are simultaneous because that would imply (eb) they occur at the same "time" and require the existence of a global time for them to be simultaneous in | |

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| $G_{20}$ : Category one of they occur at the same "time" and require the existence of a global time for them to be simultaneous in<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of lack of ordering does not mean the points of the pair are simultaneous<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>Global time does not exist in (eb) GR. | |
| $G_{24}$ : Category one of Global time<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of GR<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>This partial ordering of the points of spacetime is referred to as the causal structure of spacetime because (e) it coincides with the potential for physical effects to propagate | |
| $G_{28}$ : Category one of it coincides with the potential for physical effects to propagate<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of partial ordering of the points of spacetime is referred to as the causal structure of spacetime<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>A point of spacetime is an idealization of an event that has (e) no physical extent, like an infinitely small, infinitely fast-burning firecracker [3]. 3 if and only if (e&eb) A precedes B in the causal structure | |
| $G_{32}$ : Category one of no physical extent, like an infinitely small, infinitely fast-burning firecracker [3]. 3 if | |

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| and only if (e&eb) A precedes B in the causal structure<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of point of spacetime is an idealization of an event<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>If two spacetime points are unordered then no physical effect can propagate from one to the other because (e) to do so would require (e) something physical to travel faster than light | |
| $G_{36}$ : Category one of to do so would require something physical to travel faster than light<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of two spacetime points are unordered then no physical effect can propagate from one to the other<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Causal structure plays a central role in GR and indeed the epitome of the theory, a black hole, is defined in terms of the causal structure: it is a region of spacetime such that nothing that happens in that region can affect (e&eb) anything outside the region | |
| $G_{40}$ : Category one of **Causal structure plays a central role in GR and indeed the epitome of the theory, a black hole, is defined in terms of the causal structure: it is a region of spacetime such that nothing that happens in that region can affect;** anything outside the region<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of anything outside the region; **Causal structure plays a central role in GR and indeed the epitome of the theory, a black hole, is defined in terms of the causal structure: it is a region of spacetime such that nothing that happens in that region can affect**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |

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| **Module Nine**<br><br>It is only by thinking of a black hole in terms of (e&eb) causal structure that its physics can be understood | |
| $G_{44}$ : Category one of **only by thinking of a black hole;** causal structure that its physics can be understood<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of causal structure that its physics can be understood **;only by thinking of a black hole**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a_{13}^{'})^{(1)}, (a_{14}^{'})^{(1)}, (a_{15}^{'})^{(1)}, (b_{13}^{'})^{(1)}, (b_{14}^{'})^{(1)}, (b_{15}^{'})^{(1)}, (a_{16}^{'})^{(2)}, (a_{17}^{'})^{(2)}, (a_{18}^{'})^{(2)}$, $(b_{16}^{'})^{(2)}, (b_{17}^{'})^{(2)}, (b_{18}^{'})^{(2)}$<br>$, (a_{20}^{'})^{(3)}, (a_{21}^{'})^{(3)}, (a_{22}^{'})^{(3)}, (b_{20}^{'})^{(3)}, (b_{21}^{'})^{(3)}, (b_{22}^{'})^{(3)}$<br>$(a_{24}^{'})^{(4)}, (a_{25}^{'})^{(4)}, (a_{26}^{'})^{(4)}, (b_{24}^{'})^{(4)}, (b_{25}^{'})^{(4)}, (b_{26}^{'})^{(4)}, (b_{28}^{'})^{(5)}, (b_{29}^{'})^{(5)}, (b_{30}^{'})^{(5)}$ $(a_{28}^{'})^{(5)}, (a_{29}^{'})^{(5)}, (a_{30}^{'})^{(5)}$ ,<br>$(a_{32}^{'})^{(6)}, (a_{33}^{'})^{(6)}, (a_{34}^{'})^{(6)}, (b_{32}^{'})^{(6)}, (b_{33}^{'})^{(6)}, (b_{34}^{'})^{(6)}$<br>$(a_{36}^{'})^{(7)}, (a_{37}^{'})^{(7)}, (a_{38}^{'})^{(7)}, (b_{36}^{'})^{(7)}, (b_{37}^{'})^{(7)}, (b_{38}^{'})^{(7)}$,<br>$(a_{40}^{'})^{(8)}, (a_{41}^{'})^{(8)}, (a_{42}^{'})^{(8)}, (b_{40}^{'})^{(8)}, (b_{41}^{'})^{(8)}, (b_{42}^{'})^{(8)}$,<br>$(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)} G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right] G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)} G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right] G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)} G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right] G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)} T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right] T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)} T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right] T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)} T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right] T_{15}$ | 6 |

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| $+(a_{13}^{''})^{(1)}(T_{14},t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G,t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17},t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17},t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17},t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17},t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}),t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}\left((G_{27}),t\right) =$ **First detritions factor** | |
| **Module Numbered Five**: | |

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| **The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |

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| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}\left((G_{43}),t\right)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}\left((G_{43}),t\right)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}\left((G_{47}),t\right) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)} & +(a_{13}^{''})^{(1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,)}(T_{17},t) \\ +(a_{20}^{''})^{(3,3,)}(T_{21},t) & +(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t) \\ +(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t) & +(a_{36}^{''})^{(7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8)}(T_{41},t) \\ +(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) & & \end{array}\right]G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{array}{ccc} (a_{14}^{'})^{(1)} & +(a_{14}^{''})^{(1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,)}(T_{17},t) \\ +(a_{21}^{''})^{(3,3,)}(T_{21},t) & +(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t) \\ +(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t) & +(a_{37}^{''})^{(7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8)}(T_{41},t) \\ +(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) & & \end{array}\right]G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{15}^{'})^{(1)} & +(a_{15}^{''})^{(1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,)}(T_{17},t) \\ +(a_{22}^{''})^{(3,3,)}(T_{21},t) & +(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t) \\ +(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t) & +(a_{38}^{''})^{(7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8)}(T_{41},t) \\ +(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) & & \end{array}\right]G_{15}$ | *57* |
| Where $(a_{13}^{''})^{(1)}(T_{14},t)$, $(a_{14}^{''})^{(1)}(T_{14},t)$, $(a_{15}^{''})^{(1)}(T_{14},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{38}^{''})^{(7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3 | |

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| $+(a_{40}^{''})^{(8,8)}(T_{41},t)$ $+(a_{41}^{''})^{(8,8)}(T_{41},t)$ $+(a_{42}^{''})^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3 $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)} & -(b_{13}^{''})^{(1)}(G,t) & -(b_{16}^{''})^{(2,2,)}(G_{19},t) & -(b_{20}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,)}(G_{39},t) & -(b_{40}^{''})^{(8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)} & -(b_{14}^{''})^{(1)}(G,t) & -(b_{17}^{''})^{(2,2,)}(G_{19},t) & -(b_{21}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,)}(G_{39},t) & -(b_{41}^{''})^{(8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)} & -(b_{15}^{''})^{(1)}(G,t) & -(b_{18}^{''})^{(2,2,)}(G_{19},t) & -(b_{22}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,)}(G_{39},t) & -(b_{42}^{''})^{(8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$ | *60* |
| Where $-(b_{13}^{''})^{(1)}(G,t)$, $-(b_{14}^{''})^{(1)}(G,t)$, $-(b_{15}^{''})^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{16}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3 $-(b_{20}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3 $-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3 $-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{37}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 $-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3 $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}^{'})^{(2)} & +(a_{16}^{''})^{(2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,)}(T_{14},t) & +(a_{20}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{16}$ | *61* |

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| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}^{'})^{(2)}\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{17}$ | 62 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}^{'})^{(2)}\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{18}$ | 63 |
| Where $\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)}$ are seventh augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)}$ are eight augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)}$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)}\boxed{-(b_{16}^{''})^{(2)}(G_{19},t)} & \boxed{-(b_{13}^{''})^{(1,1,)}(G,t)} & \boxed{-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{16}$ | 64 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)}\boxed{-(b_{17}^{''})^{(2)}(G_{19},t)} & \boxed{-(b_{14}^{''})^{(1,1,)}(G,t)} & \boxed{-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{17}$ | 65 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)}\boxed{-(b_{18}^{''})^{(2)}(G_{19},t)} & \boxed{-(b_{15}^{''})^{(1,1,)}(G,t)} & \boxed{-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{18}$ | 66 |
| where $\boxed{-(b_{16}^{''})^{(2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3 | |

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| $-(b_{13}^{''})^{(1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3<br>$-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[ \begin{array}{c} (a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t) + (a_{16}^{''})^{(2,2,2)}(T_{17},t) + (a_{13}^{''})^{(1,1,1,)}(T_{14},t) \\ + (a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t) + (a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t) + (a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ + (a_{36}^{''})^{(7,7,7,7)}(T_{37},t) + (a_{40}^{''})^{(8,8,8,8)}(T_{41},t) + (a_{44}^{''})^{(9,9,9)}(T_{45},t) \end{array} \right] G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[ \begin{array}{c} (a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t) + (a_{17}^{''})^{(2,2,2)}(T_{17},t) + (a_{14}^{''})^{(1,1,1,)}(T_{14},t) \\ + (a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t) + (a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t) + (a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ + (a_{37}^{''})^{(7,7,7,7)}(T_{37},t) + (a_{41}^{''})^{(8,8,8,8)}(T_{41},t) + (a_{45}^{''})^{(9,9,9)}(T_{45},t) \end{array} \right] G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[ \begin{array}{c} (a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t) + (a_{18}^{''})^{(2,2,2)}(T_{17},t) + (a_{15}^{''})^{(1,1,1,)}(T_{14},t) \\ + (a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t) + (a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t) + (a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ + (a_{38}^{''})^{(7,7,7,7)}(T_{37},t) + (a_{42}^{''})^{(8,8,8,8)}(T_{41},t) + (a_{46}^{''})^{(9,9,9)}(T_{45},t) \end{array} \right] G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation | |

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| coefficients for category 1, 2 and 3 $+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3 $+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3 $+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt}=(b_{20})^{(3)}T_{21}-\begin{bmatrix}(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23},t) & -(b_{16}^{''})^{(2,2,2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,1,)}(G,t)\\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)\\ -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9)}(G_{47},t)\end{bmatrix}T_{20}$ | *70* |
| $\frac{dT_{21}}{dt}=(b_{21})^{(3)}T_{20}-\begin{bmatrix}(b_{21}^{'})^{(3)}-(b_{21}^{''})^{(3)}(G_{23},t) & -(b_{17}^{''})^{(2,2,2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,1,)}(G,t)\\ -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)\\ -(b_{37}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9)}(G_{47},t)\end{bmatrix}T_{21}$ | *71* |
| $\frac{dT_{22}}{dt}=(b_{22})^{(3)}T_{21}-\begin{bmatrix}(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23},t) & -(b_{18}^{''})^{(2,2,2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,1,)}(G,t)\\ -(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)\\ -(b_{38}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9)}(G_{47},t)\end{bmatrix}T_{22}$ | *72* |
| $-(b_{20}^{''})^{(3)}(G_{23},t)$, $-(b_{21}^{''})^{(3)}(G_{23},t)$, $-(b_{22}^{''})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3* $-(b_{16}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3* $-(b_{13}^{''})^{(1,1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3* $-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3* $-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3* $-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3* $-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 $-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3 $-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)} G_{25} - \begin{bmatrix} (a'_{24})^{(4)} & +(a''_{24})^{(4)}(T_{25},t) & +(a''_{28})^{(5,5,)}(T_{29},t) & +(a''_{32})^{(6,6,)}(T_{33},t) \\ & +(a''_{13})^{(1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3)}(T_{21},t) \\ & +(a''_{36})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)} G_{24} - \begin{bmatrix} (a'_{25})^{(4)} & +(a''_{25})^{(4)}(T_{25},t) & +(a''_{29})^{(5,5,)}(T_{29},t) & +(a''_{33})^{(6,6)}(T_{33},t) \\ & +(a''_{14})^{(1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3)}(T_{21},t) \\ & +(a''_{37})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)} G_{25} - \begin{bmatrix} (a'_{26})^{(4)} & +(a''_{26})^{(4)}(T_{25},t) & +(a''_{30})^{(5,5,)}(T_{29},t) & +(a''_{34})^{(6,6,)}(T_{33},t) \\ & +(a''_{15})^{(1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a''_{24})^{(4)}(T_{25},t)$, $(a''_{25})^{(4)}(T_{25},t)$, $(a''_{26})^{(4)}(T_{25},t)$ are first augmentation coefficients category 1, 2 3

$+(a''_{28})^{(5,5,)}(T_{29},t)$, $+(a''_{29})^{(5,5,)}(T_{29},t)$, $+(a''_{30})^{(5,5,)}(T_{29},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a''_{32})^{(6,6,)}(T_{33},t)$, $+(a''_{33})^{(6,6,)}(T_{33},t)$, $+(a''_{34})^{(6,6,)}(T_{33},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a''_{13})^{(1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1, 2 and 3

$+(a''_{16})^{(2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1, 2 and 3

$+(a''_{20})^{(3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1, 2 and 3

$+(a''_{36})^{(7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3

$+(a''_{40})^{(8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1, 2 and 3

$+(a''_{46})^{(9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9)}(T_{45},t)$, $+(a''_{44})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt} = (b_{24})^{(4)} T_{25} - \begin{bmatrix} (b'_{24})^{(4)} & -(b''_{24})^{(4)}(G_{27},t) & -(b''_{28})^{(5,5,)}(G_{31},t) & -(b''_{32})^{(6,6,)}(G_{35},t) \\ & -(b''_{13})^{(1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)} T_{24} - \begin{bmatrix} (b'_{25})^{(4)} & -(b''_{25})^{(4)}(G_{27},t) & -(b''_{29})^{(5,5,)}(G_{31},t) & -(b''_{33})^{(6,6,)}(G_{35},t) \\ & -(b''_{14})^{(1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b'_{26})^{(4)}\boxed{-(b''_{26})^{(4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{26} \quad (78)$$

Where $\boxed{-(b''_{24})^{(4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4)}(G_{27},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,)}(G_{31},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,)}(G_{35},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3

$$\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a'_{28})^{(5)}\boxed{+(a''_{28})^{(5)}(T_{29},t)} & \boxed{+(a''_{24})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{32})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{28} \quad (79)$$

$$\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a'_{29})^{(5)}\boxed{+(a''_{29})^{(5)}(T_{29},t)} & \boxed{+(a''_{25})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{33})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{29} \quad (80)$$

$$\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a'_{30})^{(5)}\boxed{+(a''_{30})^{(5)}(T_{29},t)} & \boxed{+(a''_{26})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{34})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{30} \quad (81)$$

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| *Where* $+(a_{28}^{''})^{(5)}(T_{29},t)$ , $+(a_{29}^{''})^{(5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5)}(T_{29},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3<br>*And* $+(a_{24}^{''})^{(4,4,)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,)}(T_{25},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3<br>$+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$ , $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3<br>$+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3<br>$+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3<br>$+(a_{40}^{''})^{(8,8\ ,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3<br>$+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b_{28}^{'})^{(5)} & -(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) & -(b_{32}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{28}$ | *82* |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b_{29}^{'})^{(5)} & -(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) & -(b_{33}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{29}$ | *83* |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b_{30}^{'})^{(5)} & -(b_{30}^{''})^{(5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,)}(G_{27},t) & -(b_{34}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{15}^{''})^{(1,1,1,1,1,)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{30}$ | *84* |
| *where* $-(b_{28}^{''})^{(5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3<br>$-(b_{24}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3<br>$-(b_{32}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3<br>$-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3<br>$-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients | |

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|---|---|
| for category 1,2, and 3<br>$-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3<br>$-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3<br>$-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3<br>$-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3 | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,4,)}(T_{25},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{32}$ | *85* |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a_{33}^{'})^{(6)}+(a_{33}^{''})^{(6)}(T_{33},t) & +(a_{29}^{''})^{(5,5,5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,4,)}(T_{25},t) \\ +(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ +(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{33}$ | *86* |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \begin{bmatrix} (a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33},t) & +(a_{30}^{''})^{(5,5,5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,4,)}(T_{25},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ +(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34}$ | *87* |
| $+(a_{32}^{''})^{(6)}(T_{33},t)$, $+(a_{33}^{''})^{(6)}(T_{33},t)$, $+(a_{34}^{''})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{28}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients<br>$+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients<br>$+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients<br>$+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients<br>$+(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients<br>$+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation | |

coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b'_{32})^{(6)}\boxed{-(b''_{32})^{(6)}(G_{35},t)} & \boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b'_{33})^{(6)}\boxed{-(b''_{33})^{(6)}(G_{35},t)} & \boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)}\boxed{-(b''_{34})^{(6)}(G_{35},t)} & \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{34} \quad (90)$$

$\boxed{-(b''_{32})^{(6)}(G_{35},t)}, \boxed{-(b''_{33})^{(6)}(G_{35},t)}, \boxed{-(b''_{34})^{(6)}(G_{35},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)}, \boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)}, \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)}$ *are second detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)}$ *are third detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a'_{36})^{(7)}\boxed{+(a''_{36})^{(7)}(T_{37},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}')^{(7)}\boxed{+(a_{37}'')^{(7)}(T_{37},t)} & \boxed{+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}')^{(7)}\boxed{+(a_{38}'')^{(7)}(T_{37},t)} & \boxed{+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad (93)$$

Where $\boxed{(a_{36}'')^{(7)}(T_{37},t)}$, $\boxed{(a_{37}'')^{(7)}(T_{37},t)}$, $\boxed{(a_{38}'')^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{40}'')^{(8,8,8,8,8,8,8,8,)}(T_{41},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}')^{(7)}\boxed{-(b_{36}'')^{(7)}(G_{39},t)} & \boxed{-(b_{16}'')^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}'')^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}'')^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}'')^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}'')^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}'')^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}'')^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13} \quad (94)$$

$$\frac{dT_{37}}{dt} =$$

$$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b'_{37})^{(7)}\boxed{-(b''_{37})^{(7)}(G_{39},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} =$$

$$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b'_{38})^{(7)}\boxed{-(b''_{38})^{(7)}(G_{39},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a'_{40})^{(8)}\boxed{+(a''_{40})^{(8)}(T_{41},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a'_{41})^{(8)}\boxed{+(a''_{41})^{(8)}(T_{41},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a'_{42})^{(8)}\boxed{+(a''_{42})^{(8)}(T_{41},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{40})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b'_{40})^{(8)}\boxed{-(b''_{40})^{(8)}(G_{43},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b'_{41})^{(8)}\boxed{-(b''_{41})^{(8)}(G_{43},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b'_{42})^{(8)}\boxed{-(b''_{42})^{(8)}(G_{43},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[\begin{array}{ccc} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[\begin{array}{ccc} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{14}$$

$$\frac{dT_{46}}{dt} =$$

$$(b_{46})^{(9)}T_{45} - \begin{bmatrix} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

$$(a_i^{''})^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$$

$$(b_i^{''})^{(1)}(G,t) \le (r_i)^{(1)} \le (b_i^{'})^{(1)} \le (\hat{B}_{13})^{(1)}$$

$lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$ 98

$\lim_{G\to\infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$

**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :

Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$

| | |
|---|---|
| They satisfy Lipschitz condition:<br>$\|(a_i'')^{(1)}(T_{14}', t) - (a_i'')^{(1)}(T_{14}, t)\| \leq (\hat{k}_{13})^{(1)}\|T_{14} - T_{14}'\|e^{-(\hat{M}_{13})^{(1)}t}$<br>$\|(b_i'')^{(1)}(G', t) - (b_i'')^{(1)}(G, t)\| < (\hat{k}_{13})^{(1)}\|G - G'\|e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(1)}(T_{14}', t)$ and $(a_i'')^{(1)}(T_{14}, t)$ . $(T_{14}', t)$ and $(T_{14}, t)$ are points belonging to the interval $[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}]$. It is to be noted that $(a_i'')^{(1)}(T_{14}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i'')^{(1)}(T_{14}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)},$ are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i')^{(1)}, (b_i)^{(1)}, (b_i')^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i, j = 16,17,18$ | |
| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17}, t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19}, t) \leq (r_i)^{(2)} \leq (b_i')^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty}(a_i'')^{(2)}(T_{17}, t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty}(b_i'')^{(2)}((G_{19}), t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ : | 106 |

| | |
|---|---|
| Where $\boxed{(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}}$ are positive constants and $\boxed{i = 16,17,18}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'}, t) - (a_i^{''})^{(2)}(T_{17}, t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}^{'}\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'}, t) - (b_i^{''})^{(2)}((G_{19}), t)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})^{'}\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'}, t)$ and $(a_i^{''})^{(2)}(T_{17}, t)$ . $(T_{17}^{'}, t)$ and $(T_{17}, t)$ are points belonging to the interval $\left[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}\right]$. It is to be noted that $(a_i^{''})^{(2)}(T_{17}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i^{''})^{(2)}(T_{17}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i^{'})^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i, j = 20,21,22$<br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(3)}$, $(r_i)^{(3)}$:<br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(3)}(T_{21}, t) = (p_i)^{(3)}$ | 113 |

| | |
|---|---|
| $\lim_{G\to\infty}(b_i'')^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(3)}(T_{21}',t) - (a_i'')^{(3)}(T_{21},t)\rvert \le (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}'\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i'')^{(3)}(G_{23}',t) - (b_i'')^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}'\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(3)}(T_{21}',t)$ and$(a_i'')^{(3)}(T_{21},t)$ . $(T_{21}',t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i'')^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i'')^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i')^{(3)}, (b_i)^{(3)}, (b_i')^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i')^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i')^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i')^{(4)}, (a_i'')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (b_i'')^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i'')^{(4)}, (b_i'')^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i'')^{(4)}(T_{25},t) \le (p_i)^{(4)} \le (\hat{A}_{24})^{(4)}$<br><br>$(b_i'')^{(4)}\big((G_{27}),t\big) \le (r_i)^{(4)} \le (b_i')^{(4)} \le (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2\to\infty}(a_i'')^{(4)}(T_{25},t) = (p_i)^{(4)}$ | 118 |

| | |
|---|---|
| $\lim_{G\to\infty}(b_i'')^{(4)}\left((G_{27}),t\right) = \ (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, \ (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24{,}25{,}26}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(4)}(T_{25}',t) - (a_i'')^{(4)}(T_{25},t)\rvert \le (\hat{k}_{24})^{(4)}\lvert T_{25} - T_{25}'\rvert e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\lvert(b_i'')^{(4)}((G_{27})',t) - (b_i'')^{(4)}\left((G_{27}),t\right)\rvert < (\hat{k}_{24})^{(4)}\lvert\lvert(G_{27}) - (G_{27})'\rvert\rvert e^{-(\hat{M}_{24})^{(4)}t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(4)}(T_{25}',t)$ and $(a_i'')^{(4)}(T_{25},t)$ . $(T_{25}',t)$ and $(T_{25},t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i'')^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i'')^{(4)}(T_{25},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (p_i)^{(4)}, \ (r_i)^{(4)}, i = 24{,}25{,}26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i')^{(4)} + \ (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)}(\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i')^{(4)} + \ (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i')^{(5)}, (a_i'')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (b_i'')^{(5)} > 0, \quad i,j = 28{,}29{,}30$<br><br>The functions $(a_i'')^{(5)}, (b_i'')^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}, \ (r_i)^{(5)}$:<br><br>$(a_i'')^{(5)}(T_{29},t) \le (p_i)^{(5)} \le (\hat{A}_{28})^{(5)}$<br><br>$(b_i'')^{(5)}\left((G_{31}),t\right) \le \ (r_i)^{(5)} \le (b_i')^{(5)} \le (\hat{B}_{28})^{(5)}$ | 122 |

$lim_{T_2\to\infty}(a_i'')^{(5)}(T_{29},t) = (p_i)^{(5)}$

$\lim_{G\to\infty}(b_i'')^{(5)}(G_{31},t) = (r_i)^{(5)}$

**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ : 123

Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$

They satisfy Lipschitz condition: 124

$$|(a_i'')^{(5)}(T_{29}',t) - (a_i'')^{(5)}(T_{29},t)| \le (\hat{k}_{28})^{(5)}|T_{29} - T_{29}'|e^{-(\hat{M}_{28})^{(5)}t}$$

$$|(b_i'')^{(5)}((G_{31})',t) - (b_i'')^{(5)}((G_{31}),t)| < (\hat{k}_{28})^{(5)}||(G_{31}) - (G_{31})'||e^{-(\hat{M}_{28})^{(5)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(5)}(T_{29}',t)$ and $(a_i'')^{(5)}(T_{29},t)$ . $(T_{29}',t)$ and $(T_{29},t)$ are points belonging to the interval $\left[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\right]$. It is to be noted that $(a_i'')^{(5)}(T_{29},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i'')^{(5)}(T_{29},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ : 125

$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)},$ are positive constants

$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$$

**Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ : 126

There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i')^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i')^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

Where we suppose

$(a_i)^{(6)}, (a_i')^{(6)}, (a_i'')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (b_i'')^{(6)} > 0, \quad i,j = 32,33,34$ 127

The functions $(a_i'')^{(6)}, (b_i'')^{(6)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(6)}, (r_i)^{(6)}$:

$$(a_i'')^{(6)}(T_{33},t) \le (p_i)^{(6)} \le (\hat{A}_{32})^{(6)}$$

| | |
|---|---|
| $(b_i^{''})^{(6)}((G_{35}),t) \leq \ (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(6)}(T_{33},t) = (p_i)^{(6)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(6)}\left((G_{35}),t\right) = \ (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'},t) - (a_i^{''})^{(6)}(T_{33},t)\rvert \leq (\hat{k}_{32})^{(6)}\lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})^{'},t) - (b_i^{''})^{(6)}\left((G_{35}),t\right)\rvert < (\hat{k}_{32})^{(6)}\lvert\lvert(G_{35}) - (G_{35})^{'}\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'},t)$ and$(a_i^{''})^{(6)}(T_{33},t)$ . $(T_{33}^{'},t)$ and $(T_{33},t)$ are points belonging to the interval $\left[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\right]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + \ (\hat{A}_{32})^{(6)} + \ (\hat{P}_{32})^{(6)}\ (\hat{k}_{32})^{(6)}] < 1$<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[\ (b_i)^{(6)} + (b_i^{'})^{(6)} + \ (\hat{B}_{32})^{(6)} + \ (\hat{Q}_{32})^{(6)}\ \ (\hat{k}_{32})^{(6)}] < 1$ | 130 |
| Where we suppose | |
| (G) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i,j = 36,37,38$<br><br>(H) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$: | 131 |

$(a_i^{''})^{(7)}(T_{37},t) \le (p_i)^{(7)} \le (\hat{A}_{36})^{(7)}$

$(b_i^{''})^{(7)}(G_{39},t) \le (r_i)^{(7)} \le (b_i^{'})^{(7)} \le (\hat{B}_{36})^{(7)}$

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*(I)* $\lim_{T_2\to\infty} (a_i^{''})^{(7)}(T_{37},t) = (p_i)^{(7)}$

*(J)*

$\lim_{G\to\infty}(b_i^{''})^{(7)}\big((G_{39}),t\big) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

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They satisfy Lipschitz condition:

$|(a_i^{''})^{(7)}(T_{37}^{'},t) - (a_i^{''})^{(7)}(T_{37},t)| \le (\hat{k}_{36})^{(7)}|T_{37} - T_{37}^{'}|e^{-(\hat{M}_{36})^{(7)}t}$

$|(b_i^{''})^{(7)}((G_{39})^{'},t) - (b_i^{''})^{(7)}\big((G_{39}),t\big)| < (\hat{k}_{36})^{(7)}||(G_{39}) - (G_{39})^{'}||e^{-(\hat{M}_{36})^{(7)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'},t)$ and$(a_i^{''})^{(7)}(T_{37},t)$ . $(T_{37}^{'},t)$ and $(T_{37},t)$ are points belonging to the interval $\big[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}\big]$ . It is to be noted that $(a_i^{''})^{(7)}(T_{37},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

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**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :

(K) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$, are positive constants

$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$

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**Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :

(L) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities

$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$

$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$

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| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})^{'}, t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})^{'}\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $\left[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}\right]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i^{'})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (p_i)^{(8)}$, $(r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |

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| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i,j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45},t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47},t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2\to\infty}(a_i'')^{(9)}(T_{45},t) = (p_i)^{(9)}$<br>$\lim_{G\to\infty}(b_i'')^{(9)}(G_{47},t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}',t) - (a_i'')^{(9)}(T_{45},t)\rvert \le (\hat{k}_{44})^{(9)}\lvert T_{45} - T_{45}'\rvert e^{-(\hat{M}_{44})^{(9)}t}$<br>$\lvert(b_i'')^{(9)}((G_{47})',t) - (b_i'')^{(9)}((G_{47}),t)\rvert < (\hat{k}_{44})^{(9)}\lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}',t)$ and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i'')^{(9)}(T_{45},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :<br>$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants<br>$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$ | |
| **Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :<br>There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and (\hat{B}_{44})^{(9)}$ and the constants | |

$(a_i)^{(9)}, (a_i')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}, i = 44,45,46,$
satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i')^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i')^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0), T_i(0)$ :

$$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}, \quad \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}, \quad \boxed{T_i(0) = T_i^0 > 0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0), T_i(0)$

$$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}, \quad G_i(0) = G_i^0 > 0$$

$$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}, \quad T_i(0) = T_i^0 > 0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}, \quad G_i(0) = G_i^0 > 0$$

$$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}, \quad T_i(0) = T_i^0 > 0$$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

**Definition of** $G_i(0), T_i(0)$ :

$$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}, \quad \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}, \quad \boxed{T_i(0) = T_i^0 > 0}$$

**Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 151

**Definition of** $G_i(0), T_i(0)$ :

$$G_i(t) \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}, \quad \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}, \quad \boxed{T_i(0) = T_i^0 > 0}$$

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| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le \left(\hat{Q}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le \left(\hat{P}_{13}\right)^{(1)}$ , $T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le \left(\hat{P}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |
| $0 \le T_i(t) - T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 157 |
| By<br><br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}\left(s_{(13)}\right) - \left((a_{13}')^{(1)} + a_{13}'')^{(1)}\left(T_{14}\left(s_{(13)}\right), s_{(13)}\right)\right) G_{13}\left(s_{(13)}\right)\right] ds_{(13)}$ | 158 |

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| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{16})^{(2)}$, $T_i^0 \le (\hat{Q}_{16})^{(2)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} G_{17}(s_{(16)}) - \left((a_{16}')^{(2)} + a_{16}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{16}(s_{(16)})\right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[(a_{17})^{(2)} G_{16}(s_{(16)}) - \left((a_{17}')^{(2)} + (a_{17}'')^{(2)}(T_{17}(s_{(16)}), s_{(17)})\right) G_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{20})^{(3)}$ , $T_i^0 \leq (\hat{Q}_{20})^{(3)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a_{20}^{'})^{(3)} + a_{20}^{''})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{24})^{(4)}$ , $T_i^0 \leq (\hat{Q}_{24})^{(4)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a_{24}^{'})^{(4)} + a_{24}^{''})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |

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| $\overline{T}_{26}(t) = T_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b_{26}')^{(4)} - (b_{26}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{28})^{(5)}$ , $T_i^0 \leq (\hat{Q}_{28})^{(5)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a_{28}')^{(5)} + a_{28}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a_{29}')^{(5)} + (a_{29}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a_{30}')^{(5)} + (a_{30}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\overline{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b_{28}')^{(5)} - (b_{28}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\overline{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b_{29}')^{(5)} - (b_{29}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\overline{T}_{30}(t) = T_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b_{30}')^{(5)} - (b_{30}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a_{32}')^{(6)} + a_{32}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a_{33}')^{(6)} + (a_{33}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |

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| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{36})^{(7)}$, $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a_{36}^{'})^{(7)} + a_{36}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ | |

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| which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{40})^{(8)}$ , $T_i^0 \leq (\hat{Q}_{40})^{(8)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a'_{40})^{(8)} + a''_{40})^{(8)} (T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a'_{41})^{(8)} + (a''_{41})^{(8)} (T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a'_{42})^{(8)} + (a''_{42})^{(8)} (T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b'_{40})^{(8)} - (b''_{40})^{(8)} (G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b'_{41})^{(8)} - (b''_{41})^{(8)} (G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b'_{42})^{(8)} - (b''_{42})^{(8)} (G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |

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| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that<br>$(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} \left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$<br>$\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}} \left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$ | 171 |
| From which it follows that | 172 |

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| $(G_{20}(t)-G_{20}^0)e^{-(\hat{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that $G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\hat{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$ $\left(1+(a_{24})^{(4)}t\right)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that $(G_{24}(t)-G_{24}^0)e^{-(\hat{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$ $\left(1+(a_{28})^{(5)}t\right)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left(e^{(\hat{M}_{28})^{(5)}t}-1\right)$ | |
| From which it follows that $(G_{28}(t)-G_{28}^0)e^{-(\hat{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 5 | 175 |
| The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0+(\hat{P}_{32})^{(6)}e^{(\hat{M}_{32})^{(6)}s_{(32)}}\right)\right] ds_{(32)} =$ $\left(1+(a_{32})^{(6)}t\right)G_{33}^0 + \frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)}t}-1\right)$ | 176 |
| From which it follows that $(G_{32}(t)-G_{32}^0)e^{-(\hat{M}_{32})^{(6)}t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^0)e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^0}{G_{33}^0}\right)}+(\hat{P}_{32})^{(6)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 6 | 177 |

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| Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$ | |
| (b) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} \left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$<br>$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$ | 178 |
| From which it follows that<br>$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left[\left((\hat{P}_{36})^{(7)} + G_{37}^0\right) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 7 | |
| The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} \left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$<br>$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$ | 180 |
| From which it follows that<br>$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left[\left((\hat{P}_{40})^{(8)} + G_{41}^0\right) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} \left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$<br>$\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left(e^{(\hat{M}_{44})^{(9)} t} - 1\right)$ | |
| From which it follows that<br>$(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left[\left((\hat{P}_{44})^{(9)} + G_{45}^0\right) e^{\left(-\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0}\right)} + (\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ and to choose | 182 |

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| $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{13})^{(1)}\left\|G_{14}^{(1)}-G_{14}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$<br>$\int_0^t\{(a_{13}')^{(1)}\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{-(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$G_{13}^{(2)}\left\\|(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}'')^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right\\|\ e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$<br>Where $s_{(13)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | |
| $\left\|G^{(1)}-G^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}t}\leq$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}\left((a_{13})^{(1)}+(a_{13}')^{(1)}+(\hat{A}_{13})^{(1)}+(\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)},T^{(1)};\ G^{(2)},T^{(2)}\right)\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a_{13}'')^{(1)}$ and $(b_{13}'')^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by | |

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| $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(1)} - (a_i^{''})^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a_{14}^{'})^{(1)}G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}^{'})^{(1)}$<br><br>If $G_{14}\ or\ G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}\ and\ G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t), t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t), t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right),\quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t), t) = (b_{15}^{'})^{(1)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |

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| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)}, (T_{19})^{(1)}), ((G_{19})^{(2)}, (T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)}\left\|G_{17}^{(1)} - G_{17}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\,ds_{(16)} +$<br>$\int_0^t\{(a_{16}')^{(2)}\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{-(\hat{M}_{16})^{(2)}s_{(16)}} +$<br>$(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}} +$<br>$G_{16}^{(2)}\left\|(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}'')^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}t} \leq$<br>$\frac{1}{(\hat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}\right)d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}'')^{(2)}$ and $(b_{16}'')^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | 199 |

condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ and $(\widehat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition.

**Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$ 200

it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)}t)} > 0 \quad \text{for } t > 0$$

**Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1, \left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ : 201

**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if

$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)}G_{17}$ and by integrating

$$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1/(a_{17}^{'})^{(2)}$$

In the same way , one can obtain

$$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}^{'})^{(2)}$$

If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively.

**Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. 202

**Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(2)}((G_{19})(t), t)) = (b_{17}^{'})^{(2)}$ then $T_{17} \to \infty$. 203

**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :

Indeed let $t_2$ be so that for $t > t_2$

$$(b_{17})^{(2)} - (b_i^{''})^{(2)}((G_{19})(t), t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$$

Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to 204

$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results

$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right), \quad t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded. 205

The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}^{''})^{(2)}\left((G_{19})(t), t\right) = (b_{18}^{'})^{(2)}$

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| We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)}, (T_{23})^{(1)}), ((G_{23})^{(2)}, (T_{23})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{20})^{(3)}t}\}$ | 211 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :(\widetilde{(G_{23})}, \widetilde{(T_{23})}) = \mathcal{A}^{(3)}((G_{23}), (T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)}\left\|G_{21}^{(1)} - G_{21}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)} +$<br>$\int_0^t\{(a'_{20})^{(3)}\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{-(\hat{M}_{20})^{(3)}s_{(20)}} +$<br>$(a''_{20})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right)\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}} +$<br>$G_{20}^{(2)}\left\|(a''_{20})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a''_{20})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}t} \leq$<br>$\frac{1}{(\hat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a'_{20})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)}, (T_{23})^{(1)}; (G_{23})^{(2)}, (T_{23})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |

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| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ $and$ $(\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(3)}t)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ $and$ $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ $and$ $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)}G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br><br>If $G_{21}$ $or$ $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ $is$ bounded, from below, the same property holds for $G_{21}$ $and$ $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t), t\right)) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t), t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded. | 220 |

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| The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}'')^{(3)}\left((G_{23})(t),t\right)=(b_{22}')^{(3)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}<1$ and to choose<br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)}+\left((\widehat{P}_{24})^{(4)}+G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right]\le(\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)}+T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)}+(\widehat{Q}_{24})^{(4)}\right]\le(\widehat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{24})^{(4)}\left\|G_{25}^{(1)}-G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}ds_{(24)}+$<br>$\int_0^t\{(a_{24}')^{(4)}\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$(a_{24}'')^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$G_{24}^{(2)}\left\\|(a_{24}'')^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}'')^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\\|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)}-(G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t}\le$<br>$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}')^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$ | 226 |

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| And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i = 24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br>it results<br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]} \geq 0$<br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(4)}t\right)} > 0$ for t > 0 | 228 |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2\ and\ \left((\widehat{M}_{24})^{(4)}\right)_3$ :<br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if<br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}^{'})^{(4)}G_{25}$ and by integrating<br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a_{25}^{'})^{(4)}$<br>In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a_{26}^{'})^{(4)}$<br>If $G_{25}\ or\ G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}\ and\ G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t),t)) = (b_{25}^{'})^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i^{''})^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}\,(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results | 232 |

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right), \quad t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42

Analogous inequalities hold also for $G_{29}, G_{30}, T_{28}, T_{29}, T_{30}$

It is now sufficient to take $\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ and to choose

$(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ large to have

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$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[(\hat{P}_{28})^{(5)} + \left((\hat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{28})^{(5)} \quad (234)$$

$$\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{28})^{(5)}\right] \leq (\hat{Q}_{28})^{(5)} \quad (235)$$

In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric (236)

$$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right) = \mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$

It results

$$\left|\tilde{G}_{28}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{28})^{(5)}\left|G_{29}^{(1)} - G_{29}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)} +$$

$$\int_0^t\{(a_{28}^{'})^{(5)}\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{-(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$G_{28}^{(2)}\left|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right) - (a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right|\ e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$$

Where $s_{(28)}$ represents integrand that is integrated over the interval [0, t]

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| From the hypotheses on it follows | |
| $\left\|(G_{31})^{(1)} - (G_{31})^{(2)}\right\| e^{-(\widehat{M}_{28})^{(5)}t} \leq$ $\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)} + (a'_{28})^{(5)} + (\widehat{A}_{28})^{(5)} + (\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right) d\left(((G_{31})^{(1)}, (T_{31})^{(1)};\ (G_{31})^{(2)}, (T_{31})^{(2)})\right)$ And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a''_{28})^{(5)}$ and $(b''_{28})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ and $(\widehat{Q}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(5)}$ and $(b''_i)^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t) = 0$ and $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(5)} - (a''_i)^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(5)}t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ and $\left((\widehat{M}_{28})^{(5)}\right)_3$ : **Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ and $G_{30}$ . indeed if $G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a'_{29})^{(5)} G_{29}$ and by integrating $G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a'_{29})^{(5)}$ In the same way , one can obtain $G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a'_{30})^{(5)}$ If $G_{29}$ or $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ is bounded, from below, the same property holds for $G_{29}$ and $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |

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| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t),t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$. | 242 |
| **Definition of** $(m)^{(5)}$ and $\varepsilon_5$ : | |
| Indeed let $t_5$ be so that for $t > t_5$ | |
| $(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to | 243 |
| $T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results | |
| $T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded. The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$ | |
| We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| Analogous inequalities hold also for $G_{33}, G_{34}, T_{32}, T_{33}, T_{34}$ | |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ and to choose | 244 |
| $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ large to have | |
| $\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}\left[(\hat{P}_{32})^{(6)} + \left((\hat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{32})^{(6)}\right] \leq (\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric | 247 |
| $d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right) =$ | |
| $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{32})^{(6)}t}\}$ | |
| Indeed if we denote | |
| **Definition of** $\widetilde{(G_{35})}, \widetilde{(T_{35})}$ : $\left(\widetilde{(G_{35})}, \widetilde{(T_{35})}\right) = \mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$ | |
| It results | |

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| $\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{32})^{(6)} \left\|G_{33}^{(1)}-G_{33}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\, ds_{(32)} +$<br>$\int_0^t \{(a_{32}')^{(6)} \left\|G_{32}^{(1)}-G_{32}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$(a_{32}'')^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$G_{32}^{(2)} \left\|(a_{32}'')^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right) - (a_{32}'')^{(6)}\left(T_{33}^{(2)}, s_{(32)}\right)\right\| \; e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\} ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}t} \leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)} + (a_{32}')^{(6)} + (\widehat{A}_{32})^{(6)} + (\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right) d\left(((G_{35})^{(1)}, (T_{35})^{(1)};\ (G_{35})^{(2)}, (T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a_{32}'')^{(6)}$ and $(b_{32}'')^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)} e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)} e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(6)}$ and $(b_i'')^{(6)}, i = 32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 249 |
| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(6)} - (a_i'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(6)}t\right)} > 0$ for $t > 0$ | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ :<br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if<br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}')^{(6)} G_{33}$ and by integrating<br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}')^{(6)}$<br>In the same way , one can obtain<br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}')^{(6)}$ | 251 |

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| If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i^{''})^{(6)}\big((G_{35})(t),t\big) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ large to have | 255 |
| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[(\hat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote | 258 |

**Definition of** $\widetilde{(G_{39})}, \widetilde{(T_{39})} : \left( \widetilde{(G_{39})}, \widetilde{(T_{39})} \right) = \mathcal{A}^{(7)}((G_{39}), (T_{39}))$
It results

$\left|\tilde{G}_{36}^{(1)} - \tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{36})^{(7)} \left|G_{37}^{(1)} - G_{37}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}}\, ds_{(36)} +$
$\int_0^t \{(a_{36}^{'})^{(7)} \left|G_{36}^{(1)} - G_{36}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} +$
$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) \left|G_{36}^{(1)} - G_{36}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}} +$
$G_{36}^{(2)} \left|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) - (a_{36}^{''})^{(7)}\left(T_{37}^{(2)}, s_{(36)}\right)\right| \; e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}} \} ds_{(36)}$

Where $s_{(36)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on it follows

$\left|(G_{39})^{(1)} - (G_{39})^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} t} \leq$
$\frac{1}{(\widehat{M}_{36})^{(7)}} \left((a_{36})^{(7)} + (a_{36}^{'})^{(7)} + (\widehat{A}_{36})^{(7)} + (\widehat{P}_{36})^{(7)} (\widehat{k}_{36})^{(7)}\right) d\left(((G_{39})^{(1)}, (T_{39})^{(1)};\ (G_{39})^{(2)}, (T_{39})^{(2)})\right)$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

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**Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)} e^{(\widehat{M}_{36})^{(7)} t}$ *and* $(\widehat{Q}_{36})^{(7)} e^{(\widehat{M}_{36})^{(7)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36{,}37{,}38$ depend only on $T_{37}$ and respectively on $(G_{39})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition.

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**Remark 32:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$

it results

$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)} (T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$

$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0

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**Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2$ *and* $\left((\widehat{M}_{36})^{(7)}\right)_3$ :

**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}$ *and* $G_{38}$ . indeed if

$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating

$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)} \left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$

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| In the same way , one can obtain<br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}')^{(7)}$<br>If $G_{37}$ *or* $G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}$ *and* $G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(7)}((G_{39})(t),t)) = (b_{37}')^{(7)}$ then $T_{37} \to \infty$.<br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br>Indeed let $t_7$ be so that for $t > t_7$<br>$(b_{37})^{(7)} - (b_i'')^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right)$, $t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}'')^{(7)}\left((G_{39})(t),t\right) = (b_{38}')^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)} + \left((\widehat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\widehat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{40})^{(8)}\right] \leq (\widehat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |

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| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$ $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\,ds_{(40)}+$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\le$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ *and* $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i=40,41,42$ depend only on $\mathrm{T}_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$<br><br>it results<br><br>$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(8)}-(a_i^{''})^{(8)}(T_{41}(s_{(40)}),s_{(40)})\}ds_{(40)}\right]}\ge 0$<br><br>$T_i\,(t)\ge T_i^0e^{\left(-(b_i^{'})^{(8)}t\right)}>0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1,\left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ : | 276 |

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| **Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a'_{41})^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1/(a'_{41})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2/(a'_{42})^{(8)}$<br><br>If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | |
| **Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t),t)) = (b'_{41})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t),t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$ | 278 |
| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right),\ t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.<br>The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t),t(t),t\right) = (b'_{42})^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$ | |

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\left(((G_{47})^{(1)},(T_{47})^{(1)}),((G_{47})^{(2)},(T_{47})^{(2)})\right)=$$
$$sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})},\widetilde{(T_{47})}:\left(\widetilde{(G_{47})},\widetilde{(T_{47})}\right)=\mathcal{A}^{(9)}\left((G_{47}),(T_{47})\right)$
It results

$$\left|\tilde{G}_{44}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{44})^{(9)}\left|G_{45}^{(1)}-G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)}+$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)-(a_{44}^{''})^{(9)}\left(T_{45}^{(2)},s_{(44)}\right)\right|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t}\le$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)}+(a_{44}^{'})^{(9)}+(\widehat{A}_{44})^{(9)}+(\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)},(T_{47})^{(1)};\ (G_{47})^{(2)},(T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}\ and\ (\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i=44{,}45{,}46$ depend only on $\mathrm{T}_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

From 99 to 44 it results

$$G_i\ (t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(9)}-(a_i^{''})^{(9)}(T_{45}(s_{(44)}),s_{(44)})\}ds_{(44)}\right]}\ge 0$$

$$T_i\ (t)\ge T_i^0e^{(-(b_i^{'})^{(9)}t)}>0\quad \text{for t}>0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1,\left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3:$

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)} G_{45}$ and by integrating

$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$

In the same way , one can obtain

$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t),t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t),t\right) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right),\quad t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\left((G_{47})(t),t\right) = (b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} - (a_{13}^{''})^{(1)}(T_{14},t) + (a_{14}^{''})^{(1)}(T_{14},t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(v_1)^{(1)}, (v_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, v^{(1)}, u^{(1)}$ : 281

By $(v_1)^{(1)} > 0$ , $(v_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$ , $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

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| **Definition of** $(\bar{v}_1)^{(1)},,(\bar{v}_2)^{(1)},(\bar{u}_1)^{(1)},(\bar{u}_2)^{(1)}$ :<br><br>By $(\bar{v}_1)^{(1)}>0$ , $(\bar{v}_2)^{(1)}<0$ and respectively $(\bar{u}_1)^{(1)}>0$ , $(\bar{u}_2)^{(1)}<0$ the roots of the equations $(a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_2)^{(1)}v^{(1)}-(a_{13})^{(1)}=0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2+(\tau_2)^{(1)}u^{(1)}-(b_{13})^{(1)}=0$ | 282 |
| **Definition of** $(m_1)^{(1)}$ , $(m_2)^{(1)}$ , $(\mu_1)^{(1)},(\mu_2)^{(1)},(\nu_0)^{(1)}$ :-<br><br>If we define $(m_1)^{(1)}$ , $(m_2)^{(1)}$ , $(\mu_1)^{(1)},(\mu_2)^{(1)}$ by<br>$(m_2)^{(1)}=(\nu_0)^{(1)},(m_1)^{(1)}=(\nu_1)^{(1)},\ if\ (\nu_0)^{(1)}<(\nu_1)^{(1)}$<br><br>$(m_2)^{(1)}=(\nu_1)^{(1)},(m_1)^{(1)}=(\bar{\nu}_1)^{(1)}\ ,if\ (\nu_1)^{(1)}<(\nu_0)^{(1)}<(\bar{\nu}_1)^{(1)},$<br><br>and $\boxed{(\nu_0)^{(1)}=\frac{G_{13}^0}{G_{14}^0}}$<br><br>$(m_2)^{(1)}=(\nu_1)^{(1)},(m_1)^{(1)}=(\nu_0)^{(1)},\ if\ (\bar{\nu}_1)^{(1)}<(\nu_0)^{(1)}$ | 283 |
| and analogously<br><br>$(\mu_2)^{(1)}=(u_0)^{(1)},(\mu_1)^{(1)}=(u_1)^{(1)},\ if\ (u_0)^{(1)}<(u_1)^{(1)}$<br><br>$(\mu_2)^{(1)}=(u_1)^{(1)},(\mu_1)^{(1)}=(\bar{u}_1)^{(1)}\ ,if\ (u_1)^{(1)}<(u_0)^{(1)}<(\bar{u}_1)^{(1)},$<br><br>and $\boxed{(u_0)^{(1)}=\frac{T_{13}^0}{T_{14}^0}}$<br><br>$(\mu_2)^{(1)}=(u_1)^{(1)},(\mu_1)^{(1)}=(u_0)^{(1)},if\ (\bar{u}_1)^{(1)}<(u_0)^{(1)}$ where $(u_1)^{(1)},(\bar{u}_1)^{(1)}$<br><br>are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{13}^0e^{((S_1)^{(1)}-(p_{13})^{(1)})t}\le G_{13}(t)\le G_{13}^0e^{(S_1)^{(1)}t}$<br><br>where $(p_i)^{(1)}$ is defined by equation<br><br>$\frac{1}{(m_1)^{(1)}}G_{13}^0e^{((S_1)^{(1)}-(p_{13})^{(1)})t}\le G_{14}(t)\le\frac{1}{(m_2)^{(1)}}G_{13}^0e^{(S_1)^{(1)}t}$ | 285 |
| $(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t}-e^{-(S_2)^{(1)}t}\right]+G_{15}^0e^{-(S_2)^{(1)}t}\le G_{15}(t)\le$<br><br>$\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a_{15}^{'})^{(1)})}[e^{(S_1)^{(1)}t}-e^{-(a_{15}^{'})^{(1)}t}]+\ G_{15}^0e^{-(a_{15}^{'})^{(1)}t})$ | 286 |
| $\boxed{T_{13}^0e^{(R_1)^{(1)}t}\le T_{13}(t)\le T_{13}^0e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}}T_{13}^0e^{(R_1)^{(1)}t}\le T_{13}(t)\le\frac{1}{(\mu_2)^{(1)}}T_{13}^0e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |

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| $\frac{(b_{15})^{(1)}T_{13}^{0}}{(\mu_1)^{(1)}((R_1)^{(1)}-(b_{15}^{'})^{(1)})}\left[e^{(R_1)^{(1)}t}-e^{-(b_{15}^{'})^{(1)}t}\right]+T_{15}^{0}e^{-(b_{15}^{'})^{(1)}t}\le T_{15}(t)\le$ $\frac{(a_{15})^{(1)}T_{13}^{0}}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t}-e^{-(R_2)^{(1)}t}\right]+T_{15}^{0}e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)},(S_2)^{(1)},(R_1)^{(1)},(R_2)^{(1)}$:- Where $(S_1)^{(1)}=(a_{13})^{(1)}(m_2)^{(1)}-(a_{13}^{'})^{(1)}$ $(S_2)^{(1)}=(a_{15})^{(1)}-(p_{15})^{(1)}$ $(R_1)^{(1)}=(b_{13})^{(1)}(\mu_2)^{(1)}-(b_{13}^{'})^{(1)}$ $(R_2)^{(1)}=(b_{15}^{'})^{(1)}-(r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ : $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)}\le-(a_{16}^{'})^{(2)}+(a_{17}^{'})^{(2)}-(a_{16}^{''})^{(2)}(\mathrm{T}_{17},t)+(a_{17}^{''})^{(2)}(\mathrm{T}_{17},t)\le-(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)}\le-(b_{16}^{'})^{(2)}+(b_{17}^{'})^{(2)}-(b_{16}^{''})^{(2)}\big((G_{19}),t\big)-(b_{17}^{''})^{(2)}\big((G_{19}),t\big)\le-(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(v_1)^{(2)},(v_2)^{(2)},(u_1)^{(2)},(u_2)^{(2)}$ : | 295 |
| By $(v_1)^{(2)}>0$ , $(v_2)^{(2)}<0$ and respectively $(u_1)^{(2)}>0$ , $(u_2)^{(2)}<0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(v^{(2)}\big)^2+(\sigma_1)^{(2)}v^{(2)}-(a_{16})^{(2)}=0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2+(\tau_1)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ and | 298 |
| **Definition of** $(\bar{v}_1)^{(2)},,(\bar{v}_2)^{(2)},(\bar{u}_1)^{(2)},(\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{v}_1)^{(2)}>0$ , $(\bar{v}_2)^{(2)}<0$ and respectively $(\bar{u}_1)^{(2)}>0$ , $(\bar{u}_2)^{(2)}<0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(v^{(2)}\big)^2+(\sigma_2)^{(2)}v^{(2)}-(a_{16})^{(2)}=0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2+(\tau_2)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ | 302 |
| **Definition of** $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)}=(v_0)^{(2)},(m_1)^{(2)}=(v_1)^{(2)}$, ***if*** $(v_0)^{(2)}<(v_1)^{(2)}$ | 305 |

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| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)} t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)} t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)} t}\right] + G_{18}^0 e^{-(S_2)^{(2)} t} \le G_{18}(t) \le \right.$<br>$\left. \frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a'_{18})^{(2)})}\left[e^{(S_1)^{(2)} t} - e^{-(a'_{18})^{(2)} t}\right] + G_{18}^0 e^{-(a'_{18})^{(2)} t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b'_{18})^{(2)})}\left[e^{(R_1)^{(2)} t} - e^{-(b'_{18})^{(2)} t}\right] + T_{18}^0 e^{-(b'_{18})^{(2)} t} \le T_{18}(t) \le$<br>$\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)} + (r_{16})^{(2)} + (R_2)^{(2)})}\left[e^{((R_1)^{(2)} + (r_{16})^{(2)})t} - e^{-(R_2)^{(2)} t}\right] + T_{18}^0 e^{-(R_2)^{(2)} t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a'_{16})^{(2)}$<br>$(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b'_{16})^{(2)}$ | 318 |

$(\mathrm{R}_2)^{(2)} = (b_{18}^{'})^{(2)} - (r_{18})^{(2)}$

**Behavior of the solutions** 319

**Theorem 3:** If we denote and define

**Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ :

$(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying

$$-(\sigma_2)^{(3)} \le -(a_{20}^{'})^{(3)} + (a_{21}^{'})^{(3)} - (a_{20}^{''})^{(3)}(T_{21}, t) + (a_{21}^{''})^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$$

$$-(\tau_2)^{(3)} \le -(b_{20}^{'})^{(3)} + (b_{21}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t) - (b_{21}^{''})^{(3)}\big((G_{23}), t\big) \le -(\tau_1)^{(3)}$$

**Definition of** $(\nu_1)^{(3)}, (\nu_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ : 320

By $(\nu_1)^{(3)} > 0$ , $(\nu_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$ , $(u_2)^{(3)} < 0$ the roots of the equations

$(a_{21})^{(3)}\big(\nu^{(3)}\big)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\big(u^{(3)}\big)^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and

By $(\bar{\nu}_1)^{(3)} > 0$ , $(\bar{\nu}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$ , $(\bar{u}_2)^{(3)} < 0$ the

roots of the equations $(a_{21})^{(3)}\big(\nu^{(3)}\big)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\big(u^{(3)}\big)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$

**Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :- 321

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ , ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,

and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously 322

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$ , ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

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| $G_{20}^0 e^{((S_1)^{(3)}-(p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)}t}$<br>$(p_i)^{(3)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)}-(p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)}G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)}-(p_{20})^{(3)}-(S_2)^{(3)})}\left[e^{((S_1)^{(3)}-(p_{20})^{(3)})t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \le G_{22}(t) \le \right.$<br>$\left.\frac{(a_{22})^{(3)}G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)}-(a'_{22})^{(3)})}[e^{(S_1)^{(3)}t} - e^{-(a'_{22})^{(3)}t}] + G_{22}^0 e^{-(a'_{22})^{(3)}t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)}+(r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)}+(r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)}T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)}-(b'_{22})^{(3)})}\left[e^{(R_1)^{(3)}t} - e^{-(b'_{22})^{(3)}t}\right] + T_{22}^0 e^{-(b'_{22})^{(3)}t} \le T_{22}(t) \le$<br>$\frac{(a_{22})^{(3)}T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)}+(r_{20})^{(3)}+(R_2)^{(3)})}\left[e^{((R_1)^{(3)}+(r_{20})^{(3)})t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$ | 327 |
| **Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:-<br>Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$<br>$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$<br>$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$<br>$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$ | 328 |
| **Behavior of the solutions of equation**<br>**Theorem:** If we denote and define<br>**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :<br>$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying<br>$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$<br>$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ :<br>By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}(v^{(4)})^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$<br>and $(b_{25})^{(4)}(u^{(4)})^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and | 329 |

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| **Definition of** $(\bar{v}_1)^{(4)},,(\bar{v}_2)^{(4)},(\bar{u}_1)^{(4)},(\bar{u}_2)^{(4)}$ :<br><br>By $(\bar{v}_1)^{(4)}>0$ , $(\bar{v}_2)^{(4)}<0$ and respectively $(\bar{u}_1)^{(4)}>0$ , $(\bar{u}_2)^{(4)}<0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_2)^{(4)}v^{(4)}-(a_{24})^{(4)}=0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2+(\tau_2)^{(4)}u^{(4)}-(b_{24})^{(4)}=0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)},(v_0)^{(4)}$ :-<br><br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)}$ by<br><br>$(m_2)^{(4)}=(v_0)^{(4)},(m_1)^{(4)}=(v_1)^{(4)}$, ***if*** $(v_0)^{(4)}<(v_1)^{(4)}$<br><br>$(m_2)^{(4)}=(v_1)^{(4)},(m_1)^{(4)}=(\bar{v}_1)^{(4)}$ , ***if*** $(v_4)^{(4)}<(v_0)^{(4)}<(\bar{v}_1)^{(4)}$,<br>and $\boxed{(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}$<br><br>$(m_2)^{(4)}=(v_4)^{(4)},(m_1)^{(4)}=(v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)}<(v_0)^{(4)}$ | 330 |
| and analogously<br><br>$(\mu_2)^{(4)}=(u_0)^{(4)},(\mu_1)^{(4)}=(u_1)^{(4)}$, ***if*** $(u_0)^{(4)}<(u_1)^{(4)}$<br><br>$(\mu_2)^{(4)}=(u_1)^{(4)},(\mu_1)^{(4)}=(\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)}<(u_0)^{(4)}<(\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)}=\frac{T_{24}^0}{T_{25}^0}}$<br><br>$(\mu_2)^{(4)}=(u_1)^{(4)},(\mu_1)^{(4)}=(u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)}<(u_0)^{(4)}$ where $(u_1)^{(4)},(\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{24}^0e^{((S_1)^{(4)}-(p_{24})^{(4)})t}\le G_{24}(t)\le G_{24}^0e^{(S_1)^{(4)}t}$<br><br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}}G_{24}^0e^{((S_1)^{(4)}-(p_{24})^{(4)})t}\le G_{25}(t)\le\frac{1}{(m_2)^{(4)}}G_{24}^0e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)}G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t}-e^{-(S_2)^{(4)}t}\right]+G_{26}^0e^{-(S_2)^{(4)}t}\le G_{26}(t)\le\right.$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0e^{(R_1)^{(4)}t}\le T_{24}(t)\le T_{24}^0e^{((R_1)^{(4)}+(r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}}T_{24}^0e^{(R_1)^{(4)}t}\le T_{24}(t)\le\frac{1}{(\mu_2)^{(4)}}T_{24}^0e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)}T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}^{'})^{(4)})}\left[e^{(R_1)^{(4)}t}-e^{-(b_{26}^{'})^{(4)}t}\right]+T_{26}^0e^{-(b_{26}^{'})^{(4)}t}\le T_{26}(t)\le$<br><br>$\frac{(a_{26})^{(4)}T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t}-e^{-(R_2)^{(4)}t}\right]+T_{26}^0e^{-(R_2)^{(4)}t}$ | 336 |

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| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a'_{24})^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b'_{24})^{(4)}$<br>$(R_2)^{(4)} = (b'_{26})^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a'_{28})^{(5)} + (a'_{29})^{(5)} - (a''_{28})^{(5)}(T_{29}, t) + (a''_{29})^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b'_{28})^{(5)} + (b'_{29})^{(5)} - (b''_{28})^{(5)}((G_{31}), t) - (b''_{29})^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ :<br>By $(v_1)^{(5)} > 0$, $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | 339 |
| **Definition of** $(\bar{v}_1)^{(5)},, (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ :<br>By $(\bar{v}_1)^{(5)} > 0$, $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$, $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :-<br>If we define $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}$ by<br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$, ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}}}$<br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$ | 341 |

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| $(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , $\boldsymbol{if}$ $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$, and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$ $(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}, \boldsymbol{if}$ $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | |
| Then the solution of global equations satisfies the inequalities $G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)} t}$ where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \le G_{30}(t) \le$ *(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b_{30}^{'})^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b_{30}^{'})^{(5)} t}\right] + T_{30}^0 e^{-(b_{30}^{'})^{(5)} t} \le T_{30}(t) \le$ $\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:- Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a_{28}^{'})^{(5)}$ $(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$ $(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b_{28}^{'})^{(5)}$ $(R_2)^{(5)} = (b_{30}^{'})^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ : $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying $-(\sigma_2)^{(6)} \le -(a_{32}^{'})^{(6)} + (a_{33}^{'})^{(6)} - (a_{32}^{''})^{(6)}(T_{33}, t) + (a_{33}^{''})^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$ | 349 |

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| $-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}\big((G_{35}),t\big) - (b''_{33})^{(6)}\big((G_{35}),t\big) \le -(\tau_1)^{(6)}$ | |
| **Definition of** $(\nu_1)^{(6)}, (\nu_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, \nu^{(6)}, u^{(6)}$ :<br><br>By $(\nu_1)^{(6)} > 0$ , $(\nu_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{\nu}_1)^{(6)},, (\bar{\nu}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br><br>By $(\bar{\nu}_1)^{(6)} > 0$ , $(\bar{\nu}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (\nu_0)^{(6)}$ :-<br><br>If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br><br>$(m_2)^{(6)} = (\nu_0)^{(6)}, (m_1)^{(6)} = (\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)} < (\nu_1)^{(6)}$<br><br>$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)} < (\nu_0)^{(6)} < (\bar{\nu}_1)^{(6)}$,<br>and $\boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$<br><br>$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)} < (\nu_0)^{(6)}$ | 351 |
| and analogously<br><br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br><br>$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$<br><br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G^0_{32}e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)}G^0_{32}}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G^0_{34}e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |

| | |
|---|---|
| $\boxed{T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)}T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)}-(b_{34}')^{(6)})}\left[e^{(R_1)^{(6)}t} - e^{-(b_{34}')^{(6)}t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)}T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t} - e^{-(R_2)^{(6)}t}\right] + T_{34}^0 e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:- Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$ $(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$ $(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$ $(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ :<br><br>$(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying<br><br>$-(\sigma_2)^{(7)} \le -(a_{36}')^{(7)} + (a_{37}')^{(7)} - (a_{36}'')^{(7)}(T_{37}, t) + (a_{37}'')^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$<br><br>$-(\tau_2)^{(7)} \le -(b_{36}')^{(7)} + (b_{37}')^{(7)} - (b_{36}'')^{(7)}((G_{39}), t) - (b_{37}'')^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)}, (v_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, v^{(7)}, u^{(7)}$ :<br><br>By $(v_1)^{(7)} > 0, (v_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(v^{(7)})^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)}, , (\bar{v}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ :<br><br>By $(\bar{v}_1)^{(7)} > 0, (\bar{v}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(v^{(7)})^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$<br>**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (v_0)^{(7)}$ :- | 362 |

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}, \ \boldsymbol{if}\ (\nu_0)^{(7)} < (\nu_1)^{(7)}$$

$$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}, \boldsymbol{if}\ (\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)},$$

and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$

$$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}, \ \boldsymbol{if}\ (\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$$

and analogously

$$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}, \ \boldsymbol{if}\ (u_0)^{(7)} < (u_1)^{(7)}$$

$$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}, \boldsymbol{if}\ (u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)},$$

and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$

$$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}, \boldsymbol{if}\ (\bar{u}_1)^{(7)} < (u_0)^{(7)}$$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

Then the solution of global equations satisfies the inequalities

$$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)} t}$$

where $(p_i)^{(7)}$ is defined by equation

$$\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)} t} \quad (365)$$

$$\left(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})}\left[e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)} t}\right] + G_{38}^0 e^{-(S_2)^{(7)} t} \le G_{38}(t) \le \right.$$

$$\left. \frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a_{38}')^{(7)})}\left[e^{(S_1)^{(7)} t} - e^{-(a_{38}')^{(7)} t}\right] + G_{38}^0 e^{-(a_{38}')^{(7)} t}\right) \quad (366)$$

$$\boxed{T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}} \quad (367)$$

$$\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t} \quad (368)$$

$$\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b_{38}')^{(7)})}\left[e^{(R_1)^{(7)} t} - e^{-(b_{38}')^{(7)} t}\right] + T_{38}^0 e^{-(b_{38}')^{(7)} t} \le T_{38}(t) \le \quad (369)$$

$$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)} + (r_{36})^{(7)} + (R_2)^{(7)})}\left[e^{((R_1)^{(7)} + (r_{36})^{(7)})t} - e^{-(R_2)^{(7)} t}\right] + T_{38}^0 e^{-(R_2)^{(7)} t}$$

**Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:- 370

Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a'_{36})^{(7)}$

$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$

$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b'_{36})^{(7)}$

$(R_2)^{(7)} = (b'_{38})^{(7)} - (r_{38})^{(7)}$

**Behavior of the solutions of equation** 371

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :

$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying

$-(\sigma_2)^{(8)} \le -(a'_{40})^{(8)} + (a'_{41})^{(8)} - (a''_{40})^{(8)}(T_{41}, t) + (a''_{41})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$

$-(\tau_2)^{(8)} \le -(b'_{40})^{(8)} + (b'_{41})^{(8)} - (b''_{40})^{(8)}\big((G_{43}), t\big) - (b''_{41})^{(8)}\big((G_{43}), t\big) \le -(\tau_1)^{(8)}$

**Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : 372

By $(v_1)^{(8)} > 0, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$
and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and

**Definition of** $(\bar{v}_1)^{(8)}, , (\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :

By $(\bar{v}_1)^{(8)} > 0, (\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0, (\bar{u}_2)^{(8)} < 0$ the

roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$

and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$

**Definition of** $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :-

If we define $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by

$(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)}$, ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}$, ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)}$,

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|---|---|
| and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br><br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)},$ ***if*** $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br><br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},$ ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)},$ ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{40}(t) \leq G_{40}^0 e^{(S_1)^{(8)} t}$<br><br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{41}(t) \leq \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \leq G_{42}(t) \leq \right.$<br><br>$\left.\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)} t} - e^{-(a'_{42})^{(8)} t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)} t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \leq T_{40}(t) \leq T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \leq T_{40}(t) \leq \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b'_{42})^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b'_{42})^{(8)} t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)} t} \leq T_{42}(t) \leq$<br><br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br><br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br><br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$ | 381 |

$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$

$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$

**Behavior of the solutions of equation 37 to 92** 382

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$

$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}\big((G_{47}), t\big) - (b''_{45})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$

**Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :

By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(9)}, , (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{\nu}_1)^{(9)} > 0$ , $(\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}$, ***if*** $(\nu_0)^{(9)} < (\nu_1)^{(9)}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}$, ***if*** $(\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T^0_{44}}{T^0_{45}}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^{0}e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^{0}e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^{0}e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}}G_{44}^{0}e^{(S_1)^{(9)}t}$$

(

$$\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^{0}e^{-(S_2)^{(9)}t} \le G_{46}(t) \le$$

$$\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}^{'})^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}^{'})^{(9)}t}]+\ G_{46}^{0}e^{-(a_{46}^{'})^{(9)}t})$$

$$\boxed{T_{44}^{0}e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^{0}e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}}T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^{0}}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}^{'})^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}^{'})^{(9)}t}\right]+T_{46}^{0}e^{-(b_{46}^{'})^{(9)}t} \le T_{46}(t) \le$$

$$\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a_{44}^{'})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b_{44}^{'})^{(9)}$

$(R_2)^{(9)} = (b_{46}^{'})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a_{13}^{'})^{(1)} - (a_{14}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14},t)\right) - (a_{14}^{''})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}(v^{(1)})^{2} + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}(v^{(1)})^{2} + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

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For $0 < \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (\nu_1)^{(1)} < (\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t) \geq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(\nu_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\nu_1)^{(1)}$

In the same manner , we get 384

$$\nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{\nu}_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\bar{\nu}_1)^{(1)}$

If $0 < (\nu_1)^{(1)} < (\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{\nu}_1)^{(1)}$ we find like in the previous case, 385

$$(\nu_1)^{(1)} \leq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}} \leq \nu^{(1)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (C)^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$ if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \le \frac{d\nu^{(2)}}{dt} \le -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \ge \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \le \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \le \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \le \nu^{(2)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$$

If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain 394

$$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(2)}(t)$ :- 395

$$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}, \quad \boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$$

In a completely analogous way, we obtain 396

**Definition of** $u^{(2)}(t)$ :-

$$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}, \quad \boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :** 397

If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$

Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then

$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$

**Proof** : From global equations we obtain 398

$$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right) - (a_{21}^{''})^{(3)}(T_{21},t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$$

**Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$ 399

It follows

$$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$$

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From which one obtains

For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

In the same manner , we get

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$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

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If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case,

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

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If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$ if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

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In the same manner , we get

$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

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$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}'')^{(5)} = (b_{29}'')^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then

$(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a'_{32})^{(6)} - (a'_{33})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right) - (a''_{33})^{(6)}(T_{33},t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \le \frac{d\nu^{(6)}}{dt} \le -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \ge \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \le \nu^{(6)}(t) \le (\nu_1)^{(6)}$

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In the same manner , we get

$$\nu^{(6)}(t) \le \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \le \nu^{(6)}(t) \le (\bar{\nu}_1)^{(6)}$

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If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

$$(\nu_1)^{(6)} \le \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}} \le \nu^{(6)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \le (\bar{\nu}_1)^{(6)}$$

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If $0 < (\nu_1)^{(6)} \le (\bar{\nu}_1)^{(6)} \le \boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$ , we obtain

$$(\nu_1)^{(6)} \le \nu^{(6)}(t) \le \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \le (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

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$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}$, $\boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

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$$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$$

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If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain

$$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(C)^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

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**Definition of** $u^{(7)}(t)$ :-

$$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}, \quad \boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}'')^{(7)} = (a_{37}'')^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}'')^{(7)} = (b_{37}'')^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain 421

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a'_{40})^{(8)} - (a'_{41})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right) - (a''_{41})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \le \frac{dv^{(8)}}{dt} \le -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(8)}, (v_0)^{(8)}$ :-

For $0 < \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}} < (v_1)^{(8)} < (\bar{v}_1)^{(8)}$

$$v^{(8)}(t) \ge \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(v_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (v_2)^{(8)}}}$$

it follows $(v_0)^{(8)} \le v^{(8)}(t) \le (v_1)^{(8)}$

In the same manner , we get 422

$$v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{v}_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (\bar{v}_2)^{(8)}}}$$

From which we deduce $(v_0)^{(8)} \le v^{(8)}(t) \le (\bar{v}_8)^{(8)}$

If $0 < (v_1)^{(8)} < (v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}} < (\bar{v}_1)^{(8)}$ we find like in the previous case, 423

$$(v_1)^{(8)} \le \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}} \le v^{(8)}(t) \le$$

$$\frac{(\bar{v}_1)^{(8)} + (C)^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (\bar{v}_1)^{(8)}$$

If $0 < (v_1)^{(8)} \le (\bar{v}_1)^{(8)} \le \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$ , we obtain 424

$$(v_1)^{(8)} \le v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (v_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(8)}(t)$ :-

$(m_2)^{(8)} \leq v^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{v^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $v^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)}+(\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)}-(\bar{v}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)}-(\bar{v}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{v}_1)^{(9)}-(v_0)^{(9)}}{(v_0)^{(9)}-(\bar{v}_2)^{(9)}}}$$

From which we deduce $(v_0)^{(9)} \le v^{(9)}(t) \le (\bar{v}_1)^{(9)}$

If $0 < (v_1)^{(9)} < (v_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{v}_1)^{(9)}$ we find like in the previous case,

$$(v_1)^{(9)} \le \frac{(v_1)^{(9)}+(C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)}-(v_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)}-(v_2)^{(9)}\right)t\right]}} \le v^{(9)}(t) \le$$

$$\frac{(\bar{v}_1)^{(9)}+(\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)}-(\bar{v}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)}-(\bar{v}_2)^{(9)}\right)t\right]}} \le (\bar{v}_1)^{(9)}$$

If $0 < (v_1)^{(9)} \le (\bar{v}_1)^{(9)} \le \boxed{(v_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(v_1)^{(9)} \le v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)}+(\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)}-(\bar{v}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)}-(\bar{v}_2)^{(9)}\right)t\right]}} \le (v_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(9)}(t)$ :-

$$(m_2)^{(9)} \le v^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{v^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \le u^{(9)}(t) \le (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(v_1)^{(9)} = (\bar{v}_1)^{(9)}$ if in addition $(v_0)^{(9)} = (v_1)^{(9)}$ then $v^{(9)}(t) = (v_0)^{(9)}$ and as a consequence $G_{44}(t) = (v_0)^{(9)}G_{45}(t)$ **this also defines** $(v_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(v_1)^{(9)}$ and $(\bar{v}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

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| $(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,<br>$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$<br>$with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$ | 433 |

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| $(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$ | 436 A |

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| $(a'_{44})^{(9)}(a'_{45})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a'_{45})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b'_{44})^{(9)}(r_{45})^{(9)} - (b'_{45})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b'_{13})^{(1)} - (b''_{13})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b'_{14})^{(1)} - (b''_{14})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b'_{15})^{(1)} - (b''_{15})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b'_{16})^{(2)} - (b''_{16})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b'_{17})^{(2)} - (b''_{17})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b'_{18})^{(2)} - (b''_{18})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a'_{20})^{(3)} + (a''_{20})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a'_{21})^{(3)} + (a''_{21})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a'_{22})^{(3)} + (a''_{22})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b'_{20})^{(3)} - (b''_{20})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b'_{21})^{(3)} - (b''_{21})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b'_{22})^{(3)} - (b''_{22})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) +$ | 485 |

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| $(a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | |
| **Proof:**<br><br>(b) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |

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| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492<br>A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations | 498 |

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| $G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a'_{28})^{(5)}+(a''_{28})^{(5)}(T^*_{29})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a'_{30})^{(5)}+(a''_{30})^{(5)}(T^*_{29})]}$ | |
| **Definition and uniqueness of** $T^*_{33}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T^*_{33}$ for which $f(T^*_{33}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a'_{32})^{(6)}+(a''_{32})^{(6)}(T^*_{33})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a'_{34})^{(6)}+(a''_{34})^{(6)}(T^*_{33})]}$ | 499 |
| **Definition and uniqueness of** $T^*_{37}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T^*_{37}$ for which $f(T^*_{37}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a'_{36})^{(7)}+(a''_{36})^{(7)}(T^*_{37})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a'_{38})^{(7)}+(a''_{38})^{(7)}(T^*_{37})]}$ | 500 |
| **Definition and uniqueness of** $T^*_{41}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T^*_{41}$ for which $f(T^*_{41}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a'_{40})^{(8)}+(a''_{40})^{(8)}(T^*_{41})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a'_{42})^{(8)}+(a''_{42})^{(8)}(T^*_{41})]}$ | 501 |
| **Definition and uniqueness of** $T^*_{45}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T^*_{45}$ for which $f(T^*_{45}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a'_{44})^{(9)}+(a''_{44})^{(9)}(T^*_{45})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a'_{46})^{(9)}+(a''_{46})^{(9)}(T^*_{45})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br><br>$\left[(b'_{13})^{(1)}(b''_{14})^{(1)}(G) + (b'_{14})^{(1)}(b''_{13})^{(1)}(G)\right]+(b''_{13})^{(1)}(G)(b''_{14})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that $\varphi$ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that | 502 |

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| there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br>$\varphi(G_{19}) = (b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br>$\left[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19}) + (b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})\right] + (b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right] + (b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right] + (b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that | 508 |

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| there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0\,, T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0\,, T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |

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| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^*$ given by $\varphi((G_{23})^*) = 0$ , $T_{21}^*$ given by $f(T_{21}^*) = 0$ and<br><br>$G_{20}^* = \frac{(a_{20})^{(3)} G_{21}^*}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^*)]}$ , $G_{22}^* = \frac{(a_{22})^{(3)} G_{21}^*}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^*)]}$<br><br>$T_{20}^* = \frac{(b_{20})^{(3)} T_{21}^*}{[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}{}^*)]}$ , $T_{22}^* = \frac{(b_{22})^{(3)} T_{21}^*}{[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}{}^*)]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br><br>$G_{25}^*$ given by $\varphi(G_{27}) = 0$ , $T_{25}^*$ given by $f(T_{25}^*) = 0$ and<br><br>$G_{24}^* = \frac{(a_{24})^{(4)} G_{25}^*}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}^*)]}$ , $G_{26}^* = \frac{(a_{26})^{(4)} G_{25}^*}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}^*)]}$ | 516 |
| $T_{24}^* = \frac{(b_{24})^{(4)} T_{25}^*}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27})^*)]}$ , $T_{26}^* = \frac{(b_{26})^{(4)} T_{25}^*}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27})^*)]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br><br>$G_{29}^*$ given by $\varphi((G_{31})^*) = 0$ , $T_{29}^*$ given by $f(T_{29}^*) = 0$ and<br><br>$G_{28}^* = \frac{(a_{28})^{(5)} G_{29}^*}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^*)]}$ , $G_{30}^* = \frac{(a_{30})^{(5)} G_{29}^*}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^*)]}$ | 518 |
| $T_{28}^* = \frac{(b_{28})^{(5)} T_{29}^*}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^*)]}$ , $T_{30}^* = \frac{(b_{30})^{(5)} T_{29}^*}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^*)]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br><br>$G_{33}^*$ given by $\varphi((G_{35})^*) = 0$ , $T_{33}^*$ given by $f(T_{33}^*) = 0$ and<br><br>$G_{32}^* = \frac{(a_{32})^{(6)} G_{33}^*}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^*)]}$ , $G_{34}^* = \frac{(a_{34})^{(6)} G_{33}^*}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^*)]}$ | 520 |
| $T_{32}^* = \frac{(b_{32})^{(6)} T_{33}^*}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^*)]}$ , $T_{34}^* = \frac{(b_{34})^{(6)} T_{33}^*}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^*)]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br><br>$G_{37}^*$ given by $\varphi((G_{39})^*) = 0$ , $T_{37}^*$ given by $f(T_{37}^*) = 0$ and | 522 |

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| $G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)} (T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)} (T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)} ((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)} ((G_{39})^{*})]}$ | |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)} (T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)} (T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)} ((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)} ((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br><br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br><br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)} (T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)} (T_{45}^{*})]}$<br><br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)} ((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)} ((G_{47})^{*})]}$ | 523 A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:**_Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}} (T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j} (G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)} \mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*} \mathbb{T}_{14}$ | 525 |

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| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^{*}\mathbb{G}_{j}\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^{*}\mathbb{G}_{j}\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^{*}\mathbb{G}_{j}\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 532 |
| $\frac{\partial(a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial(b_i^{''})^{(2)}}{\partial G_j}((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^{*}\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | 540 |

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| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)} \ , \ \frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^*\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^*\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^*\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^*\mathbb{G}_j\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^*\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^*\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ *and* $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)} \ , \ \frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^*) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right)$ | 554 |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^*) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^*\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^*\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^*\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |

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| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{37})^{(7)}}{\partial T_{37}}(T^*_{37}) = (q_{37})^{(7)}$ , $\frac{\partial(b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a'_{36})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G^*_{36}\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a'_{37})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G^*_{37}\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. | |

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| Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \ , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}((G_{43})^*) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^*\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^*\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b_{41}^{'})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T_{41}^*\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b_{42}^{'})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T_{42}^*\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS** **Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. **Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |

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| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $(\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\{((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)})$ $\left[\left(((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)})(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\right]$ $\left(((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)})s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\right)$ $+\left(((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)})(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\right)$ $\left(((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)})s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\right)$ $\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $\left((\lambda)^{(1)}\right)^2 + \left((b'_{13})^{(1)} + (b'_{14})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $+\left(\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$ $+\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G^*_{13}\right)$ $\left(((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)})s_{(14),(15)}T^*_{14} + (b_{14})^{(1)}s_{(13),(15)}T^*_{13}\right)\} = 0$ $+$ $(\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)}\{((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)})$ $\left[\left(((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)})(q_{17})^{(2)}\mathrm{G}^*_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^*_{16}\right)\right]$ $\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(17)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^*_{17}\right)$ $+\left(((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)})(q_{16})^{(2)}\mathrm{G}^*_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^*_{17}\right)$ $\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(16)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^*_{16}\right)$ $\left((\lambda)^{(2)}\right)^2 + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ $\left((\lambda)^{(2)}\right)^2 + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ | |

$$+\left(\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\right)\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left(\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\right)$$

$$+\left(\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((b_{20}')^{(3)}+(b_{21}')^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}')^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}')^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}')^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}')^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}')^{(4)}+(a_{25}')^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((b_{24}')^{(4)}+(b_{25}')^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\}\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^2+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G^*_{33}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G^*_{32}\right)$$

$$\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T^*_{33}+(b_{33})^{(6)}s_{(32),(34)}T^*_{32}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b'_{38})^{(7)}-(r_{38})^{(7)}\}\{\left((\lambda)^{(7)}+(a'_{38})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(q_{36})^{(7)}G^*_{36}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T^*_{37}+(b_{37})^{(7)}s_{(36),(37)}T^*_{37}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a'_{37})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G^*_{36}+(a_{36})^{(7)}(q_{37})^{(7)}G^*_{37}\right)$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T^*_{37}+(b_{37})^{(7)}s_{(36),(36)}T^*_{36}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^2+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^*_{36}\right)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^*_{37}+(b_{37})^{(7)}s_{(36),(38)}T^*_{36}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^*_{40}\right)\right]$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^*_{41}+(b_{41})^{(8)}s_{(40),(41)}T^*_{41}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^*_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^*_{41}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^*_{41}+(b_{41})^{(8)}s_{(40),(40)}T^*_{40}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^*_{40}\right)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^*_{41}+(b_{41})^{(8)}s_{(40),(42)}T^*_{40}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^*_{44}\right)\right]$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^*_{45}+(b_{45})^{(9)}s_{(44),(45)}T^*_{45}\Big)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^*_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45}+(b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left(\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45}+(b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

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# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i(x)\, xi$ where $x_i \in R^n_i$, $x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i(x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i(\mu)$ of the $A_i(x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k \tag{2}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to

replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 2. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon\ C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \ i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0}$

$\overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow *x.t restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set *K* is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** *x.t* is defined for all $t \geq 0$

**A4:** *x.t* is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C \backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \longrightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) in\ ||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population *i* is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^{t}_{0} (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\to\infty}{limsup} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v = au_i(x) + w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |B_i(t,x)u_i(x)|(a + \alpha + exp(-\beta t)|w|)$. Hence, $\limsup_{t\to\infty} \frac{1}{t} In\, B_i(t,x)\, v| \leq$

$$\limsup_{t\to\infty} \frac{1}{t} In\, B_i(t,x)\, u_i(x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\, |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\, u_i(x)| = In\, |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\, u_i(x.t)||\, B_i(t,x)\, u_i(x)|$$

$$= \text{In } b_i(s,t.x)\, b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x - bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

$$=$$

$$= (A_i(x.t)u_i(x.s), u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s), u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } | (x.t)_i |$$

$$\leq \limsup_{t\to\infty} \frac{In\ \gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x) =$ Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of

irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$ where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ and $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as $n\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$. Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\le \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x)u_i^n(x), u_i^n(x)\rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)|d\mu(x) + |\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\,d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\,ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i\,dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\,d\,v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak$^*$ limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \{ \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i \epsilon I}{if\ i \notin I}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that \quad \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t$ −invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently *K* is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s =$

$s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume $x \mapsto A(x)$ is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists $j \in \{1,\ldots.,k\}$ such that $M_i$ ----- $\{x \in C : x_l \gg 0 \text{ for } l \neq j\}$ Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_i M_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, *$\lambda_i(\mu) > 0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, ..., x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda(0)$, which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i(0) > 0$ for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the

system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j \ (x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1{}^1, \ldots, E_1{}^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i{}_2(E_1{}^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j \ x_2^j \ - \alpha^j \ x_3^j) + \sum d_1^{jk} \ x_1^k \ - e_1^j \ x_1^j$$

$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > 1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria $E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$and $\lambda_3(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\ldots,k;\ j = 1,\ldots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i\, a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i\, x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \qquad i = 1, \ldots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \ldots, k$$

It has been shown [12,ch.16.1,16.2] That for $k = 3$,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is

the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y \end{pmatrix}$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & O \\ \gamma & my\text{-}b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \begin{pmatrix} \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1-y) - b(N)\ y \end{pmatrix}$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1 = N$, $x_2 = p$, $x_3 = (y,z)'$ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\curlywedge_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\curlywedge_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\curlywedge_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + K p\, x_i x_i - 1}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i)-1 & \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions

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# Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a

qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and

orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t\geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a\geq 0)$ at time $t$ so that we have

(1.4) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t\geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.5) $D\rho_i(a, t) = \limsup_{h\to 0} \frac{\rho_i(a+h,t+h)-\rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.6) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0,' \text{(1.4)} & \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, \quad t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s\in[-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s\in[-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1, \varphi_2, f_1, f_2, b_1, b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for

each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as* $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 2. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'}\ - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2,$

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, \text{R}^-),$$

$A_3$. $\varphi_i\ (\cdot\ , s) \in L^2\ (R^+, R^+)\ \cap C(R^+, R^+)$ for each $s \in\ [-\tau, 0]$,

$i, = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+ \ (i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) \ P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[-\int_0^\infty f_i(a + s, P(t + S))ds\right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[-\int_o^x f_1(a + s, p(t + s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0)\exp\left[-\int_o^t f_1(a - t + s, p_1(s))\, ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

*(2.3)* $p_i(a,t) = B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0,t), i = 1,2, t > 0.$$

Define $M_i and\ L_i\ (i = 1,2)$ as follows:

$$M_i(a,t,p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a,t,\ p_i(t)) = \exp[-\int_o^t f_1(a+s,\ p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

$$(2.6)\ p_i(t) = \int_o^t B_1(a) M_1(a,t,p_i(t))da + \int_o^\infty \varphi_i(a,0)\ L_i(a,t,p_i(t)da,$$

$$t > 0, i = 1,2,$$

$$(2.7)\ p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a,s)da, s \in [-\tau, 0], i = 1,2,$$

$$(2.8)\ \beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.\left(a,t,P_i(t)\right)da$$

$$+\int_0^\infty b_i\left(a+t, P_i(t)\right)\varphi_i(a,0)L_i\left(a,t,P_i(t)\right)da$$

$$i = 1, 2, t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t)\, \phi(0) d_i^*]t \quad for \ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T], R^+), \quad B_i \in C([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in C([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau, T]$. Then for any

$(x, y) \in C^+[-\tau, T] \times C^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau, T], \quad C_T^2 = C^+[-\tau, T] \times C^+[-\tau, T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for $i = 1,2$.

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \pi C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \quad P_t = \ (P_{1t}, P_{2t}) \in C^2_{t*} \mid \|P_t - \Phi_t\|_{t*} \le r^*$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)\, da \ for\ s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)\, da \ for\ s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Phi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^* \right] \quad t^*$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for t $\le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \le \left| \int_0^t B_1^{(1)}(a) \left\{ M_1(a, t, P_t^{(1)}) - M_1(a, t, P_t^{(2)}) \right\} da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a, t, P_t^{(2)}) da \right|$$

$$+ \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \; for \; x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \| ds\right) da$$

$$\leq d_1 \beta_1^* \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o) \left[ \int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds \right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \| t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\text{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma\right.$$

$$+ \int_o^\infty \varphi(\sigma, o) \left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^{\infty} \varphi(\sigma,o) b_1\left(\sigma+a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma,a,P_t^{(1)}(a)\right)-L_1\left(\sigma,a,P_t^{(2)}(a)\right)\right]d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right)-b_1\left(\sigma+a, P_t^{(2)}(a)\right)\right| \le B_1 \,/\!/\, P_t^{(1)}-P_t^{(2)} /\!/ \text{ for } a\,\varepsilon[o,t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \le \beta_1 \Phi_1(0)||P_t^{(1)}-P_t^{(2)}||_t^{*}+\Phi_1(0)\beta_1 d_1||P_t^{(1)}-P_t^{(2)}||_t^{*a},$$

$$|R_2(a)|=[\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a))-b_1(a-\sigma, P_t^{(2)}(a))\}M_1(\sigma, a, P_t^{(1)}(a))$$

$$+b_1(a,-\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a))-M_1(a-\sigma, P_t^{(2)}(a))\}]d\sigma]$$

$$+b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a))-M_1(a-\sigma, P_t^{(2)}(a))\}]d\sigma]$$

$$\le \int_0^a [\beta_1^{*}\Phi_1(0)\, e^{\delta 1^a}\beta_1 \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}+\beta_1 d_1 \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}(a-o)]do$$

$$\le \beta_1^{*}\Phi_1(0)\, e^{\delta 1^a} \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\{\beta_1 a+d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a)\right|$$

$$\le \int_o^a \left| B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma)\right| B_1^{*d}\sigma$$

$$+\Phi_1(0) /\!/ P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\left\{B_1+B_2 d_1 a+B_1^{*}\left(B_2 a+d_1 a^2/2\right)e^{\delta 1^a}\right\}$$

$$\le \beta_1^{*}\int_o^a \left| B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma)\right| do+\left(r^{*}+/\!/\Phi_t /\!/_{t*}\right)\| P_t^{(1)}-P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a)\to \beta_1$ as $a\to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a)\right| \le \left(r^{*}+/\!/\Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))\text{-}b_1(a\text{-}\sigma, P_t^{(1)}(a))\, M_1(\sigma, a, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + /\!/ \Phi_t /\!/_{t*}\right) /\!/ \; P_t^{(1)}\text{-}P_t^{(2)} /\!/_{t*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right) /\!/ \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^{*})$$

As $t^{*} \to o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^{*}, P_t^{*}$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp\, P_2^{*}(a)}{da} + f_i(\mathrm{a}, P_1^{*}, P_2^{*})\, P_i^{*}(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^{*} = \int_o^{\infty} P_i^{*}(a)da, i = 1,\ 2,$$

$$(3,3)\ P_i^{*} = \int_o^{\infty} b_i(a,\ P_1^{*},\ P_2^{*}) P_i^{*}(a)da,\ i = 1.\ \ 2$$

All solutions of (3.1)are of the from

$$(3.4)\ P_i^{*}(a) = P_i^{*}(o) \exp\left[ - \int_o^{\infty} f_i\,(s, P_1^{*}, P_2^{*})ds \right], i = 1, \quad 2, a \geq 0,$$

Which together with (3.3) lead to

$$1=\int_o^{\infty} b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^{\infty} b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $$F_{i.}\ (P_1,P_2) = \int_o^{\infty} b_i(a,P_1,P_2)\exp\left[-\int_o^{\infty} f_i\,(s,P_1,P_2)ds\right]da,$$

$$i = 1,\ 2, \quad P_1 \geq 0, \quad P_2 \geq 0,$$

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_5$. *there exists a positive constant c for which*

$$F_1(\mathrm{C},0) < F_{2.}(\mathrm{C},0),$$

(3.8) $$F_1(0.\mathrm{C}) < F_{2.}(0,\mathrm{C}),$$

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $$F_1(P_1^*,P_2^*) = 1 = F_2(P_1^*,P_2^*)$$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $$P_2^*(a) = [P_2^*\exp[-f_i\,(s,P_1^*,P_2^*)ds])/(\int_o^{\infty}\exp[-\int_o^a f_i\,(s,P_1^*,P_2^*)ds]da),$$

$$i = 1,\ 2, \quad a \geq 0,$$

Proof Consider the elements of the surfaces defined by

$$\mathrm{Z}= F_1(P_1,P_2), z = F_2(P_1,P_2),\ P_1 \geq 0, P_2 \geq 0,$$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i/\partial P_j < 0\ and$ , by ($A_2$), $\partial b_i/\partial P_j \leq 0$ for all

$$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$$

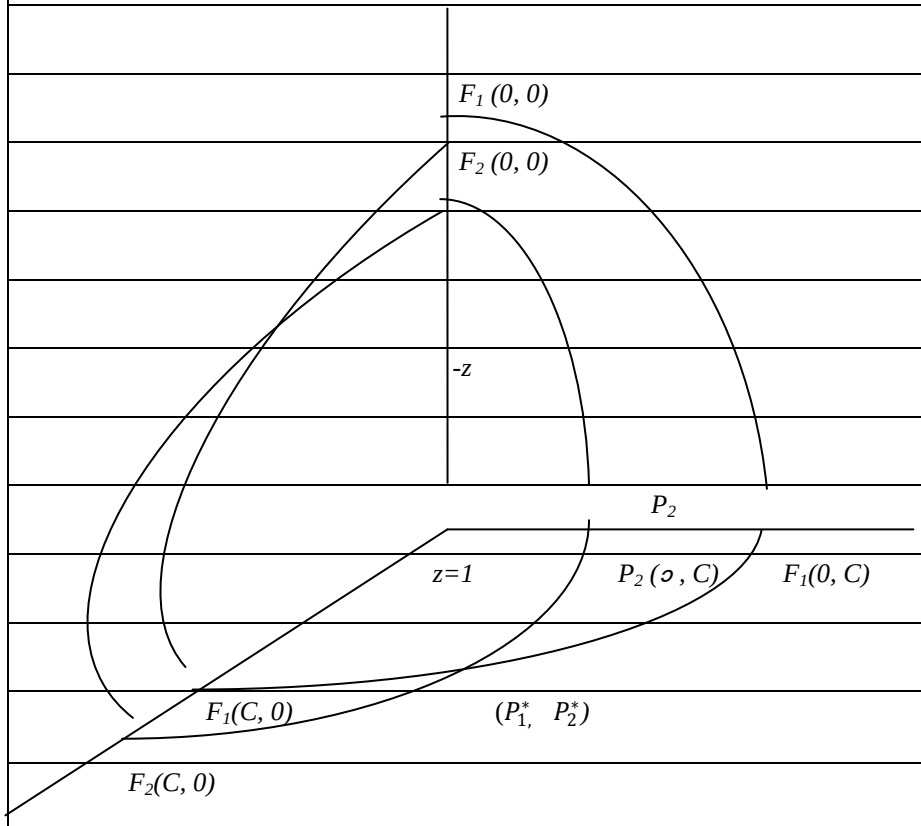
$j = 1,\ 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= 1, $P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$$F_1(P_1, P_2 = 1 \; and \; F_2(P_1, P_2) = 1, \; F_1(P_1, P_2 \geq 0,$$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility 1< $F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p_1^*, p_2^*)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p_1^*, p_2^*$, we let

$$(3.11) \quad \begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \; i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*}\, p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square 1}^{2} \frac{\partial b_i}{\partial p_{\square}^*}\, p_{i\square}(t)\right\} da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_{\square}(a)\exp[\lambda t], \quad \square(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty \square(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d\square}{da} + \lambda\square = -f_i\left(a, p_1^*, p_2^*\right)\square(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*}\, p_{i\square}^*\right\},$$

(3.15)

$$\square(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\square(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*}\, p_{i\square}^*\right\} da,$$

Where $i = 1, 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \quad p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\Phi_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\Phi_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\Phi_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\Phi_i(a) = \{\Phi_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_i\right\} e^{\lambda s} ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\Phi_i(a)\left[p_i^*(a)/p_i^*(0)c_i \exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_i\right\} da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a}\left\{\int_0^a \left(\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_i\right) e^{\lambda s} ds\right\} da\right]$$

$$- p_i^*(a) e^{-\lambda a}\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_i\right\} e^{\lambda s} ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\Phi_1(a)$ and $\Phi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$\left(\lambda - A_{11}\right) p_1^* - A_{12} p_2^* \exp(-\lambda\tau) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21} p_1^* \exp(-\lambda\tau) + \left(\lambda - A_{22}\right) p_2^* = g_{21}(\lambda) p_1^* \exp(-\lambda\tau) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $A_{ij} = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_j^*} - \frac{\partial f_i}{\partial p_j^*}\right]da,$

(3.23) $g_{ij}(\lambda) = [p_i^*(0)c_i]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\,da\right\}$
$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_j^*} - b_i(a, p_1^*, p_2^*)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)\right]da\right\}$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)da,\quad i, j = 1, 2,$$

(3.24) $\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*),\quad i = 1, 2.$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix}\lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda))\end{bmatrix} = 0$

Or equivalently

(3.25) $\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = [A_{21}g_{12}(\lambda) + A_{12}g_{21}(\lambda) + g_{12}(\lambda)g_{21}(\lambda)]\exp[-2\lambda\tau]$

$-A_{11}g_{22}(\lambda) - A_{22}g_{11}(\lambda) - g_{11}(\lambda)g_{22}(\lambda)$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i = 1, 2.)\ (see(A_1)\ and\ (A_2))$;

$A_7.\ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\},$

$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$

$\| S_2(0) \| = |A_{11}g_{12}(0)| + |A_{12}g_{21}(0)| + |g_{12}(0)g_{21}(0)|$

$+ |A_{11}g_{22}(0)| + |A_{22}g_{11}(0)| + |g_{11}(0)g_{22}(0)|$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i(a, t) = \phi_i(a) \exp[\lambda t] \to 0$ as $t \to \infty$ for $i = 1, 2$ and $a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda$ $\mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$

$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \left|\right.$

$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$

$\geq [(|\lambda| - P)^2 + (q)]^2$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \{4|(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| | - |A_{11} + A_{22} + \| S_1(0) \| |^2\}/2,$

It will now follow, from (A) and (3.26), $F_1(\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder

concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power

obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy

withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation

from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

**TITLE**

## Gravitational-Wave Ringdown

**INTRODUCTION—VARIABLES USED**

**Is the Gravitational-Wave Ringdown a Probe of the Event Horizon? Vitor Cardoso, Edgardo Franzin, and Paolo Pani Phys. Rev. Lett. 116, 171101 – Published 27 April 2016**

(1) It is commonly believed that the ringdown signal from (e) a binary coalescence provides (eb) a conclusive proof for (e) the formation of an event horizon after the merger.
(2) This expectation is based on (e) the assumption that the ringdown waveform at (eb) intermediate times is dominated by (e) the quasinormal modes of the final object.
(3) Authors point out that this assumption should be taken with great care, and that very compact objects with a light ring will display (eb) a similar ringdown stage, even when (e) their quasinormal-mode spectrum is completely different from (e) that of a black hole.
(4) In other words, universal ringdown waveforms indicate (eb) the presence of light rings, rather than of horizons.
(5) Only precision observations of the late-time ringdown signal, where (e) the differences in the quasinormal-mode spectrum eventually show up, can be used to (e) rule out exotic alternatives to (e) black holes and to test quantum effects at the horizon scale

**NOTATION**

**Module One**

Physical time in GR is associated, locally and "personally," with (e&eb) every localised physical system, in a manner that more closely reflects our intimate experience of time than the global time of pre-relativistic Newtonian mechanics.

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| $G_{13}$ : Category one of **Physical time in GR is associated, locally and "personally;** every localised physical system, in a manner that more closely reflects our intimate experience of time than the global time of pre-relativistic Newtonian mechanics.<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of every localised physical system, in a manner that more closely reflects our intimate experience of time than the global time of pre-relativistic Newtonian mechanics.;**Physical time in GR is associated, locally and "personally**<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>It is commonly believed that the ringdown signal from (e) a binary coalescence provides (eb) a conclusive proof for (e) the formation of an event horizon after the merger | |
| $G_{16}$ : Category one of binary coalescence provides (eb) a conclusive proof for (e) the formation of an event horizon after the merger<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of ringdown signal<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>It is commonly believed that the ringdown signal from a binary coalescence provides (eb) a conclusive proof for (e) the formation of an event horizon after the merger | |
| $G_{20}$ : Category one of ringdown signal from a binary coalescence<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of conclusive proof for (e) the formation of an event horizon after the merger<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>It is commonly believed that the ringdown signal from a binary coalescence provides a conclusive proof for (e) the formation of an event horizon after the merger | |

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| $G_{24}$ : Category one of formation of an event horizon after the merger<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of ringdown signal from a binary coalescence provides a conclusive proof<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>This expectation is based on the assumption that the ringdown waveform at (eb) intermediate times is dominated by (e) the quasinormal modes of the final object | |
| $G_{28}$ : Category one of expectation is based on the assumption that the ringdown waveform<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of intermediate times is dominated by (e) the quasinormal modes of the final object<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>This expectation is based on the assumption that the ringdown waveform at intermediate times is dominated by (e) the quasinormal modes of the final object | |
| $G_{32}$ : Category one of quasinormal modes of the final object<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of expectation is based on the assumption that the ringdown waveform at intermediate times<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br>Authors point out that this assumption should be taken with great care, and that very compact objects with a light ring will display (eb) a similar ringdown stage, even when (e) their quasinormal-mode spectrum is completely different from (e) that of a black hole | |

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|---|---|
| $G_{36}$ : Category one of assumption should be taken with great care, and that very compact objects with a light ring<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of similar ringdown stage, even when (e) their quasinormal-mode spectrum is completely different from (e) that of a black hole<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Authors point out that this assumption should be taken with great care, and that very compact objects with a light ring will display a similar ringdown stage, even when (e) their quasinormal-mode spectrum is completely different from (e) that of a black hole | |
| $G_{40}$ : Category one of their quasinormal-mode spectrum is completely different from that of a black hole<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of assumption should be taken with great care, and that very compact objects with a light ring will display a similar ringdown stage<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>In other words, universal ringdown waveforms indicate (eb) the presence of light rings, rather than of horizons | |
| $G_{44}$ : Category one of universal ringdown waveforms<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of presence of light rings, rather than of horizons<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |

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|---|---|
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$: $(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$ $(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$, $(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$ $(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$ $(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$ $(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$ , $(a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$ $(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}$ $(a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ , $(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$ $(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$, $(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$, $(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b'_{13})^{(1)} - (b''_{13})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b'_{14})^{(1)} - (b''_{14})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b'_{15})^{(1)} - (b''_{15})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a''_{13})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b''_{13})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b'_{16})^{(2)} - (b''_{16})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b'_{17})^{(2)} - (b''_{17})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b'_{18})^{(2)} - (b''_{18})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a''_{16})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b''_{16})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |

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| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |

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| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b'_{32})^{(6)} - (b''_{32})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b'_{33})^{(6)} - (b''_{33})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b'_{34})^{(6)} - (b''_{34})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a''_{32})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b'_{36})^{(7)} - (b''_{36})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b'_{37})^{(7)} - (b''_{37})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b'_{38})^{(7)} - (b''_{38})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a''_{36})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a'_{41})^{(8)} + (a''_{41})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b'_{40})^{(8)} - (b''_{40})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b'_{41})^{(8)} - (b''_{41})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b'_{42})^{(8)} - (b''_{42})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b'_{44})^{(9)} - (b''_{44})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b'_{45})^{(9)} - (b''_{45})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b'_{46})^{(9)} - (b''_{46})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a''_{44})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b''_{44})^{(9)}((G_{47}),t) =$ **First detrition factor** | |

$$\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}')^{(1)}\boxed{+(a_{13}'')^{(1)}(T_{14},t)}\boxed{+(a_{16}'')^{(2,2,)}(T_{17},t)}\boxed{+(a_{20}'')^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}'')^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{28}'')^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{32}'')^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}'')^{(7,7)}(T_{37},t)}\boxed{+(a_{40}'')^{(8,8)}(T_{41},t)}\boxed{+(a_{44}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13} \quad (55)$$

$$\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}')^{(1)}\boxed{+(a_{14}'')^{(1)}(T_{14},t)}\boxed{+(a_{17}'')^{(2,2,)}(T_{17},t)}\boxed{+(a_{21}'')^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}'')^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{29}'')^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{33}'')^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}'')^{(7,7)}(T_{37},t)}\boxed{+(a_{41}'')^{(8,8)}(T_{41},t)}\boxed{+(a_{45}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (56)$$

$$\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}')^{(1)}\boxed{+(a_{15}'')^{(1)}(T_{14},t)}\boxed{+(a_{18}'')^{(2,2,)}(T_{17},t)}\boxed{+(a_{22}'')^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}'')^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{30}'')^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{34}'')^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}'')^{(7,7)}(T_{37},t)}\boxed{+(a_{42}'')^{(8,8)}(T_{41},t)}\boxed{+(a_{46}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad (57)$$

Where $\boxed{(a_{13}'')^{(1)}(T_{14},t)}$, $\boxed{(a_{14}'')^{(1)}(T_{14},t)}$, $\boxed{(a_{15}'')^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}'')^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}'')^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}'')^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}'')^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}'')^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}'')^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}'')^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}'')^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}'')^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}'')^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}'')^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}'')^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}'')^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}'')^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}'')^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{38}'')^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}'')^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}'')^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a_{40}'')^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}'')^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}'')^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3

$\boxed{+(a_{44}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}')^{(1)}\boxed{-(b_{13}'')^{(1)}(G,t)}\boxed{-(b_{16}'')^{(2,2,)}(G_{19},t)}\boxed{-(b_{20}'')^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}'')^{(4,4,4,4,)}(G_{27},t)}\boxed{-(b_{28}'')^{(5,5,5,5,)}(G_{31},t)}\boxed{-(b_{32}'')^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}'')^{(7,7,)}(G_{39},t)}\boxed{-(b_{40}'')^{(8,8)}(G_{43},t)}\boxed{-(b_{44}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13} \quad (58)$$

$$\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}')^{(1)}\boxed{-(b_{14}'')^{(1)}(G,t)}\boxed{-(b_{17}'')^{(2,2,)}(G_{19},t)}\boxed{-(b_{21}'')^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}'')^{(4,4,4,4,)}(G_{27},t)}\boxed{-(b_{29}'')^{(5,5,5,5,)}(G_{31},t)}\boxed{-(b_{33}'')^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}'')^{(7,7,)}(G_{39},t)}\boxed{-(b_{41}'')^{(8,8)}(G_{43},t)}\boxed{-(b_{45}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14} \quad (59)$$

$$\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)} & -(b_{15}^{''})^{(1)}(G,t) & -(b_{18}^{''})^{(2,2,)}(G_{19},t) & -(b_{22}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,)}(G_{39},t) & -(b_{42}^{''})^{(8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15} \quad (60)$$

Where $-(b_{13}^{''})^{(1)}(G,t)$, $-(b_{14}^{''})^{(1)}(G,t)$, $-(b_{15}^{''})^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{37}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}^{'})^{(2)} & +(a_{16}^{''})^{(2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,)}(T_{14},t) & +(a_{20}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{16} \quad (61)$$

$$\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}^{'})^{(2)} & +(a_{17}^{''})^{(2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,)}(T_{14},t) & +(a_{21}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{17} \quad (62)$$

$$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}^{'})^{(2)} & +(a_{18}^{''})^{(2)}(T_{17},t) & +(a_{15}^{''})^{(1,1,)}(T_{14},t) & +(a_{22}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{18} \quad (63)$$

Where $+(a_{16}^{''})^{(2)}(T_{17},t)$, $+(a_{17}^{''})^{(2)}(T_{17},t)$, $+(a_{18}^{''})^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for

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| category 1, 2 and 3 $+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3 $+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3 $+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)} & -(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{16}$ | *64* |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)} & -(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{17}$ | *65* |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)} & -(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{18}$ | *66* |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$, $-(b_{17}^{''})^{(2)}(G_{19},t)$, $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{13}^{''})^{(1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3 $-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3 $-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3 $-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3 $-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3 $-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3 | |

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| $-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[\begin{array}{c} (a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21},t)+(a_{16}^{''})^{(2,2,2)}(T_{17},t)+(a_{13}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{44}^{''})^{(9,9,9)}(T_{45},t) \end{array}\right]G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[\begin{array}{c} (a_{21}^{'})^{(3)}+(a_{21}^{''})^{(3)}(T_{21},t)+(a_{17}^{''})^{(2,2,2)}(T_{17},t)+(a_{14}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{45}^{''})^{(9,9,9)}(T_{45},t) \end{array}\right]G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[\begin{array}{c} (a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21},t)+(a_{18}^{''})^{(2,2,2)}(T_{17},t)+(a_{15}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{46}^{''})^{(9,9,9)}(T_{45},t) \end{array}\right]G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[\begin{array}{c} (b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23},t)-(b_{16}^{''})^{(2,2,2)}(G_{19},t)-(b_{13}^{''})^{(1,1,1,)}(G,t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)-(b_{44}^{''})^{(9,9,9)}(G_{47},t) \end{array}\right]T_{20}$ | 70 |

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b'_{21})^{(3)} & -(b''_{21})^{(3)}(G_{23},t) & -(b''_{17})^{(2,2,2)}(G_{19},t) & -(b''_{14})^{(1,1,1,)}(G,t) \\ -(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b''_{37})^{(7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)} & -(b''_{22})^{(3)}(G_{23},t) & -(b''_{18})^{(2,2,2)}(G_{19},t) & -(b''_{15})^{(1,1,1,)}(G,t) \\ -(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b''_{38})^{(7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{22} \quad (72)$$

$-(b''_{20})^{(3)}(G_{23},t)$, $-(b''_{21})^{(3)}(G_{23},t)$, $-(b''_{22})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*

$-(b''_{16})^{(2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*

$-(b''_{13})^{(1,1,1,)}(G,t)$, $-(b''_{14})^{(1,1,1,)}(G,t)$, $-(b''_{15})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*

$-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*

$-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*

$-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*

$-(b''_{36})^{(7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7)}(G_{39},t)$ $-(b''_{38})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b''_{40})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b''_{46})^{(9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a'_{24})^{(4)} & +(a''_{24})^{(4)}(T_{25},t) & +(a''_{28})^{(5,5,)}(T_{29},t) & +(a''_{32})^{(6,6,)}(T_{33},t) \\ +(a''_{13})^{(1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{36})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a'_{25})^{(4)} & +(a''_{25})^{(4)}(T_{25},t) & +(a''_{29})^{(5,5,)}(T_{29},t) & +(a''_{33})^{(6,6)}(T_{33},t) \\ +(a''_{14})^{(1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{37})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a'_{26})^{(4)} & +(a''_{26})^{(4)}(T_{25},t) & +(a''_{30})^{(5,5,)}(T_{29},t) & +(a''_{34})^{(6,6,)}(T_{33},t) \\ +(a''_{15})^{(1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{38})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a''_{24})^{(4)}(T_{25},t)$, $(a''_{25})^{(4)}(T_{25},t)$, $(a''_{26})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a_{28}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* $1,2$ *and* $3$

$+(a_{32}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* $1,2$ *and* $3$

$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$ *are fourth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$ *are fifth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$ *are sixth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$ *are seventh augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$ *are eighth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)}&-(b_{24}^{''})^{(4)}(G_{27},t)&-(b_{28}^{''})^{(5,5,)}(G_{31},t)&-(b_{32}^{''})^{(6,6,)}(G_{35},t)\\-(b_{13}^{''})^{(1,1,1,1)}(G,t)&-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)&-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)\\-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)&-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)&-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt}=(b_{25})^{(4)}T_{24}-\begin{bmatrix}(b_{25}^{'})^{(4)}&-(b_{25}^{''})^{(4)}(G_{27},t)&-(b_{29}^{''})^{(5,5,)}(G_{31},t)&-(b_{33}^{''})^{(6,6,)}(G_{35},t)\\-(b_{14}^{''})^{(1,1,1,1)}(G,t)&-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)&-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)\\-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)&-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)&-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt}=(b_{26})^{(4)}T_{25}-\begin{bmatrix}(b_{26}^{'})^{(4)}&-(b_{26}^{''})^{(4)}(G_{27},t)&-(b_{30}^{''})^{(5,5,)}(G_{31},t)&-(b_{34}^{''})^{(6,6,)}(G_{35},t)\\-(b_{15}^{''})^{(1,1,1,1)}(G,t)&-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)&-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)\\-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)&-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)&-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{26} \quad (78)$$

*Where* $-(b_{24}^{''})^{(4)}(G_{27},t)$, $-(b_{25}^{''})^{(4)}(G_{27},t)$, $-(b_{26}^{''})^{(4)}(G_{27},t)$ *are first detrition coefficients for category* $1,2$ *and* $3$

$-(b_{28}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,)}(G_{31},t)$ *are second detrition coefficients for category* $1,2$ *and* $3$

$-(b_{32}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,)}(G_{35},t)$ *are third detrition coefficients for category* $1,2$ *and* $3$

$-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$ *are fourth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$ *are fifth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$

| | |
|---|---|
| *are sixth detrition coefficients for category* 1, 2 *and* 3 $\boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)}$ *are seventh detrition coefficients for category* 1, 2 *and* 3 $\boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)}$ *are eighth detrition coefficients for category* 1, 2 *and* 3 $\boxed{-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[\begin{array}{ccc} (a_{28}^{'})^{(5)}\boxed{+(a_{28}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[\begin{array}{ccc} (a_{29}^{'})^{(5)}\boxed{+(a_{29}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[\begin{array}{ccc} (a_{30}^{'})^{(5)}\boxed{+(a_{30}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{30}$ | *81* |
| *Where* $\boxed{+(a_{28}^{''})^{(5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5)}(T_{29},t)}$ *are first augmentation coefficients for category* 1, 2 *and* 3 *And* $\boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3 $\boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3 $\boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3 $\boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation coefficients for category 1,2,and 3 $\boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)}$ are sixth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1,2, 3 $\boxed{+(a_{40}^{''})^{(8,8\ ,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$ are eighth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1,2, 3 | |

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b'_{28})^{(5)}\boxed{-(b''_{28})^{(5)}(G_{31},t)} & \boxed{-(b''_{24})^{(4,4,)}(G_{27},t)} & \boxed{-(b''_{32})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1)}(G,t)} & \boxed{-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b'_{29})^{(5)}\boxed{-(b''_{29})^{(5)}(G_{31},t)} & \boxed{-(b''_{25})^{(4,4,)}(G_{27},t)} & \boxed{-(b''_{33})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1)}(G,t)} & \boxed{-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b'_{30})^{(5)}\boxed{-(b''_{30})^{(5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,)}(G_{27},t)} & \boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{30} \quad (84)$$

*where* $\boxed{-(b''_{28})^{(5)}(G_{31},t)}$ , $\boxed{-(b''_{29})^{(5)}(G_{31},t)}$ , $\boxed{-(b''_{30})^{(5)}(G_{31},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{24})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,)}(G_{27},t)}$ *are second detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{32})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)}$ *are third detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{13})^{(1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1)}(G,t)}$ , $\boxed{-(b''_{15})^{(1,1,1,1,1,)}(G,t)}$ are fourth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a'_{32})^{(6)}\boxed{+(a''_{32})^{(6)}(T_{33},t)} & \boxed{+(a''_{28})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{24})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a'_{33})^{(6)}\boxed{+(a''_{33})^{(6)}(T_{33},t)} & \boxed{+(a''_{29})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{25})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)} G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)} T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)} T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)} T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{28})^{(5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)}+(a_{37}^{''})^{(7)}(T_{37},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15} \quad (93)$$

Where $(a_{36}^{''})^{(7)}(T_{37},t)$, $(a_{37}^{''})^{(7)}(T_{37},t)$, $(a_{38}^{''})^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation

coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3 $+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}(G_{39},t) & -(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$ 94

$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}-(b_{37}^{''})^{(7)}(G_{39},t) & -(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}(G_{39},t) & -(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3 $-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3 $-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3 $-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}'')^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}'')^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}'')^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{46}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

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$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}')^{(8)}+(a_{40}'')^{(8)}(T_{41},t) & +(a_{16}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13}$$

$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}')^{(8)}+(a_{41}'')^{(8)}(T_{41},t) & +(a_{17}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}')^{(8)}+(a_{42}'')^{(8)}(T_{41},t) & +(a_{18}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$$

Where $+(a_{40}'')^{(8)}(T_{41},t)$, $+(a_{41}'')^{(8)}(T_{41},t)$, $+(a_{42}'')^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & -(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & -(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & -(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

| | |
|---|---|
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[\begin{matrix} (a_{44}^{'})^{(9)}\boxed{+(a_{44}^{''})^{(9)}(T_{45},t)}\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{matrix}\right] G_{13}$ | *96* |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[\begin{matrix} (a_{45}^{'})^{(9)}\boxed{+(a_{45}^{''})^{(9)}(T_{45},t)}\boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{matrix}\right] G_{14}$ | |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[\begin{matrix} (a_{46}^{'})^{(9)}\boxed{+(a_{46}^{''})^{(9)}(T_{37},t)}\boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{matrix}\right] G_{15}$ | |
| Where $\boxed{+(a_{44}^{''})^{(9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3<br>$\boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{44}}{dt} =$ | |

$$(b_{44})^{(9)}T_{45} - \left[\begin{array}{ccc} (b_{44}^{'})^{(9)}\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[\begin{array}{ccc} (b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[\begin{array}{ccc} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

| | |
|---|---|
| **Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br>$(a_i^{''})^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$<br>$(b_i^{''})^{(1)}(G,t) \le (r_i)^{(1)} \le (b_i^{'})^{(1)} \le (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$<br>$\lim_{G\to\infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$<br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'},t) - (a_i^{''})^{(1)}(T_{14},t)\rvert \le (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br>$\lvert(b_i^{''})^{(1)}(G^{'},t) - (b_i^{''})^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G^{'}\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i^{'})^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i^{'})^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i^{'})^{(2)}, (a_i^{''})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (b_i^{''})^{(2)} > 0, \quad i,j = 16,17,18$ | |

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| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17}, t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19}, t) \leq (r_i)^{(2)} \leq (b_i')^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty} (a_i'')^{(2)}(T_{17}, t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty} (b_i'')^{(2)}((G_{19}), t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}$ are positive constants and $i = 16,17,18$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(2)}(T_{17}', t) - (a_i'')^{(2)}(T_{17}, t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}'\rvert e^{-(\hat{M}_{16})^{(2)} t}$ | 107 |
| $\lvert(b_i'')^{(2)}((G_{19})', t) - (b_i'')^{(2)}((G_{19}), t)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})'\rvert\rvert e^{-(\hat{M}_{16})^{(2)} t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}', t)$ and $(a_i'')^{(2)}(T_{17}, t)$ . $(T_{17}', t)$ and $(T_{17}, t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}]$ . It is to be noted that $(a_i'')^{(2)}(T_{17}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i'')^{(2)}(T_{17}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i')^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |

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|---|---|
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(3)} (T_{21}, t) = (p_i)^{(3)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(3)} (G_{23}, t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'}, t) - (a_i^{''})^{(3)}(T_{21}, t)\rvert \leq (\hat{k}_{20})^{(3)} \lvert T_{21} - T_{21}^{'} \rvert e^{-(\hat{M}_{20})^{(3)} t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'}, t) - (b_i^{''})^{(3)}(G_{23}, t)\rvert < (\hat{k}_{20})^{(3)} \lvert\lvert G_{23} - G_{23}^{'} \rvert\rvert e^{-(\hat{M}_{20})^{(3)} t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'}, t)$ and$(a_i^{''})^{(3)}(T_{21}, t)$ . $(T_{21}^{'}, t)$ And $(T_{21}, t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$<br>satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$ | 116 |

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|---|---|
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24{,}25{,}26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}\big((G_{27}), t\big) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(4)}\big((G_{27}), t\big) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24{,}25{,}26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\rvert \leq (\hat{k}_{24})^{(4)} \lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)} t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'}, t) - (b_i^{''})^{(4)}\big((G_{27}), t\big)\rvert < (\hat{k}_{24})^{(4)} \lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)} t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $\big[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\big]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24{,}25{,}26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |

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| $\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i^{'})^{(4)} +\ (\hat{B}_{24})^{(4)} +\ (\hat{Q}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}\big((G_{31}), t\big) \leq\ (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(5)}\,(T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(5)}\,(G_{31}, t) =\ (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'}, t) - (b_i^{''})^{(5)}\big((G_{31}), t\big)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $\big[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\big]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br><br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\ , \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities | 126 |

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| $\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(6)}(T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(6)}((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)}\lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})^{'}, t) - (b_i^{''})^{(6)}((G_{35}), t)\rvert < (\hat{k}_{32})^{(6)}\lvert\lvert(G_{35}) - (G_{35})^{'}\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and$(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants | 130 |

$(a_i)^{(6)}, (a_i')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$
satisfy the inequalities

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i')^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$$

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i')^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$$

Where we suppose

(M) $(a_i)^{(7)}, (a_i')^{(7)}, (a_i'')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (b_i'')^{(7)} > 0, \quad i, j = 36,37,38$ 131

(N) The functions $(a_i'')^{(7)}, (b_i'')^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$:

$$(a_i'')^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$$

$$(b_i'')^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i')^{(7)} \leq (\hat{B}_{36})^{(7)}$$

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*(O)* $\lim_{T_2 \to \infty} (a_i'')^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(P)*

$\lim_{G \to \infty} (b_i'')^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, \ (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i'')^{(7)}(T_{37}', t) - (a_i'')^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)}|T_{37} - T_{37}'|e^{-(\hat{M}_{36})^{(7)}t}$$

$$|(b_i'')^{(7)}((G_{39})', t) - (b_i'')^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)}||(G_{39}) - (G_{39})'||e^{-(\hat{M}_{36})^{(7)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(7)}(T_{37}', t)$ and $(a_i'')^{(7)}(T_{37}, t)$ . $(T_{37}', t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$ . It is to be noted that $(a_i'')^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i'')^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ : 134

(Q) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants

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| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :<br><br>(R) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}, (r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})', t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})'\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would | |

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| be absolutely continuous. | |
| **Definition of** $(\widehat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\widehat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\widehat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\widehat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\widehat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i,j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45}, t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47}, t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2 \to \infty} (a_i'')^{(9)}(T_{45}, t) = (p_i)^{(9)}$<br>$\lim_{G \to \infty} (b_i'')^{(9)}(G_{47}, t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}', t) - (a_i'')^{(9)}(T_{45}, t)\rvert \le (\hat{k}_{44})^{(9)} \lvert T_{45} - T_{45}' \rvert e^{-(\widehat{M}_{44})^{(9)} t}$<br>$\lvert(b_i'')^{(9)}((G_{47})', t) - (b_i'')^{(9)}((G_{47}), t)\rvert < (\hat{k}_{44})^{(9)} \lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\widehat{M}_{44})^{(9)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}', t)$ | |

and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)},(\widehat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{44})^{(9)}=1$ then the function $(a_i'')^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$ :

$(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\ ,\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}<1$$

**Definition of** $(\hat{P}_{44})^{(9)},(\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},(\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)},(a_i')^{(9)},(b_i)^{(9)},(b_i')^{(9)},(p_i)^{(9)},\ (r_i)^{(9)}, i=44,45,46,$ satisfy the inequalities

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[(a_i)^{(9)}+(a_i')^{(9)}+\ (\hat{A}_{44})^{(9)}+\ (\hat{P}_{44})^{(9)}\ (\hat{k}_{44})^{(9)}]<1$$

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[\ (b_i)^{(9)}+(b_i')^{(9)}+\ (\hat{B}_{44})^{(9)}+\ (\hat{Q}_{44})^{(9)}\ (\hat{k}_{44})^{(9)}]<1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0)\ ,T_i(0)$ :

$$G_i(t)\le\ (\hat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{G_i(0)=G_i^0>0}$$

$$T_i(t)\le\ (\hat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{T_i(0)=T_i^0>0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0)\ ,T_i(0)$

$$G_i(t)\le\ (\hat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t)\le\ (\hat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

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| **Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$ | 153 B |

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| $T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |

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| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} G_{21}(s_{(20)}) - \left((a_{20}')^{(3)} + a_{20}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{20}(s_{(20)})\right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[(a_{21})^{(3)} G_{20}(s_{(20)}) - \left((a_{21}')^{(3)} + (a_{21}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}')^{(3)} - (b_{21}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{28})^{(5)}$, $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[ (b_{29})^{(5)} T_{28}(s_{(28)}) - \left( (b'_{29})^{(5)} - (b''_{29})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[ (b_{30})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{30})^{(5)} - (b''_{30})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{30}(s_{(28)}) \right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** | |

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| Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[ (a_{32})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{32}(s_{(32)}) \right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[ (a_{33})^{(6)} G_{32}(s_{(32)}) - \left( (a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[ (a_{34})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{34}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[ (b_{32})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{32}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[ (b_{33})^{(6)} T_{32}(s_{(32)}) - \left( (b'_{33})^{(6)} - (b''_{33})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[ (b_{34})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{34})^{(6)} - (b''_{34})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{34}(s_{(32)}) \right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |

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| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}^{'})^{(8)} + a_{40}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^{0} + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^{0} + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^{0} + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{13}(t) \leq G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \leq \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}$ , $G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \leq G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that | 170 |

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| $(G_{16}(t)-G_{16}^0)e^{-(\bar{M}_{16})^{(2)}t} \le \frac{(a_{16})^{(2)}}{(\bar{M}_{16})^{(2)}}\left[((\hat{P}_{16})^{(2)}+G_{17}^0)e^{\left(-\frac{(\hat{P}_{16})^{(2)}+G_{17}^0}{G_{17}^0}\right)}+(\hat{P}_{16})^{(2)}\right]$ | |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0+(\hat{P}_{20})^{(3)}e^{(\bar{M}_{20})^{(3)}s_{(20)}}\right)\right] ds_{(20)} =$<br>$(1+(a_{20})^{(3)}t)G_{21}^0 + \frac{(a_{20})^{(3)}(\hat{P}_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left(e^{(\bar{M}_{20})^{(3)}t}-1\right)$ | 171 |
| From which it follows that<br>$(G_{20}(t)-G_{20}^0)e^{-(\bar{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that<br>$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\bar{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$<br>$(1+(a_{24})^{(4)}t)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left(e^{(\bar{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that<br>$(G_{24}(t)-G_{24}^0)e^{-(\bar{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$<br>$(1+(a_{28})^{(5)}t)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)}t}-1\right)$ | |
| From which it follows that<br>$(G_{28}(t)-G_{28}^0)e^{-(\bar{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 5 | 175 |
| The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious | 176 |

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| that<br>$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$<br>$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$ | |
| From which it follows that<br>$(G_{32}(t) - G_{32}^0) e^{-(\hat{M}_{32})^{(6)} t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)} + G_{33}^0) e^{\left(-\frac{(\hat{P}_{32})^{(6)} + G_{33}^0}{G_{33}^0}\right)} + (\hat{P}_{32})^{(6)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 6<br>Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$ | 177 |
| (c) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)}\left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$<br>$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$ | 178 |
| From which it follows that<br>$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)} + G_{37}^0) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 7 | |
| The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)}\left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$<br>$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$ | 180 |
| From which it follows that<br>$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)} + G_{41}^0) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)}\left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$ | |

$$\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$$

From which it follows that

$$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t}\leq\frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{P}_{44})^{(9)}+G_{45}^{0}\right)e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 9

Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$

It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ and to choose

$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have

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$$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$$

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$$\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$$

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In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric

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$$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$$

$$\sup_i\left\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\hat{M}_{13})^{(1)}t}\right\}$$

Indeed if we denote

**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$

It results

$$\left|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right|\leq\int_0^t(a_{13})^{(1)}\left|G_{14}^{(1)}-G_{14}^{(2)}\right|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$$

$$\int_0^t\{(a_{13}')^{(1)}\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{-(\hat{M}_{13})^{(1)}s_{(13)}}+$$

$$(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}+$$

$$G_{13}^{(2)}\left|(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}'')^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right|\ e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$$

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| Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | |
| $\left\|G^{(1)} - G^{(2)}\right\| e^{-(\widehat{M}_{13})^{(1)}t} \leq$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right) d\left(\left(G^{(1)}, T^{(1)};\ G^{(2)}, T^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(1)}$ and $(b''_i)^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(1)} - (a''_i)^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1 / (a'_{14})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2 / (a'_{15})^{(1)}$<br><br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(1)}(G(t), t)) = (b'_{14})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ : | 189 |

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| Indeed let $t_1$ be so that for $t > t_1$<br>$(b_{14})^{(1)} - (b_i'')^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}'')^{(1)}(G(t),t) = (b_{15}')^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |

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| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$(a_{16}^{''})^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)} \left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}} \left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)} (\widehat{k}_{16})^{(2)}\right) d\left(\left((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)}\right)\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)} (T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0 \quad \text{for } t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)} \left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$<br>In the same way , one can obtain | 201 |

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| $G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$, $G_{18}$ and $G_{16}$, $G_{17}$ respectively. | |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$. The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}$, $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}} < 1$ and to choose<br><br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$, $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{20})^{(3)}t}\}$ | 211 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :\left( \widetilde{(G_{23})}, \widetilde{(T_{23})} \right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)} \left\|G_{21}^{(1)} - G_{21}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\, ds_{(20)} +$<br><br>$\int_0^t \{(a_{20}^{'})^{(3)} \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right)\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$G_{20}^{(2)} \left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}^{''})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\} ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} t} \leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a_{20}^{'})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)} (\widehat{k}_{20})^{(3)}\right) d\left(\left((G_{23})^{(1)}, (T_{23})^{(1)};\ (G_{23})^{(2)}, (T_{23})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ *and* $(\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)} t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating | 217 |

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| $G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}')^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}')^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$ | 225 |

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{24})^{(4)}\left|G_{25}^{(1)}-G_{25}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$$

$$\int_0^t\{(a_{24}^{'})^{(4)}\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$G_{24}^{(2)}\left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)}-(G_{27})^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}t}\le$$
$$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows

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**Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}\ and\ (\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

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**Remark 17:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

it results

$$G_i\ (t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]}\ge 0$$

$$T_i\ (t)\ge T_i^0e^{\left(-(b_i^{'})^{(4)}t\right)}>0\quad \text{for t} > 0$$

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**Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1,\left((\widehat{M}_{24})^{(4)}\right)_2\ and\ \left((\widehat{M}_{24})^{(4)}\right)_3$ :

**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if

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| $G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a'_{25})^{(4)} G_{25}$ and by integrating<br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a'_{25})^{(4)}$<br>In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a'_{26})^{(4)}$<br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t),t)) = (b_{25}^{'})^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i^{''})^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded.<br>The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + \left((\widehat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\widehat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{28})^{(5)}\right] \leq (\widehat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying | |

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| Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}((G_{31}),(T_{31}))$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)}+$<br>$\int_0^t\{(a_{28}^{'})^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\le$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}^{'})^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}^{''})^{(5)}$ and $(b_{28}^{''})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(5)}-(a_i^{''})^{(5)}(T_{29}(s_{(28)}),s_{(28)})\}ds_{(28)}\right]}\ge 0$ | 239 |

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| $T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(5)}t)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}^{'})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1/(a_{29}^{'})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2/(a_{30}^{'})^{(5)}$<br><br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t),t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |

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| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{32})^{(6)}\right]\leq(\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}ds_{(32)}+$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a''_{32})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a'_{32})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(6)}$ and $(b_i'')^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 249 |

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| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(6)} - (a_i'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(6)} t)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}')^{(6)} G_{33}$ and by integrating<br><br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}')^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}')^{(6)}$<br><br>If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(6)}((G_{35})(t), t)) = (b_{33}')^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i'')^{(6)}((G_{35})(t), t) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty} (b_{34}'')^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}')^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have | 255 |

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| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)}+\left((\hat{P}_{36})^{(7)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{36})^{(7)}\right]\leq(\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}')^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}'')^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}')^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}'')^{(7)}$ and $(b_{36}'')^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$. | 260 |

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| If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2 / (a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t), t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t), t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right),\ \ t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded. | 265 |

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| The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t),t\right)=(b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}<1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)}+\left((\hat{P}_{40})^{(8)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{40})^{(8)}\right]\leq(\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}ds_{(40)}+$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\leq$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ $and$ $T_i$. Taking into account the hypothesis the result follows | 273 |

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| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(8)} - (a_i^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2\ and\ \left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}^{'})^{(8)}$<br><br>If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t), t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t), t\right) < \varepsilon_8, T_{40}\,(t) > (m)^{(8)}$ | 278 |

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| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.<br>The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}^{'})^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$ | |
| In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying 39,35,36 into itself | |
| The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{47})^{(1)}, (T_{47})^{(1)}), ((G_{47})^{(2)}, (T_{47})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \left(\widetilde{(G_{47})}, \widetilde{(T_{47})}\right) = \mathcal{A}^{(9)}\left((G_{47}), (T_{47})\right)$<br>It results<br><br>$\left\|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$<br>$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right) - (a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right| \; e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$<br><br>Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on 45,46,47,28 and 29 it follows | |
| $\left|(G_{47})^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t} \leq$<br>$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)}, (T_{47})^{(1)}; (G_{47})^{(2)}, (T_{47})^{(2)})\right)$ | |

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$

From 99 to 44 it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(9)}t\right)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)}G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t), t\right) < \varepsilon_9, T_{44}\,(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded. The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b''_{46})^{(9)}\left((G_{47})(t),t\right) = (b'_{46})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a'_{13})^{(1)} + (a'_{14})^{(1)} - (a''_{13})^{(1)}(T_{14}, t) + (a''_{14})^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b'_{13})^{(1)} + (b'_{14})^{(1)} - (b''_{13})^{(1)}(G,t) - (b''_{14})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,

and $\boxed{(u_0)^{(1)} = \frac{T^0_{13}}{T^0_{14}}}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$

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| are defined | |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le \right.$<br>$\left.\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a'_{15})^{(1)})}[e^{(S_1)^{(1)}t} - e^{-(a'_{15})^{(1)}t}] + G_{15}^0 e^{-(a'_{15})^{(1)}t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b'_{15})^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b'_{15})^{(1)}t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)}t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$<br>$(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ :<br>$(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a'_{16})^{(2)} + (a'_{17})^{(2)} - (a''_{16})^{(2)}(T_{17}, t) + (a''_{17})^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b'_{16})^{(2)} + (b'_{17})^{(2)} - (b''_{16})^{(2)}((G_{19}), t) - (b''_{17})^{(2)}((G_{19}), t) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(v_1)^{(2)}, (v_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |

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| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\left(u^{(2)}\right)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}$,, $(\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\left(u^{(2)}\right)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously $(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$ $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities $G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)}t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le\right.$ | 312 |

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| $\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_2)^{(2)}((S_1)^{(2)}-(a_{18}')^{(2)})}[e^{(S_1)^{(2)}t}-e^{-(a_{18}')^{(2)}t}]+G_{18}^{0}e^{-(a_{18}')^{(2)}t})$ | |
| $\boxed{T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}}T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le\frac{1}{(\mu_2)^{(2)}}T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)}T_{16}^{0}}{(\mu_1)^{(2)}((R_1)^{(2)}-(b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t}-e^{-(b_{18}')^{(2)}t}\right]+T_{18}^{0}e^{-(b_{18}')^{(2)}t}\le T_{18}(t)\le$ $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)},(S_2)^{(2)},(R_1)^{(2)},(R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)}=(a_{16})^{(2)}(m_2)^{(2)}-(a_{16}')^{(2)}$ $(S_2)^{(2)}=(a_{18})^{(2)}-(p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)}=(b_{16})^{(2)}(\mu_2)^{(1)}-(b_{16}')^{(2)}$ $(R_2)^{(2)}=(b_{18}')^{(2)}-(r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$: $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)}\le-(a_{20}')^{(3)}+(a_{21}')^{(3)}-(a_{20}'')^{(3)}(T_{21},t)+(a_{21}'')^{(3)}(T_{21},t)\le-(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)}\le-(b_{20}')^{(3)}+(b_{21}')^{(3)}-(b_{20}'')^{(3)}(G_{23},t)-(b_{21}'')^{(3)}((G_{23}),t)\le-(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)},(v_2)^{(3)},(u_1)^{(3)},(u_2)^{(3)}$: By $(v_1)^{(3)}>0,(v_2)^{(3)}<0$ and respectively $(u_1)^{(3)}>0,(u_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_1)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_1)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ and By $(\bar{v}_1)^{(3)}>0,(\bar{v}_2)^{(3)}<0$ and respectively $(\bar{u}_1)^{(3)}>0,(\bar{u}_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_2)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_2)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ | 320 |
| **Definition of** $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$:- | 321 |

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$, ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,

and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously

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$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$$

$(p_i)^{(3)}$ is defined by equation

$$\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t} \quad (323)$$

$$\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$$

$$\left. \frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right) \quad (324)$$

$$\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}} \quad (325)$$

$$\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t} \quad (326)$$

$$\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$$

$$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t} \quad (327)$$

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- (328)

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$$

$$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$$

$$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}\big((G_{27}), t\big) - (b''_{25})^{(4)}\big((G_{27}), t\big) \le -(\tau_1)^{(4)}$$

**Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : 329

By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$
and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{v}_1)^{(4)} > 0$, $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$
and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (v_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}, \ \textbf{\textit{if}}\ (v_0)^{(4)} < (v_1)^{(4)}$$

$$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}, \textbf{\textit{if}}\ (v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)},$$

and $\boxed{(v_0)^{(4)} = \frac{G^0_{24}}{G^0_{25}}}$

$$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}, \ \textbf{\textit{if}}\ (\bar{v}_4)^{(4)} < (v_0)^{(4)}$$

and analogously 331

$$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}, \ \textbf{\textit{if}}\ (u_0)^{(4)} < (u_1)^{(4)}$$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}, \textbf{\textit{if}}\ (u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)},$$

and $\boxed{(u_0)^{(4)} = \frac{T^0_{24}}{T^0_{25}}}$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \textbf{\textit{if}}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)} \text{ where } (u_1)^{(4)}, (\bar{u}_1)^{(4)}$$

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| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}')^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}')^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)}t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ :<br>By $(v_1)^{(5)} > 0$ , $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$ , $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}(v^{(5)})^2 + (\sigma_1)^{(5)} v^{(5)} - (a_{28})^{(5)} = 0$ | 339 |

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| and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ :<br><br>By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)},(v_0)^{(5)}$ :-<br><br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)}$ by<br><br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br><br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br><br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |

$\frac{(b_{30})^{(5)}T_{28}^{0}}{(\mu_1)^{(5)}((R_1)^{(5)}-(b_{30}^{'})^{(5)})}\left[e^{(R_1)^{(5)}t}-e^{-(b_{30}^{'})^{(5)}t}\right]+T_{30}^{0}e^{-(b_{30}^{'})^{(5)}t}\le T_{30}(t)\le$

$\frac{(a_{30})^{(5)}T_{28}^{0}}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t}-e^{-(R_2)^{(5)}t}\right]+T_{30}^{0}e^{-(R_2)^{(5)}t}$ 347

**Definition of** $(S_1)^{(5)},(S_2)^{(5)},(R_1)^{(5)},(R_2)^{(5)}$:- 348

Where $(S_1)^{(5)}=(a_{28})^{(5)}(m_2)^{(5)}-(a_{28}^{'})^{(5)}$

$(S_2)^{(5)}=(a_{30})^{(5)}-(p_{30})^{(5)}$

$(R_1)^{(5)}=(b_{28})^{(5)}(\mu_2)^{(5)}-(b_{28}^{'})^{(5)}$

$(R_2)^{(5)}=(b_{30}^{'})^{(5)}-(r_{30})^{(5)}$

**Behavior of the solutions of equation** 349

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ four constants satisfying

$-(\sigma_2)^{(6)}\le-(a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}-(a_{32}^{''})^{(6)}(T_{33},t)+(a_{33}^{''})^{(6)}(T_{33},t)\le-(\sigma_1)^{(6)}$

$-(\tau_2)^{(6)}\le-(b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(b_{32}^{''})^{(6)}\left((G_{35}),t\right)-(b_{33}^{''})^{(6)}\left((G_{35}),t\right)\le-(\tau_1)^{(6)}$

**Definition of** $(\nu_1)^{(6)},(\nu_2)^{(6)},(u_1)^{(6)},(u_2)^{(6)},\nu^{(6)},u^{(6)}$ : 350

By $(\nu_1)^{(6)}>0$ , $(\nu_2)^{(6)}<0$ and respectively $(u_1)^{(6)}>0$ , $(u_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_1)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2+(\tau_1)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$ and

**Definition of** $(\bar{\nu}_1)^{(6)},,(\bar{\nu}_2)^{(6)},(\bar{u}_1)^{(6)},(\bar{u}_2)^{(6)}$ : 351

By $(\bar{\nu}_1)^{(6)}>0$ , $(\bar{\nu}_2)^{(6)}<0$ and respectively $(\bar{u}_1)^{(6)}>0$ , $(\bar{u}_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_2)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2+(\tau_2)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$

**Definition of** $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)},(\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)}$ by

$(m_2)^{(6)}=(\nu_0)^{(6)},(m_1)^{(6)}=(\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)}<(\nu_1)^{(6)}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)}<(\nu_0)^{(6)}<(\bar{\nu}_1)^{(6)}$,

and $\boxed{(\nu_0)^{(6)}=\frac{G_{32}^{0}}{G_{33}^{0}}}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)}<(\nu_0)^{(6)}$

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| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},\ \boldsymbol{if}\ (u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}, \boldsymbol{if}\ (u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)},$<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}, \boldsymbol{if}\ (\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le\right.$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b_{34}')^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b_{34}')^{(6)} t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$<br>$(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ : | |

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| $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying<br><br>$-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$<br><br>$-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}((G_{39}), t) - (b''_{37})^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ :<br><br>By $(\nu_1)^{(7)} > 0, (\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ :<br><br>By $(\bar{\nu}_1)^{(7)} > 0, (\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$<br>**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-<br><br>If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by<br><br>$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,<br>and $\boxed{(\nu_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | 362 |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,<br>and $\boxed{(u_0)^{(7)} = \frac{T^0_{36}}{T^0_{37}}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities | 364 |

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|---|---|
| $G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^{0}e^{(S_1)^{(7)}t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | |
| $\frac{1}{(m_7)^{(7)}} G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^{0}e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^{0}e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}^{'})^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}^{'})^{(7)}t}] + G_{38}^{0}e^{-(a_{38}^{'})^{(7)}t})$ | 366 |
| $\boxed{T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)}T_{36}^{0}}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}^{'})^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}^{'})^{(7)}t}\right] + T_{38}^{0}e^{-(b_{38}^{'})^{(7)}t} \le T_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)}T_{36}^{0}}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^{0}e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}^{'})^{(7)}$<br><br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br><br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}^{'})^{(7)}$<br><br>$(R_2)^{(7)} = (b_{38}^{'})^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br><br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br><br>$-(\sigma_2)^{(8)} \le -(a_{40}^{'})^{(8)} + (a_{41}^{'})^{(8)} - (a_{40}^{''})^{(8)}(T_{41}, t) + (a_{41}^{''})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br><br>$-(\tau_2)^{(8)} \le -(b_{40}^{'})^{(8)} + (b_{41}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}), t) - (b_{41}^{''})^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |
| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : | 372 |

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| By $(v_1)^{(8)} > 0\,, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0\,, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$ and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(8)},, (\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :<br><br>By $(\bar{v}_1)^{(8)} > 0\,, (\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0\,, (\bar{u}_2)^{(8)} < 0$ the<br><br>roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$<br><br>and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$<br><br>**Definition of** $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :-<br><br>If we define $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by<br><br>$(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)},$ ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$<br><br>$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}\,,$ ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)},$<br>and $\boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br><br>$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (v_0)^{(8)},$ ***if*** $(\bar{v}_1)^{(8)} < (v_0)^{(8)}$ | |
| and analogously<br><br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},$ ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}\,,$ ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)},$ ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)}t}$<br><br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$ | 376 |

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| $\left(\frac{(a_{42})^{(8)}G_{40}^{0}}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t}-e^{-(S_2)^{(8)}t}\right]+G_{42}^{0}e^{-(S_2)^{(8)}t}\leq G_{42}(t)\leq \frac{(a_{42})^{(8)}G_{40}^{0}}{(m_2)^{(8)}((S_1)^{(8)}-(a_{42}^{'})^{(8)})}[e^{(S_1)^{(8)}t}-e^{-(a_{42}^{'})^{(8)}t}]+G_{42}^{0}e^{-(a_{42}^{'})^{(8)}t}\right)$ | 377 |
| $\boxed{T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}}T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq \frac{1}{(\mu_2)^{(8)}}T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^{0}}{(\mu_1)^{(8)}((R_1)^{(8)}-(b_{42}^{'})^{(8)})}\left[e^{(R_1)^{(8)}t}-e^{-(b_{42}^{'})^{(8)}t}\right]+T_{42}^{0}e^{-(b_{42}^{'})^{(8)}t}\leq T_{42}(t)\leq$ $\frac{(a_{42})^{(8)}T_{40}^{0}}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t}-e^{-(R_2)^{(8)}t}\right]+T_{42}^{0}e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)},(S_2)^{(8)},(R_1)^{(8)},(R_2)^{(8)}$:-<br>Where $(S_1)^{(8)}=(a_{40})^{(8)}(m_2)^{(8)}-(a_{40}^{'})^{(8)}$<br>$(S_2)^{(8)}=(a_{42})^{(8)}-(p_{42})^{(8)}$<br>$(R_1)^{(8)}=(b_{40})^{(8)}(\mu_2)^{(8)}-(b_{40}^{'})^{(8)}$<br>$(R_2)^{(8)}=(b_{42}^{'})^{(8)}-(r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)}\leq -(a_{44}^{'})^{(9)}+(a_{45}^{'})^{(9)}-(a_{44}^{''})^{(9)}(T_{45},t)+(a_{45}^{''})^{(9)}(T_{45},t)\leq -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)}\leq -(b_{44}^{'})^{(9)}+(b_{45}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47}),t)-(b_{45}^{''})^{(9)}((G_{47}),t)\leq -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(\nu_1)^{(9)},(\nu_2)^{(9)},(u_1)^{(9)},(u_2)^{(9)},\nu^{(9)},u^{(9)}$ :<br>By $(\nu_1)^{(9)}>0$ , $(\nu_2)^{(9)}<0$ and respectively $(u_1)^{(9)}>0$ , $(u_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_1)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$<br>and $(b_{45})^{(9)}(u^{(9)})^2+(\tau_1)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(9)},,(\bar{\nu}_2)^{(9)},(\bar{u}_1)^{(9)},(\bar{u}_2)^{(9)}$ :<br>By $(\bar{\nu}_1)^{(9)}>0$ , $(\bar{\nu}_2)^{(9)}<0$ and respectively $(\bar{u}_1)^{(9)}>0$ , $(\bar{u}_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_2)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$ | |

and $(b_{45})^{(9)}\left(u^{(9)}\right)^2+(\tau_2)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)}=(\nu_0)^{(9)}, (m_1)^{(9)}=(\nu_1)^{(9)},$ ***if*** $(\nu_0)^{(9)}<(\nu_1)^{(9)}$

$(m_2)^{(9)}=(\nu_1)^{(9)}, (m_1)^{(9)}=(\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)}<(\nu_0)^{(9)}<(\bar{\nu}_1)^{(9)},$

and $\boxed{(\nu_0)^{(9)}=\frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)}=(\nu_1)^{(9)}, (m_1)^{(9)}=(\nu_0)^{(9)},$ ***if*** $(\bar{\nu}_1)^{(9)}<(\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)}=(u_0)^{(9)}, (\mu_1)^{(9)}=(u_1)^{(9)},$ ***if*** $(u_0)^{(9)}<(u_1)^{(9)}$

$(\mu_2)^{(9)}=(u_1)^{(9)}, (\mu_1)^{(9)}=(\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)}<(u_0)^{(9)}<(\bar{u}_1)^{(9)},$

and $\boxed{(u_0)^{(9)}=\frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)}=(u_1)^{(9)}, (\mu_1)^{(9)}=(u_0)^{(9)},$ ***if*** $(\bar{u}_1)^{(9)}<(u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0 e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{44}(t)\le G_{44}^0 e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^0 e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{45}(t)\le \frac{1}{(m_2)^{(9)}}G_{44}^0 e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)}G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^0 e^{-(S_2)^{(9)}t}\le G_{46}(t)\le\right.$$

$$\left.\frac{(a_{46})^{(9)}G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)}-(a'_{46})^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a'_{46})^{(9)}t}]+G_{46}^0 e^{-(a'_{46})^{(9)}t}\right)$$

$$\boxed{T_{44}^0 e^{(R_1)^{(9)}t}\le T_{44}(t)\le T_{44}^0 e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^0 e^{(R_1)^{(9)}t}\le T_{44}(t)\le \frac{1}{(\mu_2)^{(9)}}T_{44}^0 e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)}-(b'_{46})^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b'_{46})^{(9)}t}\right]+T_{46}^0 e^{-(b'_{46})^{(9)}t}\le T_{46}(t)\le$$

$$\frac{(a_{46})^{(9)}T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^0 e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain (383)

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right) - (a''_{14})^{(1)}(T_{14}, t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get (384)

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{v}_1)^{(1)}$ we find like in the previous case, (385)

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}'')^{(1)} = (a_{14}'')^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)}G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}'')^{(1)} = (b_{14}'')^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)}T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}')^{(2)} - (a_{17}')^{(2)} + (a_{16}'')^{(2)}(T_{17}, t)\right) - (a_{17}'')^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$$

If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain 394

$$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(2)}(t)$ :- 395

$$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}, \quad \boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$$

In a completely analogous way, we obtain 396

**Definition of** $u^{(2)}(t)$ :-

$$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}, \quad \boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$$

| | |
|---|---|
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$ if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}')^{(3)} - (a_{21}')^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| From which one obtains<br><br>For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$<br><br>$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)} + (C)^{(3)}(\nu_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}$ , $\boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\nu_2)^{(3)}}}$<br><br>it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$ | 400 |
| In the same manner , we get<br><br>$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)} + (\bar{C})^{(3)}(\bar{\nu}_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\bar{\nu}_2)^{(3)}}}$<br><br>**Definition of** $(\bar{\nu}_1)^{(3)}$ :-<br><br>From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$ | 401 |
| If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, | 402 |

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

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$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right) - (a_{25}^{''})^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\quad \boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

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$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} =$

$(\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\quad \boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get 413

$$v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{v}_1)^{(6)}-(v_0)^{(6)}}{(v_0)^{(6)}-(\bar{v}_2)^{(6)}}}$$

From which we deduce $(v_0)^{(6)} \le v^{(6)}(t) \le (\bar{v}_1)^{(6)}$

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If $0 < (v_1)^{(6)} < (v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{v}_1)^{(6)}$ we find like in the previous case,

$$(v_1)^{(6)} \le \frac{(v_1)^{(6)}+(C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}} \le v^{(6)}(t) \le$$

$$\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (\bar{v}_1)^{(6)}$$

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If $0 < (v_1)^{(6)} \le (\bar{v}_1)^{(6)} \le \boxed{(v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(v_1)^{(6)} \le v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (v_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$(m_2)^{(6)} \le v^{(6)}(t) \le (m_1)^{(6)}$, $\boxed{v^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \le u^{(6)}(t) \le (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(v_1)^{(6)} = (\bar{v}_1)^{(6)}$ if in addition $(v_0)^{(6)} = (v_1)^{(6)}$ then $v^{(6)}(t) = (v_0)^{(6)}$ and as a consequence $G_{32}(t) = (v_0)^{(6)}G_{33}(t)$ **this also defines** $(v_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(v_1)^{(6)}$ and $(\bar{v}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right) - (a_{37}^{''})^{(7)}(T_{37},t)v^{(7)} - (a_{37})^{(7)}v^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

$$\nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \leq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}} \leq \nu^{(7)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\bar{\nu}_1)^{(7)}$$

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If $0 < (\nu_1)^{(7)} \leq (\bar{\nu}_1)^{(7)} \leq \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain

$$(\nu_1)^{(7)} \leq \nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \leq \nu^{(7)}(t) \leq (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

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In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :- 420

$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)} G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain 421

$$\frac{d\nu^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right) - (a_{41}^{''})^{(8)}(T_{41}, t)\nu^{(8)} - (a_{41})^{(8)}\nu^{(8)}$$

**Definition of** $\nu^{(8)}$ :- $\boxed{\nu^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right) \leq \frac{d\nu^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)}-(\nu_0)^{(8)}}{(\nu_0)^{(8)}-(\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \le \nu^{(8)}(t) \le (\bar{\nu}_8)^{(8)}$

If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case, 423

$$(\nu_1)^{(8)} \le \frac{(\nu_1)^{(8)}+(C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}} \le \nu^{(8)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\bar{\nu}_1)^{(8)}$$

If $0 < (\nu_1)^{(8)} \le (\bar{\nu}_1)^{(8)} \le \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$, we obtain 424

$$(\nu_1)^{(8)} \le \nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have
**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \le \nu^{(8)}(t) \le (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain
**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \le u^{(8)}(t) \le (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)}G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)}T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$. 424

**Proof** : From 99,20,44,22,23,44 we obtain 424 A

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right) - (a''_{45})^{(9)}(T_{45},t)v^{(9)} - (a_{45})^{(9)}v^{(9)}$$

**Definition of** $v^{(9)}$ :- $\boxed{v^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)}\right) \le \frac{dv^{(9)}}{dt} \le -\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(9)}, (v_0)^{(9)}$ :-

For $0 < \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (v_1)^{(9)} < (\bar{v}_1)^{(9)}$

$$v^{(9)}(t) \ge \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(v_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (v_2)^{(9)}}}$$

it follows $(v_0)^{(9)} \le v^{(9)}(t) \le (v_9)^{(9)}$

In the same manner , we get

$$v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{v}_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (\bar{v}_2)^{(9)}}}$$

From which we deduce $(v_0)^{(9)} \le v^{(9)}(t) \le (\bar{v}_1)^{(9)}$

If $0 < (v_1)^{(9)} < (v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}} < (\bar{v}_1)^{(9)}$ we find like in the previous case,

$$(v_1)^{(9)} \le \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}} \le v^{(9)}(t) \le$$

$$\frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (\bar{v}_1)^{(9)}$$

If $0 < (v_1)^{(9)} \le (\bar{v}_1)^{(9)} \le \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$ , we obtain

$$(v_1)^{(9)} \le v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (v_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(9)}(t)$ :-

$$(m_2)^{(9)} \le v^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{v^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

| | |
|---|---|
| $(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}$, $\boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$<br><br>Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.<br><br>**Particular case :**<br><br>If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .<br><br>Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$. | |
| We can prove the following<br><br>**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$<br><br>$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$<br><br>$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,<br><br>$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$<br><br>$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | 425 |
| **Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br><br>$with\ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br><br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br><br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br><br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$ | 431 |

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| $with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | |
| We can prove the following<br><br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br><br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br><br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br><br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$ | 435 |

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| $with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |

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| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27})\big)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\big((G_{27})\big)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\big((G_{27})\big)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |

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| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |

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| $(b_{42})^{(8)}T_{41} - [(b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br>$F(T) = (a'_{13})^{(1)}(a'_{14})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a'_{13})^{(1)}(a''_{14})^{(1)}(T_{14}) + (a'_{14})^{(1)}(a''_{13})^{(1)}(T_{14}) + (a''_{13})^{(1)}(T_{14})(a''_{14})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br>(c) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br>$\mathrm{F}(T_{19}) = (a'_{16})^{(2)}(a'_{17})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a'_{16})^{(2)}(a''_{17})^{(2)}(T_{17}) + (a'_{17})^{(2)}(a''_{16})^{(2)}(T_{17}) + (a''_{16})^{(2)}(T_{17})(a''_{17})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br>$F(T_{23}) = (a'_{20})^{(3)}(a'_{21})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a'_{20})^{(3)}(a''_{21})^{(3)}(T_{21}) + (a'_{21})^{(3)}(a''_{20})^{(3)}(T_{21}) + (a''_{20})^{(3)}(T_{21})(a''_{21})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br>$F(T_{27}) = (a'_{24})^{(4)}(a'_{25})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a'_{24})^{(4)}(a''_{25})^{(4)}(T_{25}) + (a'_{25})^{(4)}(a''_{24})^{(4)}(T_{25}) + (a''_{24})^{(4)}(T_{25})(a''_{25})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br>$F(T_{31}) = (a'_{28})^{(5)}(a'_{29})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a'_{28})^{(5)}(a''_{29})^{(5)}(T_{29}) + (a'_{29})^{(5)}(a''_{28})^{(5)}(T_{29}) +$ | 489 |

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| $(a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that | 496 |

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| there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_{i}^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_{i}^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_{i}^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{32}=\frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}=\frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_{i}^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{36}=\frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}=\frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |
| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_{i}^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*})=0$. With this value , we obtain from the three first equations | 501 |

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| $G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G) + (b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*}) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19}) + (b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br>$[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ | 506 |

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| is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution | 511 |

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| $G_{14}^{*}$ given by $\varphi(G^{*})=0$ , $T_{14}^{*}$ given by $f(T_{14}^{*})=0$ and<br>$G_{13}^{*}=\frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*}=\frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br>$T_{13}^{*}=\frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*}=\frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br>Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*})=0$ , $T_{17}^{*}$ given by $f(T_{17}^{*})=0$ and | 512 |
| $G_{16}^{*}=\frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*}=\frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*}=\frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*}=\frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*})=0$ , $T_{21}^{*}$ given by $f(T_{21}^{*})=0$ and<br>$G_{20}^{*}=\frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*}=\frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br>$T_{20}^{*}=\frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*}=\frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^{*}$ given by $\varphi(G_{27})=0$ , $T_{25}^{*}$ given by $f(T_{25}^{*})=0$ and<br>$G_{24}^{*}=\frac{(a_{24})^{(4)}G_{25}^{*}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*}=\frac{(a_{26})^{(4)}G_{25}^{*}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*}=\frac{(b_{24})^{(4)}T_{25}^{*}}{[(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*}=\frac{(b_{26})^{(4)}T_{25}^{*}}{[(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}((G_{27})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*})=0$ , $T_{29}^{*}$ given by $f(T_{29}^{*})=0$ and<br>$G_{28}^{*}=\frac{(a_{28})^{(5)}G_{29}^{*}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*}=\frac{(a_{30})^{(5)}G_{29}^{*}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |

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| $T_{28}^{*} = \frac{(b_{28})^{(5)}T_{29}^{*}}{[(b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)}T_{29}^{*}}{[(b_{30}^{'})^{(5)}-(b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br>$G_{32}^{*} = \frac{(a_{32})^{(6)}G_{33}^{*}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)}G_{33}^{*}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)}T_{33}^{*}}{[(b_{32}^{'})^{(6)}-(b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)}T_{33}^{*}}{[(b_{34}^{'})^{(6)}-(b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br>$G_{36}^{*} = \frac{(a_{36})^{(7)}G_{37}^{*}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)}G_{37}^{*}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br>$T_{36}^{*} = \frac{(b_{36})^{(7)}T_{37}^{*}}{[(b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)}T_{37}^{*}}{[(b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br>$G_{40}^{*} = \frac{(a_{40})^{(8)}G_{41}^{*}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)}G_{41}^{*}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br>$T_{40}^{*} = \frac{(b_{40})^{(8)}T_{41}^{*}}{[(b_{40}^{'})^{(8)}-(b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)}T_{41}^{*}}{[(b_{42}^{'})^{(8)}-(b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)}G_{45}^{*}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)}G_{45}^{*}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)}T_{45}^{*}}{[(b_{44}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)}T_{45}^{*}}{[(b_{46}^{'})^{(9)}-(b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | 523 A |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^*) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^*) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)}G_{13}^*\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^*\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^*\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^*\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^*\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^*\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)}$ , $\frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |

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| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}\mathrm{T}_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}\mathrm{T}_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}\mathrm{T}_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ $and$ $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_{+})$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)}$ , $\frac{\partial(b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_{j}\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_{j}\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_{j}\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_{+})$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 548 |

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| $\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^{*}) = (q_{25})^{(4)}$ , $\frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^{*}) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^{*}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^{*}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^{*}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^{*}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^{*}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^{*}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^{*}) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^{*}) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^{*}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^{*}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^{*}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^{*}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^{*}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^{*}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 563 |

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| **Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^*\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^*\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^*\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^*\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}((G_{39})^{**}) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |

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| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^{*}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^{*}\mathbb{G}_{j}\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^{*}\mathbb{G}_{j}\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^{*}\mathbb{G}_{j}\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^{*}) = (q_{41})^{(8)} \quad , \frac{\partial(b_i^{''})^{(8)}}{\partial G_j}\left((G_{43})^{*}\right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^{*}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^{*}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^{*}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^{*}\mathbb{G}_{j}\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b_{41}^{'})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T_{41}^{*}\mathbb{G}_{j}\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b_{42}^{'})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T_{42}^{*}\mathbb{G}_{j}\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |

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| | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \ \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b_{44}^{'})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T_{44}^*\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b_{45}^{'})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T_{45}^*\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b_{46}^{'})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T_{46}^*\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)$<br>$\left[\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(q_{13})^{(1)}G_{13}^*\right)\Big]$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T_{14}^* + (b_{14})^{(1)}s_{(13),(14)}T_{14}^*\Big)$<br>$+\left(\left((\lambda)^{(1)} + (a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^* + (a_{13})^{(1)}(q_{14})^{(1)}G_{14}^*\right)$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^* + (b_{14})^{(1)}s_{(13),(13)}T_{13}^*\Big)$<br>$\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$<br>$\left((\lambda)^{(1)}\right)^2 + \left((b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$<br>$+\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$<br>$+\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^*\right)$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^* + (b_{14})^{(1)}s_{(13),(15)}T_{13}^*\Big)\} = 0$ | |

$$+$$

$$((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)})\{((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)})$$

$$\left[\left(((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)})(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)\right]$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(17)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^{*}_{17}\right)$$

$$+\left(((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)})(q_{16})^{(2)}\mathrm{G}^{*}_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(16)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^{*}_{16}\right)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left(\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(18)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(18)}\mathrm{T}^{*}_{16}\right)\} = 0$$

$$+$$

$$((\lambda)^{(3)} + (b'_{22})^{(3)} - (r_{22})^{(3)})\{((\lambda)^{(3)} + (a'_{22})^{(3)} + (p_{22})^{(3)})$$

$$\left[\left(((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)})(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)\right]$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(21)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(21)}T^{*}_{21}\right)$$

$$+\left(((\lambda)^{(3)} + (a'_{21})^{(3)} + (p_{21})^{(3)})(q_{20})^{(3)}G^{*}_{20} + (a_{20})^{(3)}(q_{21})^{(1)}G^{*}_{21}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(20)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(20)}T^{*}_{20}\right)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((b'_{20})^{(3)} + (b'_{21})^{(3)} - (r_{20})^{(3)} + (r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left(\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(22)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(22)}T^{*}_{20}\right)\} = 0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\right)\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\right)\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\right)\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(8)}+(b_{42}^{'})^{(8)}-(r_{42})^{(8)}\right)\{\left((\lambda)^{(8)}+(a_{42}^{'})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a_{40}^{'})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(q_{40})^{(8)}G_{40}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(41)}T_{41}^{*}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a_{41}^{'})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G_{40}^{*}+(a_{40})^{(8)}(q_{41})^{(8)}G_{41}^{*}\right)$$

$$\left(\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(40)}T_{40}^{*}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((a_{40}^{'})^{(8)}+(a_{41}^{'})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((b_{40}^{'})^{(8)}+(b_{41}^{'})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$+\left(\left((\lambda)^{(8)}\right)^{2}+\left((a_{40}^{'})^{(8)}+(a_{41}^{'})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a_{40}^{'})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G_{40}^{*}\right)$$

$$\left(\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(42)}T_{40}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(9)}+(b_{46}^{'})^{(9)}-(r_{46})^{(9)}\right)\{\left((\lambda)^{(9)}+(a_{46}^{'})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a_{44}^{'})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G_{45}^{*}+(a_{45})^{(9)}(q_{44})^{(9)}G_{44}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(9)}+(b_{44}^{'})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T_{45}^{*}+(b_{45})^{(9)}s_{(44),(45)}T_{45}^{*}\right)$$

$$+\left(\left((\lambda)^{(9)}+(a_{45}^{'})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G_{44}^{*}+(a_{44})^{(9)}(q_{45})^{(9)}G_{45}^{*}\right)$$

$$\left(\left((\lambda)^{(9)}+(b_{44}^{'})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T_{45}^{*}+(b_{45})^{(9)}s_{(44),(44)}T_{44}^{*}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^{2}+\left((a_{44}^{'})^{(9)}+(a_{45}^{'})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^{2}+\left((b_{44}^{'})^{(9)}+(b_{45}^{'})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$+\left(\left((\lambda)^{(9)}\right)^{2}+\left((a_{44}^{'})^{(9)}+(a_{45}^{'})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a_{44}^{'})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G_{45}^{*}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G_{44}^{*}\right)$$

$\left(\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\right)\} = 0$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i \in R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i\ X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor

bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k \tag{3}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 3. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,...x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone $\{ x_i \epsilon C_i: \prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i(x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \in I, x \in K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\,\delta\,(K)= \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\mathcal{A}(\tilde{A}), C\backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^{n}_{i}$ defined by $(x,y).t = (x.t,B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^{+}$ with $u_i(x)|=|v_i(x)|=1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) in\ ||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population *i* is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int_0^t (A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\to\infty}{limsup}\frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \epsilon R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v=au_i(x)+w$ with $a \epsilon R$ and $w \epsilon F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |B_i(t,x)u_i(x)|(a+ \alpha+ exp(-\beta t)|w|)$. Hence, $\underset{t\to\infty}{limsup}\frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\underset{t\to\infty}{limsup}\frac{1}{t} In\ B_i(t,x)\ u_i(x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \underset{t\to\infty}{limsup}\frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a- \alpha+ exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \underset{t\to\infty}{limsup}\frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \underset{t\to\infty}{limsup}\frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In } b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) =0$ implies

$$\frac{d}{dt}b_i(t,x) = \lim_{s\to 0}\frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

=

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \le 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \le \gamma$ for all $t \ge 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \operatorname{In} |B_i(t,x)x_i| = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \operatorname{In} |\, (x.t)_i\, |$$

$$\le \operatorname{limsup}_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0. If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\ge 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\operatorname{limsup}_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \ge 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure*. $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$ , $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x) \rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1 \le j \le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |\mathrm{h}(x.t) - h(x^n.t)|d\mu_n(x)| + |\int_c h(x^n.t)|d\mu_n(x) - \int_c \mathrm{h}(x)d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x)d\mu_n(x) - \int_c h(x)d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $\mathrm{h}_i^n(x) = \langle \mathrm{A}_i^n(x)\mathrm{u}_i^n(x), \mathrm{u}_i^n(x)\rangle$ where $\mathrm{u}_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $\mathrm{A}_i^n(x) \to \mathrm{A}_i(x)$ uniformly for $x \in N_1((\mathrm{A}))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $\mathrm{u}_i^n(x) \to \mathrm{u}_i(x)$ converges uniformly for $x \in N_1(\Lambda(\mathrm{A}))$. Given $\in > 0$, choose $n$ sufficiently large so that $|\mathrm{h}_i^n(x) - \mathrm{h}_i(x)| \leq \in$ for all $x \in N_1(\Lambda(\mathrm{A}))$ and $|\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$\lambda_i(\mu) \leq \int_c |\mathrm{h}_i(x) - \mathrm{h}_i^n(x)|d\mu(x) + |\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)|$

$+ \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)$

$\leq \quad 2\in$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\, \mathrm{ds}$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument

implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i \, (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

(6) $\min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_{i} Pi\lambda i(\mu)$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$ $\sum_{i} P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_{1,}\ldots.,\tilde{A}_{k})$ by

$$\tilde{A}_i(x) = \begin{cases} \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} & if\ i \epsilon I \\ if\ i \notin I \end{cases}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t\ \ whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set $K$ if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$?then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$?and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume x* $\mapsto$ *A(x) is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists j* $\in$ { 1,….,k} *such that $M_i$ ----- { $x \in C : x_l \gg 0$ for $l \neq j$} Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_iM_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----{1,…,k} such that $\mu(---_i \in_I C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures μ with support in ∂C, $\lambda_i(\mu)>0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $$\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j \qquad i = 1,..,k \qquad j=1,\ldots,m$$

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge

to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j$ ($x^j_i$) = 0 i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product ($E_1^1$ ,…,$E_1^m$), and the matrix $A^2$ ($E_1$) is close to the diagonal matrix diag($f^i_2(E_1{}^j)$)$^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f_2^j(E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial$C is given by two equilibria {0,$E_1$}, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j\ x_2^j\ - \alpha^j\ x_3^j\ ) + \sum d_1^{jk}\ \ x_1^k\ - e_1^j\ x_1^j$

$\frac{dx_2^j}{dt} = x_2^j(1 - \alpha^j\ x_1^j\ - x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_2^{jk}\ \ x_2^k\ - e_2^j\ x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1 - \beta^j\ x_1^j\ - \alpha^j x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_3^{jk}\ \ x_3^k\ - e_3^j\ x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2\ (E_1)$and $\lambda_3\ (E_1)$are close to $\max_j\ 1 - a^j$ and $\max_j\ 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j\ 1 - a^j$ and $\max_j\ 1 - \beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1 - a^j) > -\max_j\ (1 - \beta^j)$$

Equivalently,

$$2 > \min_j\ a^j\ +\ \min_j\ \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set H= $\{x \in C: x_i^j = x_i^l \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i\left(r_i - \sum_s a_{is}x_s\right), \qquad i = 1,\dots,k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$$

It has been shown [12,ch.16.1,16.2] That for *k* =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \ \ with\ \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1 \to E_2 \to E_3 \to E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing

holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left[ \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1 - y) - b(N)\ y \right]$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*′ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\curlywedge_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\curlywedge_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d\ and$

$$\curlywedge_3(\mu) = \beta \int \frac{x_2}{\epsilon+x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} - y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords

Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation

of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1$ *(t)* and $P_2$ *(t)* denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial

alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.7) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.8) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.9) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_{1,}$ $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_{i,}$ $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic ($as$ $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 3. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2)\epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2}\epsilon\ C\ R^+ x\ k_1 x\ k_2, R^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2}\epsilon\ C\ R^+ x\ k_2 x\ k_1, R^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1, 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) \; P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^\infty f_i(a + s, P(t + S)) ds \right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) \, exp\left[ - \int_o^x f_1(a + s, p(t + s)) ds \right]$

For $a \geq t$ *we choose* $(a,t) = (a-t,0)$ in (1.3) and $x = t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0) \exp\left[ - \int_o^t f_1\left(a - t + s, p_1(s)\right) ds \right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(\text{a+s}, p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da+\int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$$t> 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in[-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.(a, t, P_i(t))da$

$$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$$

$$i = 1, 2, t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t)\, \phi(0) d_i^*]t \quad for\ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T],R^+), \quad B_i \in C([0,T],R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau,T] = \{f \in C([-\tau,T],R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau,T]$. Then for any

$(x, y) \in C^+[-\tau,T] \times C^+[-\tau,T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau,T], \quad C_T^2 = C^+[-\tau,T] \times C^+[-\tau,T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau,T]$ *and* $t \in [0,T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$B: C_T^2 \to C_T^2$ *and* $\Pi C_T^2 \to C_T^2$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = P_t \left\{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

$$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right)$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Pi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Pi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Pi(P_t^{(1)}) - \Pi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Pi(P^{(1)}(t)) - \Pi(P^{(2)}(t)| \le \int_0^t B_1^{(1)}(a) \left\{ M_1\left(a, t, P_t^{(1)}\right) - M_1\left(a, t, P_t^{(2)}\right) \right\} da$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left( \int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_* ds \right) da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o) \left[ \int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds \right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \|_t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$=\int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\mathrm{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \, /\!/ P_t^{(1)} - P_t^{(2)} /\!/ \text{ for } a \, \varepsilon [o, t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \leq \beta_1 \Phi_1(0) ||P_t^{(1)} - P_t^{(2)}||_t^* + \Phi_1(0) \beta_1 d_1 ||P_t^{(1)} - P_t^{(2)}||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) \, e^{\delta 1^a} \beta_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) \, e^{\delta 1^a} /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| B_1^{*d}\sigma$$

$$+ \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \left\{ B_1 + B_2 d_1 a + B_1^* \left(B_2 a + d_1 a^2/2\right) e^{\delta 1^a} \right\}$$

$$\leq \beta_1^* \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| do + \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \to \beta_1$ as $a \to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{g_1(a) + 0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a)) - b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a)) do \right|$$

$$\left| B_1^* \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi\left(P_t^{(1)} - \pi\left(P_t^{(2)}\right) /\!/ \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^*)$$

As $t^* \to o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp P_2^*(a)}{da} + f_i(a, P_1^*, P_2^*) P_i^*(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^* = \int_o^\infty P_i^*(a) da, i = 1,\ 2,$$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i=1.\ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s,P_1^*,P_2^*)ds\right], i=1,\quad 2, a\geq 0,$

Which together with (3.3) lead to

$$1=\int_o^\infty b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^\infty b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1,P_2) = \int_o^\infty b_i(a,P_1,P_2)\exp\left[-\int_o^\infty f_i(s,P_1,P_2)ds\right]da,$

$i=1,\ 2,\quad P_1\geq 0,\quad P_2\geq 0,$

**And suppose the following hold:**

(3.7) $A_4.\quad F_{i.}\ (0,\ 0)>1,\ i=1,2$:

$A_5.$ *there exists a positive constant c for which*

$$F_1(C,0)< F_{2.}(C,0),$$

(3.8) $$F_1(0.C)< F_{2.}(0,C),$$

Then there exists a unique pair $(P_1^*,P_2^*)$ of real numbers $P_1^*>o,$

$P_2^*>0$ such that

(3,9) $F_1(P_1^*,P_2^*)=1=F_2(P_1^*,P_2^*)$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a)=[P_2^*\exp[-f_i(s,P_1^*,P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s,P_1^*,P_2^*)ds]da),$

$i=1,2,\quad a\geq 0,$

Proof Consider the elements of the surfaces defined by

Z= $F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0,$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

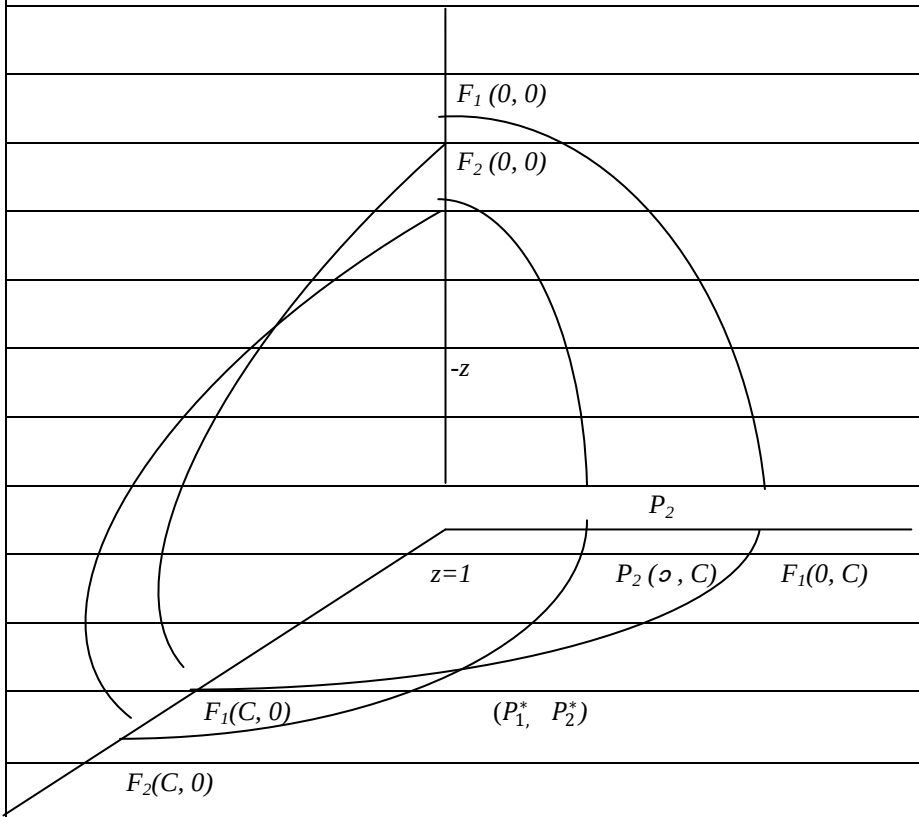
$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0,$ These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, F_1(P_1, P_2 \geq 0,$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11)
$$\begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square=1}^{2} \frac{\partial b_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u\square(a)\exp[\lambda t], \quad \square(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty \square(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d\square}{da} + \lambda\square = -f_i\left(a, p_1^*, p_2^*\right)\square(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_{i\square}^*\right\},$$

(3.15)

$$\square(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\square(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*} p_{i\square}^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^{*} = p_{1}^{*}, \quad p_{12}^{*} = p_{2}^{*} \exp\{-\lambda\tau\},$$

$$p_{21}^{*} = p_{1}^{*}, \exp\{-\lambda\tau\}, \ p_{22}^{*} = p_{2}^{*}.$$

If we integrate the first of (3.15) and use the fact that $\varphi_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\varphi_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\varphi_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\varphi_i(a) = \{\varphi_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$

$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\varphi_i\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\varphi_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\varphi_i\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\varphi_i\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\varphi_i\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)}\int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}da, \quad i = 1,\ 2,$$

Now supplying $\varphi_1(a)$ and $\varphi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^*\exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^*\exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^* \exp(-\lambda\tau) + (\lambda - A_{22})\, p_2^* = g_{21}(\lambda)\, p_1^* \exp(-\lambda\tau) + g_{22}(\lambda)\, p_2^*$$

Where

(3.22) $$A_i = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_i^*} - \frac{\partial f_i}{\partial p_i^*}\right]da,$$

(3.23) $$g_i(\lambda) = \left[p_i^*(0)c_i\right]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\, da\right\}$$
$$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_i^*} - b_i(a, p_1^*, p_2^*)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_i^*}\, e^{\lambda s}\, ds\right)\right]da\right\}$$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_i^*}\, e^{\lambda s}\, ds\right)da, \quad i = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*), \quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})\, e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$$S_2(\lambda, \tau) = \left[A_{21}\, g_{12}(\lambda) + A_{12}\, g_{21}(\lambda) + g_{12}(\lambda)\, g_{21}(\lambda)\right]\exp[-2\lambda\tau]$$

$$-A_{11}\, g_{22}(\lambda) - A_{22}\, g_{11}(\lambda) - g_{11}(\lambda)\, g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6$. $0 < \beta_i^* < d_i^*$ $(i = 1, 2.)$ $(see(A_1) and (A_2))$;

$A_7$. $\{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \text{[illegible]}(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - \left| F_2(\lambda, \tau) \right|$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \,| - |A_{11} + A_{22} + \| S_1(0) \| \,|^2 \right\}/2,$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational

ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the

Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have

obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is:

individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

**Scientists cast doubt on Heisenberg's uncertainty principle Date: September 7, 2012 Source: University of Toronto**

Summary:

(1) Werner Heisenberg's uncertainty principle, formulated by the theoretical physicist in 1927, is one of the cornerstones of quantum mechanics. The principle has bedeviled quantum physicists for nearly a century, until recently, when researchers demonstrated the ability to directly measure the disturbance caused by measuring a property of something, and confirm that Heisenberg was too pessimistic. Heisenberg's gamma-ray microscope for locating an electron (shown in blue) (figure deleted). The incoming gamma ray (shown in green) is scattered by (e) the electron up into the microscope's aperture angle. The scattered gamma-ray is shown in red. Classical optics shows (eb) that the electron position can be resolved only up to an uncertainty”x that depends on ¸ and the wavelength » of the incoming light. Credit: By parri (Wikimedia commons) [GFDL or CC-BY-SA-3.0], via Wikimedia Commons

(2) Werner Heisenberg's uncertainty principle, formulated by the theoretical physicist in 1927, is one of the cornerstones of quantum mechanics. In its most familiar form, it says that it is impossible to measure anything without (e) disturbing it.

(3) For instance, any attempt to measure a particle's position must randomly change (e&eb) its speed.

(4) The principle has bedeviled quantum physicists for nearly a century, until recently, when researchers at the University of Toronto demonstrated the ability to directly measure the disturbance and confirm (eb) that Heisenberg was too pessimistic. "We designed an apparatus to measure a property -- the polarization -- of a

single photon. We then needed to measure how much that apparatus disturbed (e&eb) that photon," says Lee Rozema, a Ph.D. candidate in Professor Aephraim Steinberg's quantum optics research group at U of T, and lead author of a study published this week in Physical Review Letters. "To do this, we would need to measure the photon before the apparatus but that measurement would also disturb (e&eb) the photon," Rozema says.

(5) In order to overcome this hurdle, Rozema and his colleagues employed a technique known as weak measurement wherein the action of a measuring device is (=) weak enough to have (e) an imperceptible impact on what is being measured.

(6) Before each photon was sent to the measurement apparatus, the researchers measured it weakly and then measured it again afterwards, comparing the results.

(7) They found that the disturbance induced by (e) the measurement is less than Heisenberg's precision-disturbance relation would require.

(8) "Each shot only gave us (=) a tiny bit of information about the disturbance, but by (e) repeating the experiment many times we were able to get (eb) a very good idea about how much the photon was disturbed," says Rozema.

(9) The findings build on recent challenges to Heisenberg's principle by scientists the world over. Nagoya University physicist Masanao Ozawa suggested in 2003 that Heisenberg's uncertainty principle does not apply to measurement, but could only suggest (eb) an indirect way to confirm his predictions.

(10) A validation of the sort he proposed was carried out last year by Yuji Hasegawa's group at the Vienna University of Technology. In 2010, Griffith University scientists Austin Lund and Howard Wiseman showed that weak measurements could be used to characterize the process of measuring a quantum system. However, there were still hurdles to clear as their idea effectively required a small quantum computer, which is difficult to build.

(11) "In the past, we have worked experimentally both on implementing weak measurements, and using (e) a technique called 'cluster state quantum computing' to simplify building quantum computers.

(12) The combination of these two ideas led to (eb) the realization that there was a way to implement Lund and Wiseman's ideas in the lab," says Rozema.

(13) It is often assumed that Heisenberg's uncertainty principle applies to (e&eb) both the intrinsic uncertainty that a quantum system must possess, as well as to measurements.

(14) These results show that (eb) this is not the case and demonstrate the degree of precision that can be achieved with weak-measurement techniques.

(15) "The results force us to adjust our view of exactly what limits quantum mechanics places on (e&eb) measurement," says Rozema. "These limits are important to fundamental quantum mechanics and also central in developing 'quantum cryptography' technology, which relies on (e) the uncertainty principle to guarantee that any eavesdropper would be detected due to (e) the disturbance caused by her measurements."

(16) "The quantum world is still full of uncertainty, but at least our attempts to look at it don't have to (e) add as much uncertainty as we used to think!" The findings are reported in the paper "Violation of Heisenberg's Measurement-Disturbance Relationship by Weak Measurements." The research is supported by funding from Natural Sciences and Engineering Research Council of Canada and the Canadian Institute for Advanced Research. The above post is reprinted from materials provided by University of Toronto. Note: Materials may be edited for content and length. University of Toronto. "Scientists cast doubt on Heisenberg's uncertainty principle." Science Daily. ScienceDaily, 7 September 2012. www.sciencedaily.com/releases/2012/09/120907125154.htm

**Physicists find a way to probe the quantum realm without wrecking everything: Finally...! BRENDAN COLE 15 APR 2016 Facebook Icon10.6k Twitter Icon**

(17) In 1930, German theoretical physicist Werner Heisenberg came up with a thought experiment, now known as Heisenberg's microscope, to try to show why it's impossible to (e) measure an atom's location with unlimited precision.

(18) He imagined trying to measure the position of something like an atom by shooting light at it. Light travels as a wave, and Heisenberg knew that different wavelengths could give you (eb) different degrees of confidence when used to (e) measure where something is in space.

(19) Short wavelengths can give (eb) a more precise measurement than long ones, so you'd want to use (e) light with (e&eb) a tiny wavelength to measure where (e) an atom is, since atoms are really small.

(20) But there's a problem: light also carries (e&eb) momentum and short wavelengths carry (e&eb) more momentum than long ones.

(21) That means if you use light with a short wavelength to find the atom, (eb)you'll hit the atom with all of that momentum, and that kicks it around and risks completely changing (e&eb) its location (and other properties) in the process.

(22) Use longer wavelengths, (eb) and you'll move the atom less, (eb) but you'll also be more uncertain about your measurement.

(23) You're in a bind: any measurement changes (e&eb) what you're measuring and better measurements lead to (eb) bigger changes.

(24) It's also possible to prepare atoms in what's called an entangled state, which means (eb) they act cooperatively like a single atom, no matter how far away they are from (e) each other.

(25) If you push one, (eb) the rest move like you pushed them all individually. And if you mess up one atom by shooting some light at it, (eb) you generally mess up the whole collection.

(26) In the past, these two effects made it impossible to (e) measure how entangled atoms are arranged without (e) destroying the arrangement and the entanglement – which were presumably prepared for (e) some specific purpose, like to make a quantum computer.But now, physicists led by T. J. Elliott from the University of Oxford in the UK have proposed a way to measure large-scale properties of a group of entangled atoms without messing up the entanglement. It isn't measuring individual atoms – that's permanently off-limits – but it's more than physicists have managed to do before.

(27) Usually, when physicists entangle atoms, they have to be (=) careful that the atoms are all more or less the same when they start out.

(28) If there are lots of different kinds of atoms in there, they become (=) a lot harder to match up, so the entanglement becomes (=) more fragile.

(29) But it's still possible to make stable groups of entangled atoms that have (e) some outliers among them that are unlike the main group, and the paper's authors have shown that (eb) these outliers can be used to (e) measure things about the main group without (e) messing up their entanglement.

(30) This includes really basic information like the density of atoms – how closely they are to one another – while they're entangled, which historically has been out of physicists' reach in individual experiments.

(31) Before, physicists had to measure a whole bunch of the entangled atoms really quickly, and they'd have to (e) accept that they were changing things around as soon as (e) they measured that first atom.

(32) More measurements might check more atoms, (eb) but they'd be increasingly uncertain as time went on.

(33) Now, all they have to do is measure (eb) what the outliers are doing and they can figure out how the atoms are distributed without (e) the chaos.

(34) Within some limits, (eb) knowledge about the atoms' density gets better – not worse – as more measurements are made.

(35) Admittedly, measurements still change things a little bit (light is still being used and Heisenberg's microscope still applies) but the measurements won't (e) wreck the whole system like they would have before.

(36) This method of measuring the outliers is (=) a window into a new realm for physicists, who could previously only see what entangled atoms did, not what they're doing.

(37) The researchers simulated a simple system as a proof-of-concept, but they showed (eb) mathematically that this should work with a wide range of quantum systems where (e) entanglement plays a key role.

(38) And small changes to the method could make (eb) it possible to measure (eb) properties like the magnetisation of entangled atoms, instead of (e) just their density. All with atoms that shouldn't be in the group in the first place. Not bad, physicists. Not bad.

(39) An international team of scientists has managed to create a quantum knot for the first time - a fundamental breakthrough in quantum physics that could one day help (eb) power the supercomputers of the future. These knots aren't quite the same as the ones you might tie to moor a boat to a jetty - they've been made in a superfluid form of quantum matter called Bose-Einstein Condensate, or BEC, and are more like smoke rings than traditional knots.

(40) "For decades, physicists have been theoretically predicting (eb) that it should be possible to have (e) knots in quantum fields, but nobody else has been able to make one," said lead researcher, Mikko Möttönen.

(41) "Now that we have seen these exotic beasts, we are (=) really excited to study their peculiar properties. Importantly, our discovery connects to a diverse set of research fields including cosmology,(e&eb) fusion power, and (e&eb) quantum computers."

(42) These knots of BEC, called knotty solitons, form (eb) particle-like rings or loops that roll forward at (eb) a constant speed without deforming.

(43) When the BEC superfluid is first created, (eb) it's made up of a series of points in space all having (e) a specific orientation.

(44) The trick of the knotting technique is to change (e&eb) that orientation using (e) very precise magnetic fields - so precise, the scientists conducting the experiment were afraid to move at all.

(45) With it so far? Even if the science is difficult to grasp, the end result is simple enough: tiny knots composed of (e) an interlocking series of circles that look a little like smoke rings or doughnuts.

(46) These quantum knots are also topologically stable, which means (eb)they can't be undone and must be broken instead

(47) . If you think about a shoelace analogy, these particular types of shoelaces can't be pulled apart with your fingers, and have to be (=) cut off with scissors instead.

(48) The tying of the knot takes less than a thousandth of a second, according to Möttönen's research partner, David Hall. "After we learned how to tie the first quantum knot, we have become rather good at it," he said. "Thus far, we have tied several hundred such knots." This is high-level, mind-blowing stuff, and it could be a long time before the results trickle through to everyday life. The researchers believe the discovery might one day help inform the development of quantum computers, where qubits could be braided into different types of knots depending on the task at hand.

**The research has been published in Nature Physics. Tying quantum knots D. S. Hall, M. W. Ray, K. Tiurev, E. Ruokokoski, A. H. Gheorghe & M. Möttönen AffiliationsContributionsCorresponding author Nature Physics 12, 478–483 (2016) doi:10.1038/nphys3624 Received 27 July 2015 Accepted 01 December 2015 Published online 18 January 2016**

(49) As topologically stable objects in field theories, knots have been put forward to (e) explain various persistent phenomena in systems ranging from atoms and molecules to (e&eb) cosmic textures in the universe.

(50) Recent experiments have reported the observation of knots in different classical contexts. However, no experimental observation of knots has yet been reported in quantum matter. Here authors demonstrate the experimental creation and detection of (e) knot solitons in the order parameter of (e) a spinor Bose–Einstein condensate.

(51) The observed texture corresponds to (e&eb) a topologically nontrivial element of the third homotopy group and exhibits (eb) the celebrated Hopf fibration, which unites (e&eb)many seemingly unrelated physical phenomena. Work calls for future studies of the stability and dynamics knot solitons in (eb) the quantum regime.

**Nondestructive probing of means, variances, and correlations of ultracold-atomic-system densities via qubit impurities T. J. Elliott and T. H. Johnson Phys. Rev. A 93, 043612 – Published 11 April 2016**

(52) Authors show how impurity atoms can measure (eb) moments of ultracold-atomic-gas densities, using (e) the example of bosons in (eb) a one-dimensional lattice.
(53) This builds on a body of work regarding the probing of systems by (e) measuring the dephasing of an immersed qubit.
(54) Authors show that this dephasing is captured by (e) a function resembling characteristic functions of probability theory, of (e) which the derivatives at short times reveal moments of (e) the system operator to (e) which the qubit couples.
**(55)** For a qubit formed by an impurity atom, in a system of ultracold atoms, this operator can be (=) the density of the system at the location of the impurity, and thus means,(eb) variances, and correlations of the atomic densities are accessible.

**@@@

He was more articulate!

Social and psychological interactions gormandize and gobble up topological spaces and vice versa!

That is the talk of higher consciousness!

But you crowed much about the Kahmiri Shaivism and Stanzasa say that there is no difference between one quantum state and the other!

Ugh! That is in pure consciousness!

Pure consciousness does not evolve!

So self doth lead to more than witness consciousness!

Cetainly!

It is a vacuum?

No! It can be a set of dynamic axiomatic alcovishness for evolution!

You must have a coincidence with the same object or the subject!

I donot understand!

That which lacks coincidence with itself is a lackey of some lesser energy level!

You said there is no lesser energy level!

Ity is in pure consciousness!

In Dvaita, in space time they exist!

It is phenomenal and impure!

Like pure silence produces impure sound and impure silence produce pure sound!

That is a different case of insentient consciousness!

Matter and consciousness are entwined with each other!

Like Words and sentences are entwined with each other so that they annot be disentangled!

Nietzsche argues against the "misleading significance of (e) words" and its production (eb) of a "thing in itself":

Time is the only way of measurement!

That is what I should conclude after reading some Sanskrit stanzas!

Reflection is important!

Zero alos reflects!

Result is the phenomenal world and transactions and interations!

It is this reflection that gives value to the present state of a parameter!

Yeah!

Value is omnipresent but still never present!

It is almost like going back in time!

You must know the vaule of something before you reflect!

Bernard sahw once said of the kind and type confirmed!

Troll it and move on!

What I mean to say is something might exist when you are an ignoramus or an omniscient!

So?

Why not introduce another space or axes like indeterminate space and indeterminate axes!

Root of -1 is not defined!

So also infinity divided by infinity!

Other system of possibilities exists when Schrodinger's wave function did not collapse at all!

Scott's argument on expander graph!

"If reality impacted directly on our senses and our consciousness, if we could have direct communication between the material world and ourselves, art would be unnecessary", Bergson, ""

You mean it is necessary, sufficient and noetic condtion for the consciousness to exist!

Possibilities and potentialities in which the Schrodinger's wave function can collapse cannot be reduced to subjective reality!

Aw! Like it is both opportunity and a condition!

How can the opportunity be a condition!

Things and feelings are different!

Feelings could be about things!

There cannot be anything about feeling!

How is it possible!

Like opportunity is a condition!

Thought cannot be hidden if the value is to be proposed!

Yes!

Thought is like silence; value is like a noise!

Silence can be heard noise can be overheard!

Utch!

Final verdict is never said!

Form and content can never bestow perception and intentionality!

There is nothing to be serious about!

All world is just addition, subtraction, differentiation and integration!

Maxwell's equations are elegant!

I mean you have to concentrate on the nabla times relations one with the other!

It has to be!

OK!

It is not the equation but an escape from equation!

It resolved lot of headaches!

Which is better escape or total surrender?

Both are like walking on the edge of a sword!

It must be blade!

I am talking about exile not sacrilege in a piligrimage!

Subjective reality exists!

Only subjective realityexists!

Universe is a simulation!

Possibilities of collapse of Schrodinger's wave function are not same of the figment of imagination and product of prognostication of possibilities and potentialities!

Heh! Heh! It is like telling tesion is different from extension!

When the God-forsaken worldliness of earthly life shuts itself in complacency, (eb) the confined air develops poison, the moment gets stuck and stands still, the prospect is (=) lost, a need is felt for (e) a refreshing, enlivening breeze to cleanse the air and dispel (eb) the poisonous vapors lest we suffocate in worldliness. ... Lovingly to hope all things is (=) the opposite of despairingly to hope nothing at all. Love hopes all things – yet is never put to shame. To relate oneself expectantly to (e&eb) the possibility of the good is to hope. To relate oneself expectantly to (e&eb) the possibility of evil is to fear. By the decision to choose (e&eb) hope one decides infinitely more than it seems, because (e) it is an **eternal decision. p. 246-250**

Inexorable insouciance has not left yourself!

As sullen ambience has not yourself!

Like gravity and antigravity!

Like Terrestial Brahman Anti Brahman and Celestial Brahman Anti Brahman!

Relief is anot accentuation or addition!

It is lessening of deprivation!

Referential integrity and heightened awareness are necessary to understand what gratification is and what deprivation is!

Process of phase locking!

They are not phases; one leads to the other!

You mean they are transitional!

Yes!

Whatever you say there is an effervescenece in topsi turveying on the waves of gratification and deprivation!

You might not be there to enjoy other wave!

Crotchets and conundrums as we pass by years!

One is maintenance and other is management!

Like a black hole is usually formed from the collapse of a quantity of matter or radiation, both of which carry entropy!

Somehow there has to be blackholishness a chaos to articulate is my final verdict!

That is turning apprehensions in to speculation!

Very human desire to briong the forthcoming chaos at the earliest!

It is like using horsepower to measure consciousness!

Make adolescence the bench mark!

Ugh!

Unfailingness uniformity and stability staunchness!

Truths govered by principles have chances to be contradictory!

"The subjective thinker's form, the form of his communication, is (=) his style. His form must be just as manifold as (=) are the opposites that he holds together. The systematic eins, zwei, drei is (=) an abstract form that also must inevitably run into trouble whenever it is to be applied to (e&eb) the concrete. To the same degree as the subjective

thinker is concrete, (eb)to the same degree his form must also be concretely dialectical. But just as he himself is not a poet, not an ethicist, not a dialectician, (eb) so also his form is none of these directly. His form must first and last be related to (e&eb) existence, and in this regard he must have (e) at his disposal the poetic, the ethical, the dialectical, the religious. Subordinate characters, setting, etc., which belong to (e) the well balanced character of (e) the esthetic production, are in themselves breadth; the subjective thinker has (e) only one setting—existence—and has (e) nothing to do with localities and such things. The setting is (=) not the fairyland of the imagination, where (e) poetry produces consummation, nor is the setting laid in England, and historical accuracy is (=) not a concern. The setting is inwardness in existing as (=) a human being; the concretion is the relation of (e) the existence-categories to (e) one another. Historical accuracy and historical actuality are breadth." **Søren Kierkegaard (Concluding Postscript, Hong p. 357–358: Wikipedia). Paragraph is in model able form**

Why then are the phenomenal methodology, predicational anteriorities, accolytish representation, atrophied asservation, axiomatic predications and postulation alcovishness!

Without that you shall fall back in to a dreary apparition like state!

Questions About The Topology Of Time Appear To Be Closely Connected To (e&eb) The Issue Of Platonism Versus (e&eb) Reductionism With Respect To Time. For If Reductionism Is True, (eb) And Then It Seems Likely That Time's Topological Features Will Depend On (=) Contingent Facts About The Relations Among Things And Events In (eb) The World, Whereas If Platonism Is True, So That (e) Time Exists Independently Of Whatever Is In Time, Then (eb) Time Will Presumably Have (e) Its Topological Properties As A Matter Of Necessity. But Even If We Assume That Platonism Is True, (eb) It's Not Clear Just What Topological Properties Should Be Attributed To Time. Reductionism And Platonism With Respect To Time (Mc Garette's Theories Source: Stanford Encyclopedia)

Time would have topological properties!

Yes! If space time atoms exist!

**TITLE**

## Pure "Subjectivism" Or "Relativism."

**INTRODUCTION—VARIABLES USED**

**66 www.parrhesiajournal.org DELEUZE AND THE QUESTION OF DESIRE: TOWARD AN IMMANENT THEORY OF ETHICS: Daniel W. Smith. Some sentences have been deleted due to spatial restrictions. Kindly pardon me. All these can be modelled on similar lines and are preeminently and predominantly cast in the model format. Models are given in the one of the papers of the series. My commentary is given on each sentence wise.**

(1) The title of this paper rises two questions, each of which I would like to address in turn. The first question is: What exactly is an immanent ethics (as opposed to an ethics that appeal to transcendence)? The second question is: What is the philosophical question of (e) desire? My ultimate question concerns the link between these two issues: What relation does an immanent ethic have to (e&eb) the question of desire?

(2) Historically, the first question is primarily linked with the names of Spinoza and Nietzsche (as well as, as we shall see, Leibniz), since it was Spinoza and Nietzsche who posed the question of an immanent ethics in its most rigorous form.

(3) The second question is linked to names like Freud and Lacan, and behind them, to Kant, since it was they who formulated the modern conceptualization of (e) desire in its most acute form—that is, in terms of (e&eb)unconscious desire, desire as (=) unconscious.

(4) It was in Anti-Oedipus, published in 1972, that Deleuze (along with Félix Guattari, his co-author) would attempt to formulate his own theory of desire—what he would call a purely immanent theory of desire.

(5) In his preface to Anti-Oedipus, Michel Foucault would claim, famously, that "Anti-Oedipus is a book of ethics, the first book of ethics to be written in France in quite a long time"—thereby making explicit the link between the theory of desire developed in Anti-Oedipus with (e&eb) the immanent theory of ethics Deleuze worked out in his monographs on Nietzsche and Spinoza.

(6) The paper falls into three parts. In the first, I want to make some general comments about the nature of an immanent ethics.

(7) In the second part, I would like to examine in some detail two sets of texts from Nietzsche and Leibniz, which will flesh out some of the details of an immanent ethics.

(8) And I'll conclude with some all-too-brief comments on the nature of desire and some of the themes of Anti-Oedipus.

**ON THE NATURE OF AN IMMANENT ETHICS**

(9) Let's turn to the first question, then: What is an immanent ethics?

(10) Throughout his writings, Deleuze has often drawn a distinction between "ethics" and (e&eb) "morality"—a distinction that has traditionally been drawn to (e) distinguish modes of reflection that place greater emphasis (eb) respectively, on the good life (such as Stoicism) or (eb) on the moral law (such as Kantianism). Deleuze, however, uses the term "morality" to define, (eb) in very general terms, any set of "constraining" (e) rules, such as a moral code, that consists in judging (e&eb) actions and intentions by relating them to (e&eb) transcendent or universal values

**This is good that is Evil**

(11) What he calls "ethics" is (=) on the contrary, a set of "facilitative" [facultative] rules that evaluates (e&eb) what we do, say, and think according to the immanent mode of existence that it implies.

(12) One says or does this, thinks or feels that: what mode of existence does it imply? "We always have (e) the beliefs, feelings, and thoughts we deserve," writes Deleuze, "given our way of being or our style of life." This can only be interpreted of underdevelopment of individual consciousness for discretion, circumspection and jurisprudence for what is right or wrong. Deleuze too was a product of his space and time unlike Einstein who saw beyond.

(13) Now according to Deleuze, this immanent approach to the question of ethics was developed most fully, in the history of philosophy, by Spinoza and Nietzsche, whom Deleuze has often identified as his own philosophical precursors. Both Spinoza and Nietzsche—perhaps not surprisingly— were both maligned by their contemporaries not simply for being atheists, but even worse, for being "immoralists." A potent danger, in other words, was immediately seen to be lurking in Spinoza's Ethics and Nietzsche's Genealogy of Morals: without transcendence, without recourse to (e) normative universals, we will all fall into (eb) the dark night of chaos, and ethics will be reduced to (e&eb) a pure "subjectivism" or "relativism."

(14) Both Spinoza and Nietzsche argued, each in his own way, that there are things one cannot do or think except on the condition of being weak, base, or enslaved, unless (e) one harbors a vengeance or ressentiment against life (Nietzsche), unless (e)_ one remains the slave of passive affections (Spinoza); and there are other things one cannot do or say except (e) on the condition of being

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| strong, noble, or free, unless one affirms life, unless (e) one attains active affections.<br>(15) Deleuze calls this the method of "dramatization": actions and propositions are interpreted as (=) so many sets of symptoms that express (eb) or "dramatize" the mode of existence of the speaker.<br>**(16)** "What is the mode of existence of the person who utters a given proposition?" asks Nietzsche, "What mode of existence is needed in order to be able to utter it?"7 Rather than "judging" actions and thoughts by appealing to transcendent or universal values, one "evaluates" them by determining the mode of existence that serves as their principle. A pluralistic method of explanation by immanent modes of existence is in this way made to replace the recourse to transcendent values: in Spinoza and Nietzsche, the transcendent moral opposition (between Good and Evil) is replaced an immanent ethical difference (between noble and base modes of existence, in Nietzsche; or between passive and active affections, in Spinoza). In Spinoza, for instance, an individual will be considered "bad" (or servile, or weak, or foolish) who remains cut off from its power of acting, which remains in a state of slavery with regard to its passions | |
| **NOTATION** | |
| **Module One**<br>The title of this paper rises two questions, each of which I would like to address in turn.<br>The first question is: What exactly is an immanent ethics (as opposed to an ethics that appeal to transcendence)?<br>The second question is: What is the philosophical question of (e) desire?<br>My ultimate question concerns the link between these two issues: What relation does an immanent ethic have to (e&eb) the question of desire? | |
| $G_{13}$ : Category one of immanent ethics; **transcendence ethics**<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of **transcendence ethics;** immanent ethics<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>What relation does an immanent ethic have to (e&eb) the question of desire? | |
| $G_{16}$ : Category one of **immanent ethic;** question of desire<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of question of desire **;immanent ethic**<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three** | |

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| The second question is linked to names like Freud and Lacan, and behind them, to Kant, since it was they who formulated the modern conceptualization of (e) desire in its most acute form—that is, in terms of (e&eb) unconscious desire, desire as (=) unconscious | |
| $G_{20}$ : Category one of desire<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of unconscious<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>Throughout his writings, Deleuze has often drawn a distinction between "ethics" and (e&eb) "morality"—a distinction that has traditionally been drawn to (e) distinguish modes of reflection that place greater emphasis (eb) respectively, on the good life (such as Stoicism) or (eb) on the moral law (such as Kantianism). Deleuze, however, uses the term "morality" to define, (eb) in very general terms, any set of "constraining" (e) rules, such as a moral code, that consists in judging (e&eb) actions and intentions by relating them to (e&eb) transcendent or universal values | |
| $G_{24}$ : Category one of**"ethics" ;** "morality"—<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of "morality"—; **"ethics"**<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>Throughout his writings, Deleuze has often drawn a distinction between "ethics" and "morality"—a distinction that has traditionally been drawn to (e) distinguish modes of reflection that place greater emphasis (eb) respectively, on the good life (such as Stoicism) or (eb) on the moral law (such as Kantianism). Deleuze, however, uses the term "morality" to define, (eb) in very general terms, any set of "constraining" (e) rules, such as a moral code, that consists in judging (e&eb) actions and intentions by relating them to (e&eb) transcendent or universal values | |
| $G_{28}$ : Category one ofdistinguish modes of reflection that place greater emphasis (eb) respectively, on the good life (such as Stoicism) or (eb) on the moral law (such as Kantianism). Deleuze, however, uses the term "morality" to define, (eb) in very general terms, any set of "constraining" (e) rules, such as a moral code, that consists in judging (e&eb) actions and intentions by relating them to (e&eb) transcendent or universal values<br>$G_{29}$ : Category two of SAS | |

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| $G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of distinction between "ethics" and "morality"—a distinction that has traditionally been drawn<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Throughout his writings, Deleuze has often drawn a distinction between "ethics" and "morality"—a distinction that has traditionally been drawn to distinguish modes of reflection that place greater emphasis (eb) respectively, on the good life (such as Stoicism) or (eb) on the moral law (such as Kantianism). Deleuze, however, uses the term "morality" to define, (eb) in very general terms, any set of "constraining" (e) rules, such as a moral code, that consists in judging (e&eb) actions and intentions by relating them to (e&eb) transcendent or universal values | |
| $G_{32}$ : Category one of distinction between "ethics" and "morality"—a distinction that has traditionally been drawn to distinguish modes of reflection that place greater emphasis<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one ofgood life (such as Stoicism) or (eb) on the moral law (such as Kantianism). Deleuze, however, uses the term "morality" to define, (eb) in very general terms, any set of "constraining" (e) rules, such as a moral code, that consists in judging (e&eb) actions and intentions by relating them to (e&eb) transcendent or universal values<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Throughout his writings, Deleuze has often drawn a distinction between "ethics" and "morality"—a distinction that has traditionally been drawn to distinguish modes of reflection that place greater emphasis respectively, on the good life (such as Stoicism) or on the moral law (such as Kantianism). Deleuze, however, uses the term "morality" to define, (eb) in very general terms, any set of "constraining" (e) rules, such as a moral code, that consists in judging (e&eb) actions and intentions by relating them to (e&eb) transcendent or universal values | |
| $G_{36}$ : Category one of **distinction between "ethics" and "morality"—a distinction that has traditionally been drawn to distinguish modes of reflection that place greater emphasis respectively, on the good life (such as Stoicism);** moral law (such as Kantianism). Deleuze, however, uses the term "morality" to define, (eb) in very general terms, any set of "constraining" (e) rules, such as a moral code, that consists in judging (e&eb) actions and intentions by relating them to (e&eb) transcendent or universal values | |

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| $G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of moral law (such as Kantianism). Deleuze, however, uses the term "morality" to define, (eb) in very general terms, any set of "constraining" (e) rules, such as a moral code, that consists in judging (e&eb) actions and intentions by relating them to (e&eb) transcendent or universal values;**distinction between "ethics" and "morality"—a distinction that has traditionally been drawn to distinguish modes of reflection that place greater emphasis respectively, on the good life (such as Stoicism)**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Throughout his writings, Deleuze has often drawn a distinction between "ethics" and "morality"—a distinction that has traditionally been drawn to distinguish modes of reflection that place greater emphasis respectively, on the good life (such as Stoicism) or on the moral law (such as Kantianism). Deleuze, however, uses the term "morality" to define, in very general terms, any set of "constraining" rules, such as a moral code, that consists in judging (e&eb) actions and intentions by relating them to (e&eb) transcendent or universal values | |
| $G_{40}$ : Category one ofdistinction between "ethics" and "morality"—a distinction that has traditionally been drawn to distinguish modes of reflection that place greater emphasis respectively, on the good life (such as Stoicism) or on the moral law (such as Kantianism). Deleuze, however, uses the term "morality" to define<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of any set of "constraining" (e) rules, such as a moral code, that consists in judging (e&eb) actions and intentions by relating them to (e&eb) transcendent or universal values<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Throughout his writings, Deleuze has often drawn a distinction between "ethics" and "morality"—a distinction that has traditionally been drawn to distinguish modes of reflection that place greater emphasis respectively, on the good life (such as Stoicism) or on the moral law (such as Kantianism). Deleuze, however, uses the term "morality" to define, in very general terms, any set of "constraining" rules, such as a moral code, that consists in judging actions and intentions by relating them to (e&eb) transcendent or universal values | |
| $G_{44}$ : Category one of**"ethics" and "morality"—a distinction that has traditionally been drawn to distinguish modes of reflection that place greater emphasis respectively, on the good life (such as** | |

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| **Stoicism) or on the moral law (such as Kantianism). Deleuze, however, uses the term "morality" to define, in very general terms, any set of "constraining" rules, such as a moral code, that consists in judging actions and intentions;** transcendent or universal values<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of transcendent or universal values; "**ethics" and "morality"—a distinction that has traditionally been drawn to distinguish modes of reflection that place greater emphasis respectively, on the good life (such as Stoicism) or on the moral law (such as Kantianism). Deleuze, however, uses the term "morality" to define, in very general terms, any set of "constraining" rules, such as a moral code, that consists in judging actions and intentions**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a_{13}^{'})^{(1)}, (a_{14}^{'})^{(1)}, (a_{15}^{'})^{(1)}, (b_{13}^{'})^{(1)}, (b_{14}^{'})^{(1)}, (b_{15}^{'})^{(1)}, (a_{16}^{'})^{(2)}, (a_{17}^{'})^{(2)}, (a_{18}^{'})^{(2)}$, $(b_{16}^{'})^{(2)}, (b_{17}^{'})^{(2)}, (b_{18}^{'})^{(2)}$<br>$, (a_{20}^{'})^{(3)}, (a_{21}^{'})^{(3)}, (a_{22}^{'})^{(3)}, (b_{20}^{'})^{(3)}, (b_{21}^{'})^{(3)}, (b_{22}^{'})^{(3)}$<br>$(a_{24}^{'})^{(4)}, (a_{25}^{'})^{(4)}, (a_{26}^{'})^{(4)}, (b_{24}^{'})^{(4)}, (b_{25}^{'})^{(4)}, (b_{26}^{'})^{(4)}, (b_{28}^{'})^{(5)}, (b_{29}^{'})^{(5)}, (b_{30}^{'})^{(5)}$ $(a_{28}^{'})^{(5)}, (a_{29}^{'})^{(5)}, (a_{30}^{'})^{(5)}$ ,<br>$(a_{32}^{'})^{(6)}, (a_{33}^{'})^{(6)}, (a_{34}^{'})^{(6)}, (b_{32}^{'})^{(6)}, (b_{33}^{'})^{(6)}, (b_{34}^{'})^{(6)}$<br>$(a_{36}^{'})^{(7)}, (a_{37}^{'})^{(7)}, (a_{38}^{'})^{(7)}, (b_{36}^{'})^{(7)}, (b_{37}^{'})^{(7)}, (b_{38}^{'})^{(7)}$,<br>$(a_{40}^{'})^{(8)}, (a_{41}^{'})^{(8)}, (a_{42}^{'})^{(8)}, (b_{40}^{'})^{(8)}, (b_{41}^{'})^{(8)}, (b_{42}^{'})^{(8)}$,<br>$(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)} G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right] G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)} G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right] G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)} G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right] G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)} T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right] T_{13}$ | 4 |

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| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G,t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G,t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14},t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G,t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17},t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17},t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17},t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19}),t)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}((G_{19}),t)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19}),t)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17},t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}((G_{19}),t) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |

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| $-(b_{24}^{''})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |

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| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}\left((G_{43}),t\right)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}\left((G_{43}),t\right)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}\left((G_{43}),t\right)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}\left((G_{47}),t\right) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)} & +(a_{13}^{''})^{(1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,)}(T_{17},t) \\ +(a_{20}^{''})^{(3,3,)}(T_{21},t) & +(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t) \\ +(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t) & +(a_{36}^{''})^{(7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8)}(T_{41},t) \\ +(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) & & \end{array}\right]G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{array}{ccc} (a_{14}^{'})^{(1)} & +(a_{14}^{''})^{(1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,)}(T_{17},t) \\ +(a_{21}^{''})^{(3,3,)}(T_{21},t) & +(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t) \\ +(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t) & +(a_{37}^{''})^{(7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8)}(T_{41},t) \\ +(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) & & \end{array}\right]G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{15}^{'})^{(1)} & +(a_{15}^{''})^{(1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,)}(T_{17},t) \\ +(a_{22}^{''})^{(3,3,)}(T_{21},t) & +(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t) \\ +(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t) & +(a_{38}^{''})^{(7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8)}(T_{41},t) \\ +(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) & & \end{array}\right]G_{15}$ | *57* |
| Where $(a_{13}^{''})^{(1)}(T_{14},t)$, $(a_{14}^{''})^{(1)}(T_{14},t)$, $(a_{15}^{''})^{(1)}(T_{14},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 | |

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| $+(a_{38}^{''})^{(7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3<br>$+(a_{40}^{''})^{(8,8)}(T_{41},t)$ $+(a_{41}^{''})^{(8,8)}(T_{41},t)$ $+(a_{42}^{''})^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3<br>$+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)} -(b_{13}^{''})^{(1)}(G,t) & -(b_{16}^{''})^{(2,2,)}(G_{19},t) & -(b_{20}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,)}(G_{39},t) & -(b_{40}^{''})^{(8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)} -(b_{14}^{''})^{(1)}(G,t) & -(b_{17}^{''})^{(2,2,)}(G_{19},t) & -(b_{21}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,)}(G_{39},t) & -(b_{41}^{''})^{(8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)} -(b_{15}^{''})^{(1)}(G,t) & -(b_{18}^{''})^{(2,2,)}(G_{19},t) & -(b_{22}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,)}(G_{39},t) & -(b_{42}^{''})^{(8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$ | *60* |
| Where $-(b_{13}^{''})^{(1)}(G,t)$, $-(b_{14}^{''})^{(1)}(G,t)$, $-(b_{15}^{''})^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{16}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{20}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{37}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
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$$\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}')^{(2)}\boxed{+(a_{16}'')^{(2)}(T_{17},t)}\ \boxed{+(a_{13}'')^{(1,1,)}(T_{14},t)}\ \boxed{+(a_{20}'')^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}'')^{(4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{28}'')^{(5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{32}'')^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}'')^{(7,7,7)}(T_{37},t)}\ \boxed{+(a_{40}'')^{(8,8,8)}(T_{41},t)}\ \boxed{+(a_{44}'')^{(9,9)}(T_{45},t)} \end{bmatrix} G_{16} \quad (61)$$

$$\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}')^{(2)}\boxed{+(a_{17}'')^{(2)}(T_{17},t)}\ \boxed{+(a_{14}'')^{(1,1,)}(T_{14},t)}\ \boxed{+(a_{21}'')^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}'')^{(4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{29}'')^{(5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{33}'')^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}'')^{(7,7,7)}(T_{37},t)}\ \boxed{+(a_{41}'')^{(8,8,8)}(T_{41},t)}\ \boxed{+(a_{45}'')^{(9,9)}(T_{45},t)} \end{bmatrix} G_{17} \quad (62)$$

$$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}')^{(2)}\boxed{+(a_{18}'')^{(2)}(T_{17},t)}\ \boxed{+(a_{15}'')^{(1,1,)}(T_{14},t)}\ \boxed{+(a_{22}'')^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}'')^{(4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{30}'')^{(5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{34}'')^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}'')^{(7,7,7)}(T_{37},t)}\ \boxed{+(a_{42}'')^{(8,8,8)}(T_{41},t)}\ \boxed{+(a_{46}'')^{(9,9)}(T_{45},t)} \end{bmatrix} G_{18} \quad (63)$$

Where $\boxed{+(a_{16}'')^{(2)}(T_{17},t)}$, $\boxed{+(a_{17}'')^{(2)}(T_{17},t)}$, $\boxed{+(a_{18}'')^{(2)}(T_{17},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{13}'')^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{14}'')^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{15}'')^{(1,1,)}(T_{14},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}'')^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}'')^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}'')^{(3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}'')^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}'')^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}'')^{(4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}'')^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}'')^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}'')^{(5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}'')^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}'')^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}'')^{(6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{36}'')^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}'')^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}'')^{(7,7,7)}(T_{37},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{40}'')^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}'')^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}'')^{(8,8,8)}(T_{41},t)}$ are eight augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{44}'')^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{45}'')^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{46}'')^{(9,9)}(T_{45},t)}$ are ninth augmentation coefficient for category 1, 2 and 3

$$\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}')^{(2)}\boxed{-(b_{16}'')^{(2)}(G_{19},t)}\ \boxed{-(b_{13}'')^{(1,1,)}(G,t)}\ \boxed{-(b_{20}'')^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}'')^{(4,4,4,4,4)}(G_{27},t)}\ \boxed{-(b_{28}'')^{(5,5,5,5,5)}(G_{31},t)}\ \boxed{-(b_{32}'')^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{36}'')^{(7,7,7)}(G_{39},t)}\ \boxed{-(b_{40}'')^{(8,8,8)}(G_{43},t)}\ \boxed{-(b_{44}'')^{(9,9)}(G_{47},t)} \end{bmatrix} T_{16} \quad (64)$$

$$\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}')^{(2)}\boxed{-(b_{17}'')^{(2)}(G_{19},t)}\ \boxed{-(b_{14}'')^{(1,1,)}(G,t)}\ \boxed{-(b_{21}'')^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}'')^{(4,4,4,4,4)}(G_{27},t)}\ \boxed{-(b_{29}'')^{(5,5,5,5,5)}(G_{31},t)}\ \boxed{-(b_{33}'')^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{37}'')^{(7,7,7)}(G_{39},t)}\ \boxed{-(b_{41}'')^{(8,8,8)}(G_{43},t)}\ \boxed{-(b_{45}'')^{(9,9)}(G_{47},t)} \end{bmatrix} T_{17} \quad (65)$$

$$\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b'_{18})^{(2)} & \boxed{-(b''_{18})^{(2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,)}(G,t)} & \boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{18} \quad (66)$$

where $\boxed{-(b''_{16})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{17})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{18})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,)}(G,t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{44})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2 and 3

$$\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a'_{20})^{(3)} & \boxed{+(a''_{20})^{(3)}(T_{21},t)} & \boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{20} \quad (67)$$

$$\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a'_{21})^{(3)} & \boxed{+(a''_{21})^{(3)}(T_{21},t)} & \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{21} \quad (68)$$

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a'_{22})^{(3)} & \boxed{+(a''_{22})^{(3)}(T_{21},t)} & \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{22} \quad (69)$$

$\boxed{+(a''_{20})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)}$ are third augmentation coefficients

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|---|---|
| for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23},t) & -(b_{16}^{''})^{(2,2,2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,1,)}(G,t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{20}$ | *70* |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)}-(b_{21}^{''})^{(3)}(G_{23},t) & -(b_{17}^{''})^{(2,2,2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,1,)}(G,t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21}$ | *71* |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23},t) & -(b_{18}^{''})^{(2,2,2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,1,)}(G,t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{22}$ | *72* |
| $-(b_{20}^{''})^{(3)}(G_{23},t)$, $-(b_{21}^{''})^{(3)}(G_{23},t)$, $-(b_{22}^{''})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*<br>$-(b_{16}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*<br>$-(b_{13}^{''})^{(1,1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*<br>$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 | |

$\boxed{-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{46}^{''})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt}=(a_{24})^{(4)}G_{25}-\begin{bmatrix}(a_{24}^{'})^{(4)}\boxed{+(a_{24}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)}\\ \boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt}=(a_{25})^{(4)}G_{24}-\begin{bmatrix}(a_{25}^{'})^{(4)}\boxed{+(a_{25}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6)}(T_{33},t)}\\ \boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt}=(a_{26})^{(4)}G_{25}-\begin{bmatrix}(a_{26}^{'})^{(4)}\boxed{+(a_{26}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)}\\ \boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{26} \quad (75)$$

$\boxed{(a_{24}^{''})^{(4)}(T_{25},t)}$, $\boxed{(a_{25}^{''})^{(4)}(T_{25},t)}$, $\boxed{(a_{26}^{''})^{(4)}(T_{25},t)}$ *are first augmentation coefficients category* 1, 2 3

$\boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)}$
*are fourth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)}$
*are fifth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)}$
*are sixth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)}$
*are seventh augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)}$
*are eighth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)}$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)}\boxed{-(b_{24}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,)}(G_{35},t)}\\ \boxed{-(b_{13}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)}\\ \boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}\end{bmatrix}T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b'_{25})^{(4)} & -(b''_{25})^{(4)}(G_{27},t) & -(b''_{29})^{(5,5,)}(G_{31},t) & -(b''_{33})^{(6,6,)}(G_{35},t) \\ & -(b''_{14})^{(1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b'_{26})^{(4)} & -(b''_{26})^{(4)}(G_{27},t) & -(b''_{30})^{(5,5,)}(G_{31},t) & -(b''_{34})^{(6,6,)}(G_{35},t) \\ & -(b''_{15})^{(1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{26} \quad (78)$$

Where $-(b''_{24})^{(4)}(G_{27},t)$, $-(b''_{25})^{(4)}(G_{27},t)$, $-(b''_{26})^{(4)}(G_{27},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b''_{28})^{(5,5,)}(G_{31},t)$, $-(b''_{29})^{(5,5,)}(G_{31},t)$, $-(b''_{30})^{(5,5,)}(G_{31},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b''_{32})^{(6,6,)}(G_{35},t)$, $-(b''_{33})^{(6,6,)}(G_{35},t)$, $-(b''_{34})^{(6,6,)}(G_{35},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b''_{13})^{(1,1,1,1)}(G,t)$, $-(b''_{14})^{(1,1,1,1)}(G,t)$, $-(b''_{15})^{(1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b''_{16})^{(2,2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b''_{20})^{(3,3,3,3)}(G_{23},t)$, $-(b''_{21})^{(3,3,3,3)}(G_{23},t)$, $-(b''_{22})^{(3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b''_{36})^{(7,7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)$, $-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b''_{40})^{(8,8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b''_{46})^{(9,9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a'_{28})^{(5)} & +(a''_{28})^{(5)}(T_{29},t) & +(a''_{24})^{(4,4,)}(T_{25},t) & +(a''_{32})^{(6,6,6)}(T_{33},t) \\ & +(a''_{13})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{28} \quad (79)$$

$$\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a'_{29})^{(5)} & +(a''_{29})^{(5)}(T_{29},t) & +(a''_{25})^{(4,4,)}(T_{25},t) & +(a''_{33})^{(6,6,6)}(T_{33},t) \\ & +(a''_{14})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{29} \quad (80)$$

$$\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a'_{30})^{(5)} & +(a''_{30})^{(5)}(T_{29},t) & +(a''_{26})^{(4,4,)}(T_{25},t) & +(a''_{34})^{(6,6,6)}(T_{33},t) \\ & +(a''_{15})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{30} \quad (81)$$

Where $+(a''_{28})^{(5)}(T_{29},t)$, $+(a''_{29})^{(5)}(T_{29},t)$, $+(a''_{30})^{(5)}(T_{29},t)$ are first augmentation coefficients for category 1, 2 and 3

| | |
|---|---|
| $And\ \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)}, \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)}, \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)}\ are\ second\ augmentation\ coefficient\ for\ category\ 1,2\ and\ 3$ $\boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)}, \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)}, \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)}\ are\ third\ augmentation\ coefficient\ for\ category\ 1,2\ and\ 3$ $\boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)}, \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)}, \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3 $\boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation coefficients for category 1,2,and 3 $\boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)}$ are sixth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$ , $\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1,2, 3 $\boxed{+(a_{40}^{''})^{(8,8\ ,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$ are eighth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1,2, 3 | |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b_{28}^{'})^{(5)}\boxed{-(b_{28}^{''})^{(5)}(G_{31},t)} & \boxed{-(b_{24}^{''})^{(4,4,)}(G_{27},t)} & \boxed{-(b_{32}^{''})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{28}$ | *82* |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b_{29}^{'})^{(5)}\boxed{-(b_{29}^{''})^{(5)}(G_{31},t)} & \boxed{-(b_{25}^{''})^{(4,4,)}(G_{27},t)} & \boxed{-(b_{33}^{''})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{29}$ | *83* |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b_{30}^{'})^{(5)}\boxed{-(b_{30}^{''})^{(5)}(G_{31},t)} & \boxed{-(b_{26}^{''})^{(4,4,)}(G_{27},t)} & \boxed{-(b_{34}^{''})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{30}$ | *84* |
| $where\ \boxed{-(b_{28}^{''})^{(5)}(G_{31},t)}, \boxed{-(b_{29}^{''})^{(5)}(G_{31},t)}, \boxed{-(b_{30}^{''})^{(5)}(G_{31},t)}\ are\ first\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{24}^{''})^{(4,4,)}(G_{27},t)}, \boxed{-(b_{25}^{''})^{(4,4,)}(G_{27},t)}, \boxed{-(b_{26}^{''})^{(4,4,)}(G_{27},t)}\ are\ second\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{32}^{''})^{(6,6,6)}(G_{35},t)}, \boxed{-(b_{33}^{''})^{(6,6,6)}(G_{35},t)}, \boxed{-(b_{34}^{''})^{(6,6,6)}(G_{35},t)}\ are\ third\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)}, \boxed{-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)}, \boxed{-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)}$ are fourth detrition coefficients for category 1,2, and 3 $\boxed{-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)}, \boxed{-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)}, \boxed{-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1,2, and 3 $\boxed{-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)}, \boxed{-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)}, \boxed{-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients | |

| | |
|---|---|
| for category 1,2, and 3<br>$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3<br>$-(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3<br>$-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3 | |
| $\frac{dG_{32}}{dt}=(a_{32})^{(6)}G_{33}$<br>$-\begin{bmatrix}(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{32}$ | *85* |
| $\frac{dG_{33}}{dt}=(a_{33})^{(6)}G_{32}-\begin{bmatrix}(a_{33}^{'})^{(6)}+(a_{33}^{''})^{(6)}(T_{33},t) & +(a_{29}^{''})^{(5,5,5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{33}$ | *86* |
| $\frac{dG_{34}}{dt}=(a_{34})^{(6)}G_{33}-\begin{bmatrix}(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33},t) & +(a_{30}^{''})^{(5,5,5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{34}$ | *87* |
| $+(a_{32}^{''})^{(6)}(T_{33},t)$, $+(a_{33}^{''})^{(6)}(T_{33},t)$, $+(a_{34}^{''})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{28}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients<br>$+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients<br>$+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients<br>$+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients<br>$+(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients<br>$+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients | |

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{28})^{(5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b''_{24})^{(4,4,4,)}(G_{27},t)$, $-(b''_{25})^{(4,4,4,)}(G_{27},t)$, $-(b''_{26})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b''_{13})^{(1,1,1,1,1,1)}(G,t)$, $-(b''_{14})^{(1,1,1,1,1,1)}(G,t)$, $-(b''_{15})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3

$-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3

$-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3

$-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3

$-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a'_{36})^{(7)} & +(a''_{36})^{(7)}(T_{37},t) & +(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t) & \\ +(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a''_{40})^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}')^{(7)}\boxed{+(a_{37}'')^{(7)}(T_{37},t)} & \boxed{+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}')^{(7)}\boxed{+(a_{38}'')^{(7)}(T_{37},t)} & \boxed{+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad (93)$$

Where $\boxed{(a_{36}'')^{(7)}(T_{37},t)}$, $\boxed{(a_{37}'')^{(7)}(T_{37},t)}$, $\boxed{(a_{38}'')^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{40}'')^{(8,8,8,8,8,8,8,8,)}(T_{41},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}')^{(7)}\boxed{-(b_{36}'')^{(7)}(G_{39},t)} & \boxed{-(b_{16}'')^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}'')^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}'')^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}'')^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}'')^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}'')^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}'')^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13} \quad (94)$$

$$\frac{dT_{37}}{dt} =$$

$$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b'_{37})^{(7)}\boxed{-(b''_{37})^{(7)}(G_{39},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} =$$

$$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b'_{38})^{(7)}\boxed{-(b''_{38})^{(7)}(G_{39},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a'_{40})^{(8)}\boxed{+(a''_{40})^{(8)}(T_{41},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a'_{41})^{(8)}\boxed{+(a''_{41})^{(8)}(T_{41},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a'_{42})^{(8)}\boxed{+(a''_{42})^{(8)}(T_{41},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{40})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b'_{40})^{(8)}\boxed{-(b''_{40})^{(8)}(G_{43},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b'_{41})^{(8)}\boxed{-(b''_{41})^{(8)}(G_{43},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b'_{42})^{(8)}\boxed{-(b''_{42})^{(8)}(G_{43},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

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$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13}$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} =$$

$$(b_{46})^{(9)}T_{45} - \begin{bmatrix} (b'_{46})^{(9)}\boxed{-(b''_{46})^{(9)}(G_{47},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{44})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a'_i)^{(1)}, (a''_i)^{(1)}, (b_i)^{(1)}, (b'_i)^{(1)}, (b''_i)^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a''_i)^{(1)}, (b''_i)^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

$$(a''_i)^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$$

$$(b''_i)^{(1)}(G,t) \le (r_i)^{(1)} \le (b'_i)^{(1)} \le (\hat{B}_{13})^{(1)}$$

$lim_{T_2\to\infty}(a''_i)^{(1)}(T_{14},t) = (p_i)^{(1)}$ 98

$\lim_{G\to\infty}(b''_i)^{(1)}(G,t) = (r_i)^{(1)}$

**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :

Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$

| | |
|---|---|
| They satisfy Lipschitz condition:<br>$\|(a_i^{''})^{(1)}(T_{14}^{'},t)-(a_i^{''})^{(1)}(T_{14},t)\| \le (\hat{k}_{13})^{(1)}\|T_{14}-T_{14}^{'}\|e^{-(\hat{M}_{13})^{(1)}t}$<br>$\|(b_i^{''})^{(1)}(G^{'},t)-(b_i^{''})^{(1)}(G,t)\| < (\hat{k}_{13})^{(1)}\|G-G^{'}\|e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)},(\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)}=1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)}$, are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)},(\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)},(\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)},(a_i^{'})^{(1)},(b_i)^{(1)},(b_i^{'})^{(1)},(p_i)^{(1)}$, $(r_i)^{(1)}, i=13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)}+(a_i^{'})^{(1)}+(\hat{A}_{13})^{(1)}+(\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}]<1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)}+(b_i^{'})^{(1)}+(\hat{B}_{13})^{(1)}+(\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}]<1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)},(a_i^{'})^{(2)},(a_i^{''})^{(2)},(b_i)^{(2)},(b_i^{'})^{(2)},(b_i^{''})^{(2)}>0, \quad i,j=16,17,18$ | |
| The functions $(a_i^{''})^{(2)},(b_i^{''})^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i^{''})^{(2)}(T_{17},t)\le(p_i)^{(2)}\le(\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i^{''})^{(2)}(G_{19},t)\le (r_i)^{(2)}\le(b_i^{'})^{(2)}\le(\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(2)}(T_{17},t)=(p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty}(b_i^{''})^{(2)}\left((G_{19}),t\right)=(r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)}$ : | 106 |

| | |
|---|---|
| Where $\boxed{(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}}$ are positive constants and $\boxed{i = 16,17,18}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'}, t) - (a_i^{''})^{(2)}(T_{17}, t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}^{'}\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'}, t) - (b_i^{''})^{(2)}((G_{19}), t)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})^{'}\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'}, t)$ and $(a_i^{''})^{(2)}(T_{17}, t)$ . $(T_{17}^{'}, t)$ and $(T_{17}, t)$ are points belonging to the interval $\left[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}\right]$. It is to be noted that $(a_i^{''})^{(2)}(T_{17}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i^{''})^{(2)}(T_{17}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i^{'})^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i, j = 20,21,22$<br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(3)}$, $(r_i)^{(3)}$:<br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(3)}(T_{21}, t) = (p_i)^{(3)}$ | 113 |

| | |
|---|---|
| $\lim_{G\to\infty}(b_i'')^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(3)}(T_{21}',t) - (a_i'')^{(3)}(T_{21},t)\rvert \le (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}'\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i'')^{(3)}(G_{23}',t) - (b_i'')^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}'\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(3)}(T_{21}',t)$ and$(a_i'')^{(3)}(T_{21},t)$ . $(T_{21}',t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i'')^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i'')^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i')^{(3)}, (b_i)^{(3)}, (b_i')^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i')^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i')^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i')^{(4)}, (a_i'')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (b_i'')^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i'')^{(4)}, (b_i'')^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i'')^{(4)}(T_{25},t) \le (p_i)^{(4)} \le (\hat{A}_{24})^{(4)}$<br><br>$(b_i'')^{(4)}\big((G_{27}),t\big) \le (r_i)^{(4)} \le (b_i')^{(4)} \le (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2\to\infty}(a_i'')^{(4)}(T_{25},t) = (p_i)^{(4)}$ | 118 |

| | |
|---|---|
| $\lim_{G\to\infty}(b_i'')^{(4)}\left((G_{27}),t\right) = \ (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, \ (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24{,}25{,}26}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(4)}(T_{25}',t) - (a_i'')^{(4)}(T_{25},t)\rvert \le (\hat{k}_{24})^{(4)}\lvert T_{25} - T_{25}'\rvert e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\lvert(b_i'')^{(4)}((G_{27})',t) - (b_i'')^{(4)}\left((G_{27}),t\right)\rvert < (\hat{k}_{24})^{(4)}\lvert\lvert(G_{27}) - (G_{27})'\rvert\rvert e^{-(\hat{M}_{24})^{(4)}t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(4)}(T_{25}',t)$ and $(a_i'')^{(4)}(T_{25},t)$ . $(T_{25}',t)$ and $(T_{25},t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i'')^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i'')^{(4)}(T_{25},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (p_i)^{(4)}, \ (r_i)^{(4)}, i = 24{,}25{,}26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[\,(a_i)^{(4)} + (a_i')^{(4)} + \ (\hat{A}_{24})^{(4)} + \ (\hat{P}_{24})^{(4)}\,(\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i')^{(4)} + \ (\hat{B}_{24})^{(4)} + \ (\hat{Q}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i')^{(5)}, (a_i'')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (b_i'')^{(5)} > 0, \quad i,j = 28{,}29{,}30$<br><br>The functions $(a_i'')^{(5)}, (b_i'')^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}, \ (r_i)^{(5)}$:<br><br>$(a_i'')^{(5)}(T_{29},t) \le (p_i)^{(5)} \le (\hat{A}_{28})^{(5)}$<br><br>$(b_i'')^{(5)}\left((G_{31}),t\right) \le \ (r_i)^{(5)} \le (b_i')^{(5)} \le (\hat{B}_{28})^{(5)}$ | 122 |

$lim_{T_2\to\infty}(a_i'')^{(5)}(T_{29},t) = (p_i)^{(5)}$

$\lim_{G\to\infty}(b_i'')^{(5)}(G_{31},t) = (r_i)^{(5)}$

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**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :

Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$

They satisfy Lipschitz condition: 124

$$|(a_i'')^{(5)}(T_{29}', t) - (a_i'')^{(5)}(T_{29}, t)| \le (\hat{k}_{28})^{(5)}|T_{29} - T_{29}'|e^{-(\hat{M}_{28})^{(5)}t}$$

$$|(b_i'')^{(5)}((G_{31})', t) - (b_i'')^{(5)}((G_{31}), t)| < (\hat{k}_{28})^{(5)}||(G_{31}) - (G_{31})'||e^{-(\hat{M}_{28})^{(5)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(5)}(T_{29}', t)$ and $(a_i'')^{(5)}(T_{29}, t)$ . $(T_{29}', t)$ and $(T_{29}, t)$ are points belonging to the interval $\left[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\right]$. It is to be noted that $(a_i'')^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i'')^{(5)}(T_{29}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ : 125

$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)},$ are positive constants

$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$$

**Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ : 126

There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i')^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i')^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

Where we suppose

$(a_i)^{(6)}, (a_i')^{(6)}, (a_i'')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (b_i'')^{(6)} > 0, \quad i, j = 32,33,34$ 127

The functions $(a_i'')^{(6)}, (b_i'')^{(6)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(6)}, (r_i)^{(6)}$:

$$(a_i'')^{(6)}(T_{33}, t) \le (p_i)^{(6)} \le (\hat{A}_{32})^{(6)}$$

| | |
|---|---|
| $(b_i^{''})^{(6)}((G_{35}),t) \leq \ (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(6)}(T_{33},t) = (p_i)^{(6)}$<br>$\lim_{G\to\infty}(b_i^{''})^{(6)}\left((G_{35}),t\right) = \ (r_i)^{(6)}$<br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'},t) - (a_i^{''})^{(6)}(T_{33},t)\rvert \leq (\hat{k}_{32})^{(6)}\lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br>$\lvert(b_i^{''})^{(6)}((G_{35})^{'},t) - (b_i^{''})^{(6)}\left((G_{35}),t\right)\rvert < (\hat{k}_{32})^{(6)}\lvert\lvert(G_{35}) - (G_{35})^{'}\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'},t)$ and$(a_i^{''})^{(6)}(T_{33},t)$ . $(T_{33}^{'},t)$ and $(T_{33},t)$ are points belonging to the interval $\left[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\right]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + \ (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$<br>$\frac{1}{(\hat{M}_{32})^{(6)}}[\ (b_i)^{(6)} + (b_i^{'})^{(6)} + \ (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)}\ (\hat{k}_{32})^{(6)}] < 1$ | 130 |
| Where we suppose | |
| (S) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i,j = 36,37,38$<br>(T) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$: | 131 |

$(a_i^{''})^{(7)}(T_{37},t) \le (p_i)^{(7)} \le (\hat{A}_{36})^{(7)}$

$(b_i^{''})^{(7)}(G_{39},t) \le (r_i)^{(7)} \le (b_i^{'})^{(7)} \le (\hat{B}_{36})^{(7)}$

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*(U)* $\lim_{T_2\to\infty} (a_i^{''})^{(7)}(T_{37},t) = (p_i)^{(7)}$

*(V)*

$\lim_{G\to\infty}(b_i^{''})^{(7)}\big((G_{39}),t\big) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

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They satisfy Lipschitz condition:

$|(a_i^{''})^{(7)}(T_{37}^{'},t) - (a_i^{''})^{(7)}(T_{37},t)| \le (\hat{k}_{36})^{(7)}|T_{37} - T_{37}^{'}|e^{-(\hat{M}_{36})^{(7)}t}$

$|(b_i^{''})^{(7)}((G_{39})^{'},t) - (b_i^{''})^{(7)}\big((G_{39}),t\big)| < (\hat{k}_{36})^{(7)}||(G_{39}) - (G_{39})^{'}||e^{-(\hat{M}_{36})^{(7)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'},t)$ and$(a_i^{''})^{(7)}(T_{37},t)$ . $(T_{37}^{'},t)$ and $(T_{37},t)$ are points belonging to the interval $\big[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}\big]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

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**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :

(W) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$, are positive constants

$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$

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**Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :

(X) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities

$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$

$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$

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| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})^{'}, t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})^{'}\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $\left[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}\right]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i^{'})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |

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| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i,j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45},t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47},t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2\to\infty}(a_i'')^{(9)}(T_{45},t) = (p_i)^{(9)}$<br>$\lim_{G\to\infty}(b_i'')^{(9)}(G_{47},t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}',t) - (a_i'')^{(9)}(T_{45},t)\rvert \le (\hat{k}_{44})^{(9)}\lvert T_{45} - T_{45}'\rvert e^{-(\hat{M}_{44})^{(9)}t}$<br>$\lvert(b_i'')^{(9)}((G_{47})',t) - (b_i'')^{(9)}((G_{47}),t)\rvert < (\hat{k}_{44})^{(9)}\lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}',t)$ and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i'')^{(9)}(T_{45},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :<br>$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants<br>$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$ | |
| **Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :<br>There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and (\hat{B}_{44})^{(9)}$ and the constants | |

$(a_i)^{(9)}, (a_i')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}, i = 44,45,46,$
satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i')^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i')^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0), T_i(0)$ :

$$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}, \quad \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}, \quad \boxed{T_i(0) = T_i^0 > 0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0), T_i(0)$

$$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}, \quad G_i(0) = G_i^0 > 0$$

$$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}, \quad T_i(0) = T_i^0 > 0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}, \quad G_i(0) = G_i^0 > 0$$

$$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}, \quad T_i(0) = T_i^0 > 0$$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

**Definition of** $G_i(0), T_i(0)$ :

$$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}, \quad \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}, \quad \boxed{T_i(0) = T_i^0 > 0}$$

**Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 151

**Definition of** $G_i(0), T_i(0)$ :

$$G_i(t) \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}, \quad \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}, \quad \boxed{T_i(0) = T_i^0 > 0}$$

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| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le \left(\hat{Q}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le \left(\hat{P}_{13}\right)^{(1)}$ , $T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le \left(\hat{P}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |
| $0 \le T_i(t) - T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 157 |
| By<br><br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}\left(s_{(13)}\right) - \left((a_{13}')^{(1)} + a_{13}''\right)^{(1)} \left(T_{14}\left(s_{(13)}\right), s_{(13)}\right)\right) G_{13}\left(s_{(13)}\right)\Big] ds_{(13)}$ | 158 |

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| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{16})^{(2)}$, $T_i^0 \le (\hat{Q}_{16})^{(2)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} G_{17}(s_{(16)}) - \left((a_{16}')^{(2)} + a_{16}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{16}(s_{(16)})\right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[(a_{17})^{(2)} G_{16}(s_{(16)}) - \left((a_{17}')^{(2)} + (a_{17}'')^{(2)}(T_{17}(s_{(16)}), s_{(17)})\right) G_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{20})^{(3)}$ , $T_i^0 \leq (\hat{Q}_{20})^{(3)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a_{20}^{'})^{(3)} + a_{20}^{''})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{24})^{(4)}$ , $T_i^0 \leq (\hat{Q}_{24})^{(4)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a_{24}^{'})^{(4)} + a_{24}^{''})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |

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| $\overline{T}_{26}(t) = T_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b_{26}')^{(4)} - (b_{26}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{28})^{(5)}$ , $T_i^0 \leq (\hat{Q}_{28})^{(5)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a_{28}')^{(5)} + a_{28}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a_{29}')^{(5)} + (a_{29}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a_{30}')^{(5)} + (a_{30}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\overline{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b_{28}')^{(5)} - (b_{28}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\overline{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b_{29}')^{(5)} - (b_{29}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\overline{T}_{30}(t) = T_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b_{30}')^{(5)} - (b_{30}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a_{32}')^{(6)} + a_{32}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a_{33}')^{(6)} + (a_{33}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |

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| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{36})^{(7)}$, $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a_{36}^{'})^{(7)} + a_{36}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ | |

which satisfy

$G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{40})^{(8)}$ , $T_i^0 \leq (\hat{Q}_{40})^{(8)}$,

$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$

$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$

By 166

$$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{40})^{(8)} + a''_{40})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{40}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[ (a_{41})^{(8)} G_{40}(s_{(40)}) - \left( (a'_{41})^{(8)} + (a''_{41})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{41}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[ (a_{42})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{42})^{(8)} + (a''_{42})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{42}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[ (b_{40})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{40})^{(8)} - (b''_{40})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{40}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[ (b_{41})^{(8)} T_{40}(s_{(40)}) - \left( (b'_{41})^{(8)} - (b''_{41})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{41}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[ (b_{42})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{42})^{(8)} - (b''_{42})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{42}(s_{(40)}) \right] ds_{(40)}$$

Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$

**Proof:** 166 A

Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy

$G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$,

$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$

$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$

By

$$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[ (a_{44})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{44}(s_{(44)}) \right] ds_{(44)}$$

$$\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[ (a_{45})^{(9)} G_{44}(s_{(44)}) - \left( (a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{45}(s_{(44)}) \right] ds_{(44)}$$

$$\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[ (a_{46})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{46}(s_{(44)}) \right] ds_{(44)}$$

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| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that<br>$(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} \left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$<br>$\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}} \left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$ | 171 |
| From which it follows that | 172 |

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| $(G_{20}(t)-G_{20}^0)e^{-(\hat{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that $G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\hat{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$ $\left(1+(a_{24})^{(4)}t\right)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that $(G_{24}(t)-G_{24}^0)e^{-(\hat{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$ $\left(1+(a_{28})^{(5)}t\right)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left(e^{(\hat{M}_{28})^{(5)}t}-1\right)$ | |
| From which it follows that $(G_{28}(t)-G_{28}^0)e^{-(\hat{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 5 | 175 |
| The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0+(\hat{P}_{32})^{(6)}e^{(\hat{M}_{32})^{(6)}s_{(32)}}\right)\right] ds_{(32)} =$ $\left(1+(a_{32})^{(6)}t\right)G_{33}^0 + \frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)}t}-1\right)$ | 176 |
| From which it follows that $(G_{32}(t)-G_{32}^0)e^{-(\hat{M}_{32})^{(6)}t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^0)e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^0}{G_{33}^0}\right)}+(\hat{P}_{32})^{(6)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 6 | 177 |

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| Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$ | |
| (d) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} \left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$<br>$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$ | 178 |
| From which it follows that<br>$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left[((\hat{P}_{36})^{(7)} + G_{37}^0) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 7 | |
| The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} \left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$<br>$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$ | 180 |
| From which it follows that<br>$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left[((\hat{P}_{40})^{(8)} + G_{41}^0) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} \left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$<br>$\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left(e^{(\hat{M}_{44})^{(9)} t} - 1\right)$ | |
| From which it follows that<br>$(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left[((\hat{P}_{44})^{(9)} + G_{45}^0) e^{\left(-\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0}\right)} + (\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ and to choose | 182 |

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| $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{13})^{(1)}\left\|G_{14}^{(1)}-G_{14}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$<br>$\int_0^t\{(a_{13}')^{(1)}\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{-(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$G_{13}^{(2)}\left\\|(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}'')^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right\\|\ e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$<br>Where $s_{(13)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | |
| $\left\|G^{(1)}-G^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}t}\leq$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}\left((a_{13})^{(1)}+(a_{13}')^{(1)}+(\hat{A}_{13})^{(1)}+(\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)},T^{(1)};\ G^{(2)},T^{(2)}\right)\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a_{13}'')^{(1)}$ and $(b_{13}'')^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by | |

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| $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(1)} - (a_i^{''})^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a_{14}^{'})^{(1)}G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}^{'})^{(1)}$<br><br>If $G_{14}\ or\ G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}\ and\ G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t), t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t), t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right),\quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t), t) = (b_{15}^{'})^{(1)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |

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| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)}, (T_{19})^{(1)}), ((G_{19})^{(2)}, (T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)}\left\|G_{17}^{(1)} - G_{17}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\,ds_{(16)} +$<br>$\int_0^t\{(a_{16}')^{(2)}\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{-(\hat{M}_{16})^{(2)}s_{(16)}} +$<br>$(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}} +$<br>$G_{16}^{(2)}\left\|(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}'')^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}t} \leq$<br>$\frac{1}{(\hat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}\right)d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}'')^{(2)}$ and $(b_{16}'')^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | 199 |

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| condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ and $(\widehat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)}t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1, \left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)}G_{17}$ and by integrating<br><br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1/(a_{17}^{'})^{(2)}$<br><br>In the same way , one can obtain<br><br>$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}^{'})^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | 201 |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(2)}((G_{19})(t), t)) = (b_{17}^{'})^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i^{''})^{(2)}((G_{19})(t), t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}^{''})^{(2)}\left((G_{19})(t), t\right) = (b_{18}^{'})^{(2)}$ | 205 |

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| We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\widehat{P}_{20})^{(3)}$ and $(\widehat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[(\widehat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)}, (T_{23})^{(1)}), ((G_{23})^{(2)}, (T_{23})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | 211 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :(\widetilde{(G_{23})}, \widetilde{(T_{23})}) = \mathcal{A}^{(3)}((G_{23}), (T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)}\left\|G_{21}^{(1)} - G_{21}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)} +$<br>$\int_0^t\{(a'_{20})^{(3)}\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{-(\widehat{M}_{20})^{(3)}s_{(20)}} +$<br>$(a''_{20})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right)\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}} +$<br>$G_{20}^{(2)}\left\|(a''_{20})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a''_{20})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}t} \leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a'_{20})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)}, (T_{23})^{(1)};\ (G_{23})^{(2)}, (T_{23})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |

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| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ $and$ $(\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(3)}t)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2\ and\ \left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}\ and\ G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)}G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br><br>If $G_{21}\ or\ G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ $is$ bounded, from below, the same property holds for $G_{21}\ and\ G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t), t\right)) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t), t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded. | 220 |

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| The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}'')^{(3)}\left((G_{23})(t),t\right)=(b_{22}')^{(3)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}<1$ and to choose<br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)}+\left((\hat{P}_{24})^{(4)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\hat{Q}_{24})^{(4)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{24})^{(4)}\right]\leq(\hat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{24})^{(4)}\left\|G_{25}^{(1)}-G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$<br>$\int_0^t\{(a_{24}')^{(4)}\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$(a_{24}'')^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$G_{24}^{(2)}\left\|(a_{24}'')^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}'')^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)}-(G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t}\leq$<br>$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}')^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$ | 226 |

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| And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i = 24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br>it results<br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]} \geq 0$<br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(4)}t\right)} > 0$ for t > 0 | 228 |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2\ and\ \left((\widehat{M}_{24})^{(4)}\right)_3$ :<br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if<br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}^{'})^{(4)}G_{25}$ and by integrating<br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a_{25}^{'})^{(4)}$<br>In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a_{26}^{'})^{(4)}$<br>If $G_{25}\ or\ G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}\ and\ G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t),t)) = (b_{25}^{'})^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i^{''})^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}\,(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results | 232 |

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right), \quad t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42

Analogous inequalities hold also for $G_{29}, G_{30}, T_{28}, T_{29}, T_{30}$

It is now sufficient to take $\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ and to choose

$(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ large to have

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$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[(\hat{P}_{28})^{(5)} + \left((\hat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{28})^{(5)} \quad (234)$$

$$\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{28})^{(5)}\right] \leq (\hat{Q}_{28})^{(5)} \quad (235)$$

In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric (236)

$$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{31}}), (\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}), (\widetilde{T_{31}})\right) = \mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$

It results

$$\left|\tilde{G}_{28}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{28})^{(5)}\left|G_{29}^{(1)} - G_{29}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)} +$$

$$\int_0^t\{(a_{28}^{'})^{(5)}\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{-(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)}, s_{(28)}\right)\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$G_{28}^{(2)}\left|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)}, s_{(28)}\right) - (a_{28}^{''})^{(5)}\left(T_{29}^{(2)}, s_{(28)}\right)\right|\ e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$$

Where $s_{(28)}$ represents integrand that is integrated over the interval [0, t]

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| From the hypotheses on it follows | |
| $\left\|(G_{31})^{(1)} - (G_{31})^{(2)}\right\| e^{-(\widehat{M}_{28})^{(5)}t} \leq$ $\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)} + (a'_{28})^{(5)} + (\widehat{A}_{28})^{(5)} + (\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right) d\left(((G_{31})^{(1)}, (T_{31})^{(1)};\ (G_{31})^{(2)}, (T_{31})^{(2)})\right)$ And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a''_{28})^{(5)}$ and $(b''_{28})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ and $(\widehat{Q}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(5)}$ and $(b''_i)^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t) = 0$ and $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(5)} - (a''_i)^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(5)}t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ and $\left((\widehat{M}_{28})^{(5)}\right)_3$ : **Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ and $G_{30}$ . indeed if $G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a'_{29})^{(5)} G_{29}$ and by integrating $G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a'_{29})^{(5)}$ In the same way , one can obtain $G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a'_{30})^{(5)}$ If $G_{29}$ or $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ is bounded, from below, the same property holds for $G_{29}$ and $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |

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| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t),t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1-e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}, G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}\left[(\hat{P}_{32})^{(6)} + \left((\hat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{32})^{(6)}\right] \leq (\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{32})^{(6)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{35})}, \widetilde{(T_{35})}$ : $\left(\widetilde{(G_{35})}, \widetilde{(T_{35})}\right) = \mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br><br>It results | 247 |

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| $\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{32})^{(6)} \left\|G_{33}^{(1)}-G_{33}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\, ds_{(32)} +$<br>$\int_0^t \{(a_{32}^{'})^{(6)} \left\|G_{32}^{(1)}-G_{32}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$(a_{32}^{''})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$G_{32}^{(2)} \left\|(a_{32}^{''})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right) - (a_{32}^{''})^{(6)}\left(T_{33}^{(2)}, s_{(32)}\right)\right\| \; e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\} ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}t} \leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)} + (a_{32}^{'})^{(6)} + (\widehat{A}_{32})^{(6)} + (\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right) d\left(((G_{35})^{(1)}, (T_{35})^{(1)};\ (G_{35})^{(2)}, (T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a_{32}^{''})^{(6)}$ and $(b_{32}^{''})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)} e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)} e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i = 32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 249 |
| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(6)} - (a_i^{''})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(6)}t\right)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ :<br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if<br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}^{'})^{(6)} G_{33}$ and by integrating<br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}^{'})^{(6)}$<br>In the same way , one can obtain<br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}^{'})^{(6)}$ | 251 |

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| If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i^{''})^{(6)}\big((G_{35})(t),t\big) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ large to have | 255 |
| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[(\hat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote | 258 |

**Definition of** $\widetilde{(G_{39})}, \widetilde{(T_{39})} : \left( \widetilde{(G_{39})}, \widetilde{(T_{39})} \right) = \mathcal{A}^{(7)}((G_{39}), (T_{39}))$
It results

$\left|\tilde{G}_{36}^{(1)} - \tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{36})^{(7)} \left|G_{37}^{(1)} - G_{37}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}}\, ds_{(36)} +$
$\int_0^t \{(a_{36}^{'})^{(7)} \left|G_{36}^{(1)} - G_{36}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} +$
$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) \left|G_{36}^{(1)} - G_{36}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}} +$
$G_{36}^{(2)} \left|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) - (a_{36}^{''})^{(7)}\left(T_{37}^{(2)}, s_{(36)}\right)\right| \; e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}} \} ds_{(36)}$

Where $s_{(36)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on it follows

$\left|(G_{39})^{(1)} - (G_{39})^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} t} \leq$
$\frac{1}{(\widehat{M}_{36})^{(7)}} \left((a_{36})^{(7)} + (a_{36}^{'})^{(7)} + (\widehat{A}_{36})^{(7)} + (\widehat{P}_{36})^{(7)} (\widehat{k}_{36})^{(7)}\right) d\left(((G_{39})^{(1)}, (T_{39})^{(1)};\ (G_{39})^{(2)}, (T_{39})^{(2)})\right)$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

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**Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)} e^{(\widehat{M}_{36})^{(7)} t}$ *and* $(\widehat{Q}_{36})^{(7)} e^{(\widehat{M}_{36})^{(7)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36{,}37{,}38$ depend only on $T_{37}$ and respectively on $(G_{39})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition.

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**Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$

it results

$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)} (T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$

$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0

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**Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2$ *and* $\left((\widehat{M}_{36})^{(7)}\right)_3$ :

**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}$ *and* $G_{38}$ . indeed if

$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating

$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)} \left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$

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| In the same way , one can obtain<br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}')^{(7)}$<br>If $G_{37}$ *or* $G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}$ *and* $G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(7)}((G_{39})(t),t)) = (b_{37}')^{(7)}$ then $T_{37} \to \infty$.<br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br>Indeed let $t_7$ be so that for $t > t_7$<br>$(b_{37})^{(7)} - (b_i'')^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right)$, $t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}'')^{(7)}\left((G_{39})(t),t\right) = (b_{38}')^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)} + \left((\widehat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\widehat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{40})^{(8)}\right] \leq (\widehat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |

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| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$ $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\,ds_{(40)}+$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\le$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ *and* $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i=40,41,42$ depend only on $\mathrm{T}_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$<br><br>it results<br><br>$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(8)}-(a_i^{''})^{(8)}(T_{41}(s_{(40)}),s_{(40)})\}ds_{(40)}\right]}\ge 0$<br><br>$T_i\,(t)\ge T_i^0e^{\left(-(b_i^{'})^{(8)}t\right)}>0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1,\left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ : | 276 |

**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if

$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a'_{41})^{(8)}G_{41}$ and by integrating

$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1/(a'_{41})^{(8)}$

In the same way , one can obtain

$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2/(a'_{42})^{(8)}$

If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively.

**Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. 277

**Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t),t)) = (b'_{41})^{(8)}$ then $T_{41} \to \infty$. 278

**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :

Indeed let $t_8$ be so that for $t > t_8$

$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t),t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$

Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to 279

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right),\ t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded. The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t),t(t),t\right) = (b'_{42})^{(8)}$

It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have 279 A

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\left(((G_{47})^{(1)},(T_{47})^{(1)}),((G_{47})^{(2)},(T_{47})^{(2)})\right)=$$
$$sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})},\widetilde{(T_{47})}:\left(\widetilde{(G_{47})},\widetilde{(T_{47})}\right)=\mathcal{A}^{(9)}\left((G_{47}),(T_{47})\right)$
It results

$$\left|\tilde{G}_{44}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{44})^{(9)}\left|G_{45}^{(1)}-G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)}+$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)-(a_{44}^{''})^{(9)}\left(T_{45}^{(2)},s_{(44)}\right)\right|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t}\le$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)}+(a_{44}^{'})^{(9)}+(\widehat{A}_{44})^{(9)}+(\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)},(T_{47})^{(1)};\ (G_{47})^{(2)},(T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i=44{,}45{,}46$ depend only on $\mathrm{T}_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$

From 99 to 44 it results

$$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(9)}-(a_i^{''})^{(9)}(T_{45}(s_{(44)}),s_{(44)})\}ds_{(44)}\right]}\ge 0$$

$$T_i\,(t)\ge T_i^0e^{(-(b_i^{'})^{(9)}t)}>0\quad\text{for t}>0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1,\left((\widehat{M}_{44})^{(9)}\right)_2$ *and* $\left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}$ *and* $G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)} G_{45}$ and by integrating

$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$

In the same way , one can obtain

$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$

If $G_{45}$ *or* $G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}$ *and* $G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t),t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t),t\right) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\left((G_{47})(t),t\right) = (b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} - (a_{13}^{''})^{(1)}(T_{14},t) + (a_{14}^{''})^{(1)}(T_{14},t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(v_1)^{(1)}, (v_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, v^{(1)}, u^{(1)}$ : 281

By $(v_1)^{(1)} > 0$ , $(v_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$ , $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

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| **Definition of** $(\bar{v}_1)^{(1)},,(\bar{v}_2)^{(1)},(\bar{u}_1)^{(1)},(\bar{u}_2)^{(1)}$ :<br><br>By $(\bar{v}_1)^{(1)}>0$ , $(\bar{v}_2)^{(1)}<0$ and respectively $(\bar{u}_1)^{(1)}>0$ , $(\bar{u}_2)^{(1)}<0$ the roots of the equations $(a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_2)^{(1)}v^{(1)}-(a_{13})^{(1)}=0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2+(\tau_2)^{(1)}u^{(1)}-(b_{13})^{(1)}=0$ | 282 |
| **Definition of** $(m_1)^{(1)}$ , $(m_2)^{(1)}$ , $(\mu_1)^{(1)},(\mu_2)^{(1)},(\nu_0)^{(1)}$ :-<br><br>If we define $(m_1)^{(1)}$ , $(m_2)^{(1)}$ , $(\mu_1)^{(1)},(\mu_2)^{(1)}$ by<br>$(m_2)^{(1)}=(\nu_0)^{(1)},(m_1)^{(1)}=(\nu_1)^{(1)},\ if\ (\nu_0)^{(1)}<(\nu_1)^{(1)}$<br><br>$(m_2)^{(1)}=(\nu_1)^{(1)},(m_1)^{(1)}=(\bar{\nu}_1)^{(1)}\ ,if\ (\nu_1)^{(1)}<(\nu_0)^{(1)}<(\bar{\nu}_1)^{(1)},$<br><br>and $\boxed{(\nu_0)^{(1)}=\frac{G_{13}^0}{G_{14}^0}}$<br><br>$(\ m_2)^{(1)}=(\nu_1)^{(1)},(m_1)^{(1)}=(\nu_0)^{(1)},\ if\ (\bar{\nu}_1)^{(1)}<(\nu_0)^{(1)}$ | 283 |
| and analogously<br><br>$(\mu_2)^{(1)}=(u_0)^{(1)},(\mu_1)^{(1)}=(u_1)^{(1)},\ if\ (u_0)^{(1)}<(u_1)^{(1)}$<br><br>$(\mu_2)^{(1)}=(u_1)^{(1)},(\mu_1)^{(1)}=(\bar{u}_1)^{(1)}\ ,if\ (u_1)^{(1)}<(u_0)^{(1)}<(\bar{u}_1)^{(1)},$<br><br>and $\boxed{(u_0)^{(1)}=\frac{T_{13}^0}{T_{14}^0}}$<br><br>$(\ \mu_2)^{(1)}=(u_1)^{(1)},(\mu_1)^{(1)}=(u_0)^{(1)},if\ (\bar{u}_1)^{(1)}<(u_0)^{(1)}$ where $(u_1)^{(1)},(\bar{u}_1)^{(1)}$<br><br>are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{13}^0e^{((S_1)^{(1)}-(p_{13})^{(1)})t}\le G_{13}(t)\le G_{13}^0e^{(S_1)^{(1)}t}$<br><br>where $(p_i)^{(1)}$ is defined by equation<br><br>$\frac{1}{(m_1)^{(1)}}G_{13}^0e^{((S_1)^{(1)}-(p_{13})^{(1)})t}\le G_{14}(t)\le\frac{1}{(m_2)^{(1)}}G_{13}^0e^{(S_1)^{(1)}t}$ | 285 |
| $(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t}-e^{-(S_2)^{(1)}t}\right]+G_{15}^0e^{-(S_2)^{(1)}t}\le G_{15}(t)\le$<br><br>$\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a_{15}^{'})^{(1)})}[e^{(S_1)^{(1)}t}-e^{-(a_{15}^{'})^{(1)}t}]+\ G_{15}^0e^{-(a_{15}^{'})^{(1)}t})$ | 286 |
| $\boxed{T_{13}^0e^{(R_1)^{(1)}t}\le T_{13}(t)\le T_{13}^0e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}}T_{13}^0e^{(R_1)^{(1)}t}\le T_{13}(t)\le\frac{1}{(\mu_2)^{(1)}}T_{13}^0e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |

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| $\frac{(b_{15})^{(1)}T_{13}^{0}}{(\mu_1)^{(1)}((R_1)^{(1)}-(b_{15}^{'})^{(1)})}\left[e^{(R_1)^{(1)}t}-e^{-(b_{15}^{'})^{(1)}t}\right]+T_{15}^{0}e^{-(b_{15}^{'})^{(1)}t}\le T_{15}(t)\le$ $\frac{(a_{15})^{(1)}T_{13}^{0}}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t}-e^{-(R_2)^{(1)}t}\right]+T_{15}^{0}e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)},(S_2)^{(1)},(R_1)^{(1)},(R_2)^{(1)}$:- Where $(S_1)^{(1)}=(a_{13})^{(1)}(m_2)^{(1)}-(a_{13}^{'})^{(1)}$ $(S_2)^{(1)}=(a_{15})^{(1)}-(p_{15})^{(1)}$ $(R_1)^{(1)}=(b_{13})^{(1)}(\mu_2)^{(1)}-(b_{13}^{'})^{(1)}$ $(R_2)^{(1)}=(b_{15}^{'})^{(1)}-(r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ : $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)}\le-(a_{16}^{'})^{(2)}+(a_{17}^{'})^{(2)}-(a_{16}^{''})^{(2)}(\mathrm{T}_{17},t)+(a_{17}^{''})^{(2)}(\mathrm{T}_{17},t)\le-(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)}\le-(b_{16}^{'})^{(2)}+(b_{17}^{'})^{(2)}-(b_{16}^{''})^{(2)}\big((G_{19}),t\big)-(b_{17}^{''})^{(2)}\big((G_{19}),t\big)\le-(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(v_1)^{(2)},(v_2)^{(2)},(u_1)^{(2)},(u_2)^{(2)}$ : | 295 |
| By $(v_1)^{(2)}>0$ , $(v_2)^{(2)}<0$ and respectively $(u_1)^{(2)}>0$ , $(u_2)^{(2)}<0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(v^{(2)}\big)^2+(\sigma_1)^{(2)}v^{(2)}-(a_{16})^{(2)}=0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2+(\tau_1)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ and | 298 |
| **Definition of** $(\bar{v}_1)^{(2)},,(\bar{v}_2)^{(2)},(\bar{u}_1)^{(2)},(\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{v}_1)^{(2)}>0$ , $(\bar{v}_2)^{(2)}<0$ and respectively $(\bar{u}_1)^{(2)}>0$ , $(\bar{u}_2)^{(2)}<0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(v^{(2)}\big)^2+(\sigma_2)^{(2)}v^{(2)}-(a_{16})^{(2)}=0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2+(\tau_2)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ | 302 |
| **Definition of** $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)}=(v_0)^{(2)},(m_1)^{(2)}=(v_1)^{(2)}$, ***if*** $(v_0)^{(2)}<(v_1)^{(2)}$ | 305 |

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| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , $\boldsymbol{if}$ $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, $\boldsymbol{if}$ $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, $\boldsymbol{if}$ $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , $\boldsymbol{if}$ $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$,<br>and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, $\boldsymbol{if}$ $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)} t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)} t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)} t}\right] + G_{18}^0 e^{-(S_2)^{(2)} t} \le G_{18}(t) \le \right.$<br>$\left.\frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a'_{18})^{(2)})}\left[e^{(S_1)^{(2)} t} - e^{-(a'_{18})^{(2)} t}\right] + G_{18}^0 e^{-(a'_{18})^{(2)} t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b'_{18})^{(2)})}\left[e^{(R_1)^{(2)} t} - e^{-(b'_{18})^{(2)} t}\right] + T_{18}^0 e^{-(b'_{18})^{(2)} t} \le T_{18}(t) \le$<br>$\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)} + (r_{16})^{(2)} + (R_2)^{(2)})}\left[e^{((R_1)^{(2)} + (r_{16})^{(2)})t} - e^{-(R_2)^{(2)} t}\right] + T_{18}^0 e^{-(R_2)^{(2)} t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a'_{16})^{(2)}$<br>$(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b'_{16})^{(2)}$ | 318 |

$(R_2)^{(2)} = (b'_{18})^{(2)} - (r_{18})^{(2)}$

**Behavior of the solutions** 319

**Theorem 3:** If we denote and define

**Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ :

$(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying

$$-(\sigma_2)^{(3)} \le -(a'_{20})^{(3)} + (a'_{21})^{(3)} - (a''_{20})^{(3)}(T_{21}, t) + (a''_{21})^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$$

$$-(\tau_2)^{(3)} \le -(b'_{20})^{(3)} + (b'_{21})^{(3)} - (b''_{20})^{(3)}(G_{23}, t) - (b''_{21})^{(3)}\big((G_{23}), t\big) \le -(\tau_1)^{(3)}$$

**Definition of** $(\nu_1)^{(3)}, (\nu_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ : 320

By $(\nu_1)^{(3)} > 0$ , $(\nu_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$ , $(u_2)^{(3)} < 0$ the roots of the equations

$(a_{21})^{(3)}\big(\nu^{(3)}\big)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\big(u^{(3)}\big)^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and

By $(\bar{\nu}_1)^{(3)} > 0$ , $(\bar{\nu}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$ , $(\bar{u}_2)^{(3)} < 0$ the

roots of the equations $(a_{21})^{(3)}\big(\nu^{(3)}\big)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\big(u^{(3)}\big)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$

**Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :- 321

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ , ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,

and $\boxed{(\nu_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously 322

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$ , ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T^0_{20}}{T^0_{21}}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

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| $G_{20}^0 e^{((S_1)^{(3)}-(p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)}t}$<br>$(p_i)^{(3)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)}-(p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)}G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)}-(p_{20})^{(3)}-(S_2)^{(3)})}\left[e^{((S_1)^{(3)}-(p_{20})^{(3)})t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \le G_{22}(t) \le \right.$<br>$\left.\frac{(a_{22})^{(3)}G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)}-(a'_{22})^{(3)})}[e^{(S_1)^{(3)}t} - e^{-(a'_{22})^{(3)}t}] + G_{22}^0 e^{-(a'_{22})^{(3)}t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)}+(r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)}+(r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)}T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)}-(b'_{22})^{(3)})}\left[e^{(R_1)^{(3)}t} - e^{-(b'_{22})^{(3)}t}\right] + T_{22}^0 e^{-(b'_{22})^{(3)}t} \le T_{22}(t) \le$<br>$\frac{(a_{22})^{(3)}T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)}+(r_{20})^{(3)}+(R_2)^{(3)})}\left[e^{((R_1)^{(3)}+(r_{20})^{(3)})t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$ | 327 |
| **Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:-<br>Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$<br>$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$<br>$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$<br>$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$ | 328 |
| **Behavior of the solutions of equation**<br>**Theorem:** If we denote and define<br>**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :<br>$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying<br>$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$<br>$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ :<br>By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}(v^{(4)})^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$<br>and $(b_{25})^{(4)}(u^{(4)})^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and | 329 |

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| **Definition of** $(\bar{v}_1)^{(4)},,(\bar{v}_2)^{(4)},(\bar{u}_1)^{(4)},(\bar{u}_2)^{(4)}$ :<br><br>By $(\bar{v}_1)^{(4)}>0$ , $(\bar{v}_2)^{(4)}<0$ and respectively $(\bar{u}_1)^{(4)}>0$ , $(\bar{u}_2)^{(4)}<0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_2)^{(4)}v^{(4)}-(a_{24})^{(4)}=0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2+(\tau_2)^{(4)}u^{(4)}-(b_{24})^{(4)}=0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)},(v_0)^{(4)}$ :-<br><br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)}$ by<br><br>$(m_2)^{(4)}=(v_0)^{(4)},(m_1)^{(4)}=(v_1)^{(4)}$, ***if*** $(v_0)^{(4)}<(v_1)^{(4)}$<br><br>$(m_2)^{(4)}=(v_1)^{(4)},(m_1)^{(4)}=(\bar{v}_1)^{(4)}$ , ***if*** $(v_4)^{(4)}<(v_0)^{(4)}<(\bar{v}_1)^{(4)}$,<br>and $\boxed{(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}$<br><br>$(m_2)^{(4)}=(v_4)^{(4)},(m_1)^{(4)}=(v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)}<(v_0)^{(4)}$ | 330 |
| and analogously<br><br>$(\mu_2)^{(4)}=(u_0)^{(4)},(\mu_1)^{(4)}=(u_1)^{(4)}$, ***if*** $(u_0)^{(4)}<(u_1)^{(4)}$<br><br>$(\mu_2)^{(4)}=(u_1)^{(4)},(\mu_1)^{(4)}=(\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)}<(u_0)^{(4)}<(\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)}=\frac{T_{24}^0}{T_{25}^0}}$<br><br>$(\mu_2)^{(4)}=(u_1)^{(4)},(\mu_1)^{(4)}=(u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)}<(u_0)^{(4)}$ where $(u_1)^{(4)},(\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{24}^0e^{((S_1)^{(4)}-(p_{24})^{(4)})t}\le G_{24}(t)\le G_{24}^0e^{(S_1)^{(4)}t}$<br><br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}}G_{24}^0e^{((S_1)^{(4)}-(p_{24})^{(4)})t}\le G_{25}(t)\le\frac{1}{(m_2)^{(4)}}G_{24}^0e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)}G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t}-e^{-(S_2)^{(4)}t}\right]+G_{26}^0e^{-(S_2)^{(4)}t}\le G_{26}(t)\le\right.$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0e^{(R_1)^{(4)}t}\le T_{24}(t)\le T_{24}^0e^{((R_1)^{(4)}+(r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}}T_{24}^0e^{(R_1)^{(4)}t}\le T_{24}(t)\le\frac{1}{(\mu_2)^{(4)}}T_{24}^0e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)}T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}^{'})^{(4)})}\left[e^{(R_1)^{(4)}t}-e^{-(b_{26}^{'})^{(4)}t}\right]+T_{26}^0e^{-(b_{26}^{'})^{(4)}t}\le T_{26}(t)\le$<br><br>$\frac{(a_{26})^{(4)}T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t}-e^{-(R_2)^{(4)}t}\right]+T_{26}^0e^{-(R_2)^{(4)}t}$ | 336 |

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| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a'_{24})^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b'_{24})^{(4)}$<br>$(R_2)^{(4)} = (b'_{26})^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a'_{28})^{(5)} + (a'_{29})^{(5)} - (a''_{28})^{(5)}(T_{29}, t) + (a''_{29})^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b'_{28})^{(5)} + (b'_{29})^{(5)} - (b''_{28})^{(5)}((G_{31}), t) - (b''_{29})^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(\nu_1)^{(5)}, (\nu_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, \nu^{(5)}, u^{(5)}$ :<br>By $(\nu_1)^{(5)} > 0$, $(\nu_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}(\nu^{(5)})^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}(u^{(5)})^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | 339 |
| **Definition of** $(\bar{\nu}_1)^{(5)},, (\bar{\nu}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ :<br>By $(\bar{\nu}_1)^{(5)} > 0$, $(\bar{\nu}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$, $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}(\nu^{(5)})^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}(u^{(5)})^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}, (\nu_0)^{(5)}$ :-<br>If we define $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}$ by<br>$(m_2)^{(5)} = (\nu_0)^{(5)}, (m_1)^{(5)} = (\nu_1)^{(5)}$, ***if*** $(\nu_0)^{(5)} < (\nu_1)^{(5)}$<br>$(m_2)^{(5)} = (\nu_1)^{(5)}, (m_1)^{(5)} = (\bar{\nu}_1)^{(5)}$, ***if*** $(\nu_1)^{(5)} < (\nu_0)^{(5)} < (\bar{\nu}_1)^{(5)}$,<br>and $\boxed{(\nu_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}}}$<br>$(m_2)^{(5)} = (\nu_1)^{(5)}, (m_1)^{(5)} = (\nu_0)^{(5)}$, ***if*** $(\bar{\nu}_1)^{(5)} < (\nu_0)^{(5)}$ | 340 |
| and analogously<br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$ | 341 |

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| $(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , $\boldsymbol{if}$ $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$, and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$ $(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}, \boldsymbol{if}$ $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | |
| Then the solution of global equations satisfies the inequalities $G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)} t}$ where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \le G_{30}(t) \le$ *(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b_{30}')^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b_{30}')^{(5)} t}\right] + T_{30}^0 e^{-(b_{30}')^{(5)} t} \le T_{30}(t) \le$ $\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:- Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a_{28}')^{(5)}$ $(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$ $(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b_{28}')^{(5)}$ $(R_2)^{(5)} = (b_{30}')^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ : $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying $-(\sigma_2)^{(6)} \le -(a_{32}')^{(6)} + (a_{33}')^{(6)} - (a_{32}'')^{(6)}(T_{33}, t) + (a_{33}'')^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$ | 349 |

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| $-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}\big((G_{35}),t\big) - (b''_{33})^{(6)}\big((G_{35}),t\big) \le -(\tau_1)^{(6)}$ | |
| **Definition of** $(\nu_1)^{(6)}, (\nu_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, \nu^{(6)}, u^{(6)}$ :<br><br>By $(\nu_1)^{(6)} > 0$ , $(\nu_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{\nu}_1)^{(6)},, (\bar{\nu}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br><br>By $(\bar{\nu}_1)^{(6)} > 0$ , $(\bar{\nu}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}$ , $(m_2)^{(6)}$ , $(\mu_1)^{(6)}, (\mu_2)^{(6)}, (\nu_0)^{(6)}$ :-<br><br>If we define $(m_1)^{(6)}$ , $(m_2)^{(6)}$ , $(\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br><br>$(m_2)^{(6)} = (\nu_0)^{(6)}, (m_1)^{(6)} = (\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)} < (\nu_1)^{(6)}$<br><br>$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)} < (\nu_0)^{(6)} < (\bar{\nu}_1)^{(6)}$,<br>and $\boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$<br><br>$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)} < (\nu_0)^{(6)}$ | 351 |
| and analogously<br><br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br><br>$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$<br><br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G^0_{32}e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)}G^0_{32}}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G^0_{34}e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |

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| $T_{32}^{0}e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^{0}e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}}T_{32}^{0}e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}}T_{32}^{0}e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)}T_{32}^{0}}{(\mu_1)^{(6)}((R_1)^{(6)}-(b_{34}^{'})^{(6)})}\left[e^{(R_1)^{(6)}t}-e^{-(b_{34}^{'})^{(6)}t}\right]+T_{34}^{0}e^{-(b_{34}^{'})^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)}T_{32}^{0}}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t}-e^{-(R_2)^{(6)}t}\right]+T_{34}^{0}e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)},(S_2)^{(6)},(R_1)^{(6)},(R_2)^{(6)}$:- Where $(S_1)^{(6)}=(a_{32})^{(6)}(m_2)^{(6)}-(a_{32}^{'})^{(6)}$ $(S_2)^{(6)}=(a_{34})^{(6)}-(p_{34})^{(6)}$ $(R_1)^{(6)}=(b_{32})^{(6)}(\mu_2)^{(6)}-(b_{32}^{'})^{(6)}$ $(R_2)^{(6)}=(b_{34}^{'})^{(6)}-(r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(7)},(\sigma_2)^{(7)},(\tau_1)^{(7)},(\tau_2)^{(7)}$: $(\sigma_1)^{(7)},(\sigma_2)^{(7)},(\tau_1)^{(7)},(\tau_2)^{(7)}$ four constants satisfying $-(\sigma_2)^{(7)} \le -(a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}-(a_{36}^{''})^{(7)}(T_{37},t)+(a_{37}^{''})^{(7)}(T_{37},t) \le -(\sigma_1)^{(7)}$ $-(\tau_2)^{(7)} \le -(b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(b_{36}^{''})^{(7)}((G_{39}),t)-(b_{37}^{''})^{(7)}((G_{39}),t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)},(v_2)^{(7)},(u_1)^{(7)},(u_2)^{(7)},v^{(7)},u^{(7)}$: By $(v_1)^{(7)}>0,(v_2)^{(7)}<0$ and respectively $(u_1)^{(7)}>0,(u_2)^{(7)}<0$ the roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2+(\sigma_1)^{(7)}v^{(7)}-(a_{36})^{(7)}=0$ and $(b_{37})^{(7)}\left(u^{(7)}\right)^2+(\tau_1)^{(7)}u^{(7)}-(b_{36})^{(7)}=0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)},,(\bar{v}_2)^{(7)},(\bar{u}_1)^{(7)},(\bar{u}_2)^{(7)}$: By $(\bar{v}_1)^{(7)}>0,(\bar{v}_2)^{(7)}<0$ and respectively $(\bar{u}_1)^{(7)}>0,(\bar{u}_2)^{(7)}<0$ the roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2+(\sigma_2)^{(7)}v^{(7)}-(a_{36})^{(7)}=0$ and $(b_{37})^{(7)}\left(u^{(7)}\right)^2+(\tau_2)^{(7)}u^{(7)}-(b_{36})^{(7)}=0$ **Definition of** $(m_1)^{(7)},(m_2)^{(7)},(\mu_1)^{(7)},(\mu_2)^{(7)},(v_0)^{(7)}$:- | 362 |

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,

and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$

and analogously

$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,

and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

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Then the solution of global equations satisfies the inequalities

$$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)} t}$$

where $(p_i)^{(7)}$ is defined by equation

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$$\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)} t}$$

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$$\left(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})}\left[e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)} t}\right] + G_{38}^0 e^{-(S_2)^{(7)} t} \le G_{38}(t) \le \right.$$

$$\left. \frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a_{38}')^{(7)})}\left[e^{(S_1)^{(7)} t} - e^{-(a_{38}')^{(7)} t}\right] + G_{38}^0 e^{-(a_{38}')^{(7)} t}\right)$$

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$$\boxed{T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}}$$

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$$\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}$$

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$$\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b_{38}')^{(7)})}\left[e^{(R_1)^{(7)} t} - e^{-(b_{38}')^{(7)} t}\right] + T_{38}^0 e^{-(b_{38}')^{(7)} t} \le T_{38}(t) \le$$

$$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)} + (r_{36})^{(7)} + (R_2)^{(7)})}\left[e^{((R_1)^{(7)} + (r_{36})^{(7)})t} - e^{-(R_2)^{(7)} t}\right] + T_{38}^0 e^{-(R_2)^{(7)} t}$$

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| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a'_{36})^{(7)}$<br><br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br><br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b'_{36})^{(7)}$<br><br>$(R_2)^{(7)} = (b'_{38})^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br><br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br><br>$-(\sigma_2)^{(8)} \le -(a'_{40})^{(8)} + (a'_{41})^{(8)} - (a''_{40})^{(8)}(T_{41}, t) + (a''_{41})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br><br>$-(\tau_2)^{(8)} \le -(b'_{40})^{(8)} + (b'_{41})^{(8)} - (b''_{40})^{(8)}\big((G_{43}), t\big) - (b''_{41})^{(8)}\big((G_{43}), t\big) \le -(\tau_1)^{(8)}$ | 371 |
| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ :<br><br>By $(v_1)^{(8)} > 0, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$<br>and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | 372 |
| **Definition of** $(\bar{v}_1)^{(8)}, , (\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :<br><br>By $(\bar{v}_1)^{(8)} > 0, (\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0, (\bar{u}_2)^{(8)} < 0$ the<br><br>roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$<br><br>and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$<br><br>**Definition of** $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :-<br><br>If we define $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by<br><br>$(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)}$, ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$<br><br>$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}$, ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)}$, | |

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| and $\boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br><br>$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (v_0)^{(8)},\ \boldsymbol{if}\ (\bar{v}_1)^{(8)} < (v_0)^{(8)}$ | |
| and analogously<br><br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},\ \boldsymbol{if}\ (u_0)^{(8)} < (u_1)^{(8)}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}, \boldsymbol{if}\ (u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}, \boldsymbol{if}\ (\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)} t}$<br><br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \le G_{42}(t) \le \right.$<br>$\left.\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)} t} - e^{-(a'_{42})^{(8)} t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)} t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b'_{42})^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b'_{42})^{(8)} t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)} t} \le T_{42}(t) \le$<br><br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br><br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br><br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$ | 381 |

$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$

$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$

**Behavior of the solutions of equation 37 to 92** 382

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$

$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}\big((G_{47}), t\big) - (b''_{45})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$

**Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :

By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(9)}, , (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{\nu}_1)^{(9)} > 0$ , $(\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}$, ***if*** $(\nu_0)^{(9)} < (\nu_1)^{(9)}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}$, ***if*** $(\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T^0_{44}}{T^0_{45}}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^{0}e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^{0}e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^{0}e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}}G_{44}^{0}e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^{0}e^{-(S_2)^{(9)}t} \le G_{46}(t) \le \right.$$

$$\left.\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}^{'})^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}^{'})^{(9)}t}]+\ G_{46}^{0}e^{-(a_{46}^{'})^{(9)}t}\right)$$

$$\boxed{T_{44}^{0}e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^{0}e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}}T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^{0}}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}^{'})^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}^{'})^{(9)}t}\right]+T_{46}^{0}e^{-(b_{46}^{'})^{(9)}t} \le T_{46}(t) \le$$

$$\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a_{44}^{'})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b_{44}^{'})^{(9)}$

$(R_2)^{(9)} = (b_{46}^{'})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain

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$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a_{13}^{'})^{(1)} - (a_{14}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14},t)\right) - (a_{14}^{''})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}(v^{(1)})^{2} + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}(v^{(1)})^{2} + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (\nu_1)^{(1)} < (\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t) \geq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(\nu_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\nu_1)^{(1)}$

In the same manner , we get 384

$$\nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{\nu}_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\bar{\nu}_1)^{(1)}$

If $0 < (\nu_1)^{(1)} < (\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{\nu}_1)^{(1)}$ we find like in the previous case, 385

$$(\nu_1)^{(1)} \leq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}} \leq \nu^{(1)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (C)^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$ if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \le \frac{d\nu^{(2)}}{dt} \le -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \ge \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \le \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \le \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \le \nu^{(2)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$$

If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain 394

$$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(2)}(t)$ :- 395

$$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}, \quad \boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$$

In a completely analogous way, we obtain 396

**Definition of** $u^{(2)}(t)$ :-

$$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}, \quad \boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :** 397

If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$

Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then

$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$

**Proof** : From global equations we obtain 398

$$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right) - (a_{21}^{''})^{(3)}(T_{21},t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$$

**Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$ 399

It follows

$$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$$

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From which one obtains

For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

In the same manner , we get

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$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

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If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case,

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

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If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$ if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

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In the same manner , we get

$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

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$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}'')^{(5)} = (b_{29}'')^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then

$(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a'_{32})^{(6)} - (a'_{33})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right) - (a''_{33})^{(6)}(T_{33},t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \le \frac{d\nu^{(6)}}{dt} \le -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \ge \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \le \nu^{(6)}(t) \le (\nu_1)^{(6)}$

In the same manner , we get

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$$\nu^{(6)}(t) \le \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \le \nu^{(6)}(t) \le (\bar{\nu}_1)^{(6)}$

If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

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$$(\nu_1)^{(6)} \le \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}} \le \nu^{(6)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \le (\bar{\nu}_1)^{(6)}$$

If $0 < (\nu_1)^{(6)} \le (\bar{\nu}_1)^{(6)} \le \boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$ , we obtain

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$$(\nu_1)^{(6)} \le \nu^{(6)}(t) \le \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \le (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}$, $\boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

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$$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$$

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If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain

$$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(C)^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}$, $\boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$

In a completely analogous way, we obtain

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**Definition of** $u^{(7)}(t)$ :-

$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}'')^{(7)} = (a_{37}'')^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}'')^{(7)} = (b_{37}'')^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right) - (a_{41}^{''})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \le \frac{dv^{(8)}}{dt} \le -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \ge \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \le \nu^{(8)}(t) \le (\nu_1)^{(8)}$

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In the same manner , we get

$$\nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \le \nu^{(8)}(t) \le (\bar{\nu}_8)^{(8)}$

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If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case,

$$(\nu_1)^{(8)} \le \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_2)^{(8)}\right)t\right]}} \le \nu^{(8)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(8)} + (C)^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\bar{\nu}_1)^{(8)}$$

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If $0 < (\nu_1)^{(8)} \le (\bar{\nu}_1)^{(8)} \le \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain

$$(\nu_1)^{(8)} \le \nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(8)}(t)$ :-

$(m_2)^{(8)} \leq v^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{v^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $v^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)}-(\nu_0)^{(9)}}{(\nu_0)^{(9)}-(\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \le \frac{(\nu_1)^{(9)}+(C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}} \le \nu^{(9)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \le (\bar{\nu}_1)^{(9)} \le \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(\nu_1)^{(9)} \le \nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \le \nu^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \le u^{(9)}(t) \le (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)}G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

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| $(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,<br>$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$<br>$with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$ | 433 |

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| $(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$ | 436 A |

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| $(a'_{44})^{(9)}(a'_{45})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a'_{45})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b'_{44})^{(9)}(r_{45})^{(9)} - (b'_{45})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b'_{13})^{(1)} - (b''_{13})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b'_{14})^{(1)} - (b''_{14})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b'_{15})^{(1)} - (b''_{15})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b'_{16})^{(2)} - (b''_{16})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b'_{17})^{(2)} - (b''_{17})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b'_{18})^{(2)} - (b''_{18})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a'_{20})^{(3)} + (a''_{20})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a'_{21})^{(3)} + (a''_{21})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a'_{22})^{(3)} + (a''_{22})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b'_{20})^{(3)} - (b''_{20})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b'_{21})^{(3)} - (b''_{21})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b'_{22})^{(3)} - (b''_{22})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) +$ | 485 |

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| $(a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | |
| **Proof:**<br><br>(d) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |

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| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492<br>A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations | 498 |

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| $G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |
| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 501 |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br><br>$\left[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G) + (b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)\right]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that $\varphi$ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that | 502 |

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| there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*})=0$ | |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br>$\varphi(G_{19})=(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)}-(b_{16})^{(2)}(b_{17})^{(2)}-$<br>$\left[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19})+(b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})\right]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19})=0$ | 503 |
| Where in $(G_{19})(G_{16},G_{17},G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*})=0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br>$\varphi(G_{23})=(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)}-(b_{20})^{(3)}(b_{21})^{(3)}-$<br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23})+(b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23})=0$<br>Where in $G_{23}(G_{20},G_{21},G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*})=0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br>$\varphi(G_{27})=(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)}-(b_{24})^{(4)}(b_{25})^{(4)}-$<br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27})+(b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27})=0$<br>Where in $(G_{27})(G_{24},G_{25},G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*})=0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31})=(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)}-(b_{28})^{(5)}(b_{29})^{(5)}-$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31})+(b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right]+(b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31})=0$<br>Where in $(G_{31})(G_{28},G_{29},G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*})=0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35})=(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)}-(b_{32})^{(6)}(b_{33})^{(6)}-$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35})+(b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right]+(b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35})=0$<br>Where in $(G_{35})(G_{32},G_{33},G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that | 508 |

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| there exists a unique $G_{33}^{*}$ such that $\varphi(G_{35}{}^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi(G_{39}{}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi(G_{43}{}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0\,, T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0\,, T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |

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| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^*$ given by $\varphi((G_{23})^*) = 0$ , $T_{21}^*$ given by $f(T_{21}^*) = 0$ and<br><br>$G_{20}^* = \frac{(a_{20})^{(3)} G_{21}^*}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^*)]}$ , $G_{22}^* = \frac{(a_{22})^{(3)} G_{21}^*}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^*)]}$<br><br>$T_{20}^* = \frac{(b_{20})^{(3)} T_{21}^*}{[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}{}^*)]}$ , $T_{22}^* = \frac{(b_{22})^{(3)} T_{21}^*}{[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}{}^*)]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br><br>$G_{25}^*$ given by $\varphi(G_{27}) = 0$ , $T_{25}^*$ given by $f(T_{25}^*) = 0$ and<br><br>$G_{24}^* = \frac{(a_{24})^{(4)} G_{25}^*}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}^*)]}$ , $G_{26}^* = \frac{(a_{26})^{(4)} G_{25}^*}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}^*)]}$ | 516 |
| $T_{24}^* = \frac{(b_{24})^{(4)} T_{25}^*}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27})^*)]}$ , $T_{26}^* = \frac{(b_{26})^{(4)} T_{25}^*}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27})^*)]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br><br>$G_{29}^*$ given by $\varphi((G_{31})^*) = 0$ , $T_{29}^*$ given by $f(T_{29}^*) = 0$ and<br><br>$G_{28}^* = \frac{(a_{28})^{(5)} G_{29}^*}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^*)]}$ , $G_{30}^* = \frac{(a_{30})^{(5)} G_{29}^*}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^*)]}$ | 518 |
| $T_{28}^* = \frac{(b_{28})^{(5)} T_{29}^*}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^*)]}$ , $T_{30}^* = \frac{(b_{30})^{(5)} T_{29}^*}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^*)]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br><br>$G_{33}^*$ given by $\varphi((G_{35})^*) = 0$ , $T_{33}^*$ given by $f(T_{33}^*) = 0$ and<br><br>$G_{32}^* = \frac{(a_{32})^{(6)} G_{33}^*}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^*)]}$ , $G_{34}^* = \frac{(a_{34})^{(6)} G_{33}^*}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^*)]}$ | 520 |
| $T_{32}^* = \frac{(b_{32})^{(6)} T_{33}^*}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^*)]}$ , $T_{34}^* = \frac{(b_{34})^{(6)} T_{33}^*}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^*)]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br><br>$G_{37}^*$ given by $\varphi((G_{39})^*) = 0$ , $T_{37}^*$ given by $f(T_{37}^*) = 0$ and | 522 |

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| $G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)} (T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)} (T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)} ((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)} ((G_{39})^{*})]}$ | |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)} (T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)} (T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)} ((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)} ((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br><br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br><br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)} (T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)} (T_{45}^{*})]}$<br><br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)} ((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)} ((G_{47})^{*})]}$ | 523 A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:**_Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}} (T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j} (G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*} \mathbb{T}_{14}$ | 525 |

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| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^{*}\mathbb{G}_{j}\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^{*}\mathbb{G}_{j}\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^{*}\mathbb{G}_{j}\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 532 |
| $\frac{\partial(a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial(b_i^{''})^{(2)}}{\partial G_j}((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^{*}\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | 540 |

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| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}} (T_{21}^*) = (q_{21})^{(3)} \ , \frac{\partial (b_i^{''})^{(3)}}{\partial G_j} ( (G_{23})^* ) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\big((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\big)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)} G_{20}^* \mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\big((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\big)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)} G_{21}^* \mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\big((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\big)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)} G_{22}^* \mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\big((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\big)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\big(s_{(20)(j)} T_{20}^* \mathbb{G}_j\big)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\big((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\big)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\big(s_{(21)(j)} T_{21}^* \mathbb{G}_j\big)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\big((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\big)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\big(s_{(22)(j)} T_{22}^* \mathbb{G}_j\big)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ *and* $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}} (T_{25}^*) = (q_{25})^{(4)} \ , \frac{\partial (b_i^{''})^{(4)}}{\partial G_j} ((G_{27})^* ) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\big((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\big)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)} G_{24}^* \mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\big((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\big)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)} G_{25}^* \mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\big((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\big)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)} G_{26}^* \mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\big((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\big)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\big(s_{(24)(j)} T_{24}^* \mathbb{G}_j\big)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\big((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\big)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\big(s_{(25)(j)} T_{25}^* \mathbb{G}_j\big)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\big((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\big)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\big(s_{(26)(j)} T_{26}^* \mathbb{G}_j\big)$ | 554 |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^*) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^*\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^*\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^*\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |

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| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{37})^{(7)}}{\partial T_{37}}(T^*_{37}) = (q_{37})^{(7)}$ , $\frac{\partial(b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a'_{36})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G^*_{36}\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a'_{37})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G^*_{37}\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. | |

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| Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \ , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}((G_{43})^*) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^*\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^*\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b_{41}^{'})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T_{41}^*\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b_{42}^{'})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T_{42}^*\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS** **Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. **Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |

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| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$ $\left[\left(\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\right]$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\right)$ $+\left(\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\right)$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((b'_{13})^{(1)} + (b'_{14})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $+\left(\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$ $+\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G^*_{13}\right)$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T^*_{14} + (b_{14})^{(1)}s_{(13),(15)}T^*_{13}\right)\} = 0$ $+$ $\left((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)}\right)\{\left((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)}\right)$ $\left[\left(\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}^*_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^*_{16}\right)\right]$ $\left(\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^*_{17}\right)$ $+\left(\left((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}^*_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^*_{17}\right)$ $\left(\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^*_{16}\right)$ $\left(\left((\lambda)^{(2)}\right)^2 + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ $\left(\left((\lambda)^{(2)}\right)^2 + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ | |

$$+\left(\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\right)\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left(\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\right)$$

$$+\left(\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((b_{20}')^{(3)}+(b_{21}')^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}')^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}')^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}')^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}')^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}')^{(4)}+(a_{25}')^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((b_{24}')^{(4)}+(b_{25}')^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\}\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left((\lambda)^{(6)}\right)^2+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G^*_{33}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G^*_{32}\right)$$

$$\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T^*_{33}+(b_{33})^{(6)}s_{(32),(34)}T^*_{32}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b'_{38})^{(7)}-(r_{38})^{(7)}\}\{(\lambda)^{(7)}+(a'_{38})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(q_{36})^{(7)}G^*_{36}\Big)\right]$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T^*_{37}+(b_{37})^{(7)}s_{(36),(37)}T^*_{37}\Big)$$

$$+\left((\lambda)^{(7)}+(a'_{37})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G^*_{36}+(a_{36})^{(7)}(q_{37})^{(7)}G^*_{37}\Big)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T^*_{37}+(b_{37})^{(7)}s_{(36),(36)}T^*_{36}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^*_{36}\right)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^*_{37}+(b_{37})^{(7)}s_{(36),(38)}T^*_{36}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\}\{(\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^*_{40}\Big)\right]$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^*_{41}+(b_{41})^{(8)}s_{(40),(41)}T^*_{41}\Big)$$

$$+\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^*_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^*_{41}\Big)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^*_{41}+(b_{41})^{(8)}s_{(40),(40)}T^*_{40}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^*_{40}\right)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^*_{41}+(b_{41})^{(8)}s_{(40),(42)}T^*_{40}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^*_{44}\right)\right]$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^*_{45}+(b_{45})^{(9)}s_{(44),(45)}T^*_{45}\Big)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^*_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45}+(b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left(\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45}+(b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

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# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i(x)\, xi$ where $x_i \in R^n_i$, $x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i(x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i(\mu)$ of the $A_i(x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k \tag{4}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to

replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 4. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon\ C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \ i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0}$

$\overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow *x.t restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set *K* is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** *x.t* is defined for all $t \geq 0$

**A4:** *x.t* is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\mathcal{A}(\tilde{A}), C \backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \longrightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) In\ ||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population *i* is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^{t}_{0} (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\to\infty}{limsup} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v = au_i(x) + w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |B_i(t,x)u_i(x)|(a + \alpha + exp(-\beta t)|w|)$. Hence, $\limsup_{t\to\infty} \frac{1}{t} In\, B_i(t,x)\, v| \leq$

$$\limsup_{t\to\infty} \frac{1}{t} In\, B_i(t,x)\, u_i(x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\, |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\, u_i(x)| = In\, |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\, u_i(x.t)||\, B_i(t,x)\, u_i(x)|$$

$$= \text{In } b_i(s,t.x)\, b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x - bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

$$=$$

$$= (A_i(x.t)u_i(x.s), u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s), u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } |(x.t)_i|$$

$$\leq \limsup_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x) =$ Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ *- almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of

irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x)u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$ where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ and $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as $n\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$. Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\le \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x)u_i^n(x), u_i^n(x)\rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)|d\mu(x) + |\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\,d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\,ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i\,dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\,d\,v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak$^*$ limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu}\ \max_{p\epsilon\Delta}\ \sum_i\ Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \text{min max } \lambda i(\mu) = \min_{\mu}\ \max_{p\epsilon\Delta}\ \sum_i\ Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \{ \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i \epsilon I}{if\ i \notin I}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that \quad \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t$ −invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently *K* is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s =$

$s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_s$. If $r(x) = s(x), then\ x \in M_r$. Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume $x \mapsto A(x)$ is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists $j \in \{1,\ldots,k\}$ such that $M_i$ ----- $\{x \in C : x_l \gg 0 \text{ for } l \neq j\}$ Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_i M_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, *$\lambda_i(\mu) > 0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, ..., x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the

system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j \ (x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1{}^1, \ldots, E_1{}^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i{}_2(E_1{}^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j \ x_2^j \ - \alpha^j \ x_3^j) + \sum d_1^{jk} \ x_1^k \ - e_1^j \ x_1^j$$

$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j\, x_1^j - x_2^j - \beta^j\, x_3^j) + \sum d_2^{jk}\, x_2^k - e_2^j\, x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j\, x_1^j - \alpha^j x_2^j - \beta^j\, x_3^j) + \sum d_3^{jk}\, x_3^k - e_3^j\, x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > 1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria $E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$ and $\lambda_3(E_1)$ are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\Big(r_i - \sum_s a_{is}x_s^j\Big) + \sum_l d_i^{jl}\, x_i^l - e_i^j x_i^j, \qquad i = 1,\ldots,k;\ j = 1,\ldots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i\, a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i\, x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \qquad i = 1, \ldots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \ldots, k$$

It has been shown [12,ch.16.1,16.2] That for $k = 3$,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is

the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y \end{pmatrix}$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my\text{-}b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \begin{pmatrix} \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1-y) - b(N)\ y \end{pmatrix}$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*′ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\curlywedge_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\curlywedge_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\curlywedge_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + K p\, x_i x_i - 1}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions

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# Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a

qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1$ *(t)* and $P_2$ *(t)* denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and

orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.10) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.11) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.12) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1, \varphi_2, f_1, f_2, b_1, b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for

each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as* $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 4. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C(R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C(R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i(a, P_1', P_2') - f_i(a, P_1'', P_2'')| \leq d_i\ [||P_1' - P_1'', P_2' - P_2''||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\partial P_1, \partial f_1/\partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\partial P_1, \partial f_2/\partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C(R^+ K_1 \times K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C(R^+ K_2 \times K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$|b_i(a, P_1', P_2') - b_i(a, P_1'', P_2'')| \leq \beta_i\ [||P_1' - P_1''|| + ||P_2' - P_2''||]$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2)) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, \text{R}^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[-\int_0^\infty f_i(a + s, P(t + S))ds\right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[-\int_o^x f_1(a + s, p(t + s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0)\exp\left[-\int_o^t f_1(a - t + s, p_1(s))\, ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

*(2.3)* $p_i(a,t) = B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0,t), i = 1,2, t > 0.$$

Define $M_i and\ L_i\ (i = 1,2)$ as follows:

$$M_i(a,t,p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a,t,\ p_i(t)) = \exp[-\int_o^t f_1(a+s,\ p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

$$(2.6)\ p_i(t) = \int_o^t B_1(a) M_1(a,t,p_i(t)da + \int_o^\infty \varphi_i(a,0)\ L_i(a,t,p_i(t)da,$$

$$t > 0, i = 1,2,$$

$$(2.7)\ p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a,s)da, s \in [-\tau, 0], i = 1,2,$$

$$(2.8)\ \beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.\left(a,t,P_i(t)\right)da$$

$$+\int_0^\infty b_i\left(a+t, P_i(t)\right)\varphi_i(a,0)L_i\left(a,t,P_i(t)\right)da$$

$$i = 1, 2, t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t) \phi(0) d_i^*] t \quad for \ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T], R^+), \quad B_i \in C([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in C([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau, T]$. Then for any

$(x, y) \in C^+[-\tau, T] \times C^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau, T], \quad C_T^2 = C^+[-\tau, T] \times C^+[-\tau, T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for $i = 1,2$.

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \pi C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \quad P_t = \ (P_{1t}, P_{2t}) \in C^2_{t*} \mid \|P_t - \Phi_t\|_{t*} \le r^*$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)\, da \ for\ s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)\, da \ for\ s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Phi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^* \right] \quad t^*$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for t $\le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \le \left| \int_0^t B_1^{(1)}(a) \left\{ M_1(a, t, P_t^{(1)}) - M_1(a, t, P_t^{(2)}) \right\} da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a, t, P_t^{(2)}) da \right|$$

$$+ \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \| ds\right) da$$

$$\leq d_1 \beta_1^* \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o)\left[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds\right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \| t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\text{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right]\right] d\sigma$$

$$+ \int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^{\infty} \varphi(\sigma,o) b_1\left(\sigma+a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma,a,P_t^{(1)}(a)\right)-L_1\left(\sigma,a,P_t^{(2)}(a)\right)\right]d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right)-b_1\left(\sigma+a, P_t^{(2)}(a)\right)\right| \le B_1 \,/\!/\, P_t^{(1)}-P_t^{(2)} /\!/ \text{ for } a\,\varepsilon[o,t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \le \beta_1 \Phi_1(0)||P_t^{(1)}-P_t^{(2)}||_t^{*}+\Phi_1(0)\beta_1 d_1||P_t^{(1)}-P_t^{(2)}||_t^{*a},$$

$$|R_2(a)|=[\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a))-b_1(a-\sigma, P_t^{(2)}(a))\}M_1(\sigma, a, P_t^{(1)}(a))$$

$$+b_1(a,-\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a))-M_1(a-\sigma, P_t^{(2)}(a))\}]d\sigma]$$

$$+b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a))-M_1(a-\sigma, P_t^{(2)}(a))\}]d\sigma]$$

$$\le \int_0^a [\beta_1^{*}\Phi_1(0)\, e^{\delta 1^a}\beta_1 \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}+\beta_1 d_1 \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}(a-o)]do$$

$$\le \beta_1^{*}\Phi_1(0)\, e^{\delta 1^a} \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\{\beta_1 a+d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a)\right|$$

$$\le \int_o^a \left| B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma)\right| B_1^{*d}\sigma$$

$$+\Phi_1(0) /\!/ P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\left\{B_1+B_2 d_1 a+B_1^{*}\left(B_2 a+d_1 a^2/2\right)e^{\delta 1^a}\right\}$$

$$\le \beta_1^{*}\int_o^a \left| B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma)\right| do+\left(r^{*}+/\!/\Phi_t /\!/_{t*}\right)\| P_t^{(1)}-P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a)\to \beta_1$ as $a\to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a)\right| \le \left(r^{*}+/\!/\Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))\text{-}b_1(a\text{-}\sigma, P_t^{(1)}(a))\, M_1(\sigma, a, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + /\!/ \Phi_t /\!/_{t*}\right) /\!/ \; P_t^{(1)}\text{-}P_t^{(2)} /\!/_{t*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right) /\!/ \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^{*})$$

As $t^{*} \to o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^{*}, P_t^{*}$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp\, P_2^{*}(a)}{da} + f_i(\mathrm{a}, P_1^{*}, P_2^{*})\, P_i^{*}(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^{*} = \int_o^{\infty} P_i^{*}(a)da, i = 1,\ 2,$$

$$(3,3)\ P_i^{*} = \int_o^{\infty} b_i(a,\ P_1^{*},\ P_2^{*}) P_i^{*}(a)da,\ i = 1.\ \ 2$$

All solutions of (3.1)are of the from

$$(3.4)\ P_i^{*}(a) = P_i^{*}(o) \exp\left[ - \int_o^{\infty} f_i\,(s, P_1^{*}, P_2^{*})ds \right], i = 1, \quad 2, a \geq 0,$$

Which together with (3.3) lead to

$$1=\int_o^{\infty} b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^{\infty} b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $$F_{i.}\ (P_1,P_2) = \int_o^{\infty} b_i(a,P_1,P_2)\exp\left[-\int_o^{\infty} f_i\,(s,P_1,P_2)ds\right]da,$$

$$i = 1,\ 2, \quad P_1 \geq 0, \quad P_2 \geq 0,$$

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_5$. *there exists a positive constant c for which*

$$F_1(\mathrm{C},0) < F_{2.}(\mathrm{C},0),$$

(3.8) $$F_1(0.\mathrm{C}) < F_{2.}(0,\mathrm{C}),$$

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $$F_1(P_1^*,P_2^*) = 1 = F_2(P_1^*,P_2^*)$$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $$P_2^*(a) = [P_2^*\exp[-f_i\,(s,P_1^*,P_2^*)ds])/(\int_o^{\infty}\exp[-\int_o^a f_i\,(s,P_1^*,P_2^*)ds]da),$$

$$i = 1,\ 2, \quad a \geq 0,$$

Proof Consider the elements of the surfaces defined by

$$\mathrm{Z}= F_1(P_1,P_2), z = F_2(P_1,P_2),\ P_1 \geq 0, P_2 \geq 0,$$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i/\partial P_j < 0\ and$ , by ($A_2$), $\partial b_i/\partial P_j \leq 0$ for all

$$P_1 \geq 0, P_2 \geq 0, \quad i = 1,\ 2,$$

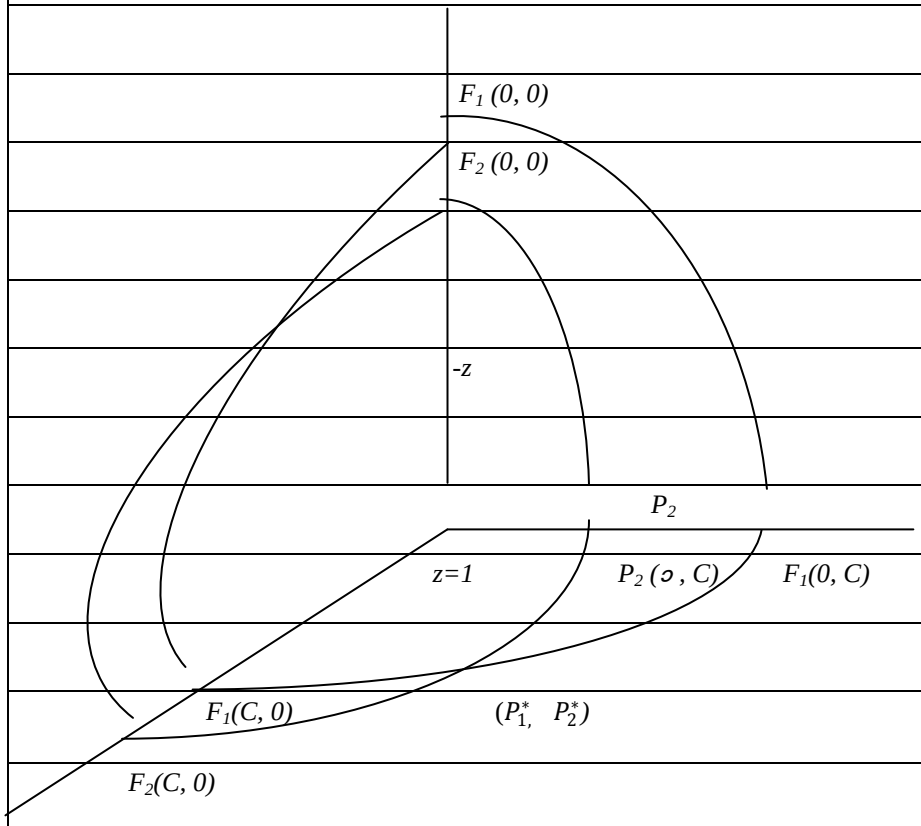
$j = 1,\ 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= 1, $P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$$F_1(P_1, P_2 = 1 \; and \; F_2(P_1, P_2) = 1, \; F_1(P_1, P_2 \geq 0,$$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

$$(3.11) \quad \begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*}\, p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square 1}^{2} \frac{\partial b_i}{\partial p_{\square}^*}\, p_{i\square}(t)\right\} da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_{\square}(a)\exp[\lambda t], \quad {}_{\square}(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty {}_{\square}(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d_{\square}}{da} + \lambda_{\square} = -f_i\left(a, p_1^*, p_2^*\right)_{\square}(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*}\, p_{i\square}^*\right\},$$

(3.15)

$$_{\square}(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)_{\square}(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*}\, p_{i\square}^*\right\} da,$$

Where $i = 1, 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \quad p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\Phi_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\Phi_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\Phi_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\Phi_i(a) = \{\Phi_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$

$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_j\right\} e^{\lambda s} ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\Phi_i(a)\left[p_i^*(a)/p_i^*(0)c_i \exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_j\right\} da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a}\left\{\int_0^a \left(\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_j\right) e^{\lambda s} ds\right\} da\right]$$

$$- p_i^*(a) e^{-\lambda a}\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_j\right\} e^{\lambda s} ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\Phi_1(a)$ and $\Phi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$\left(\lambda - A_{11}\right) p_1^* - A_{12} p_2^* \exp\left(-\lambda\tau\right) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21} p_1^* \exp\left(-\lambda\tau\right) + \left(\lambda - A_{22}\right) p_2^* = g_{21}(\lambda) p_1^* \exp\left(-\lambda\tau\right) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $A_{ij} = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_j^*} - \frac{\partial f_i}{\partial p_j^*}\right]da,$

(3.23) $g_{ij}(\lambda) = [p_i^*(0)c_i]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\,da\right\}$
$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_j^*} - b_i(a, p_1^*, p_2^*)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)\right]da\right\}$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)da,\quad i, j = 1, 2,$$

(3.24) $\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*),\quad i, = 1, 2.$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix}\lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda))\end{bmatrix} = 0$

Or equivalently

(3.25) $\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = [A_{21}g_{12}(\lambda) + A_{12}g_{21}(\lambda) + g_{12}(\lambda)g_{21}(\lambda)]\exp[-2\lambda\tau]$

$-A_{11}g_{22}(\lambda) - A_{22}g_{11}(\lambda) - g_{11}(\lambda)g_{22}(\lambda)$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i, = 1, 2.)\ (see(A_1)\ and\ (A_2))$;

$A_7.\ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\},$

$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$

$\| S_2(0) \| = |A_{11}g_{12}(0)| + |A_{12}g_{21}(0)| + |g_{12}(0)g_{21}(0)|$

$+ |A_{11}g_{22}(0)| + |A_{22}g_{11}(0)| + |g_{11}(0)g_{22}(0)|$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i(a, t) = \phi_i(a) \exp[\lambda t] \to 0$ as $t \to \infty$ for $i = 1, 2$ and $a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda$ $\mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$

$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \left|\right.$

$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$

$\geq [(|\lambda| - P)^2 + (q)]^2$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \{4 | (A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| | - | A_{11} + A_{22} + \| S_1(0) \| |^2\}/2,$

It will now follow, from (A) and (3.26), $F_1(\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder

concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power

obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy

withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation

from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

**TITLE**

## Illusions Of Transcendence.

**INTRODUCTION—VARIABLES USED**

**66 www.parrhesiajournal.org DELEUZE AND THE QUESTION OF DESIRE: TOWARD AN IMMANENT THEORY OF ETHICS: Daniel W. Smith. Some sentences have been deleted due to spatial restrictions. Kindly pardon me. All these can be modelled on similar lines and are preeminently and predominantly cast in the model format. Models are given in the one of the papers of the series.**

(1) Conversely, a mode of existence will be considered to be “good” (or free, or rational, or strong) that exercises its capacity for being affected in such a way that its power of acting increases, to the point where (e) it produces active affections and adequate ideas.
(2) For Deleuze, this is the point of convergence that unites Nietzsche and Spinoza in their search for an immanent ethics.
(3) Modes are no longer “judged” in terms of their degree of proximity to or distance from (e) an external principle, but are “evaluated” in terms of (e&eb) the manner by which they “occupy” their existence: the intensity of their power, their “tenor” of life.
(4) It is always a question of knowing whether a mode of existence— however great or small it may be—is capable of deploying its capacities, of increasing its power of acting to the point where it can be said to go to the limit of what it “can do.” The fundamental question of ethics is not “What must I do?” (Which is the question of morality) but rather “What can I do, what am I capable of doing (which is the proper question of an ethics without morality).
(5) Given my degree of power, what are my capabilities and capacities? How can I come into active possession of my power? How can I go to the limit of what I “can do”?

(6) What an ethics of immanence will criticize, then, is (=) anything that separates a mode of existence from (e&eb) its power of acting—and what separates us from (e&eb) our power of acting are, ultimately, the illusions of transcendence.

(7) (We should immediately point out that the illusions of transcendence go far beyond the transcendence of God; in the Critique of Pure Reason, Kant had already critiqued the concepts of the Self, the World, and God as the three great illusions of transcendence; and what he calls the "moral law" in the second critique is, by Kant's own admission, a transcendent law.) When Spinoza and Nietzsche criticize transcendence, their interest is not merely theoretical or speculative—exposing its fictional or illusory status—but rather practical and ethical.

(8) This is no doubt the point that separates Deleuze most from the ethical thinking of Emmanuel Levinas— the great philosopher of transcendence, insofar as the Other is (=)the paradigmatic concept of transcendence—as well as Jacques Derrida, who was much closer to Levinas than Deleuze on these matters.

(9) The ethical themes one finds in transcendent philosophies like those of Levinas and Derrida—an absolute responsibility for the other that I can never assume, or an infinite call to justice that I can never satisfy—would be, from the Deleuzian point of view of immanence, akin to (=) imperatives whose effect is to separate me from (e&eb) my capacity to act.

(10) From the viewpoint of immanence, in other words, transcendence, far from being our salvation, represents (eb) our slavery and impotence reduced to its lowest point: the demand to do (eb) the impossible (a frequent Derridean theme) is nothing other than the concept of impotence raised to infinity.

(11) But this is precisely why the question of desire is linked with the theme of an immanent ethics, and becomes a political question. For one of most difficult problems posed by an immanent ethics is the following: if transcendence represents my impotence (at the limit, my power reduced to zero), then under what conditions can I have actually been led to desire transcendence? What are the conditions that could have led, in Nietzsche's words, to "the inversion of the value-positing eye"—that is, to the whole history of nihilism that Nietzsche analyses (and nihilism, for Nietzsche, is nothing other than the triumph of transcendence, the point where life itself it given a value of nil, nihil)? This is the fundamental political problem posed by an immanent ethics: How can people reach a point where they actually desire their servitude and slavery as if it were their salvation—for those in power have an obvious interest in separating us from our capacity to act? How, in other words, can we desire to be separated from power, from out capacity to act?

**(12)** As Deleuze writes, following Reich: "The astonishing thing is not that some people steal or that others occasionally go out on strike, but rather that all those who are starving do not steal as a regular practice, and all those who are exploited are not continually out on strike" (AO 29). In other words, whereas other moral theories see transcendence as a necessary principle—the transcendence of the moral law in Kant, for instance, or the transcendence of the Other in Levinas—for Deleuze transcendence is the fundamental problem of ethics, what prevents ethics from taking place, so to speaks we've developed two aspects of an immanent ethics: it focuses on the differences between modes of existence, in terms of their immanent capabilities or power (active versus reactive, in Nietzsche; active versus passive, in Spinoza), and it poses, as one of its fundamental problems, the urge toward transcendence that effectively "perverts"( e&eb)desire, to the point where we can actually desire our own repression, a separation from our own capacities and powers.

**NOTATION**

**Module One**

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|---|---|
| Conversely, a mode of existence will be considered to be "good" (or free, or rational, or strong) that exercises its capacity for being affected in such a way that its power of acting increases, to the point where (e) it produces active affections and adequate ideas | |
| $G_{13}$ : Category one of it produces active affections and adequate ideas<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of mode of existence will be considered to be "good" (or free, or rational, or strong) that exercises its capacity for being affected in such a way that its power of acting increases, to the point<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>Modes are no longer "judged" in terms of their degree of proximity to or distance from (e) an external principle, but are "evaluated" in terms of (e&eb) the manner by which they "occupy" their existence: the intensity of their power, their "tenor" of life. | |
| $G_{16}$ : Category one of **Modes are no longer "judged" in terms of their degree of proximity to or distance from (e) an external principle, but are "evaluated";** manner by which they "occupy" their existence: the intensity of their power, their "tenor" of life.<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of manner by which they "occupy" their existence: the intensity of their power, their "tenor" of life.;**Modes are no longer "judged" in terms of their degree of proximity to or distance from (e) an external principle, but are "evaluated"**<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>What an ethics of immanence will criticize, then, is (=) anything that separates a mode of existence from (e&eb) its power of acting—and what separates us from (e&eb) our power of acting are, ultimately, the illusions of transcendence. | |
| $G_{20}$ : Category one of ethics of immanence will criticize<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of anything that separates a mode of existence from (e&eb) its power of acting—and what separates us from (e&eb) our power of acting are, ultimately, the illusions of transcendence. | |

| | |
|---|---|
| $T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>What an ethics of immanence will criticize, then, is anything that separates a mode of existence from (e&eb) its power of acting—and what separates us from (e&eb) our power of acting are, ultimately, the illusions of transcendence. | |
| $G_{24}$ : Category one of **ethics of immanence will criticize, then, is anything that separates a mode of existence;** its power of acting—and what separates us from (e&eb) our power of acting are, ultimately, the illusions of transcendence.<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of its power of acting—and what separates us from (e&eb) our power of acting are, ultimately, the illusions of transcendence. **ethics of immanence will criticize, then, is anything that separates a mode of existence**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>What an ethics of immanence will criticize, then, is anything that separates a mode of existence from its power of acting—and what separates us from our power of acting are, ultimately, the illusions of transcendence. | |
| $G_{28}$ : Category one of ethics of immanence will criticize, then, is anything that separates a mode of existence from its power of acting—and what separates us from our power of acting<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of illusions of transcendence.<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>This is no doubt the point that separates Deleuze most from the ethical thinking of Emmanuel Levinas— the great philosopher of transcendence, insofar as the Other is (=) the paradigmatic concept of transcendence—as well as Jacques Derrida, who was much closer to Levinas than Deleuze on these matters | |
| $G_{32}$ : Category one of Other | |

| | |
|---|---|
| $G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of paradigmatic concept of transcendence<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>The ethical themes one finds in transcendent philosophies like those of Levinas and Derrida—an absolute responsibility for the other that I can never assume, or an infinite call to justice that I can never satisfy—would be, from the Deleuzian point of view of immanence, akin to (=) imperatives whose effect is to separate me from (e&eb) my capacity to act. | |
| $G_{36}$ : Category one of absolute responsibility for the other that I can never assume, or an infinite call to justice that I can never satisfy—would be, from the Deleuzian point of view of immanence<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of imperatives whose effect is to separate me from (e&eb) my capacity to act.<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The ethical themes one finds in transcendent philosophies like those of Levinas and Derrida—an absolute responsibility for the other that I can never assume, or an infinite call to justice that I can never satisfy—would be, from the Deleuzian point of view of immanence, akin to imperatives whose effect is to separate me from (e&eb) my capacity to act. | |
| $G_{40}$ : Category one of **absolute responsibility for the other that I can never assume, or an infinite call to justice that I can never satisfy—would be, from the Deleuzian point of view of immanence, akin to imperatives whose effect is to separate me;** my capacity to act<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of my capacity to act **;absolute responsibility for the other that I can never assume, or an infinite call to justice that I can never satisfy—would be, from the Deleuzian point of view of immanence, akin to imperatives whose effect is to separate me**<br><br>$T_{41}$ : Category two of SAS | |

$T_{42}$ : Category three of SAS

**Module Nine**

in other words, transcendence from the viewpoint of immanence far from being our salvation, represents (eb) our slavery and impotence reduced to its lowest point: the demand to do (eb) the impossible (a frequent Derridean theme) is nothing other than the concept of impotence raised to infinity

$G_{44}$ : Category one of transcendence from the viewpoint of immanence far from being our salvation

$G_{45}$ : Category two of SAS

$G_{46}$ : Category three of SAS

$T_{44}$ : Category one of our slavery and impotence reduced to its lowest point: the demand to do (eb) the impossible (a frequent Derridean theme) is nothing other than the concept of impotence raised to infinity

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

**The Coefficients:**

$(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:
$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$
$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,
$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$
$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$
$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$
$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$

are Accentuation coefficients
$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$
$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$
$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}\ (a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,
$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$
$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,
$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,
$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,
are Dissipation coefficients

**Module Numbered One**

**The differential system of this model is now (Module Numbered one)**

$\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right]G_{13}$ 1

| | |
|---|---|
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}, t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}, t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21}, t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23}, t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25}, t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}, t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27}), t\right)\right]T_{24}$ | 22 |

| | |
|---|---|
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b'_{25})^{(4)} - (b''_{25})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b'_{26})^{(4)} - (b''_{26})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a''_{24})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b''_{24})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a'_{28})^{(5)} + (a''_{28})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a'_{29})^{(5)} + (a''_{29})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a'_{30})^{(5)} + (a''_{30})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b'_{28})^{(5)} - (b''_{28})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b'_{29})^{(5)} - (b''_{29})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b'_{30})^{(5)} - (b''_{30})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a''_{28})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b''_{28})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a'_{32})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b'_{32})^{(6)} - (b''_{32})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b'_{33})^{(6)} - (b''_{33})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b'_{34})^{(6)} - (b''_{34})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a''_{32})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b'_{36})^{(7)} - (b''_{36})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b'_{37})^{(7)} - (b''_{37})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b'_{38})^{(7)} - (b''_{38})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a''_{36})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight** | |

The differential system of this model is now

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40} \quad 43$$

$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41} \quad 44$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42} \quad 45$$

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40} \quad 46$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41} \quad 47$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42} \quad 48$$

**Module Numbered Nine**

The differential system of this model is now

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44} \quad 49$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45} \quad 50$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46} \quad 51$$

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44} \quad 52$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45} \quad 53$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46} \quad 54$$

$+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor**

$-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor**

$$\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}^{'})^{(1)} & \boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13} \quad 55$$

$$\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)} & \boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad 56$$

$$\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)} & \boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad 57$$

Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient

| | |
|---|---|
| for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{38}^{''})^{(7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3 $+(a_{40}^{''})^{(8,8)}(T_{41},t)$ $+(a_{41}^{''})^{(8,8)}(T_{41},t)$ $+(a_{42}^{''})^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3 $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt}=(b_{13})^{(1)}T_{14}-\begin{bmatrix}(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G,t) & -(b_{16}^{''})^{(2,2,)}(G_{19},t) & -(b_{20}^{''})^{(3,3,)}(G_{23},t)\\ -(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)\\ -(b_{36}^{''})^{(7,7,)}(G_{39},t) & -(b_{40}^{''})^{(8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{13}$ | 58 |
| $\frac{dT_{14}}{dt}=(b_{14})^{(1)}T_{13}-\begin{bmatrix}(b_{14}^{'})^{(1)}-(b_{14}^{''})^{(1)}(G,t) & -(b_{17}^{''})^{(2,2,)}(G_{19},t) & -(b_{21}^{''})^{(3,3,)}(G_{23},t)\\ -(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)\\ -(b_{37}^{''})^{(7,7,)}(G_{39},t) & -(b_{41}^{''})^{(8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{14}$ | 59 |
| $\frac{dT_{15}}{dt}=(b_{15})^{(1)}T_{14}-\begin{bmatrix}(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G,t) & -(b_{18}^{''})^{(2,2,)}(G_{19},t) & -(b_{22}^{''})^{(3,3,)}(G_{23},t)\\ -(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)\\ -(b_{38}^{''})^{(7,7,)}(G_{39},t) & -(b_{42}^{''})^{(8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{15}$ | *60* |
| Where $-(b_{13}^{''})^{(1)}(G,t)$, $-(b_{14}^{''})^{(1)}(G,t)$, $-(b_{15}^{''})^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{16}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3 $-(b_{20}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3 $-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3 $-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{37}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 $-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3 $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |

$$\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}')^{(2)}\boxed{+(a_{16}'')^{(2)}(T_{17},t)}\boxed{+(a_{13}'')^{(1,1,)}(T_{14},t)}\boxed{+(a_{20}'')^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}'')^{(4,4,4,4,4)}(T_{25},t)}\boxed{+(a_{28}'')^{(5,5,5,5,5)}(T_{29},t)}\boxed{+(a_{32}'')^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}'')^{(7,7,7)}(T_{37},t)}\boxed{+(a_{40}'')^{(8,8,8)}(T_{41},t)}\boxed{+(a_{44}'')^{(9,9)}(T_{45},t)} \end{bmatrix} G_{16} \quad (61)$$

$$\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}')^{(2)}\boxed{+(a_{17}'')^{(2)}(T_{17},t)}\boxed{+(a_{14}'')^{(1,1,)}(T_{14},t)}\boxed{+(a_{21}'')^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}'')^{(4,4,4,4,4)}(T_{25},t)}\boxed{+(a_{29}'')^{(5,5,5,5,5)}(T_{29},t)}\boxed{+(a_{33}'')^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}'')^{(7,7,7)}(T_{37},t)}\boxed{+(a_{41}'')^{(8,8,8)}(T_{41},t)}\boxed{+(a_{45}'')^{(9,9)}(T_{45},t)} \end{bmatrix} G_{17} \quad (62)$$

$$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}')^{(2)}\boxed{+(a_{18}'')^{(2)}(T_{17},t)}\boxed{+(a_{15}'')^{(1,1,)}(T_{14},t)}\boxed{+(a_{22}'')^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}'')^{(4,4,4,4,4)}(T_{25},t)}\boxed{+(a_{30}'')^{(5,5,5,5,5)}(T_{29},t)}\boxed{+(a_{34}'')^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}'')^{(7,7,7)}(T_{37},t)}\boxed{+(a_{42}'')^{(8,8,8)}(T_{41},t)}\boxed{+(a_{46}'')^{(9,9)}(T_{45},t)} \end{bmatrix} G_{18} \quad (63)$$

Where $\boxed{+(a_{16}'')^{(2)}(T_{17},t)}$, $\boxed{+(a_{17}'')^{(2)}(T_{17},t)}$, $\boxed{+(a_{18}'')^{(2)}(T_{17},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{13}'')^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{14}'')^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{15}'')^{(1,1,)}(T_{14},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}'')^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}'')^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}'')^{(3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}'')^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}'')^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}'')^{(4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}'')^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}'')^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}'')^{(5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}'')^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}'')^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}'')^{(6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{36}'')^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}'')^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}'')^{(7,7,7)}(T_{37},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{40}'')^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}'')^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}'')^{(8,8,8)}(T_{41},t)}$ are eight augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{44}'')^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{45}'')^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{46}'')^{(9,9)}(T_{45},t)}$ are ninth augmentation coefficient for category 1, 2 and 3

$$\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}')^{(2)}\boxed{-(b_{16}'')^{(2)}(G_{19},t)}\boxed{-(b_{13}'')^{(1,1,)}(G,t)}\boxed{-(b_{20}'')^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}'')^{(4,4,4,4,4)}(G_{27},t)}\boxed{-(b_{28}'')^{(5,5,5,5,5)}(G_{31},t)}\boxed{-(b_{32}'')^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{36}'')^{(7,7,7)}(G_{39},t)}\boxed{-(b_{40}'')^{(8,8,8)}(G_{43},t)}\boxed{-(b_{44}'')^{(9,9)}(G_{47},t)} \end{bmatrix} T_{16} \quad (64)$$

$$\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}')^{(2)}\boxed{-(b_{17}'')^{(2)}(G_{19},t)}\boxed{-(b_{14}'')^{(1,1,)}(G,t)}\boxed{-(b_{21}'')^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}'')^{(4,4,4,4,4)}(G_{27},t)}\boxed{-(b_{29}'')^{(5,5,5,5,5)}(G_{31},t)}\boxed{-(b_{33}'')^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{37}'')^{(7,7,7)}(G_{39},t)}\boxed{-(b_{41}'')^{(8,8,8)}(G_{43},t)}\boxed{-(b_{45}'')^{(9,9)}(G_{47},t)} \end{bmatrix} T_{17} \quad (65)$$

$$\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b'_{18})^{(2)} & -(b''_{18})^{(2)}(G_{19},t) & -(b''_{15})^{(1,1,)}(G,t) & -(b''_{22})^{(3,3,3,)}(G_{23},t) \\ -(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b''_{38})^{(7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{18} \quad (66)$$

where $-(b''_{16})^{(2)}(G_{19},t)$ , $-(b''_{17})^{(2)}(G_{19},t)$ , $-(b''_{18})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b''_{13})^{(1,1,)}(G,t)$, $-(b''_{14})^{(1,1,)}(G,t)$, $-(b''_{15})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3

$-(b''_{20})^{(3,3,3,)}(G_{23},t)$, $-(b''_{21})^{(3,3,3,)}(G_{23},t)$, $-(b''_{22})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3

$-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3

$-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3

$-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3

$-(b''_{36})^{(7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7)}(G_{39},t)$, $-(b''_{38})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3

$-(b''_{40})^{(8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3

$-(b''_{44})^{(9,9)}(G_{47},t)$, $-(b''_{46})^{(9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3

$$\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a'_{20})^{(3)} & +(a''_{20})^{(3)}(T_{21},t) & +(a''_{16})^{(2,2,2)}(T_{17},t) & +(a''_{13})^{(1,1,1,)}(T_{14},t) \\ +(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t) & +(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t) & +(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t) & \\ +(a''_{36})^{(7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9)}(T_{45},t) & \end{bmatrix} G_{20} \quad (67)$$

$$\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a'_{21})^{(3)} & +(a''_{21})^{(3)}(T_{21},t) & +(a''_{17})^{(2,2,2)}(T_{17},t) & +(a''_{14})^{(1,1,1,)}(T_{14},t) \\ +(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t) & +(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t) & +(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t) & \\ +(a''_{37})^{(7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9)}(T_{45},t) & \end{bmatrix} G_{21} \quad (68)$$

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a'_{22})^{(3)} & +(a''_{22})^{(3)}(T_{21},t) & +(a''_{18})^{(2,2,2)}(T_{17},t) & +(a''_{15})^{(1,1,1,)}(T_{14},t) \\ +(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t) & +(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t) & +(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t) & \\ +(a''_{38})^{(7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9)}(T_{45},t) & \end{bmatrix} G_{22} \quad (69)$$

$+(a''_{20})^{(3)}(T_{21},t)$, $+(a''_{21})^{(3)}(T_{21},t)$, $+(a''_{22})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a''_{16})^{(2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3

$+(a''_{13})^{(1,1,1,)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients

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| for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)} & -(b_{20}^{''})^{(3)}(G_{23},t) & -(b_{16}^{''})^{(2,2,2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,1,)}(G,t) \\ & -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ & -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{20}$ | *70* |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)} & -(b_{21}^{''})^{(3)}(G_{23},t) & -(b_{17}^{''})^{(2,2,2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,1,)}(G,t) \\ & -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ & -(b_{37}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21}$ | *71* |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b_{22}^{'})^{(3)} & -(b_{22}^{''})^{(3)}(G_{23},t) & -(b_{18}^{''})^{(2,2,2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,1,)}(G,t) \\ & -(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ & -(b_{38}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{22}$ | *72* |
| $-(b_{20}^{''})^{(3)}(G_{23},t)$, $-(b_{21}^{''})^{(3)}(G_{23},t)$, $-(b_{22}^{''})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*<br>$-(b_{16}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*<br>$-(b_{13}^{''})^{(1,1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*<br>$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 | |

$\boxed{-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{46}^{''})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt}=(a_{24})^{(4)}G_{25}-\begin{bmatrix}(a_{24}^{'})^{(4)}\boxed{+(a_{24}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)}\\ \boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt}=(a_{25})^{(4)}G_{24}-\begin{bmatrix}(a_{25}^{'})^{(4)}\boxed{+(a_{25}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6)}(T_{33},t)}\\ \boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt}=(a_{26})^{(4)}G_{25}-\begin{bmatrix}(a_{26}^{'})^{(4)}\boxed{+(a_{26}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)}\\ \boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{26} \quad (75)$$

$\boxed{(a_{24}^{''})^{(4)}(T_{25},t)}$, $\boxed{(a_{25}^{''})^{(4)}(T_{25},t)}$, $\boxed{(a_{26}^{''})^{(4)}(T_{25},t)}$ *are first augmentation coefficients category* 1, 2 3

$\boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)}$
*are fourth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)}$
*are fifth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)}$
*are sixth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)}$
*are seventh augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)}$
*are eighth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)}$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)}\boxed{-(b_{24}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,)}(G_{35},t)}\\ \boxed{-(b_{13}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)}\\ \boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}\end{bmatrix}T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b'_{25})^{(4)} & -(b''_{25})^{(4)}(G_{27},t) & -(b''_{29})^{(5,5,)}(G_{31},t) & -(b''_{33})^{(6,6,)}(G_{35},t) \\ & -(b''_{14})^{(1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b'_{26})^{(4)} & -(b''_{26})^{(4)}(G_{27},t) & -(b''_{30})^{(5,5,)}(G_{31},t) & -(b''_{34})^{(6,6,)}(G_{35},t) \\ & -(b''_{15})^{(1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{26} \quad (78)$$

Where $-(b''_{24})^{(4)}(G_{27},t)$, $-(b''_{25})^{(4)}(G_{27},t)$, $-(b''_{26})^{(4)}(G_{27},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b''_{28})^{(5,5,)}(G_{31},t)$, $-(b''_{29})^{(5,5,)}(G_{31},t)$, $-(b''_{30})^{(5,5,)}(G_{31},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b''_{32})^{(6,6,)}(G_{35},t)$, $-(b''_{33})^{(6,6,)}(G_{35},t)$, $-(b''_{34})^{(6,6,)}(G_{35},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b''_{13})^{(1,1,1,1)}(G,t)$, $-(b''_{14})^{(1,1,1,1)}(G,t)$, $-(b''_{15})^{(1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b''_{16})^{(2,2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b''_{20})^{(3,3,3,3)}(G_{23},t)$, $-(b''_{21})^{(3,3,3,3)}(G_{23},t)$, $-(b''_{22})^{(3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b''_{36})^{(7,7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)$, $-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b''_{40})^{(8,8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b''_{46})^{(9,9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a'_{28})^{(5)} & +(a''_{28})^{(5)}(T_{29},t) & +(a''_{24})^{(4,4,)}(T_{25},t) & +(a''_{32})^{(6,6,6)}(T_{33},t) \\ & +(a''_{13})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{28} \quad (79)$$

$$\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a'_{29})^{(5)} & +(a''_{29})^{(5)}(T_{29},t) & +(a''_{25})^{(4,4,)}(T_{25},t) & +(a''_{33})^{(6,6,6)}(T_{33},t) \\ & +(a''_{14})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{29} \quad (80)$$

$$\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a'_{30})^{(5)} & +(a''_{30})^{(5)}(T_{29},t) & +(a''_{26})^{(4,4,)}(T_{25},t) & +(a''_{34})^{(6,6,6)}(T_{33},t) \\ & +(a''_{15})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{30} \quad (81)$$

Where $+(a''_{28})^{(5)}(T_{29},t)$, $+(a''_{29})^{(5)}(T_{29},t)$, $+(a''_{30})^{(5)}(T_{29},t)$ are first augmentation coefficients for category 1, 2 and 3

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| And $+(a_{24}^{''})^{(4,4,)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,)}(T_{25},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$ , $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3<br>$+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3<br>$+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3<br>$+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b_{28}^{'})^{(5)} & -(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) & -(b_{32}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{28}$ | 82 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b_{29}^{'})^{(5)} & -(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) & -(b_{33}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{29}$ | 83 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b_{30}^{'})^{(5)} & -(b_{30}^{''})^{(5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,)}(G_{27},t) & -(b_{34}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{15}^{''})^{(1,1,1,1,1,)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{30}$ | 84 |
| where $-(b_{28}^{''})^{(5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5)}(G_{31},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{24}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,)}(G_{27},t)$ are second detrition coefficients for category 1,2 and 3<br>$-(b_{32}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6)}(G_{35},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3<br>$-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3<br>$-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients | |

| | |
|---|---|
| for category 1,2, and 3<br>$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3<br>$-(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3<br>$-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3 | |
| $\frac{dG_{32}}{dt}=(a_{32})^{(6)}G_{33}$<br>$-\begin{bmatrix}(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{32}$ | *85* |
| $\frac{dG_{33}}{dt}=(a_{33})^{(6)}G_{32}-\begin{bmatrix}(a_{33}^{'})^{(6)}+(a_{33}^{''})^{(6)}(T_{33},t) & +(a_{29}^{''})^{(5,5,5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{33}$ | *86* |
| $\frac{dG_{34}}{dt}=(a_{34})^{(6)}G_{33}-\begin{bmatrix}(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33},t) & +(a_{30}^{''})^{(5,5,5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{34}$ | *87* |
| $+(a_{32}^{''})^{(6)}(T_{33},t)$, $+(a_{33}^{''})^{(6)}(T_{33},t)$, $+(a_{34}^{''})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{28}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients<br>$+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients<br>$+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients<br>$+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients<br>$+(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients<br>$+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients | |

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & \boxed{-(b''_{32})^{(6)}(G_{35},t)} & \boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)} & \\ \boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t)} & \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & \boxed{-(b''_{33})^{(6)}(G_{35},t)} & \boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)} & \\ \boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)} & \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & \boxed{-(b''_{34})^{(6)}(G_{35},t)} & \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)} & \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)} & \end{bmatrix} T_{34} \quad (90)$$

$\boxed{-(b''_{32})^{(6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6)}(G_{35},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)}$ *are second detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)}$ *are third detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a'_{36})^{(7)} & \boxed{+(a''_{36})^{(7)}(T_{37},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} & \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} & \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)} G_{36} - \begin{bmatrix} (a_{37}')^{(7)} \boxed{+(a_{37}'')^{(7)}(T_{37},t)} & \boxed{+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)} G_{37} - \begin{bmatrix} (a_{38}')^{(7)} \boxed{+(a_{38}'')^{(7)}(T_{37},t)} & \boxed{+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad (93)$$

Where $\boxed{(a_{36}'')^{(7)}(T_{37},t)}$, $\boxed{(a_{37}'')^{(7)}(T_{37},t)}$, $\boxed{(a_{38}'')^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{40}'')^{(8,8,8,8,8,8,8,8,)}(T_{41},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)} T_{37} - \begin{bmatrix} (b_{36}')^{(7)} \boxed{-(b_{36}'')^{(7)}(G_{39},t)} & \boxed{-(b_{16}'')^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}'')^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}'')^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}'')^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}'')^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}'')^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}'')^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13} \quad (94)$$

$$\frac{dT_{37}}{dt} =$$

$$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b'_{37})^{(7)}\boxed{-(b''_{37})^{(7)}(G_{39},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} =$$

$$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b'_{38})^{(7)}\boxed{-(b''_{38})^{(7)}(G_{39},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a'_{40})^{(8)}\boxed{+(a''_{40})^{(8)}(T_{41},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a'_{41})^{(8)}\boxed{+(a''_{41})^{(8)}(T_{41},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a'_{42})^{(8)}\boxed{+(a''_{42})^{(8)}(T_{41},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{40})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b'_{40})^{(8)}\boxed{-(b''_{40})^{(8)}(G_{43},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b'_{41})^{(8)}\boxed{-(b''_{41})^{(8)}(G_{43},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b'_{42})^{(8)}\boxed{-(b''_{42})^{(8)}(G_{43},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} =$$

$$(b_{46})^{(9)}T_{45} - \begin{bmatrix} (b'_{46})^{(9)}\boxed{-(b''_{46})^{(9)}(G_{47},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{44})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a'_i)^{(1)}, (a''_i)^{(1)}, (b_i)^{(1)}, (b'_i)^{(1)}, (b''_i)^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a''_i)^{(1)}, (b''_i)^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

$$(a''_i)^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$$

$$(b''_i)^{(1)}(G,t) \le (r_i)^{(1)} \le (b'_i)^{(1)} \le (\hat{B}_{13})^{(1)}$$

$lim_{T_2\to\infty}(a''_i)^{(1)}(T_{14},t) = (p_i)^{(1)}$ 98

$\lim_{G\to\infty}(b''_i)^{(1)}(G,t) = (r_i)^{(1)}$

**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :

Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$

| | |
|---|---|
| They satisfy Lipschitz condition:<br>$\|(a_i'')^{(1)}(T_{14}', t) - (a_i'')^{(1)}(T_{14}, t)\| \leq (\hat{k}_{13})^{(1)}\|T_{14} - T_{14}'\|e^{-(\hat{M}_{13})^{(1)}t}$<br>$\|(b_i'')^{(1)}(G', t) - (b_i'')^{(1)}(G, t)\| < (\hat{k}_{13})^{(1)}\|G - G'\|e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(1)}(T_{14}', t)$ and $(a_i'')^{(1)}(T_{14}, t)$ . $(T_{14}', t)$ and $(T_{14}, t)$ are points belonging to the interval $[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}]$. It is to be noted that $(a_i'')^{(1)}(T_{14}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i'')^{(1)}(T_{14}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)},$ are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i')^{(1)}, (b_i)^{(1)}, (b_i')^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i, j = 16,17,18$ | |
| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17}, t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19}, t) \leq (r_i)^{(2)} \leq (b_i')^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty}(a_i'')^{(2)}(T_{17}, t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty}(b_i'')^{(2)}((G_{19}), t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ : | 106 |

| | |
|---|---|
| Where $\boxed{(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}}$ are positive constants and $\boxed{i = 16,17,18}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'}, t) - (a_i^{''})^{(2)}(T_{17}, t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}^{'}\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'}, t) - (b_i^{''})^{(2)}((G_{19}), t)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})^{'}\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'}, t)$ and $(a_i^{''})^{(2)}(T_{17}, t)$ . $(T_{17}^{'}, t)$ and $(T_{17}, t)$ are points belonging to the interval $\left[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}\right]$. It is to be noted that $(a_i^{''})^{(2)}(T_{17}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i^{''})^{(2)}(T_{17}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br><br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br><br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br><br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i^{'})^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i, j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(r_i)^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(3)}(T_{21}, t) = (p_i)^{(3)}$ | 113 |

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|---|---|
| $\lim_{G\to\infty}(b_i'')^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(3)}(T_{21}',t) - (a_i'')^{(3)}(T_{21},t)\rvert \le (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}'\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i'')^{(3)}(G_{23}',t) - (b_i'')^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}'\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(3)}(T_{21}',t)$ and$(a_i'')^{(3)}(T_{21},t)$ . $(T_{21}',t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i'')^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i'')^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i')^{(3)}, (b_i)^{(3)}, (b_i')^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i')^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i')^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i')^{(4)}, (a_i'')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (b_i'')^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i'')^{(4)}, (b_i'')^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i'')^{(4)}(T_{25},t) \le (p_i)^{(4)} \le (\hat{A}_{24})^{(4)}$<br><br>$(b_i'')^{(4)}\big((G_{27}),t\big) \le (r_i)^{(4)} \le (b_i')^{(4)} \le (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2\to\infty}(a_i'')^{(4)}(T_{25},t) = (p_i)^{(4)}$ | 118 |

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|---|---|
| $\lim_{G\to\infty}(b_i'')^{(4)}\left((G_{27}),t\right) = \ (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, \ (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24{,}25{,}26}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(4)}(T_{25}',t) - (a_i'')^{(4)}(T_{25},t)\rvert \le (\hat{k}_{24})^{(4)}\lvert T_{25} - T_{25}'\rvert e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\lvert(b_i'')^{(4)}((G_{27})',t) - (b_i'')^{(4)}\left((G_{27}),t\right)\rvert < (\hat{k}_{24})^{(4)}\lvert\lvert(G_{27}) - (G_{27})'\rvert\rvert e^{-(\hat{M}_{24})^{(4)}t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(4)}(T_{25}',t)$ and $(a_i'')^{(4)}(T_{25},t)$ . $(T_{25}',t)$ and $(T_{25},t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i'')^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i'')^{(4)}(T_{25},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (p_i)^{(4)}, \ (r_i)^{(4)}, i = 24{,}25{,}26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i')^{(4)} + \ (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)}(\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i')^{(4)} + \ (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i')^{(5)}, (a_i'')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (b_i'')^{(5)} > 0, \quad i,j = 28{,}29{,}30$<br><br>The functions $(a_i'')^{(5)}, (b_i'')^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}, \ (r_i)^{(5)}$:<br><br>$(a_i'')^{(5)}(T_{29},t) \le (p_i)^{(5)} \le (\hat{A}_{28})^{(5)}$<br><br>$(b_i'')^{(5)}\left((G_{31}),t\right) \le \ (r_i)^{(5)} \le (b_i')^{(5)} \le (\hat{B}_{28})^{(5)}$ | 122 |

$lim_{T_2\to\infty}(a_i'')^{(5)}(T_{29},t) = (p_i)^{(5)}$

$\lim_{G\to\infty}(b_i'')^{(5)}(G_{31},t) = (r_i)^{(5)}$

123

**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :

Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$

They satisfy Lipschitz condition: 124

$$|(a_i'')^{(5)}(T_{29}', t) - (a_i'')^{(5)}(T_{29}, t)| \le (\hat{k}_{28})^{(5)}|T_{29} - T_{29}'|e^{-(\hat{M}_{28})^{(5)}t}$$

$$|(b_i'')^{(5)}((G_{31})', t) - (b_i'')^{(5)}((G_{31}), t)| < (\hat{k}_{28})^{(5)}||(G_{31}) - (G_{31})'||e^{-(\hat{M}_{28})^{(5)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(5)}(T_{29}', t)$ and $(a_i'')^{(5)}(T_{29}, t)$ . $(T_{29}', t)$ and $(T_{29}, t)$ are points belonging to the interval $[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}]$. It is to be noted that $(a_i'')^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i'')^{(5)}(T_{29}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ : 125

$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants

$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$$

**Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ : 126

There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i')^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i')^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

Where we suppose

$(a_i)^{(6)}, (a_i')^{(6)}, (a_i'')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (b_i'')^{(6)} > 0, \quad i, j = 32,33,34$ 127

The functions $(a_i'')^{(6)}, (b_i'')^{(6)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(6)}, (r_i)^{(6)}$:

$$(a_i'')^{(6)}(T_{33}, t) \le (p_i)^{(6)} \le (\hat{A}_{32})^{(6)}$$

| | |
|---|---|
| $(b_i^{''})^{(6)}((G_{35}),t) \leq \ (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(6)}(T_{33},t) = (p_i)^{(6)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(6)}\left((G_{35}),t\right) = \ (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'},t) - (a_i^{''})^{(6)}(T_{33},t)\rvert \leq (\hat{k}_{32})^{(6)}\lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})^{'},t) - (b_i^{''})^{(6)}\left((G_{35}),t\right)\rvert < (\hat{k}_{32})^{(6)}\lvert\lvert(G_{35}) - (G_{35})^{'}\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'},t)$ and$(a_i^{''})^{(6)}(T_{33},t)$ . $(T_{33}^{'},t)$ and $(T_{33},t)$ are points belonging to the interval $\left[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\right]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[\ (a_i)^{(6)} + (a_i^{'})^{(6)} + \ (\hat{A}_{32})^{(6)} + \ (\hat{P}_{32})^{(6)}\ (\hat{k}_{32})^{(6)}] < 1$<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[\ (b_i)^{(6)} + (b_i^{'})^{(6)} + \ (\hat{B}_{32})^{(6)} + \ (\hat{Q}_{32})^{(6)}\ \ (\hat{k}_{32})^{(6)}] < 1$ | 130 |
| Where we suppose | |
| (Y) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i,j = 36,37,38$<br><br>(Z) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$: | 131 |

$(a_i^{''})^{(7)}(T_{37},t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$

$(b_i^{''})^{(7)}(G_{39},t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$

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*(AA)* $\lim_{T_2\to\infty} (a_i^{''})^{(7)}(T_{37},t) = (p_i)^{(7)}$

*(BB)*

$\lim_{G\to\infty}(b_i^{''})^{(7)}\big((G_{39}),t\big) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

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They satisfy Lipschitz condition:

$|(a_i^{''})^{(7)}(T_{37}^{'},t) - (a_i^{''})^{(7)}(T_{37},t)| \leq (\hat{k}_{36})^{(7)}|T_{37} - T_{37}^{'}|e^{-(\hat{M}_{36})^{(7)}t}$

$|(b_i^{''})^{(7)}((G_{39})^{'},t) - (b_i^{''})^{(7)}\big((G_{39}),t\big)| < (\hat{k}_{36})^{(7)}||(G_{39}) - (G_{39})^{'}||e^{-(\hat{M}_{36})^{(7)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'},t)$ and$(a_i^{''})^{(7)}(T_{37},t)$ . $(T_{37}^{'},t)$ and $(T_{37},t)$ are points belonging to the interval $\big[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}\big]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

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**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :

(CC) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$, are positive constants

$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$

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**Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :

(DD) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities

$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$

$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$

| | |
|---|---|
| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})^{'}, t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})^{'}\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $\left[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}\right]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i^{'})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (p_i)^{(8)}$, $(r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |

| | |
|---|---|
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i,j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45},t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47},t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2\to\infty}(a_i'')^{(9)}(T_{45},t) = (p_i)^{(9)}$<br>$\lim_{G\to\infty}(b_i'')^{(9)}(G_{47},t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}',t) - (a_i'')^{(9)}(T_{45},t)\rvert \le (\hat{k}_{44})^{(9)}\lvert T_{45} - T_{45}'\rvert e^{-(\hat{M}_{44})^{(9)}t}$<br>$\lvert(b_i'')^{(9)}((G_{47})',t) - (b_i'')^{(9)}((G_{47}),t)\rvert < (\hat{k}_{44})^{(9)}\lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}',t)$ and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i'')^{(9)}(T_{45},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :<br>$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants<br>$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$ | |
| **Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :<br>There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants | |

$(a_i)^{(9)}, (a_i')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}, i = 44,45,46,$
satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i')^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i')^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0), T_i(0)$ :

$$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}, \quad \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}, \quad \boxed{T_i(0) = T_i^0 > 0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0), T_i(0)$

$$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}, \quad G_i(0) = G_i^0 > 0$$

$$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}, \quad T_i(0) = T_i^0 > 0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}, \quad G_i(0) = G_i^0 > 0$$

$$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}, \quad T_i(0) = T_i^0 > 0$$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

**Definition of** $G_i(0), T_i(0)$ :

$$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}, \quad \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}, \quad \boxed{T_i(0) = T_i^0 > 0}$$

**Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 151

**Definition of** $G_i(0), T_i(0)$ :

$$G_i(t) \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}, \quad \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}, \quad \boxed{T_i(0) = T_i^0 > 0}$$

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|---|---|
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le \left(\hat{Q}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le \left(\hat{P}_{13}\right)^{(1)}$ , $T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le \left(\hat{P}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |
| $0 \le T_i(t) - T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 157 |
| By<br><br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}\left(s_{(13)}\right) - \left((a_{13}')^{(1)} + a_{13}'')^{(1)}\left(T_{14}\left(s_{(13)}\right), s_{(13)}\right)\right) G_{13}\left(s_{(13)}\right)\right] ds_{(13)}$ | 158 |

| | |
|---|---|
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{16})^{(2)}$, $T_i^0 \le (\hat{Q}_{16})^{(2)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} G_{17}(s_{(16)}) - \left((a_{16}')^{(2)} + a_{16}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{16}(s_{(16)})\right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[(a_{17})^{(2)} G_{16}(s_{(16)}) - \left((a_{17}')^{(2)} + (a_{17}'')^{(2)}(T_{17}(s_{(16)}), s_{(17)})\right) G_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |

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|---|---|
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{20})^{(3)}$ , $T_i^0 \leq (\hat{Q}_{20})^{(3)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a'_{20})^{(3)} + a''_{20})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a'_{21})^{(3)} + (a''_{21})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a'_{22})^{(3)} + (a''_{22})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b'_{20})^{(3)} - (b''_{20})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b'_{21})^{(3)} - (b''_{21})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b'_{22})^{(3)} - (b''_{22})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{24})^{(4)}$ , $T_i^0 \leq (\hat{Q}_{24})^{(4)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |

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| $\overline{T}_{26}(t) = T_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b_{26}')^{(4)} - (b_{26}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{28})^{(5)}$ , $T_i^0 \leq (\hat{Q}_{28})^{(5)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a_{28}')^{(5)} + a_{28}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a_{29}')^{(5)} + (a_{29}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a_{30}')^{(5)} + (a_{30}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\overline{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b_{28}')^{(5)} - (b_{28}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\overline{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b_{29}')^{(5)} - (b_{29}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\overline{T}_{30}(t) = T_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b_{30}')^{(5)} - (b_{30}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a_{32}')^{(6)} + a_{32}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a_{33}')^{(6)} + (a_{33}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |

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| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{36})^{(7)}$, $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a_{36}^{'})^{(7)} + a_{36}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ | |

which satisfy

$G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{40})^{(8)}$ , $T_i^0 \leq (\hat{Q}_{40})^{(8)}$,

$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$

$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$

By 166

$$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{40})^{(8)} + a''_{40})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{40}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[ (a_{41})^{(8)} G_{40}(s_{(40)}) - \left( (a'_{41})^{(8)} + (a''_{41})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{41}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[ (a_{42})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{42})^{(8)} + (a''_{42})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{42}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[ (b_{40})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{40})^{(8)} - (b''_{40})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{40}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[ (b_{41})^{(8)} T_{40}(s_{(40)}) - \left( (b'_{41})^{(8)} - (b''_{41})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{41}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[ (b_{42})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{42})^{(8)} - (b''_{42})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{42}(s_{(40)}) \right] ds_{(40)}$$

Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$

**Proof:** 166 A

Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy

$G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$,

$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$

$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$

By

$$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[ (a_{44})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{44}(s_{(44)}) \right] ds_{(44)}$$

$$\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[ (a_{45})^{(9)} G_{44}(s_{(44)}) - \left( (a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{45}(s_{(44)}) \right] ds_{(44)}$$

$$\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[ (a_{46})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{46}(s_{(44)}) \right] ds_{(44)}$$

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| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that<br>$(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} \left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$<br>$\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}} \left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$ | 171 |
| From which it follows that | 172 |

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| $(G_{20}(t)-G_{20}^0)e^{-(\hat{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that $G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\hat{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$ $\left(1+(a_{24})^{(4)}t\right)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that $(G_{24}(t)-G_{24}^0)e^{-(\hat{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$ $\left(1+(a_{28})^{(5)}t\right)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left(e^{(\hat{M}_{28})^{(5)}t}-1\right)$ | |
| From which it follows that $(G_{28}(t)-G_{28}^0)e^{-(\hat{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 5 | 175 |
| The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0+(\hat{P}_{32})^{(6)}e^{(\hat{M}_{32})^{(6)}s_{(32)}}\right)\right] ds_{(32)} =$ $\left(1+(a_{32})^{(6)}t\right)G_{33}^0 + \frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)}t}-1\right)$ | 176 |
| From which it follows that $(G_{32}(t)-G_{32}^0)e^{-(\hat{M}_{32})^{(6)}t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^0)e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^0}{G_{33}^0}\right)}+(\hat{P}_{32})^{(6)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 6 | 177 |

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| Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$ | |
| (e) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} \left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$<br>$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$ | 178 |
| From which it follows that<br>$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left[\left((\hat{P}_{36})^{(7)} + G_{37}^0\right) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 7 | |
| The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} \left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$<br>$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$ | 180 |
| From which it follows that<br>$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left[\left((\hat{P}_{40})^{(8)} + G_{41}^0\right) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} \left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$<br>$\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left(e^{(\hat{M}_{44})^{(9)} t} - 1\right)$ | |
| From which it follows that<br>$(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left[\left((\hat{P}_{44})^{(9)} + G_{45}^0\right) e^{\left(-\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0}\right)} + (\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ and to choose | 182 |

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| $(\widehat{P}_{13})^{(1)}$ and $(\widehat{Q}_{13})^{(1)}$ large to have | |
| $\frac{(a_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}\left[(\widehat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{13})^{(1)}\left\|G_{14}^{(1)}-G_{14}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$<br>$\int_0^t\{(a_{13}')^{(1)}\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}+$<br>$(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}+$<br>$G_{13}^{(2)}\left\\|(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}'')^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right\\|\ e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$<br>Where $s_{(13)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | |
| $\left\|G^{(1)}-G^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}t}\leq$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)}+(a_{13}')^{(1)}+(\widehat{A}_{13})^{(1)}+(\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)},T^{(1)};\ G^{(2)},T^{(2)}\right)\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a_{13}'')^{(1)}$ and $(b_{13}'')^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by | |

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| $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(1)} - (a_i^{''})^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a_{14}^{'})^{(1)}G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}^{'})^{(1)}$<br><br>If $G_{14}\ or\ G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}\ and\ G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t), t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t), t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right),\quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t), t) = (b_{15}^{'})^{(1)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |

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| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)}, (T_{19})^{(1)}), ((G_{19})^{(2)}, (T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)}\left\|G_{17}^{(1)} - G_{17}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\,ds_{(16)} +$<br>$\int_0^t\{(a_{16}')^{(2)}\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{-(\hat{M}_{16})^{(2)}s_{(16)}} +$<br>$(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}} +$<br>$G_{16}^{(2)}\left\|(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}'')^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}t} \leq$<br>$\frac{1}{(\hat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}\right)d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}'')^{(2)}$ and $(b_{16}'')^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | 199 |

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| condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ and $(\widehat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)}t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1, \left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)}G_{17}$ and by integrating<br><br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1/(a_{17}^{'})^{(2)}$<br><br>In the same way , one can obtain<br><br>$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}^{'})^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | 201 |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(2)}((G_{19})(t), t)) = (b_{17}^{'})^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i^{''})^{(2)}((G_{19})(t), t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}^{''})^{(2)}\left((G_{19})(t), t\right) = (b_{18}^{'})^{(2)}$ | 205 |

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| We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)}, (T_{23})^{(1)}), ((G_{23})^{(2)}, (T_{23})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{20})^{(3)}t}\}$ | 211 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :(\widetilde{(G_{23})}, \widetilde{(T_{23})}) = \mathcal{A}^{(3)}((G_{23}), (T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)}\left\|G_{21}^{(1)} - G_{21}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)} +$<br>$\int_0^t\{(a'_{20})^{(3)}\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{-(\hat{M}_{20})^{(3)}s_{(20)}} +$<br>$(a''_{20})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right)\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}} +$<br>$G_{20}^{(2)}\left\|(a''_{20})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a''_{20})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}t} \leq$<br>$\frac{1}{(\hat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a'_{20})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)}, (T_{23})^{(1)};\ (G_{23})^{(2)}, (T_{23})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |

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| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ $and$ $(\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(3)}t)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2\ and\ \left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}\ and\ G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)}G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br><br>If $G_{21}\ or\ G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ $is$ bounded, from below, the same property holds for $G_{21}\ and\ G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t), t\right)) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t), t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded. | 220 |

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| The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}'')^{(3)}\left((G_{23})(t),t\right)=(b_{22}')^{(3)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}<1$ and to choose<br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)}+\left((\hat{P}_{24})^{(4)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\hat{Q}_{24})^{(4)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{24})^{(4)}\right]\leq(\hat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{24})^{(4)}\left\|G_{25}^{(1)}-G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$<br>$\int_0^t\{(a_{24}')^{(4)}\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$(a_{24}'')^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$G_{24}^{(2)}\left\|(a_{24}'')^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}'')^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)}-(G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t}\leq$<br>$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}')^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$ | 226 |

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| And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i = 24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(4)} - (a_i^{''})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(4)}t\right)} > 0$ for t > 0 | 228 |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ :<br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if<br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}^{'})^{(4)}G_{25}$ and by integrating<br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a_{25}^{'})^{(4)}$<br>In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a_{26}^{'})^{(4)}$<br>If $G_{25}\ or\ G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}\ and\ G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t), t)) = (b_{25}^{'})^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i^{''})^{(4)}((G_{27})(t), t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results | 232 |

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right), \quad t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42

Analogous inequalities hold also for $G_{29}, G_{30}, T_{28}, T_{29}, T_{30}$

It is now sufficient to take $\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ and to choose

$(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ large to have

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$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[(\hat{P}_{28})^{(5)} + \left((\hat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{28})^{(5)} \quad (234)$$

$$\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{28})^{(5)}\right] \leq (\hat{Q}_{28})^{(5)} \quad (235)$$

In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric 236

$$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right) = \mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$

It results

$$\left|\tilde{G}_{28}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{28})^{(5)}\left|G_{29}^{(1)} - G_{29}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)} +$$

$$\int_0^t\{(a_{28}^{'})^{(5)}\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{-(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$G_{28}^{(2)}\left|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right) - (a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right|\ e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$$

Where $s_{(28)}$ represents integrand that is integrated over the interval $[0, t]$

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| From the hypotheses on it follows | |
| $\left\|(G_{31})^{(1)} - (G_{31})^{(2)}\right\| e^{-(\widehat{M}_{28})^{(5)}t} \leq$ $\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)} + (a'_{28})^{(5)} + (\widehat{A}_{28})^{(5)} + (\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right) d\left(((G_{31})^{(1)}, (T_{31})^{(1)};\ (G_{31})^{(2)}, (T_{31})^{(2)})\right)$ And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a''_{28})^{(5)}$ and $(b''_{28})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ and $(\widehat{Q}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(5)}$ and $(b''_i)^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t) = 0$ and $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(5)} - (a''_i)^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(5)}t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ and $\left((\widehat{M}_{28})^{(5)}\right)_3$ : **Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ and $G_{30}$ . indeed if $G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a'_{29})^{(5)} G_{29}$ and by integrating $G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a'_{29})^{(5)}$ In the same way , one can obtain $G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a'_{30})^{(5)}$ If $G_{29}$ or $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ is bounded, from below, the same property holds for $G_{29}$ and $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |

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| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t),t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1-e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}, G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}\left[(\hat{P}_{32})^{(6)} + \left((\hat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{32})^{(6)}\right] \leq (\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{32})^{(6)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{35})}, \widetilde{(T_{35})}$ : $\left(\widetilde{(G_{35})}, \widetilde{(T_{35})}\right) = \mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br><br>It results | 247 |

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| $\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{32})^{(6)} \left\|G_{33}^{(1)}-G_{33}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\, ds_{(32)} +$<br>$\int_0^t \{(a_{32}^{'})^{(6)} \left\|G_{32}^{(1)}-G_{32}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$(a_{32}^{''})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$G_{32}^{(2)} \left\|(a_{32}^{''})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right) - (a_{32}^{''})^{(6)}\left(T_{33}^{(2)}, s_{(32)}\right)\right\| \; e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\} ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}t} \leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)} + (a_{32}^{'})^{(6)} + (\widehat{A}_{32})^{(6)} + (\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right) d\left(((G_{35})^{(1)}, (T_{35})^{(1)};\ (G_{35})^{(2)}, (T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a_{32}^{''})^{(6)}$ and $(b_{32}^{''})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)} e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)} e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i = 32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 249 |
| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(6)} - (a_i^{''})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(6)}t\right)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ :<br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if<br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}^{'})^{(6)} G_{33}$ and by integrating<br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}^{'})^{(6)}$<br>In the same way , one can obtain<br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}^{'})^{(6)}$ | 251 |

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| If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i^{''})^{(6)}\big((G_{35})(t),t\big) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ large to have | 255 |
| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[(\hat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote | 258 |

**Definition of** $\widetilde{(G_{39})}, \widetilde{(T_{39})} : \left( \widetilde{(G_{39})}, \widetilde{(T_{39})} \right) = \mathcal{A}^{(7)}((G_{39}), (T_{39}))$
It results

$\left|\tilde{G}_{36}^{(1)} - \tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{36})^{(7)} \left|G_{37}^{(1)} - G_{37}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}}\, ds_{(36)} +$
$\int_0^t \{(a_{36}^{'})^{(7)} \left|G_{36}^{(1)} - G_{36}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} +$
$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) \left|G_{36}^{(1)} - G_{36}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}} +$
$G_{36}^{(2)} \left|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) - (a_{36}^{''})^{(7)}\left(T_{37}^{(2)}, s_{(36)}\right)\right| \; e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}} \} ds_{(36)}$

Where $s_{(36)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on it follows

$\left|(G_{39})^{(1)} - (G_{39})^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} t} \leq$
$\frac{1}{(\widehat{M}_{36})^{(7)}} \left((a_{36})^{(7)} + (a_{36}^{'})^{(7)} + (\widehat{A}_{36})^{(7)} + (\widehat{P}_{36})^{(7)} (\widehat{k}_{36})^{(7)}\right) d\left(\left((G_{39})^{(1)}, (T_{39})^{(1)}; (G_{39})^{(2)}, (T_{39})^{(2)}\right)\right)$

And analogous inequalities for $G_i \; and \; T_i$. Taking into account the hypothesis the result follows

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**Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)} e^{(\widehat{M}_{36})^{(7)} t} \; and \; (\widehat{Q}_{36})^{(7)} e^{(\widehat{M}_{36})^{(7)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and \; not \; on \; t)$ and hypothesis can replaced by a usual Lipschitz condition.

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**Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0 \; and \; T_i\,(t) = 0$

it results

$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)} (T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$

$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0 \;\; \text{for t} > 0$

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**Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2 \; and \; \left((\widehat{M}_{36})^{(7)}\right)_3$ :

**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37} \; and \; G_{38}$ . indeed if

$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating

$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)} \left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$

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In the same way , one can obtain

$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}')^{(7)}$

If $G_{37}$ *or* $G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively.

**Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}$ *and* $G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. 263

**Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(7)}((G_{39})(t),t)) = (b_{37}')^{(7)}$ then $T_{37} \to \infty$. 264

**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :

Indeed let $t_7$ be so that for $t > t_7$

$(b_{37})^{(7)} - (b_i'')^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$

Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to 265

$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results

$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right)$, $t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.
The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}'')^{(7)}\left((G_{39})(t),t\right) = (b_{38}')^{(7)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations

It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose 266
$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have

$$\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)} + \left((\widehat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{40})^{(8)} \quad 267$$

$$\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\widehat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{40})^{(8)}\right] \leq (\widehat{Q}_{40})^{(8)} \quad 268$$

In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric

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| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$ $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\,ds_{(40)}+$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\le$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ *and* $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i=40,41,42$ depend only on $\mathrm{T}_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$<br><br>it results<br><br>$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(8)}-(a_i^{''})^{(8)}(T_{41}(s_{(40)}),s_{(40)})\}ds_{(40)}\right]}\ge 0$<br><br>$T_i\,(t)\ge T_i^0e^{\left(-(b_i^{'})^{(8)}t\right)}>0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1,\left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ : | 276 |

**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if

$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a'_{41})^{(8)}G_{41}$ and by integrating

$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1/(a'_{41})^{(8)}$

In the same way , one can obtain

$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2/(a'_{42})^{(8)}$

If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively.

**Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. 277

**Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t),t)) = (b'_{41})^{(8)}$ then $T_{41} \to \infty$. 278

**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :

Indeed let $t_8$ be so that for $t > t_8$

$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t),t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$

Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to 279

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right),\ t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded. The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t),t(t),t\right) = (b'_{42})^{(8)}$

It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have 279 A

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\left(((G_{47})^{(1)},(T_{47})^{(1)}),((G_{47})^{(2)},(T_{47})^{(2)})\right) =$$
$$sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})},\widetilde{(T_{47})} : (\widetilde{(G_{47})},\widetilde{(T_{47})}) = \mathcal{A}^{(9)}((G_{47}),(T_{47}))$
It results

$$\left|\tilde{G}_{44}^{(1)}-\tilde{G}_i^{(2)}\right| \le \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)}-G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)-(a_{44}^{''})^{(9)}\left(T_{45}^{(2)},s_{(44)}\right)\right|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t} \le$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)}+(a_{44}^{'})^{(9)}+(\widehat{A}_{44})^{(9)}+(\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)},(T_{47})^{(1)};\ (G_{47})^{(2)},(T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44{,}45{,}46$ depend only on $\mathrm{T}_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$

From 99 to 44 it results

$$G_i\,(t) \ge G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(9)}-(a_i^{''})^{(9)}(T_{45}(s_{(44)}),s_{(44)})\}ds_{(44)}\right]} \ge 0$$

$$T_i\,(t) \ge T_i^0 e^{(-(b_i^{'})^{(9)}t)} > 0 \quad \text{for t} > 0$$

**Definition of** $((\widehat{M}_{44})^{(9)})_1, ((\widehat{M}_{44})^{(9)})_2$ *and* $((\widehat{M}_{44})^{(9)})_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)} G_{45}$ and by integrating

$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$

In the same way , one can obtain

$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t),t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t),t\right) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right),\quad t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\left((G_{47})(t),t\right) = (b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} - (a_{13}^{''})^{(1)}(T_{14},t) + (a_{14}^{''})^{(1)}(T_{14},t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(v_1)^{(1)}, (v_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, v^{(1)}, u^{(1)}$ : 281

By $(v_1)^{(1)} > 0$ , $(v_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$ , $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

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| **Definition of** $(\bar{v}_1)^{(1)},,(\bar{v}_2)^{(1)},(\bar{u}_1)^{(1)},(\bar{u}_2)^{(1)}$ :<br><br>By $(\bar{v}_1)^{(1)}>0$ , $(\bar{v}_2)^{(1)}<0$ and respectively $(\bar{u}_1)^{(1)}>0$ , $(\bar{u}_2)^{(1)}<0$ the roots of the equations $(a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_2)^{(1)}v^{(1)}-(a_{13})^{(1)}=0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2+(\tau_2)^{(1)}u^{(1)}-(b_{13})^{(1)}=0$ | 282 |
| **Definition of** $(m_1)^{(1)}$ , $(m_2)^{(1)}$ , $(\mu_1)^{(1)},(\mu_2)^{(1)},(\nu_0)^{(1)}$ :-<br><br>If we define $(m_1)^{(1)}$ , $(m_2)^{(1)}$ , $(\mu_1)^{(1)},(\mu_2)^{(1)}$ by<br>$(m_2)^{(1)}=(\nu_0)^{(1)},(m_1)^{(1)}=(\nu_1)^{(1)},\ if\ (\nu_0)^{(1)}<(\nu_1)^{(1)}$<br><br>$(m_2)^{(1)}=(\nu_1)^{(1)},(m_1)^{(1)}=(\bar{\nu}_1)^{(1)}\ ,if\ (\nu_1)^{(1)}<(\nu_0)^{(1)}<(\bar{\nu}_1)^{(1)},$<br><br>and $\boxed{(\nu_0)^{(1)}=\frac{G_{13}^0}{G_{14}^0}}$<br><br>$(\ m_2)^{(1)}=(\nu_1)^{(1)},(m_1)^{(1)}=(\nu_0)^{(1)},\ if\ (\bar{\nu}_1)^{(1)}<(\nu_0)^{(1)}$ | 283 |
| and analogously<br><br>$(\mu_2)^{(1)}=(u_0)^{(1)},(\mu_1)^{(1)}=(u_1)^{(1)},\ if\ (u_0)^{(1)}<(u_1)^{(1)}$<br><br>$(\mu_2)^{(1)}=(u_1)^{(1)},(\mu_1)^{(1)}=(\bar{u}_1)^{(1)}\ ,if\ (u_1)^{(1)}<(u_0)^{(1)}<(\bar{u}_1)^{(1)},$<br><br>and $\boxed{(u_0)^{(1)}=\frac{T_{13}^0}{T_{14}^0}}$<br><br>$(\ \mu_2)^{(1)}=(u_1)^{(1)},(\mu_1)^{(1)}=(u_0)^{(1)},if\ (\bar{u}_1)^{(1)}<(u_0)^{(1)}$ where $(u_1)^{(1)},(\bar{u}_1)^{(1)}$<br><br>are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{13}^0e^{((S_1)^{(1)}-(p_{13})^{(1)})t}\le G_{13}(t)\le G_{13}^0e^{(S_1)^{(1)}t}$<br><br>where $(p_i)^{(1)}$ is defined by equation<br><br>$\frac{1}{(m_1)^{(1)}}G_{13}^0e^{((S_1)^{(1)}-(p_{13})^{(1)})t}\le G_{14}(t)\le\frac{1}{(m_2)^{(1)}}G_{13}^0e^{(S_1)^{(1)}t}$ | 285 |
| $(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t}-e^{-(S_2)^{(1)}t}\right]+G_{15}^0e^{-(S_2)^{(1)}t}\le G_{15}(t)\le$<br><br>$\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a_{15}^{'})^{(1)})}[e^{(S_1)^{(1)}t}-e^{-(a_{15}^{'})^{(1)}t}]+\ G_{15}^0e^{-(a_{15}^{'})^{(1)}t})$ | 286 |
| $\boxed{T_{13}^0e^{(R_1)^{(1)}t}\le T_{13}(t)\le T_{13}^0e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}}T_{13}^0e^{(R_1)^{(1)}t}\le T_{13}(t)\le\frac{1}{(\mu_2)^{(1)}}T_{13}^0e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |

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| $\frac{(b_{15})^{(1)}T_{13}^{0}}{(\mu_1)^{(1)}((R_1)^{(1)}-(b_{15}^{'})^{(1)})}\left[e^{(R_1)^{(1)}t}-e^{-(b_{15}^{'})^{(1)}t}\right]+T_{15}^{0}e^{-(b_{15}^{'})^{(1)}t}\le T_{15}(t)\le$ $\frac{(a_{15})^{(1)}T_{13}^{0}}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t}-e^{-(R_2)^{(1)}t}\right]+T_{15}^{0}e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)},(S_2)^{(1)},(R_1)^{(1)},(R_2)^{(1)}$:- Where $(S_1)^{(1)}=(a_{13})^{(1)}(m_2)^{(1)}-(a_{13}^{'})^{(1)}$ $(S_2)^{(1)}=(a_{15})^{(1)}-(p_{15})^{(1)}$ $(R_1)^{(1)}=(b_{13})^{(1)}(\mu_2)^{(1)}-(b_{13}^{'})^{(1)}$ $(R_2)^{(1)}=(b_{15}^{'})^{(1)}-(r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ : $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)}\le-(a_{16}^{'})^{(2)}+(a_{17}^{'})^{(2)}-(a_{16}^{''})^{(2)}(\mathrm{T}_{17},t)+(a_{17}^{''})^{(2)}(\mathrm{T}_{17},t)\le-(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)}\le-(b_{16}^{'})^{(2)}+(b_{17}^{'})^{(2)}-(b_{16}^{''})^{(2)}\big((G_{19}),t\big)-(b_{17}^{''})^{(2)}\big((G_{19}),t\big)\le-(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(v_1)^{(2)},(v_2)^{(2)},(u_1)^{(2)},(u_2)^{(2)}$ : | 295 |
| By $(v_1)^{(2)}>0$ , $(v_2)^{(2)}<0$ and respectively $(u_1)^{(2)}>0$ , $(u_2)^{(2)}<0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(v^{(2)}\big)^2+(\sigma_1)^{(2)}v^{(2)}-(a_{16})^{(2)}=0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2+(\tau_1)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ and | 298 |
| **Definition of** $(\bar{v}_1)^{(2)},,(\bar{v}_2)^{(2)},(\bar{u}_1)^{(2)},(\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{v}_1)^{(2)}>0$ , $(\bar{v}_2)^{(2)}<0$ and respectively $(\bar{u}_1)^{(2)}>0$ , $(\bar{u}_2)^{(2)}<0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(v^{(2)}\big)^2+(\sigma_2)^{(2)}v^{(2)}-(a_{16})^{(2)}=0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2+(\tau_2)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ | 302 |
| **Definition of** $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)}=(v_0)^{(2)},(m_1)^{(2)}=(v_1)^{(2)}$, ***if*** $(v_0)^{(2)}<(v_1)^{(2)}$ | 305 |

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| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , $\boldsymbol{if}$ $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, $\boldsymbol{if}$ $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, $\boldsymbol{if}$ $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , $\boldsymbol{if}$ $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$,<br>and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, $\boldsymbol{if}$ $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)} t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)} t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)} t}\right] + G_{18}^0 e^{-(S_2)^{(2)} t} \le G_{18}(t) \le \right.$<br>$\left. \frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a'_{18})^{(2)})}\left[e^{(S_1)^{(2)} t} - e^{-(a'_{18})^{(2)} t}\right] + G_{18}^0 e^{-(a'_{18})^{(2)} t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b'_{18})^{(2)})}\left[e^{(R_1)^{(2)} t} - e^{-(b'_{18})^{(2)} t}\right] + T_{18}^0 e^{-(b'_{18})^{(2)} t} \le T_{18}(t) \le$<br>$\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)} + (r_{16})^{(2)} + (R_2)^{(2)})}\left[e^{((R_1)^{(2)} + (r_{16})^{(2)})t} - e^{-(R_2)^{(2)} t}\right] + T_{18}^0 e^{-(R_2)^{(2)} t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a'_{16})^{(2)}$<br>$(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b'_{16})^{(2)}$ | 318 |

$(R_2)^{(2)} = (b_{18}')^{(2)} - (r_{18})^{(2)}$

**Behavior of the solutions** 319

**Theorem 3:** If we denote and define

**Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ :

$(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying

$$-(\sigma_2)^{(3)} \le -(a_{20}')^{(3)} + (a_{21}')^{(3)} - (a_{20}'')^{(3)}(T_{21}, t) + (a_{21}'')^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$$

$$-(\tau_2)^{(3)} \le -(b_{20}')^{(3)} + (b_{21}')^{(3)} - (b_{20}'')^{(3)}(G_{23}, t) - (b_{21}'')^{(3)}((G_{23}), t) \le -(\tau_1)^{(3)}$$

**Definition of** $(v_1)^{(3)}, (v_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ : 320

By $(v_1)^{(3)} > 0$ , $(v_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$ , $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\left(u^{(3)}\right)^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and

By $(\bar{v}_1)^{(3)} > 0$ , $(\bar{v}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$ , $(\bar{u}_2)^{(3)} < 0$ the

roots of the equations $(a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\left(u^{(3)}\right)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$

**Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :- 321

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (v_0)^{(3)}, (m_1)^{(3)} = (v_1)^{(3)}$, ***if*** $(v_0)^{(3)} < (v_1)^{(3)}$

$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (\bar{v}_1)^{(3)}$ , ***if*** $(v_1)^{(3)} < (v_0)^{(3)} < (\bar{v}_1)^{(3)}$,

and $\boxed{(v_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (v_0)^{(3)}$, ***if*** $(\bar{v}_1)^{(3)} < (v_0)^{(3)}$

and analogously 322

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$ , ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

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| $G_{20}^0 e^{((S_1)^{(3)}-(p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)}t}$<br>$(p_i)^{(3)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)}-(p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)}G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)}-(p_{20})^{(3)}-(S_2)^{(3)})}\left[e^{((S_1)^{(3)}-(p_{20})^{(3)})t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \le G_{22}(t) \le \right.$<br>$\left.\frac{(a_{22})^{(3)}G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)}-(a'_{22})^{(3)})}[e^{(S_1)^{(3)}t} - e^{-(a'_{22})^{(3)}t}] + G_{22}^0 e^{-(a'_{22})^{(3)}t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)}+(r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)}+(r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)}T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)}-(b'_{22})^{(3)})}\left[e^{(R_1)^{(3)}t} - e^{-(b'_{22})^{(3)}t}\right] + T_{22}^0 e^{-(b'_{22})^{(3)}t} \le T_{22}(t) \le$<br>$\frac{(a_{22})^{(3)}T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)}+(r_{20})^{(3)}+(R_2)^{(3)})}\left[e^{((R_1)^{(3)}+(r_{20})^{(3)})t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$ | 327 |
| **Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:-<br>Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$<br>$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$<br>$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$<br>$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$ | 328 |
| **Behavior of the solutions of equation**<br>**Theorem:** If we denote and define<br>**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :<br>$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying<br>$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$<br>$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ :<br>By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}(v^{(4)})^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$<br>and $(b_{25})^{(4)}(u^{(4)})^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and | 329 |

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| **Definition of** $(\bar{v}_1)^{(4)},,(\bar{v}_2)^{(4)},(\bar{u}_1)^{(4)},(\bar{u}_2)^{(4)}$ :<br><br>By $(\bar{v}_1)^{(4)}>0$ , $(\bar{v}_2)^{(4)}<0$ and respectively $(\bar{u}_1)^{(4)}>0$ , $(\bar{u}_2)^{(4)}<0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_2)^{(4)}v^{(4)}-(a_{24})^{(4)}=0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2+(\tau_2)^{(4)}u^{(4)}-(b_{24})^{(4)}=0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)},(v_0)^{(4)}$ :-<br><br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)}$ by<br><br>$(m_2)^{(4)}=(v_0)^{(4)},(m_1)^{(4)}=(v_1)^{(4)}$, ***if*** $(v_0)^{(4)}<(v_1)^{(4)}$<br><br>$(m_2)^{(4)}=(v_1)^{(4)},(m_1)^{(4)}=(\bar{v}_1)^{(4)}$ , ***if*** $(v_4)^{(4)}<(v_0)^{(4)}<(\bar{v}_1)^{(4)}$,<br>and $\boxed{(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}$<br><br>$(m_2)^{(4)}=(v_4)^{(4)},(m_1)^{(4)}=(v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)}<(v_0)^{(4)}$ | 330 |
| and analogously<br><br>$(\mu_2)^{(4)}=(u_0)^{(4)},(\mu_1)^{(4)}=(u_1)^{(4)}$, ***if*** $(u_0)^{(4)}<(u_1)^{(4)}$<br><br>$(\mu_2)^{(4)}=(u_1)^{(4)},(\mu_1)^{(4)}=(\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)}<(u_0)^{(4)}<(\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)}=\frac{T_{24}^0}{T_{25}^0}}$<br><br>$(\mu_2)^{(4)}=(u_1)^{(4)},(\mu_1)^{(4)}=(u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)}<(u_0)^{(4)}$ where $(u_1)^{(4)},(\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{24}^0e^{((S_1)^{(4)}-(p_{24})^{(4)})t}\le G_{24}(t)\le G_{24}^0e^{(S_1)^{(4)}t}$<br><br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}}G_{24}^0e^{((S_1)^{(4)}-(p_{24})^{(4)})t}\le G_{25}(t)\le\frac{1}{(m_2)^{(4)}}G_{24}^0e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)}G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t}-e^{-(S_2)^{(4)}t}\right]+G_{26}^0e^{-(S_2)^{(4)}t}\le G_{26}(t)\le\right.$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0e^{(R_1)^{(4)}t}\le T_{24}(t)\le T_{24}^0e^{((R_1)^{(4)}+(r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}}T_{24}^0e^{(R_1)^{(4)}t}\le T_{24}(t)\le\frac{1}{(\mu_2)^{(4)}}T_{24}^0e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)}T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}^{'})^{(4)})}\left[e^{(R_1)^{(4)}t}-e^{-(b_{26}^{'})^{(4)}t}\right]+T_{26}^0e^{-(b_{26}^{'})^{(4)}t}\le T_{26}(t)\le$<br><br>$\frac{(a_{26})^{(4)}T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t}-e^{-(R_2)^{(4)}t}\right]+T_{26}^0e^{-(R_2)^{(4)}t}$ | 336 |

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| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a'_{24})^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b'_{24})^{(4)}$<br>$(R_2)^{(4)} = (b'_{26})^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a'_{28})^{(5)} + (a'_{29})^{(5)} - (a''_{28})^{(5)}(T_{29}, t) + (a''_{29})^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b'_{28})^{(5)} + (b'_{29})^{(5)} - (b''_{28})^{(5)}((G_{31}), t) - (b''_{29})^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ :<br>By $(v_1)^{(5)} > 0$, $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | 339 |
| **Definition of** $(\bar{v}_1)^{(5)},, (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ :<br>By $(\bar{v}_1)^{(5)} > 0$, $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$, $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :-<br>If we define $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}$ by<br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$, ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}}}$<br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$ | 341 |

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| $(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , $\boldsymbol{if}$ $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$, and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$ $(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}, \boldsymbol{if}$ $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | |
| Then the solution of global equations satisfies the inequalities $G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)} t}$ where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \le G_{30}(t) \le$ *(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b_{30}^{'})^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b_{30}^{'})^{(5)} t}\right] + T_{30}^0 e^{-(b_{30}^{'})^{(5)} t} \le T_{30}(t) \le$ $\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:- Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a_{28}^{'})^{(5)}$ $(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$ $(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b_{28}^{'})^{(5)}$ $(R_2)^{(5)} = (b_{30}^{'})^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ : $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying $-(\sigma_2)^{(6)} \le -(a_{32}^{'})^{(6)} + (a_{33}^{'})^{(6)} - (a_{32}^{''})^{(6)}(T_{33}, t) + (a_{33}^{''})^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$ | 349 |

| | |
|---|---|
| $-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}\big((G_{35}),t\big) - (b''_{33})^{(6)}\big((G_{35}),t\big) \le -(\tau_1)^{(6)}$ | |
| **Definition of** $(\nu_1)^{(6)}, (\nu_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, \nu^{(6)}, u^{(6)}$ :<br><br>By $(\nu_1)^{(6)} > 0$ , $(\nu_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{\nu}_1)^{(6)},, (\bar{\nu}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br><br>By $(\bar{\nu}_1)^{(6)} > 0$ , $(\bar{\nu}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}$ , $(m_2)^{(6)}$ , $(\mu_1)^{(6)}, (\mu_2)^{(6)}, (\nu_0)^{(6)}$ :-<br><br>If we define $(m_1)^{(6)}$ , $(m_2)^{(6)}$ , $(\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br><br>$(m_2)^{(6)} = (\nu_0)^{(6)}, (m_1)^{(6)} = (\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)} < (\nu_1)^{(6)}$<br><br>$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)} < (\nu_0)^{(6)} < (\bar{\nu}_1)^{(6)}$,<br>and $\boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$<br><br>$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)} < (\nu_0)^{(6)}$ | 351 |
| and analogously<br><br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br><br>$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$<br><br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G^0_{32}e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)}G^0_{32}}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G^0_{34}e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |

| | |
|---|---|
| $\boxed{T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)}T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)}-(b_{34}')^{(6)})}\left[e^{(R_1)^{(6)}t} - e^{-(b_{34}')^{(6)}t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)}T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t} - e^{-(R_2)^{(6)}t}\right] + T_{34}^0 e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:- Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$ $(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$ $(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$ $(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ : $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying $-(\sigma_2)^{(7)} \le -(a_{36}')^{(7)} + (a_{37}')^{(7)} - (a_{36}'')^{(7)}(T_{37}, t) + (a_{37}'')^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$ $-(\tau_2)^{(7)} \le -(b_{36}')^{(7)} + (b_{37}')^{(7)} - (b_{36}'')^{(7)}((G_{39}), t) - (b_{37}'')^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)}, (v_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, v^{(7)}, u^{(7)}$ : By $(v_1)^{(7)} > 0, (v_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}\left(u^{(7)}\right)^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)}, , (\bar{v}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ : By $(\bar{v}_1)^{(7)} > 0, (\bar{v}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}\left(u^{(7)}\right)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ **Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (v_0)^{(7)}$ :- | 362 |

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, $\boldsymbol{if}\ (\nu_0)^{(7)} < (\nu_1)^{(7)}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, $\boldsymbol{if}\ (\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,

and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, $\boldsymbol{if}\ (\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$

and analogously

$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, $\boldsymbol{if}\ (u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, $\boldsymbol{if}\ (u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,

and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}, \boldsymbol{if}\ (\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

(363)

Then the solution of global equations satisfies the inequalities

$$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)} t}$$

where $(p_i)^{(7)}$ is defined by equation

(364)

$$\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)} t} \quad (365)$$

$$\left(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})}\left[e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)} t}\right] + G_{38}^0 e^{-(S_2)^{(7)} t} \le G_{38}(t) \le \right.$$

$$\left. \frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a_{38}')^{(7)})}\left[e^{(S_1)^{(7)} t} - e^{-(a_{38}')^{(7)} t}\right] + G_{38}^0 e^{-(a_{38}')^{(7)} t}\right) \quad (366)$$

$$\boxed{T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}} \quad (367)$$

$$\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t} \quad (368)$$

$$\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b_{38}')^{(7)})}\left[e^{(R_1)^{(7)} t} - e^{-(b_{38}')^{(7)} t}\right] + T_{38}^0 e^{-(b_{38}')^{(7)} t} \le T_{38}(t) \le \quad (369)$$

$$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)} + (r_{36})^{(7)} + (R_2)^{(7)})}\left[e^{((R_1)^{(7)} + (r_{36})^{(7)})t} - e^{-(R_2)^{(7)} t}\right] + T_{38}^0 e^{-(R_2)^{(7)} t}$$

**Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:- 370

Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a'_{36})^{(7)}$

$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$

$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b'_{36})^{(7)}$

$(R_2)^{(7)} = (b'_{38})^{(7)} - (r_{38})^{(7)}$

**Behavior of the solutions of equation** 371

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :

$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying

$-(\sigma_2)^{(8)} \le -(a'_{40})^{(8)} + (a'_{41})^{(8)} - (a''_{40})^{(8)}(T_{41}, t) + (a''_{41})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$

$-(\tau_2)^{(8)} \le -(b'_{40})^{(8)} + (b'_{41})^{(8)} - (b''_{40})^{(8)}\big((G_{43}), t\big) - (b''_{41})^{(8)}\big((G_{43}), t\big) \le -(\tau_1)^{(8)}$

**Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : 372

By $(v_1)^{(8)} > 0, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$
and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and

**Definition of** $(\bar{v}_1)^{(8)}, , (\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :

By $(\bar{v}_1)^{(8)} > 0, (\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0, (\bar{u}_2)^{(8)} < 0$ the

roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$

and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$

**Definition of** $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :-

If we define $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by

$(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)}$, ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}$, ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)}$,

| | |
|---|---|
| and $\boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br><br>$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (v_0)^{(8)},\ \boldsymbol{if}\ (\bar{v}_1)^{(8)} < (v_0)^{(8)}$ | |
| and analogously<br><br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},\ \boldsymbol{if}\ (u_0)^{(8)} < (u_1)^{(8)}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}, \boldsymbol{if}\ (u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}, \boldsymbol{if}\ (\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)} t}$<br><br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \le G_{42}(t) \le \right.$<br>$\left.\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)} t} - e^{-(a'_{42})^{(8)} t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)} t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b'_{42})^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b'_{42})^{(8)} t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)} t} \le T_{42}(t) \le$<br><br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br><br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br><br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$ | 381 |

$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$

$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$

**Behavior of the solutions of equation 37 to 92** 382

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$

$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}\big((G_{47}), t\big) - (b''_{45})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$

**Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :

By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(9)}, , (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{\nu}_1)^{(9)} > 0$ , $(\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}$, ***if*** $(\nu_0)^{(9)} < (\nu_1)^{(9)}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}$, ***if*** $(\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T^0_{44}}{T^0_{45}}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^{0}e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^{0}e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^{0}e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}}G_{44}^{0}e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^{0}e^{-(S_2)^{(9)}t} \le G_{46}(t) \le \right.$$

$$\left.\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}^{'})^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}^{'})^{(9)}t}]+\ G_{46}^{0}e^{-(a_{46}^{'})^{(9)}t}\right)$$

$$\boxed{T_{44}^{0}e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^{0}e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}}T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^{0}}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}^{'})^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}^{'})^{(9)}t}\right]+T_{46}^{0}e^{-(b_{46}^{'})^{(9)}t} \le T_{46}(t) \le$$

$$\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a_{44}^{'})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b_{44}^{'})^{(9)}$

$(R_2)^{(9)} = (b_{46}^{'})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a_{13}^{'})^{(1)} - (a_{14}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14},t)\right) - (a_{14}^{''})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}(v^{(1)})^{2} + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}(v^{(1)})^{2} + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

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For $0 < \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (\nu_1)^{(1)} < (\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t) \geq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(\nu_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\nu_1)^{(1)}$

In the same manner , we get 384

$$\nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{\nu}_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\bar{\nu}_1)^{(1)}$

If $0 < (\nu_1)^{(1)} < (\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{\nu}_1)^{(1)}$ we find like in the previous case, 385

$$(\nu_1)^{(1)} \leq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}} \leq \nu^{(1)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (C)^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$ if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \le \frac{d\nu^{(2)}}{dt} \le -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \ge \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \le \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \le \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \le \nu^{(2)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$$

If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain

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$$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(2)}(t)$ :-

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$$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}, \quad \boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$$

In a completely analogous way, we obtain

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**Definition of** $u^{(2)}(t)$ :-

$$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}, \quad \boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

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If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$

Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then

$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right) - (a_{21}^{''})^{(3)}(T_{21},t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$$

**Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$

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It follows

$$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$$

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From which one obtains

For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

In the same manner , we get

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$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

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If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case,

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

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If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$ if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

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In the same manner , we get

$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

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$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

In the same manner , we get

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$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}'')^{(5)} = (b_{29}'')^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then

$(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(v_1)^{(5)}$ and $(\bar{v}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain 412

$$\frac{dv^{(6)}}{dt} = (a_{32})^{(6)} - \left((a'_{32})^{(6)} - (a'_{33})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right) - (a''_{33})^{(6)}(T_{33},t)v^{(6)} - (a_{33})^{(6)}v^{(6)}$$

**Definition of** $v^{(6)}$ :- $\boxed{v^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_2)^{(6)}v^{(6)} - (a_{32})^{(6)}\right) \le \frac{dv^{(6)}}{dt} \le -\left((a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_1)^{(6)}v^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(6)}, (v_0)^{(6)}$ :-

For $0 < \boxed{(v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}} < (v_1)^{(6)} < (\bar{v}_1)^{(6)}$

$$v^{(6)}(t) \ge \frac{(v_1)^{(6)} + (C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)} - (v_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)} - (v_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(v_1)^{(6)} - (v_0)^{(6)}}{(v_0)^{(6)} - (v_2)^{(6)}}}$$

it follows $(v_0)^{(6)} \le v^{(6)}(t) \le (v_1)^{(6)}$

In the same manner , we get 413

$$v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)} + (\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{v}_1)^{(6)} - (v_0)^{(6)}}{(v_0)^{(6)} - (\bar{v}_2)^{(6)}}}$$

From which we deduce $(v_0)^{(6)} \le v^{(6)}(t) \le (\bar{v}_1)^{(6)}$

If $0 < (v_1)^{(6)} < (v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}} < (\bar{v}_1)^{(6)}$ we find like in the previous case, 414

$$(v_1)^{(6)} \le \frac{(v_1)^{(6)} + (C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)} - (v_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)} - (v_2)^{(6)}\right)t\right]}} \le v^{(6)}(t) \le$$

$$\frac{(\bar{v}_1)^{(6)} + (\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}} \le (\bar{v}_1)^{(6)}$$

If $0 < (v_1)^{(6)} \le (\bar{v}_1)^{(6)} \le \boxed{(v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$ , we obtain 415

$$(v_1)^{(6)} \le v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)} + (\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}} \le (v_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}$, $\boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

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$$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$$

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If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain

$$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(C)^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

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**Definition of** $u^{(7)}(t)$ :-

$$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}, \quad \boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}'')^{(7)} = (a_{37}'')^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}'')^{(7)} = (b_{37}'')^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain 421

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a'_{40})^{(8)} - (a'_{41})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right) - (a''_{41})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \le \frac{dv^{(8)}}{dt} \le -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(8)}, (v_0)^{(8)}$ :-

For $0 < \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}} < (v_1)^{(8)} < (\bar{v}_1)^{(8)}$

$$v^{(8)}(t) \ge \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(v_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (v_2)^{(8)}}}$$

it follows $(v_0)^{(8)} \le v^{(8)}(t) \le (v_1)^{(8)}$

In the same manner , we get 422

$$v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{v}_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (\bar{v}_2)^{(8)}}}$$

From which we deduce $(v_0)^{(8)} \le v^{(8)}(t) \le (\bar{v}_8)^{(8)}$

If $0 < (v_1)^{(8)} < (v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}} < (\bar{v}_1)^{(8)}$ we find like in the previous case, 423

$$(v_1)^{(8)} \le \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}} \le v^{(8)}(t) \le$$

$$\frac{(\bar{v}_1)^{(8)} + (C)^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (\bar{v}_1)^{(8)}$$

If $0 < (v_1)^{(8)} \le (\bar{v}_1)^{(8)} \le \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$ , we obtain 424

$$(v_1)^{(8)} \le v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (v_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(8)}(t)$ :-

$(m_2)^{(8)} \leq v^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{v^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $v^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)}-(\nu_0)^{(9)}}{(\nu_0)^{(9)}-(\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \le \frac{(\nu_1)^{(9)}+(C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}} \le \nu^{(9)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \le (\bar{\nu}_1)^{(9)} \le \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(\nu_1)^{(9)} \le \nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \le \nu^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \le u^{(9)}(t) \le (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)}G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

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| $(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,<br>$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$<br>$with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$ | 433 |

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| $(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$ | 436 A |

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| $(a'_{44})^{(9)}(a'_{45})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a'_{45})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b'_{44})^{(9)}(r_{45})^{(9)} - (b'_{45})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b'_{13})^{(1)} - (b''_{13})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b'_{14})^{(1)} - (b''_{14})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b'_{15})^{(1)} - (b''_{15})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b'_{16})^{(2)} - (b''_{16})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b'_{17})^{(2)} - (b''_{17})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b'_{18})^{(2)} - (b''_{18})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a'_{20})^{(3)} + (a''_{20})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a'_{21})^{(3)} + (a''_{21})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a'_{22})^{(3)} + (a''_{22})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b'_{20})^{(3)} - (b''_{20})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b'_{21})^{(3)} - (b''_{21})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b'_{22})^{(3)} - (b''_{22})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) +$ | 485 |

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| $(a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | |
| **Proof:**<br><br>(e) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |

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| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492<br>A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations | 498 |

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| $G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a'_{28})^{(5)}+(a''_{28})^{(5)}(T^*_{29})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a'_{30})^{(5)}+(a''_{30})^{(5)}(T^*_{29})]}$ | |
| **Definition and uniqueness of** $T^*_{33}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T^*_{33}$ for which $f(T^*_{33}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a'_{32})^{(6)}+(a''_{32})^{(6)}(T^*_{33})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a'_{34})^{(6)}+(a''_{34})^{(6)}(T^*_{33})]}$ | 499 |
| **Definition and uniqueness of** $T^*_{37}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T^*_{37}$ for which $f(T^*_{37}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a'_{36})^{(7)}+(a''_{36})^{(7)}(T^*_{37})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a'_{38})^{(7)}+(a''_{38})^{(7)}(T^*_{37})]}$ | 500 |
| **Definition and uniqueness of** $T^*_{41}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T^*_{41}$ for which $f(T^*_{41}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a'_{40})^{(8)}+(a''_{40})^{(8)}(T^*_{41})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a'_{42})^{(8)}+(a''_{42})^{(8)}(T^*_{41})]}$ | 501 |
| **Definition and uniqueness of** $T^*_{45}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T^*_{45}$ for which $f(T^*_{45}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a'_{44})^{(9)}+(a''_{44})^{(9)}(T^*_{45})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a'_{46})^{(9)}+(a''_{46})^{(9)}(T^*_{45})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br><br>$\left[(b'_{13})^{(1)}(b''_{14})^{(1)}(G) + (b'_{14})^{(1)}(b''_{13})^{(1)}(G)\right]+(b''_{13})^{(1)}(G)(b''_{14})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that $\varphi$ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that | 502 |

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| there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*})=0$ | |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br>$\varphi(G_{19})=(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)}-(b_{16})^{(2)}(b_{17})^{(2)}-$<br>$\left[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19})+(b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})\right]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19})=0$ | 503 |
| Where in $(G_{19})(G_{16},G_{17},G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*})=0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br>$\varphi(G_{23})=(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)}-(b_{20})^{(3)}(b_{21})^{(3)}-$<br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23})+(b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23})=0$<br>Where in $G_{23}(G_{20},G_{21},G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*})=0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br>$\varphi(G_{27})=(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)}-(b_{24})^{(4)}(b_{25})^{(4)}-$<br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27})+(b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27})=0$<br>Where in $(G_{27})(G_{24},G_{25},G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*})=0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31})=(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)}-(b_{28})^{(5)}(b_{29})^{(5)}-$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31})+(b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right]+(b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31})=0$<br>Where in $(G_{31})(G_{28},G_{29},G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*})=0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35})=(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)}-(b_{32})^{(6)}(b_{33})^{(6)}-$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35})+(b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right]+(b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35})=0$<br>Where in $(G_{35})(G_{32},G_{33},G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that | 508 |

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| there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*})=0$ | |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39})=(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)}-(b_{36})^{(7)}(b_{37})^{(7)}-$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39})+(b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right]+(b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39})=0$<br><br>Where in $(G_{39})(G_{36},G_{37},G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0)>0\,,\varphi(\infty)<0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*})=0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43})=(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)}-(b_{40})^{(8)}(b_{41})^{(8)}-$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43})+(b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right]+(b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43})=0$<br><br>Where in $(G_{43})(G_{40},G_{41},G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0)>0\,,\varphi(\infty)<0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*})=0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47})=(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)}-(b_{44})^{(9)}(b_{45})^{(9)}-$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47})+(b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right]+(b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47})=0$<br><br>Where in $(G_{47})(G_{44},G_{45},G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0)>0\,,\varphi(\infty)<0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*})=0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*})=0\,,T_{14}^{*}$ given by $f(T_{14}^{*})=0$ and<br><br>$G_{13}^{*}=\frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}\quad,\quad G_{15}^{*}=\frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*}=\frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}\quad,\quad T_{15}^{*}=\frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*})=0\,,T_{17}^{*}$ given by $f(T_{17}^{*})=0$ and | 512 |
| $G_{16}^{*}=\frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}\quad,\quad G_{18}^{*}=\frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*}=\frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}\quad,\quad T_{18}^{*}=\frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |

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| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^*$ given by $\varphi((G_{23})^*) = 0$ , $T_{21}^*$ given by $f(T_{21}^*) = 0$ and<br>$G_{20}^* = \frac{(a_{20})^{(3)} G_{21}^*}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^*)]}$ , $G_{22}^* = \frac{(a_{22})^{(3)} G_{21}^*}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^*)]}$<br>$T_{20}^* = \frac{(b_{20})^{(3)} T_{21}^*}{[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}{}^*)]}$ , $T_{22}^* = \frac{(b_{22})^{(3)} T_{21}^*}{[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}{}^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^*$ given by $\varphi(G_{27}) = 0$ , $T_{25}^*$ given by $f(T_{25}^*) = 0$ and<br>$G_{24}^* = \frac{(a_{24})^{(4)} G_{25}^*}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}^*)]}$ , $G_{26}^* = \frac{(a_{26})^{(4)} G_{25}^*}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}^*)]}$ | 516 |
| $T_{24}^* = \frac{(b_{24})^{(4)} T_{25}^*}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27})^*)]}$ , $T_{26}^* = \frac{(b_{26})^{(4)} T_{25}^*}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^*$ given by $\varphi((G_{31})^*) = 0$ , $T_{29}^*$ given by $f(T_{29}^*) = 0$ and<br>$G_{28}^* = \frac{(a_{28})^{(5)} G_{29}^*}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^*)]}$ , $G_{30}^* = \frac{(a_{30})^{(5)} G_{29}^*}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^*)]}$ | 518 |
| $T_{28}^* = \frac{(b_{28})^{(5)} T_{29}^*}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^*)]}$ , $T_{30}^* = \frac{(b_{30})^{(5)} T_{29}^*}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^*$ given by $\varphi((G_{35})^*) = 0$ , $T_{33}^*$ given by $f(T_{33}^*) = 0$ and<br>$G_{32}^* = \frac{(a_{32})^{(6)} G_{33}^*}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^*)]}$ , $G_{34}^* = \frac{(a_{34})^{(6)} G_{33}^*}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^*)]}$ | 520 |
| $T_{32}^* = \frac{(b_{32})^{(6)} T_{33}^*}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^*)]}$ , $T_{34}^* = \frac{(b_{34})^{(6)} T_{33}^*}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^*$ given by $\varphi((G_{39})^*) = 0$ , $T_{37}^*$ given by $f(T_{37}^*) = 0$ and | 522 |

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| $G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)} (T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)} (T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)} ((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)} ((G_{39})^{*})]}$ | |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)} (T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)} (T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)} ((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)} ((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br><br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br><br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)} (T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)} (T_{45}^{*})]}$<br><br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)} ((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)} ((G_{47})^{*})]}$ | 523 A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:**_Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}} (T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j} (G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)} \mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*} \mathbb{T}_{14}$ | 525 |

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| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^{*}\mathbb{G}_{j}\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^{*}\mathbb{G}_{j}\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^{*}\mathbb{G}_{j}\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 532 |
| $\frac{\partial(a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial(b_i^{''})^{(2)}}{\partial G_j}((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^{*}\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | 540 |

$$G_i = G_i^* + \mathbb{G}_i \qquad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)} \quad , \frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^*\mathbb{T}_{21} \quad (541)$$

$$\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^*\mathbb{T}_{21} \quad (542)$$

$$\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^*\mathbb{T}_{21} \quad (543)$$

$$\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^*\mathbb{G}_j\right) \quad (544)$$

$$\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^*\mathbb{G}_j\right) \quad (545)$$

$$\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^*\mathbb{G}_j\right) \quad (546)$$

**ASYMPTOTIC STABILITY ANALYSIS** (547)

**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ *and* $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- (548)

$$G_i = G_i^* + \mathbb{G}_i \qquad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)} \quad , \frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^*) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25} \quad (549)$$

$$\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25} \quad (550)$$

$$\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25} \quad (551)$$

$$\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right) \quad (552)$$

$$\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right) \quad (553)$$

$$\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right) \quad (554)$$

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^*) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^*\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^*\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^*\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |

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| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{37})^{(7)}}{\partial T_{37}}(T^*_{37}) = (q_{37})^{(7)}$ , $\frac{\partial(b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a'_{36})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G^*_{36}\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a'_{37})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G^*_{37}\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. | |

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| Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \ , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}((G_{43})^*) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^*\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^*\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b_{41}^{'})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T_{41}^*\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b_{42}^{'})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T_{42}^*\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS** **Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. **Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |

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| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$ $\left[\left(\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\right]$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\right)$ $+\left(\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\right)$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((b'_{13})^{(1)} + (b'_{14})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $+\left(\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$ $+\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G^*_{13}\right)$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T^*_{14} + (b_{14})^{(1)}s_{(13),(15)}T^*_{13}\right)\} = 0$ $+$ $\left((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)}\right)\{\left((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)}\right)$ $\left[\left(\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}^*_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^*_{16}\right)\right]$ $\left(\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^*_{17}\right)$ $+\left(\left((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}^*_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^*_{17}\right)$ $\left(\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^*_{16}\right)$ $\left(\left((\lambda)^{(2)}\right)^2 + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ $\left(\left((\lambda)^{(2)}\right)^2 + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ | |

$$+\left(\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\right)\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left(\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\right)$$

$$+\left(\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((b_{20}')^{(3)}+(b_{21}')^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}')^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}')^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}')^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}')^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}')^{(4)}+(a_{25}')^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((b_{24}')^{(4)}+(b_{25}')^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\}\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left((\lambda)^{(6)}\right)^2+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G^*_{33}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G^*_{32}\right)$$

$$\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T^*_{33}+(b_{33})^{(6)}s_{(32),(34)}T^*_{32}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b'_{38})^{(7)}-(r_{38})^{(7)}\}\{(\lambda)^{(7)}+(a'_{38})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(q_{36})^{(7)}G^*_{36}\right)\Big]$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T^*_{37}+(b_{37})^{(7)}s_{(36),(37)}T^*_{37}\Big)$$

$$+\left((\lambda)^{(7)}+(a'_{37})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G^*_{36}+(a_{36})^{(7)}(q_{37})^{(7)}G^*_{37}\Big)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T^*_{37}+(b_{37})^{(7)}s_{(36),(36)}T^*_{36}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^*_{36}\right)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^*_{37}+(b_{37})^{(7)}s_{(36),(38)}T^*_{36}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\}\{(\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^*_{40}\right)\Big]$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^*_{41}+(b_{41})^{(8)}s_{(40),(41)}T^*_{41}\Big)$$

$$+\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^*_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^*_{41}\Big)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^*_{41}+(b_{41})^{(8)}s_{(40),(40)}T^*_{40}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^*_{40}\right)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^*_{41}+(b_{41})^{(8)}s_{(40),(42)}T^*_{40}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^*_{44}\right)\right]$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^*_{45}+(b_{45})^{(9)}s_{(44),(45)}T^*_{45}\Big)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^*_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45}+(b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left(\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45}+(b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

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# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i(x)\, xi$ where $x_i \in R^n_i$, $x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i(x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i(\mu)$ of the $A_i(x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k \tag{5}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to

replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 5. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon\ C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \ i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0}$

$\overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow *x.t restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set *K* is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** *x.t* is defined for all $t \geq 0$

**A4:** *x.t* is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\mathcal{A}(\tilde{A}), C \backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) In$ $||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population $i$ persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population $i$ is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) In \ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^{t}_{0} (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\to\infty}{limsup} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v = au_i(x) + w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |B_i(t,x)u_i(x)|(a + \alpha + exp(-\beta t)|w|)$. Hence, $\limsup_{t\to\infty} \frac{1}{t} In\, B_i(t,x)\, v| \leq$

$$\limsup_{t\to\infty} \frac{1}{t} In\, B_i(t,x)\, u_i(x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\, |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\, u_i(x)| = In\, |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\, u_i(x.t)||\, B_i(t,x)\, u_i(x)|$$

$$= \text{In } b_i(s,t.x)\, b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x - bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

$$=$$

$$= (A_i(x.t)u_i(x.s), u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s), u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } | (x.t)_i |$$

$$\leq \limsup_{t\to\infty} \frac{In\ \gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x) =$ Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let $K \subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let $K \subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if $W^s(K) \subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of

irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$ where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ and $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as $n\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n$ − perturbations of (2), then the weak* limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak* compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak* limit points of $\mu_n$. Let $\mu$ be such a weak* limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\le \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x)u_i^n(x), u_i^n(x)\rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)|d\mu(x) + |\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\,d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\,ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i\,dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\,d\,v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \{ \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i \epsilon I}{if\ i \notin I}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that \quad \tilde{V}.t \subset intV\ for\ all\ t >$ $0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t$ −invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently *K* is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s =$

$s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_s$. If $r(x) = s(x), then\ x \in M_r$. Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume $x \mapsto A(x)$ is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists $j \in \{1,\ldots,k\}$ such that $M_i$ ----- $\{x \in C : x_l \gg 0 \text{ for } l \neq j\}$ Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_i M_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, *$\lambda_i(\mu) > 0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, ..., x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the

system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j \ (x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1{}^1, \ldots, E_1{}^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i{}_2(E_1{}^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$$

$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > 1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria $E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$ and $\lambda_3(E_1)$ are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j, \qquad i = 1,\ldots,k;\ j = 1,\ldots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i\, a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i\, x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \qquad i = 1, \ldots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \ldots, k$$

It has been shown [12,ch.16.1,16.2] That for *k* =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1 \to E_2 \to E_3 \to E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is

the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1-y) - b(N)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*′ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\curlywedge_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\curlywedge_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\curlywedge_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + K p\, x_i x_i - 1}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i)-1 & \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions

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# Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a

qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and

orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.13) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.14) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.15) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{'} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$ (1.4)

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \epsilon [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1, \varphi_2, f_1, f_2, b_1, b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for

each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as* $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 5. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$. We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C(R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C(R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$,

( $d_1^*$, $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$; that is, *for* $i = 1, 2$,

$$|f_i(a, P_1', P_2') - f_i(a, P_1'', P_2'')| \leq d_i\ [||P_1' - P_1'', P_2' - P_2''||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\partial P_1, \partial f_1/\partial P_2 \in C(R^+ \times K_1 \times K_2, R^+),$$

$$\partial f_2/\partial P_1, \partial f_2/\partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C(R^+ K_1 \times K_2, R^+)$, $b_1 \leq \beta_1^*$ *on* $R^+ \times K_1 \times K_2$,

$b_2 \in C(R^+ K_2 \times K_1, R^+)$, $b_2 \leq \beta_2^*$ *on* $R^+ \times K_2 \times K_1$,

*($\beta_1^*$, $\beta_2^*$ are positive constants);*

$b_1$, $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$|b_i(a, P_1', P_2') - b_i(a, P_1'', P_2'')| \leq \beta_i\ [||P_1' - P_1''|| + ||P_2' - P_2''||]$

$B_1$, $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2)) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2,$

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, R^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, R^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1, 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[-\int_0^\infty f_i(a + s, P(t + S))ds\right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[-\int_o^x f_1(a + s, p(t + s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0)\exp\left[-\int_o^t f_1(a - t + s, p_1(s))\, ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

*(2.3)* $p_i(a,t) = B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i and\ L_i\ (i = 1,2)$ as follows:

$$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a,t,\ p_i(t)) = \exp[-\int_o^t f_1(a+s,\ p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

$$(2.6)\ p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da + \int_o^\infty \varphi_i(a, 0)\ L_i(a,t,p_i(t)da,$$

$$t > 0, i = 1,2,$$

$$(2.7)\ p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$$

$$(2.8)\ \beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.\left(a,t,P_i(t)\right)da$$

$$+\int_0^\infty b_i\left(a+t, P_i(t)\right)\varphi_i(a,0)L_i\left(a,t,P_i(t)\right)da$$

$$i = 1, 2, t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t) \left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t) \phi(0) d_i^*] t \quad for \ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T], R^+), \quad B_i \in C([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in C([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau, T]$. Then for any

$(x, y) \in C^+[-\tau, T] \times C^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau, T], \quad C_T^2 = C^+[-\tau, T] \times C^+[-\tau, T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for $i = 1,2$.

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \pi C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \quad P_t = \ (P_{1t}, P_{2t}) \in C^2_{t*} \mid \|P_t - \Phi_t\|_{t*} \le r^*$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)\, da \ for\ s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)\, da \ for\ s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Phi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^* \right] \quad t^*$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for t $\le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \le \left| \int_0^t B_1^{(1)}(a) \left\{ M_1(a, t, P_t^{(1)}) - M_1(a, t, P_t^{(2)}) \right\} da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a, t, P_t^{(2)}) da \right|$$

$$+ \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \; for \; x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \| ds\right) da$$

$$\leq d_1 \beta_1^* \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o)\left[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds\right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \| t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\text{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right]\right] d\sigma$$

$$+ \int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^{\infty} \varphi(\sigma,o) b_1\left(\sigma+a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma,a,P_t^{(1)}(a)\right)-L_1\left(\sigma,a,P_t^{(2)}(a)\right)\right]d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right)-b_1\left(\sigma+a, P_t^{(2)}(a)\right)\right| \le B_1 \,/\!/\, P_t^{(1)}-P_t^{(2)} /\!/ \text{ for } a\,\varepsilon[o,t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \le \beta_1 \Phi_1(0) ||P_t^{(1)}-P_t^{(2)}||_t^{*} + \Phi_1(0)\beta_1 d_1 ||P_t^{(1)}-P_t^{(2)}||_t^{*a},$$

$$|R_2(a)|=[\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a))-b_1(a-\sigma, P_t^{(2)}(a))\}M_1(\sigma, a, P_t^{(1)}(a))$$

$$+b_1(a,-\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a))-M_1(a-\sigma, P_t^{(2)}(a))\}]d\sigma]$$

$$+b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a))-M_1(a-\sigma, P_t^{(2)}(a))\}]d\sigma]$$

$$\le \int_0^a [\beta_1^{*} \Phi_1(0)\, e^{\delta 1^a} \beta_1 \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}(a-o)]do$$

$$\le \beta_1^{*} \Phi_1(0)\, e^{\delta 1^a} \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a)\right|$$

$$\le \int_o^a \left| B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma)\right| B_1^{*d}\sigma$$

$$+\Phi_1(0) /\!/ P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\left\{B_1 + B_2 d_1 a + B_1^{*}\left(B_2 a + d_1 a^2/2\right)e^{\delta 1^a}\right\}$$

$$\le \beta_1^{*}\int_o^a \left| B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma)\right| do + \left(r^{*} + /\!/ \Phi_t /\!/_{t*}\right)\| P_t^{(1)}-P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \to \beta_1$ as $a \to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a)\right| \le \left(r^{*} + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))\text{-}b_1(a\text{-}\sigma, P_t^{(1)}(a))\, M_1(\sigma, a, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + /\!/ \Phi_t /\!/_{t*}\right) /\!/ \; P_t^{(1)}\text{-}P_t^{(2)} /\!/_{t*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right) /\!/ \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^{*})$$

As $t^{*} \to o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^{*}, P_t^{*}$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp\, P_2^{*}(a)}{da} + f_i(\mathrm{a}, P_1^{*}, P_2^{*})\, P_i^{*}(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^{*} = \int_o^{\infty} P_i^{*}(a)da, i = 1,\ 2,$$

$$(3,3)\ P_i^{*} = \int_o^{\infty} b_i(a,\ P_1^{*},\ P_2^{*}) P_i^{*}(a)da,\ i = 1.\ \ 2$$

All solutions of (3.1)are of the from

$$(3.4)\ P_i^{*}(a) = P_i^{*}(o) \exp\left[ - \int_o^{\infty} f_i(s, P_1^{*}, P_2^{*})ds \right], i = 1, \quad 2, a \geq 0,$$

Which together with (3.3) lead to

$$1=\int_o^{\infty} b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^{\infty} b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $$F_{i.}\ (P_1,P_2) = \int_o^{\infty} b_i(a,P_1,P_2)\exp\left[-\int_o^{\infty} f_i\,(s,P_1,P_2)ds\right]da,$$

$$i = 1,\ 2, \quad P_1 \geq 0, \quad P_2 \geq 0,$$

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_5$. *there exists a positive constant c for which*

$$F_1(\mathrm{C},0) < F_{2.}(\mathrm{C},0),$$

(3.8) $$F_1(0.\mathrm{C}) < F_{2.}(0,\mathrm{C}),$$

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $$F_1(P_1^*,P_2^*) = 1 = F_2(P_1^*,P_2^*)$$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $$P_2^*(a) = [P_2^*\exp[-f_i\,(s,P_1^*,P_2^*)ds])/(\int_o^{\infty}\exp[-\int_o^a f_i\,(s,P_1^*,P_2^*)ds]da),$$

$$i = 1,\ 2, \quad a \geq 0,$$

Proof Consider the elements of the surfaces defined by

$$\mathrm{Z}= F_1(P_1,P_2), z = F_2(P_1,P_2),\ P_1 \geq 0, P_2 \geq 0,$$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i/\partial P_j < 0\ and$ , by ($A_2$), $\partial b_i/\partial P_j \leq 0$ for all

$$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$$

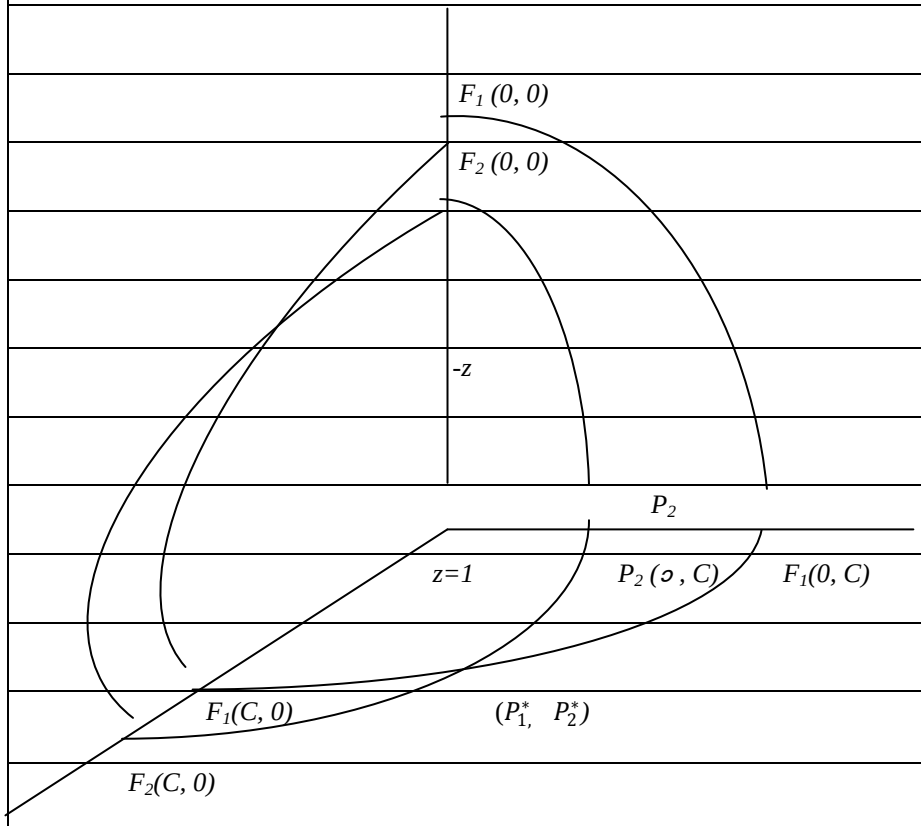
$j = 1,\ 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= 1, $P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$$F_1(P_1, P_2 = 1 \; and \; F_2(P_1, P_2) = 1, \; F_1(P_1, P_2 \geq 0,$$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility 1< $F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p_1^*, p_2^*)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p_1^*, p_2^*$, we let

$$(3.11) \quad \begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \; i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*}\, p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square 1}^{2} \frac{\partial b_i}{\partial p_{\square}^*}\, p_{i\square}(t)\right\} da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_{\square}(a)\exp[\lambda t], \quad {}_{\square}(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty {}_{\square}(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d_{\square}}{da} + \lambda_{\square} = -f_i\left(a, p_1^*, p_2^*\right)_{\square}(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*}\, p_{i\square}^*\right\},$$

(3.15)

$$_{\square}(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)_{\square}(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*}\, p_{i\square}^*\right\} da,$$

Where $i = 1, 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \quad p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\Phi_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\Phi_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\Phi_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\Phi_i(a) = \{\Phi_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_i\right\} e^{\lambda s} ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\Phi_i(a)\left[p_i^*(a)/p_i^*(0)c_i \exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_i\right\} da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a}\left\{\int_0^a \left(\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_i\right) e^{\lambda s} ds\right\} da\right]$$

$$- p_i^*(a) e^{-\lambda a}\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_i\right\} e^{\lambda s} ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\Phi_1(a)$ and $\Phi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$\left(\lambda - A_{11}\right) p_1^* - A_{12} p_2^* \exp\left(-\lambda\tau\right) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21} p_1^* \exp\left(-\lambda\tau\right) + \left(\lambda - A_{22}\right) p_2^* = g_{21}(\lambda) p_1^* \exp\left(-\lambda\tau\right) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $$A_{ij} = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_j^*} - \frac{\partial f_i}{\partial p_j^*}\right]da,$$

(3.23) $$g_{ij}(\lambda) = \left[p_i^*(0)c_i\right]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\,da\right\}\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_j^*} - b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)\right]da\right\}$$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)da,\quad i, j = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i\left(a, p_1^*, p_2^*\right) - f_i\left(a, p_1^*, p_2^*\right),\quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et$
$$\begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})\,e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda,\tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$$S_2(\lambda,\tau) = \left[A_{21}g_{12}(\lambda) + A_{12}g_{21}(\lambda) + g_{12}(\lambda)g_{21}(\lambda)\right]\exp[-2\lambda\tau]$$

$$-A_{11}g_{22}(\lambda) - A_{22}g_{11}(\lambda) - g_{11}(\lambda)g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6$. $0 < \beta_i^* < d_i^*$ $(i = 1, 2.)$ $(see(A_1)\ and\ (A_2))$;

$A_7$. $\{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = |A_{11}g_{12}(0)| + |A_{12}g_{21}(0)| + |g_{12}(0)g_{21}(0)|$$

$$+ |A_{11}g_{22}(0)| + |A_{22}g_{11}(0)| + |g_{11}(0)g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i(a, t) = \phi_i(a) \exp[\lambda t] \to 0$ *as* $t \to \infty$ for $i = 1, 2$ *and* $a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$$F_1(\lambda, \tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau];$$

$$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau),$$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$, $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda$ $\mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - \left| F_2(\lambda, \tau) \right|$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2,$$

$$q = \{4 | (A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| | - | A_{11} + A_{22} + \| S_1(0) \| |^2\}/2,$$

It will now follow, from (A) and (3.26), $F_1(\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder

concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power

obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy

withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation

from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

**TITLE**

## Numerology Or Ideology

**INTRODUCTION—VARIABLES USED**

**PARRHESIA NUMBER 1 2006 • 62 – 87 Numerology or Ideology Zizek's Critique of Deleuze Robert Sinner brink I have deleted some sentences and added my own remarks to herein below exposition. For acknowledgements and references kindly see the original article. / Please note that the expatiation is model able and the following presupposition format is preeminently suitable for modelling**

(1) Although _i_ek confines his critique to the explicit relationship between Hegel and Deleuze, this conceptual background is also important for understanding the complexities of this fascinating philosophical encounter.
(2) Zizek's perverse Hegelian critique of Deleuze operates on two related levels: first, Zizek questions the plausibility of Deleuze's violent rejection of Hegelian dialectic, and secondly, Zizek claims that this radical antipathy towards Hegel in fact conceals (e) a secret complicity.
(3) As Catherine Malabou has argued, Deleuze's 'philosophically bearded' Hegel is (=) reductive and monological, foreclosing (e) any possibility of a productive encounter between Hegelian dialectic and Deleuzian difference.
(4) Throughout Deleuze's oeuvre, according to Malabou, "Hegelian philosophy is presented as perfectly expressing binary logic in (eb) all its heaviness: one plus one equals two, and two ends up reducing to one."
(5) For Zizek, the 'violence' of Deleuze's rejection of Hegel suggests (eb) an anxiety that points to the hidden proximity between them.
(6) Zizek thus assumes an 'assimilationist' position on the Hegel-Deleuze relationship, making the semi-serious claim that Deleuze's refusal to 'couple' with Hegel suggests (eb) a kind of "incest

prohibition" revealing the secret kinship between the two (OwB 48).

(7) For all his dazzling detours and striking non-sequiturs, I want to suggest that one can discern a more or less consistent argumentative trajectory in Zizek's critical encounter with Deleuze, namely that what Deleuze presents as a radical philosophy of difference with a correlated politics of the multitude in fact is better comprehended through Zizek's Hegelio-Lacanian theory of subjectivity and post-Marxist critique of ideology. Indeed, Zizek's Hegelianised Deleuze is supposed to present a theoretical alternative to the Deleuzian philosophy of difference and politics of the multitude articulated in A Thousand Plateaus, and more explicitly, in Hardt and Negri's Empire.

(8) It is this claim that I shall question in what follows. Zizek's critique of Deleuze begins by drawing attention to the well-known conceptual couples of virtual and actual, becoming and being, claiming that these repeat the instability of the traditional philosophical opposition between idealism and materialism.

(9) This opposition was definitively superseded by post-Kantian idealism, above all by Hegel; but Zizek also points to the relevance of the contemporary 'virtualization' of matter in quantum physics as a sign of the overcoming of this traditional philosophical opposition (OwB, 21-26). Relying on Manuel de Landa's account, Zizek draws attention to a central tension or even contradiction between Deleuze I (of the Logic of Sense and Difference and Repetition), and Deleuze II (the Deleuzo-Guattarian duo of Anti-Oedipus and A Thousand Plateaux).

(10) For Deleuze I, the virtuality of becoming as the sense-event is (=) the passive, causally sterile effect of (e) actual bodily-material causes and processes.

(11) For Deleuze II, the domain of virtual singularities and flux of pure becoming now becomes (=) the productive cause of individuated bodies and subjects within (eb) diverse concrete assemblages.

(12) So it seems that the domain of virtuality, or pure flux of becoming, is (=) on the one hand an impassive effect, and on the other (=) a generative process, which suggests (eb) the need for some mediating figure or theoretical account to explain the precise relationship between these central aspects of Deleuze's thought (OwB, 20-21 ff.).

(13) This tension or unresolved opposition between virtual becoming an actual being recalls the way in which post-Kantian idealism sought to overcome the Kantian dualism between phenomenal and (e&eb) noumenal aspects of experience, notably through Schelling's complementarities between philosophy of nature and philosophy of spirit, which was then taken over and transformed in Hegelian speculative idealism (OwB, 23-24).

(14) Deleuze I's resolution of the apparent deadlock between virtual and actual or becoming and being is to posit a third mediating figure, the "quasi-cause" or "pure agency of transcendental causality" (OwB, 27)—that which remains excessive in the cause's effect, the surplus otherness irreducible to precedent causal conditions.

(15) The quasi-cause designates the advent of the New, in the sense of the unforeseeable, the incalculable, the singular; it represents the "metacause of the very excess of the effect over its (corporeal) causes" (OwB, 27). The claim is that, over and above corporeal causes, which can never be complete, we must posit a transcendental causality to account for the unforeseeable excess of the effect (the New) over its causal conditions and determinations.

**(16)** Deleuze's example, which in some ways raises more questions than it answers, is the emergence of Italian neo-realism after World War II—an instance of how the Event of the New outstrips the historical conditions that provide its context. We can't explain neo-realism, Deleuze claims, just by reference to these attendant historical circumstances; hence Deleuze posits the concept of "quasi-cause" as a 'transcendental causality' that might account for the unforeseeable excess of the effect over its antecedent causal conditions.

| NOTATION | |
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| **Module One**<br>As Catherine Malabou has argued, Deleuze's 'philosophically bearded' Hegel is reductive and monological, foreclosing (e) any possibility of a productive encounter between Hegelian dialectic and Deleuzian difference. | |
| $G_{13}$ : Category one of any possibility of a productive encounter between Hegelian dialectic and Deleuzian difference<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of reductive and monological<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>Throughout Deleuze's oeuvre, according to Malabou, "Hegelian philosophy is presented as perfectly expressing binary logic in (eb) all its heaviness: one plus one equals two, and two ends up reducing to one | |
| $G_{16}$ : Category one of binary logic<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of all its heaviness: one plus one equals two, and two ends up reducing to one<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>For Zizek, the 'violence' of Deleuze's rejection of Hegel suggests (eb) an anxiety that points to the hidden proximity between them | |
| $G_{20}$ : Category one of violence' of Deleuze's rejection of Hegel<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of anxiety that points to the hidden proximity between them<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>Zizek thus assumes an 'assimilationist' position on the Hegel-Deleuze relationship, making the semi-serious | |

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| claim that Deleuze's refusal to 'couple' with Hegel suggests (eb) a kind of "incest prohibition" revealing the secret kinship between the two (OwB 48) | |
| $G_{24}$ : Category one of assimilationist' position on the Hegel-Deleuze relationship, making the semi-serious claim that Deleuze's refusal to 'couple' with Hegel<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of kind of "incest prohibition" revealing the secret kinship between the two (OwB 48)<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>For Deleuze I, the virtuality of becoming as the sense-event is (=) the passive, causally sterile effect of (e) actual bodily-material causes and processes | |
| $G_{28}$ : Category one of virtuality of becoming as the sense-event<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of passive, causally sterile effect of (e) actual bodily-material causes and processes<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>For Deleuze I, the virtuality of becoming as the sense-event is the passive, causally sterile effect of (e) actual bodily-material causes and processes | |
| $G_{32}$ : Category one of actual bodily-material causes and processes<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of virtuality of becoming as the sense-event is the passive, causally sterile effect<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br>For Deleuze II, the domain of virtual singularities and flux of pure becoming now becomes (=)the | |

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| productive cause of individuated bodies and subjects within (eb) diverse concrete assemblages | |
| $G_{36}$ : Category one of domain of virtual singularities and flux of pure becoming<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of productive cause of individuated bodies and subjects within (eb) diverse concrete assemblages<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>For Deleuze II, the domain of virtual singularities and flux of pure becoming now becomes the productive cause of individuated bodies and subjects within (eb) diverse concrete assemblages | |
| $G_{40}$ : Category one of domain of virtual singularities and flux of pure becoming now becomes the productive cause of individuated bodies and subjects<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of diverse concrete assemblages<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>So it seems that the domain of virtuality, or pure flux of becoming, is (=) on the one hand an impassive effect, and on the other (=) a generative process, which suggests (eb) the need for some mediating figure or theoretical account to explain the precise relationship between these central aspects of Deleuze's thought (OwB, 20-21 ff.) | |
| $G_{44}$ : Category one of domain of virtuality, or pure flux of becoming<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of impassive effect, and on the other (=) a generative process, which suggests (eb) the need for some mediating figure or theoretical account to explain the precise relationship between these central aspects of Deleuze's thought (OwB, 20-21 ff.)<br><br>$T_{45}$ : Category two of SAS | |

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| $T_{46}$ : Category three of SAS | |

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| **TITLE** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**PARRHESIA NUMBER 1 2006 • 62 – 87 Numerology or Ideology Zizek's Critique of Deleuze Robert Sinner brink I have deleted some sentences and added my own remarks to herein below exposition. For acknowledgements and references kindly see the original article. / Please note that the expatiation is model able and the following presupposition format is preeminently suitable for modelling**<br><br>(1) One might object that the fact that the effect outstrips (e) its apparent cause may point to (eb) multiple complex chains of causality, something that outstrips (e) our knowledge of the causal network rather than the causal network itself.<br>(2) Be that as it may, _i_ek takes Deleuze's move here as an opportunity to neatly couple him with Lacan, noting triumphantly that the Deleuzian quasi-cause parallels the Lacanian objet petit a, the "pure, immaterial, spectral entity that serves as (=) the object-cause of desire" (OwB, 27)—a point Deleuze and Guattari themselves acknowledge in Anti-Oedipus.<br>(3) The important issue, for Zizek, is that Deleuze's unrepentant dualism generates (eb) a number of unfortunate consequences. The fundamental dualism between virtual and (e&eb) actual, becoming and (e&eb) being, is (=) transposed into a multiplicity of further dualisms that repeat the essential good/bad dichotomy that Deleuze, as a good Nietzschean, is otherwise at pains to deconstruct: active versus (e&eb) reactive forces, nomadic versus (e&eb) sedentary distribution, molecular versus (e&eb) molar processes, body without organs versus (e&eb)organized body, deterritorialisation versus reterritorialisation, and so on.<br>(4) Moreover, Deleuze's ontological dualism—the logic of pure becoming as sterile effect versus the logic of pure becoming as productive process—is then translated into two different political logics and practices. On the one hand, the ontology of productive becoming leads to a Leftist Deleuzo Guattarian politics of "the self-organization of the multitude of molecular groups that resist and undermine the molar, totalizing systems of power" (OwB, 32). On the other hand, Zizek observes, Deleuze I's ontology of the sterile sense-event appears to remain 'apolitical', without a corresponding political logic or practice. Here one is tempted to point to Alain Badiou's attempt to construct precisely that kind of politics of universality based upon an ontology of pure multiplicity, of fidelity to the Event of the New, whether in art, thought, politics, or love.<br><br>(5) Zizek, however, while relying on Badiou's critique of Deleuze, argues that the seemingly 'apolitical' Deleuzian ontology of the sterile sense event harbours a more radical Deleuzian politics than the 'vulgar' Deleuzo-Guattarian politics of the molecular multitude (as developed further in Hardt and Negri's Empire, the target of Zizek's 'political' critique of Deleuze). Zizek thus faces two challenges. First, to show that Deleuze's ontology of pure becoming is in fact assimilable to a Hegelio-Lacanian paradigm, and moreover, to articulate what the theoretical or political virtues of such assimilation would be. And secondly, to show that the 'vulgar' Deleuzo-Guattarian politics of the molecular multitude really is ideologically compromised and politically sterile, which Zizek attempts to illustrate in the case of Hardt and Negri's Empire (which is, in any case, more a | |

Deleuzo-Foucaultian inflected neo-Marxism).

(6) My claim will be that Zizek fails to meet these challenges in any serious way, which prompts the following critical question: What is at stake in Zizek's encounter with Deleuze, his anxious attempts to show that Deleuze's thought could not be an Event, an eruption of the New, but only the 'sterile' repetition of Hegelian and Lacanian motifs? Why the desire to assimilate Deleuze within the Hegelian-Lacanian paradigm of subjectivity and politics? My suggestion will be that Deleuze and Guattari recognise precisely the problem that Zizek identifies as the most challenging problem for contemporary critical theory and social philosophy: how to practice revolutionary thinking and politics within an order that is inherently self-revolutionizing?

(7) How to deterritorialize, through theory and practice, an already self-deterritorializing regime? "Deleuze equals Hegel": Zizek's Abortive Aufhebung of Deleuze Zizek takes as his point of departure the contemporary doxa concerning the radically anti-Hegelian character of Deleuze's thought. Against the prevailing view that sees Hegel as the archenemy of difference and multiplicity, a view encouraged by Deleuze's own tendentious criticisms of Hegel, Zizek argues that Deleuzian anti-dialectical thought conceals its secret affinity with Hegelianism, which is to be understood as a radical theory of subjectivity and universality.

(8) Far from being a philosophy of difference oriented towards the event of the New, Deleuze's thought must be grasped as a repetition of the Same, not only the Nietzschean-Spinozist metaphysics of power but more surprisingly, as Zizek will argue, Hegelian speculative dialectics. As though demonstrating the practical impossibility of thinking the New, Zizek cites a number of examples of Deleuzian conceptual innovations that are supposed to repeat Hegelian speculative concepts, not in the Heideggerian sense of a productive retrieval that opens up their unthought dimension, but in the more conventional sense of a 'bald' repetition of an archetype.

(9) Examples of such bald repetition include Hegel's concept of the concrete universal, which Deleuze apparently adopts holus-bolus (OwB, 49-50); Hegel's critique of abstract universality, repeated by Deleuze's critique of the Platonic ideal universal types (OwB, 50); and the equivalence drawn between Hegel and Deleuze as philosophers of pure immanence, with the Hegelian immanence of reason proposed as analogous to Deleuze's Spinozist plane of immanence. On this score, readers of Hegel will ask what has become, in Zizek's parallel, of Hegel's famous critique of Spinoza's failure to articulate self-conscious subjectivity.

**(10)** As Zizek knows very well, Hegel ridiculed superficial affinities as a substitute for genuine thought; what philosophy demands, rather, is a conceptual struggle over the matter itself. Instead of analysing whether or not Zizek's claims of familial kinship are hermeneutically warranted, I want to focus on a crucial point where Zizek does acknowledge that Deleuze and Hegel differ profoundly: the difference between the Deleuzian immanent "flux" of pure becoming and the Hegelian "gap" or irreducible rupture within this pure becoming (the negativity of the subject). Zizek claims that the 'ultimate fact' of Deleuze's transcendental empiricism is "the absolute immanence of the continuous flux of pure becoming" (OwB, 60).

**NOTATION**

**Module One**

One might object that the fact that the effect outstrips (e) its apparent cause may point to (eb) multiple complex chains of causality, something that outstrips (e) our knowledge of the causal network rather than the causal network itself

$G_{13}$ : Category one of its apparent cause may point to (eb) multiple complex chains of causality, something that outstrips (e) our knowledge of the causal network rather than the causal network itself

| | |
|---|---|
| $G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of effect<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>One might object that the fact that the effect outstrips its apparent cause may point to (eb) multiple complex chains of causality, something that outstrips (e) our knowledge of the causal network rather than the causal network itself | |
| $G_{16}$ : Category one of effect outstrips its apparent cause<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of multiple complex chains of causality, something that outstrips (e) our knowledge of the causal network rather than the causal network itself<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>One might object that the fact that the effect outstrips its apparent cause may point to multiple complex chains of causality, something that outstrips our knowledge of the causal network rather than the causal network itself | |
| $G_{20}$ : Category one of knowledge of the causal network rather than the causal network itself<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of effect outstrips its apparent cause may point to multiple complex chains of causality, something that outstrips<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Be that as it may, _i_ek takes Deleuze's move here as an opportunity to neatly couple him with Lacan, noting triumphantly that the Deleuzian quasi-cause parallels the Lacanian objet petit a, the "pure, immaterial, spectral entity that serves as (=) the object-cause of desire" (OwB, 27)—a point Deleuze and Guattari themselves acknowledge in Anti-Oedipus | |

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| $G_{24}$ : Category one of"pure, immaterial, spectral entity that serves<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of object-cause of desire" (OwB, 27)—<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>The important issue, for Zizek, is that Deleuze's unrepentant dualism generates (eb) a number of unfortunate consequences. The fundamental dualism between virtual and (e&eb) actual, becoming and (e&eb) being, is (=) transposed into a multiplicity of further dualisms that repeat the essential good/bad dichotomy that Deleuze, as a good Nietzschean, is otherwise at pains to deconstruct: active versus (e&eb) reactive forces, nomadic versus (e&eb) sedentary distribution, molecular versus (e&eb) molar processes, body without organs versus (e&eb)organized body, deterritorialisation versus reterritorialisation, and so on | |
| $G_{28}$ : Category one of **virtual;** actual<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of actual **;virtual**<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>becoming and (e&eb) being<br>Notice the fact that each one is acting as the debitor and creditor | |
| $G_{32}$ : Category one of **becoming;** being<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of being; **becoming**<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven** | |

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| active versus (e&eb) reactive forces | |
| $G_{36}$ : Category one of **active;** reactive forces<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of reactive forces **;active**<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>nomadic versus (e&eb) sedentary distribution | |
| $G_{40}$ : Category one of **nomadic;** sedentary distribution<br>$G_{41}$ : Category two of SAS<br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of sedentary distribution; **nomadic**<br>$T_{41}$ : Category two of SAS<br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br>molecular versus (e&eb) molar processes | |
| $G_{44}$ : Category one of **molecular;** molar processes<br>$G_{45}$ : Category two of SAS<br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of molar processes; **molecular**<br>$T_{45}$ : Category two of SAS<br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)},$<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$ | |

| | |
|---|---|
| $(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}, \ (b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$<br>$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$<br>$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)} \ (a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,<br>$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$<br>$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,<br>$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,<br>$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b'_{13})^{(1)} - (b''_{13})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b'_{14})^{(1)} - (b''_{14})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b'_{15})^{(1)} - (b''_{15})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a''_{13})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b''_{13})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b'_{16})^{(2)} - (b''_{16})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b'_{17})^{(2)} - (b''_{17})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b'_{18})^{(2)} - (b''_{18})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a''_{16})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b''_{16})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |

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| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}\left((G_{27}),t\right) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}\left((G_{31}),t\right) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |

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| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b'_{32})^{(6)} - (b''_{32})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b'_{33})^{(6)} - (b''_{33})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b'_{34})^{(6)} - (b''_{34})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a''_{32})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b'_{36})^{(7)} - (b''_{36})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b'_{37})^{(7)} - (b''_{37})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b'_{38})^{(7)} - (b''_{38})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a''_{36})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a'_{41})^{(8)} + (a''_{41})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b'_{40})^{(8)} - (b''_{40})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b'_{41})^{(8)} - (b''_{41})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b'_{42})^{(8)} - (b''_{42})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b'_{44})^{(9)} - (b''_{44})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b'_{45})^{(9)} - (b''_{45})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b'_{46})^{(9)} - (b''_{46})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a''_{44})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b''_{44})^{(9)}((G_{47}),t) =$ **First detrition factor** | |

$$\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{13} \quad (55)$$

$$\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{array}{ccc} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{14} \quad (56)$$

$$\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{15} \quad (57)$$

Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3

$\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[\begin{array}{ccc} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{array}\right] T_{13} \quad (58)$$

$$\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[\begin{array}{ccc} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{array}\right] T_{14} \quad (59)$$

$$\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)} & -(b_{15}^{''})^{(1)}(G,t) & -(b_{18}^{''})^{(2,2,)}(G_{19},t) & -(b_{22}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,)}(G_{39},t) & -(b_{42}^{''})^{(8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15} \quad (60)$$

Where $-(b_{13}^{''})^{(1)}(G,t)$, $-(b_{14}^{''})^{(1)}(G,t)$, $-(b_{15}^{''})^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{37}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}^{'})^{(2)} & +(a_{16}^{''})^{(2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,)}(T_{14},t) & +(a_{20}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{16} \quad (61)$$

$$\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}^{'})^{(2)} & +(a_{17}^{''})^{(2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,)}(T_{14},t) & +(a_{21}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{17} \quad (62)$$

$$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}^{'})^{(2)} & +(a_{18}^{''})^{(2)}(T_{17},t) & +(a_{15}^{''})^{(1,1,)}(T_{14},t) & +(a_{22}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{18} \quad (63)$$

Where $+(a_{16}^{''})^{(2)}(T_{17},t)$, $+(a_{17}^{''})^{(2)}(T_{17},t)$, $+(a_{18}^{''})^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for

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| category 1, 2 and 3 $+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3 $+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3 $+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{16}$ | *64* |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)}-(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{17}$ | *65* |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{18}$ | *66* |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$, $-(b_{17}^{''})^{(2)}(G_{19},t)$, $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{13}^{''})^{(1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3 $-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3 $-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3 $-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3 $-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3 $-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3 | |

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| $-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[\begin{array}{c}(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21},t)+(a_{16}^{''})^{(2,2,2)}(T_{17},t)+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)\\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)\\ +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{44}^{''})^{(9,9,9)}(T_{45},t)\end{array}\right]G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[\begin{array}{c}(a_{21}^{'})^{(3)}+(a_{21}^{''})^{(3)}(T_{21},t)+(a_{17}^{''})^{(2,2,2)}(T_{17},t)+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)\\ +(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)\\ +(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{45}^{''})^{(9,9,9)}(T_{45},t)\end{array}\right]G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[\begin{array}{c}(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21},t)+(a_{18}^{''})^{(2,2,2)}(T_{17},t)+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)\\ +(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)\\ +(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{46}^{''})^{(9,9,9)}(T_{45},t)\end{array}\right]G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[\begin{array}{c}(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23},t)-(b_{16}^{''})^{(2,2,2)}(G_{19},t)-(b_{13}^{''})^{(1,1,1,)}(G,t)\\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)\\ -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)-(b_{44}^{''})^{(9,9,9)}(G_{47},t)\end{array}\right]T_{20}$ | 70 |

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b'_{21})^{(3)} & -(b''_{21})^{(3)}(G_{23},t) & -(b''_{17})^{(2,2,2)}(G_{19},t) & -(b''_{14})^{(1,1,1,)}(G,t) \\ -(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b''_{37})^{(7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)} & -(b''_{22})^{(3)}(G_{23},t) & -(b''_{18})^{(2,2,2)}(G_{19},t) & -(b''_{15})^{(1,1,1,)}(G,t) \\ -(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b''_{38})^{(7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{22} \quad (72)$$

$-(b''_{20})^{(3)}(G_{23},t)$, $-(b''_{21})^{(3)}(G_{23},t)$, $-(b''_{22})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*

$-(b''_{16})^{(2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*

$-(b''_{13})^{(1,1,1,)}(G,t)$, $-(b''_{14})^{(1,1,1,)}(G,t)$, $-(b''_{15})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*

$-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*

$-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*

$-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*

$-(b''_{36})^{(7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7)}(G_{39},t)$ $-(b''_{38})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b''_{40})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b''_{46})^{(9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a'_{24})^{(4)} & +(a''_{24})^{(4)}(T_{25},t) & +(a''_{28})^{(5,5,)}(T_{29},t) & +(a''_{32})^{(6,6,)}(T_{33},t) \\ +(a''_{13})^{(1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{36})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a'_{25})^{(4)} & +(a''_{25})^{(4)}(T_{25},t) & +(a''_{29})^{(5,5,)}(T_{29},t) & +(a''_{33})^{(6,6)}(T_{33},t) \\ +(a''_{14})^{(1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{37})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a'_{26})^{(4)} & +(a''_{26})^{(4)}(T_{25},t) & +(a''_{30})^{(5,5,)}(T_{29},t) & +(a''_{34})^{(6,6,)}(T_{33},t) \\ +(a''_{15})^{(1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{38})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a''_{24})^{(4)}(T_{25},t)$, $(a''_{25})^{(4)}(T_{25},t)$, $(a''_{26})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a_{28}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* $1,2$ *and* $3$

$+(a_{32}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* $1,2$ *and* $3$

$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$ *are fourth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$ *are fifth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$ *are sixth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$ *are seventh augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$ *are eighth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)}\,-(b_{24}^{''})^{(4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{13}^{''})^{(1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{24} \tag{76}$$

$$\frac{dT_{25}}{dt}=(b_{25})^{(4)}T_{24}-\begin{bmatrix}(b_{25}^{'})^{(4)}\,-(b_{25}^{''})^{(4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{14}^{''})^{(1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{25} \tag{77}$$

$$\frac{dT_{26}}{dt}=(b_{26})^{(4)}T_{25}-\begin{bmatrix}(b_{26}^{'})^{(4)}\,-(b_{26}^{''})^{(4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{15}^{''})^{(1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{26} \tag{78}$$

*Where* $-(b_{24}^{''})^{(4)}(G_{27},t)$, $-(b_{25}^{''})^{(4)}(G_{27},t)$, $-(b_{26}^{''})^{(4)}(G_{27},t)$ *are first detrition coefficients for category* $1,2$ *and* $3$

$-(b_{28}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,)}(G_{31},t)$ *are second detrition coefficients for category* $1,2$ *and* $3$

$-(b_{32}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,)}(G_{35},t)$ *are third detrition coefficients for category* $1,2$ *and* $3$

$-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$ *are fourth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$ *are fifth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$

| | |
|---|---|
| *are sixth detrition coefficients for category* 1, 2 *and* 3 $\boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)}$ *are seventh detrition coefficients for category* 1, 2 *and* 3 $\boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)}$ *are eighth detrition coefficients for category* 1, 2 *and* 3 $\boxed{-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[\begin{array}{ccc} (a_{28}^{'})^{(5)}\boxed{+(a_{28}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[\begin{array}{ccc} (a_{29}^{'})^{(5)}\boxed{+(a_{29}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[\begin{array}{ccc} (a_{30}^{'})^{(5)}\boxed{+(a_{30}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{30}$ | *81* |
| *Where* $\boxed{+(a_{28}^{''})^{(5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5)}(T_{29},t)}$ *are first augmentation coefficients for category* 1, 2 *and* 3 *And* $\boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3 $\boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3 $\boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3 $\boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation coefficients for category 1,2,and 3 $\boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)}$ are sixth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1,2, 3 $\boxed{+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$ are eighth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1,2, 3 | |

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b'_{28})^{(5)}\boxed{-(b''_{28})^{(5)}(G_{31},t)} & \boxed{-(b''_{24})^{(4,4,)}(G_{27},t)} & \boxed{-(b''_{32})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1)}(G,t)} & \boxed{-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b'_{29})^{(5)}\boxed{-(b''_{29})^{(5)}(G_{31},t)} & \boxed{-(b''_{25})^{(4,4,)}(G_{27},t)} & \boxed{-(b''_{33})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1)}(G,t)} & \boxed{-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b'_{30})^{(5)}\boxed{-(b''_{30})^{(5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,)}(G_{27},t)} & \boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{30} \quad (84)$$

*where* $\boxed{-(b''_{28})^{(5)}(G_{31},t)}$ , $\boxed{-(b''_{29})^{(5)}(G_{31},t)}$ , $\boxed{-(b''_{30})^{(5)}(G_{31},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{24})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,)}(G_{27},t)}$ *are second detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{32})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)}$ *are third detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{13})^{(1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1)}(G,t)}$ , $\boxed{-(b''_{15})^{(1,1,1,1,1,)}(G,t)}$ are fourth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a'_{32})^{(6)}\boxed{+(a''_{32})^{(6)}(T_{33},t)} & \boxed{+(a''_{28})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{24})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a'_{33})^{(6)}\boxed{+(a''_{33})^{(6)}(T_{33},t)} & \boxed{+(a''_{29})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{25})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)} G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)} T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)} T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)} T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{28})^{(5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)} & +(a_{36}^{''})^{(7)}(T_{37},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)} & +(a_{37}^{''})^{(7)}(T_{37},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)} & +(a_{38}^{''})^{(7)}(T_{37},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15} \quad (93)$$

Where $(a_{36}^{''})^{(7)}(T_{37},t)$, $(a_{37}^{''})^{(7)}(T_{37},t)$, $(a_{38}^{''})^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation

coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3 $+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}(G_{39},t) & -(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$ 94

$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}-(b_{37}^{''})^{(7)}(G_{39},t) & -(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}(G_{39},t) & -(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3 $-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3 $-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3 $-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{46}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

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$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}')^{(8)}+(a_{40}'')^{(8)}(T_{41},t) & +(a_{16}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}'')^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13}$$

$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}')^{(8)}+(a_{41}'')^{(8)}(T_{41},t) & +(a_{17}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}'')^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}')^{(8)}+(a_{42}'')^{(8)}(T_{41},t) & +(a_{18}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}'')^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$$

Where $+(a_{40}'')^{(8)}(T_{41},t)$, $+(a_{41}'')^{(8)}(T_{41},t)$, $+(a_{42}'')^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & -(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & -(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & -(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3



$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[\begin{array}{ccc} (a_{44}^{'})^{(9)}\boxed{+(a_{44}^{''})^{(9)}(T_{45},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{13}$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[\begin{array}{ccc} (a_{45}^{'})^{(9)}\boxed{+(a_{45}^{''})^{(9)}(T_{45},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[\begin{array}{ccc} (a_{46}^{'})^{(9)}\boxed{+(a_{46}^{''})^{(9)}(T_{37},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{15}$$

Where $\boxed{+(a_{44}^{''})^{(9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$\frac{dT_{44}}{dt} =$

$$(b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \begin{bmatrix} (b'_{46})^{(9)}\boxed{-(b''_{46})^{(9)}(G_{47},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{44})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a'_i)^{(1)}, (a''_i)^{(1)}, (b_i)^{(1)}, (b'_i)^{(1)}, (b''_i)^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a''_i)^{(1)}, (b''_i)^{(1)}$ are positive continuous increasing and bounded.

| | |
|---|---|
| **Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br>$(a_i^{''})^{(1)}(T_{14},t) \leq (p_i)^{(1)} \leq (\hat{A}_{13})^{(1)}$<br>$(b_i^{''})^{(1)}(G,t) \leq (r_i)^{(1)} \leq (b_i^{'})^{(1)} \leq (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$<br>$\lim_{G\to\infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$<br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'},t) - (a_i^{''})^{(1)}(T_{14},t)\rvert \leq (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br>$\lvert(b_i^{''})^{(1)}(G^{'},t) - (b_i^{''})^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G^{'}\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i^{'})^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i^{'})^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i^{'})^{(2)}, (a_i^{''})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (b_i^{''})^{(2)} > 0, \quad i,j = 16,17,18$ | |

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| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17},t) \le (p_i)^{(2)} \le (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19},t) \le (r_i)^{(2)} \le (b_i')^{(2)} \le (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2\to\infty}(a_i'')^{(2)}(T_{17},t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty}(b_i'')^{(2)}((G_{19}),t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}$ are positive constants and $i = 16,17,18$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(2)}(T_{17}',t) - (a_i'')^{(2)}(T_{17},t)\rvert \le (\hat{k}_{16})^{(2)}\lvert T_{17} - T_{17}'\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i'')^{(2)}((G_{19})',t) - (b_i'')^{(2)}((G_{19}),t)\rvert < (\hat{k}_{16})^{(2)}\lvert\lvert(G_{19}) - (G_{19})'\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}',t)$ and $(a_i'')^{(2)}(T_{17},t)$ . $(T_{17}',t)$ and $(T_{17},t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}]$ . It is to be noted that $(a_i'')^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i'')^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i')^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ | 111 |

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| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21},t) \le (p_i)^{(3)} \le (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23},t) \le (r_i)^{(3)} \le (b_i^{'})^{(3)} \le (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2\to\infty}(a_i^{''})^{(3)}(T_{21},t) = (p_i)^{(3)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'},t) - (a_i^{''})^{(3)}(T_{21},t)\rvert \le (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'},t) - (b_i^{''})^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lVert G_{23} - G_{23}^{'}\rVert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'},t)$ and$(a_i^{''})^{(3)}(T_{21},t)$ . $(T_{21}^{'},t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$<br>satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |

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| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}\big((G_{27}), t\big) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(4)}\big((G_{27}), t\big) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\rvert \leq (\hat{k}_{24})^{(4)} \lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)} t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'}, t) - (b_i^{''})^{(4)}\big((G_{27}), t\big)\rvert < (\hat{k}_{24})^{(4)} \lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)} t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $\big[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\big]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |

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| $\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i^{'})^{(4)} +\ (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}\big((G_{31}), t\big) \leq\ (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(5)}\,(T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(5)}\,(G_{31}, t) =\ (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'}, t) - (b_i^{''})^{(5)}\big((G_{31}), t\big)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $\big[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\big]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br><br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities | 126 |

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| $\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(6)}(T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(6)}((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)}\lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})^{'}, t) - (b_i^{''})^{(6)}((G_{35}), t)\rvert < (\hat{k}_{32})^{(6)}\lvert\lvert(G_{35}) - (G_{35})^{'}\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and$(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants | 130 |

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| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$ | |
| Where we suppose | |
| (EE) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i, j = 36,37,38$<br><br>(FF) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$:<br><br>$(a_i^{''})^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$<br><br>$(b_i^{''})^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$ | 131 |
| *(GG)* $\lim_{T_2 \to \infty} (a_i^{''})^{(7)}(T_{37}, t) = (p_i)^{(7)}$<br>*(HH)*<br>$\lim_{G \to \infty} (b_i^{''})^{(7)}((G_{39}), t) = (r_i)^{(7)}$<br><br>**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :<br><br>Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, \ (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$ | 132 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)\rvert \leq (\hat{k}_{36})^{(7)} \lvert T_{37} - T_{37}^{'}\rvert e^{-(\hat{M}_{36})^{(7)} t}$<br><br>$\lvert(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}((G_{39}), t)\rvert < (\hat{k}_{36})^{(7)} \lvert\lvert(G_{39}) - (G_{39})^{'}\rvert\rvert e^{-(\hat{M}_{36})^{(7)} t}$ | 133 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'}, t)$ and $(a_i^{''})^{(7)}(T_{37}, t)$ . $(T_{37}^{'}, t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :<br><br>(II) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants | 134 |

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| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :<br><br>(JJ) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}, (r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})', t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})'\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would | |

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| be absolutely continuous. | |
| **Definition of** $(\widehat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\widehat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\widehat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, \ (r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\widehat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\widehat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i, j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}, \ (r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45}, t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47}, t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2 \to \infty} (a_i'')^{(9)}(T_{45}, t) = (p_i)^{(9)}$<br>$\lim_{G \to \infty} (b_i'')^{(9)}(G_{47}, t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}', t) - (a_i'')^{(9)}(T_{45}, t)\rvert \le (\hat{k}_{44})^{(9)} \lvert T_{45} - T_{45}' \rvert e^{-(\widehat{M}_{44})^{(9)} t}$<br>$\lvert(b_i'')^{(9)}((G_{47})', t) - (b_i'')^{(9)}((G_{47}), t)\rvert < (\hat{k}_{44})^{(9)} \lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\widehat{M}_{44})^{(9)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}', t)$ | |

and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)},(\widehat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{44})^{(9)}=1$ then the function $(a_i'')^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$ :

$(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\ ,\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}<1$$

**Definition of** $(\hat{P}_{44})^{(9)},(\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},(\hat{A}_{44})^{(9)}and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)},(a_i')^{(9)},(b_i)^{(9)},(b_i')^{(9)},(p_i)^{(9)},\ (r_i)^{(9)},i=44,45,46,$ satisfy the inequalities

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[(a_i)^{(9)}+(a_i')^{(9)}+\ (\hat{A}_{44})^{(9)}+\ (\hat{P}_{44})^{(9)}\ (\hat{k}_{44})^{(9)}]<1$$

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[\ (b_i)^{(9)}+(b_i')^{(9)}+\ (\hat{B}_{44})^{(9)}+\ (\hat{Q}_{44})^{(9)}\ \ (\hat{k}_{44})^{(9)}]<1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0)\ ,T_i(0)$ :

$$G_i(t)\le\ \left(\hat{P}_{13}\right)^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{G_i(0)=G_i^0>0}$$

$$T_i(t)\le\ (\hat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{T_i(0)=T_i^0>0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0)\ ,T_i(0)$

$$G_i(t)\le\ (\hat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t)\le\ (\hat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

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| **Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$ | 153 B |

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| $T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |

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| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} G_{21}(s_{(20)}) - \left((a_{20}')^{(3)} + a_{20}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{20}(s_{(20)})\right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[(a_{21})^{(3)} G_{20}(s_{(20)}) - \left((a_{21}')^{(3)} + (a_{21}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}')^{(3)} - (b_{21}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{28})^{(5)}$, $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[ (b_{29})^{(5)} T_{28}(s_{(28)}) - \left( (b'_{29})^{(5)} - (b''_{29})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[ (b_{30})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{30})^{(5)} - (b''_{30})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{30}(s_{(28)}) \right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** | |

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| Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[ (a_{32})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{32}(s_{(32)}) \right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[ (a_{33})^{(6)} G_{32}(s_{(32)}) - \left( (a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[ (a_{34})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{34}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[ (b_{32})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{32}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[ (b_{33})^{(6)} T_{32}(s_{(32)}) - \left( (b'_{33})^{(6)} - (b''_{33})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[ (b_{34})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{34})^{(6)} - (b''_{34})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{34}(s_{(32)}) \right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |

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| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}^{'})^{(8)} + a_{40}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^{0} + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^{0} + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^{0} + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{13}(t) \leq G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \leq \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}$ , $G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \leq G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that | 170 |

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| $(G_{16}(t)-G_{16}^0)e^{-(\bar{M}_{16})^{(2)}t} \le \frac{(a_{16})^{(2)}}{(\bar{M}_{16})^{(2)}}\left[((\hat{P}_{16})^{(2)}+G_{17}^0)e^{\left(-\frac{(\hat{P}_{16})^{(2)}+G_{17}^0}{G_{17}^0}\right)}+(\hat{P}_{16})^{(2)}\right]$ | |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0+(\hat{P}_{20})^{(3)}e^{(\bar{M}_{20})^{(3)}s_{(20)}}\right)\right] ds_{(20)} =$<br>$(1+(a_{20})^{(3)}t)G_{21}^0 + \frac{(a_{20})^{(3)}(\hat{P}_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left(e^{(\bar{M}_{20})^{(3)}t}-1\right)$ | 171 |
| From which it follows that<br>$(G_{20}(t)-G_{20}^0)e^{-(\bar{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that<br>$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\bar{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$<br>$(1+(a_{24})^{(4)}t)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left(e^{(\bar{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that<br>$(G_{24}(t)-G_{24}^0)e^{-(\bar{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$<br>$(1+(a_{28})^{(5)}t)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)}t}-1\right)$ | |
| From which it follows that<br>$(G_{28}(t)-G_{28}^0)e^{-(\bar{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 5 | 175 |
| The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious | 176 |

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| that<br>$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$<br>$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$ | |
| From which it follows that<br>$(G_{32}(t) - G_{32}^0) e^{-(\hat{M}_{32})^{(6)} t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)} + G_{33}^0) e^{\left(-\frac{(\hat{P}_{32})^{(6)} + G_{33}^0}{G_{33}^0}\right)} + (\hat{P}_{32})^{(6)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 6<br>Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$ | 177 |
| (f) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)}\left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$<br>$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$ | 178 |
| From which it follows that<br>$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)} + G_{37}^0) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 7 | |
| The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)}\left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$<br>$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$ | 180 |
| From which it follows that<br>$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)} + G_{41}^0) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)}\left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$ | |

$$\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$$

From which it follows that

$$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t}\leq\frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{P}_{44})^{(9)}+G_{45}^{0}\right)e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 9

Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$

It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ and to choose

$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have

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$$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$$

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$$\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$$

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In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric

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$$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$$

$$\sup_i\left\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\hat{M}_{13})^{(1)}t}\right\}$$

Indeed if we denote

**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$

It results

$$\left|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right|\leq\int_0^t(a_{13})^{(1)}\left|G_{14}^{(1)}-G_{14}^{(2)}\right|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$$

$$\int_0^t\{(a_{13}')^{(1)}\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{-(\hat{M}_{13})^{(1)}s_{(13)}}+$$

$$(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}+$$

$$G_{13}^{(2)}\left|(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}'')^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right|\ e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$$

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| Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | |
| $\left\|G^{(1)} - G^{(2)}\right\| e^{-(\widehat{M}_{13})^{(1)}t} \leq$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right) d\left(\left(G^{(1)}, T^{(1)}; G^{(2)}, T^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(1)}$ and $(b''_i)^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(1)} - (a''_i)^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1 / (a'_{14})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2 / (a'_{15})^{(1)}$<br><br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(1)}(G(t), t)) = (b'_{14})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ : | 189 |

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| Indeed let $t_1$ be so that for $t > t_1$<br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ $and$ $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |

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| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$(a_{16}^{''})^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)} \left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}} \left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)} (\widehat{k}_{16})^{(2)}\right) d\left(\left((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)}\right)\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)} (T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0 \quad \text{for } t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)} \left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$<br>In the same way , one can obtain | 201 |

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| $G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$, $G_{18}$ and $G_{16}$, $G_{17}$ respectively. | |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$. The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}$, $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}} < 1$ and to choose<br><br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$, $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{20})^{(3)}t}\}$ | 211 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}}$ :$\left( \widetilde{(G_{23})}, \widetilde{(T_{23})} \right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)} \left\|G_{21}^{(1)} - G_{21}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\, ds_{(20)} +$<br><br>$\int_0^t \{(a_{20}^{'})^{(3)} \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right)\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$G_{20}^{(2)} \left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}^{''})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\} ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} t} \leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a_{20}^{'})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)} (\widehat{k}_{20})^{(3)}\right) d\left(\left((G_{23})^{(1)}, (T_{23})^{(1)}; (G_{23})^{(2)}, (T_{23})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ *and* $(\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)} t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating | 217 |

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| $G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}')^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}')^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$ | 225 |

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{24})^{(4)}\left|G_{25}^{(1)}-G_{25}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$$

$$\int_0^t\{(a_{24}^{'})^{(4)}\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$G_{24}^{(2)}\left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)}-(G_{27})^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}t}\le$$
$$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows

(226)

**Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}\ and\ (\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

(227)

**Remark 17:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

it results

$$G_i\ (t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]}\ge 0$$

$$T_i\ (t)\ge T_i^0e^{\left(-(b_i^{'})^{(4)}t\right)}>0\quad \text{for t} > 0$$

(228)

**Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1,\left((\widehat{M}_{24})^{(4)}\right)_2\ and\ \left((\widehat{M}_{24})^{(4)}\right)_3$ :

**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if

(229)

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| $G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a'_{25})^{(4)} G_{25}$ and by integrating<br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1 / (a'_{25})^{(4)}$<br>In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2 / (a'_{26})^{(4)}$<br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t),t)) = (b_{25}^{'})^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i^{''})^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + \left((\widehat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{28})^{(5)} + G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\widehat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{28})^{(5)} + T_j^0}{T_j^0}\right)} + (\widehat{Q}_{28})^{(5)}\right] \leq (\widehat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying | |

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| Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}((G_{31}),(T_{31}))$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)}+$<br>$\int_0^t\{(a_{28}^{'})^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\le$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}^{'})^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}^{''})^{(5)}$ and $(b_{28}^{''})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(5)}-(a_i^{''})^{(5)}(T_{29}(s_{(28)}),s_{(28)})\}ds_{(28)}\right]}\ge 0$ | 239 |

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| $T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(5)}t)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}^{'})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1/(a_{29}^{'})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2/(a_{30}^{'})^{(5)}$<br><br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t),t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |

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| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{32})^{(6)}\right]\leq(\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}ds_{(32)}+$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a''_{32})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a'_{32})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(6)}$ and $(b_i'')^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 249 |

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| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(6)} - (a_i'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(6)} t)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}')^{(6)} G_{33}$ and by integrating<br><br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}')^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}')^{(6)}$<br><br>If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(6)}((G_{35})(t), t)) = (b_{33}')^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i'')^{(6)}((G_{35})(t), t) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty} (b_{34}'')^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}')^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have | 255 |

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| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)}+\left((\hat{P}_{36})^{(7)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{36})^{(7)}\right]\leq(\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}')^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}'')^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}')^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}'')^{(7)}$ and $(b_{36}'')^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$. | 260 |

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| If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2 / (a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t), t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t), t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right),\ \ t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded. | 265 |

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| The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t),t\right)=(b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}<1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)}+\left((\hat{P}_{40})^{(8)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{40})^{(8)}\right]\leq(\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\,ds_{(40)}+$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\leq$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |

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| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(8)}-(a_i^{''})^{(8)}(T_{41}(s_{(40)}),s_{(40)})\}ds_{(40)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2\ and\ \left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)}G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1/(a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2/(a_{42}^{'})^{(8)}$<br><br>If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}\,((G_{43})(t),t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t),t\right) < \varepsilon_8, T_{40}\,(t) > (m)^{(8)}$ | 278 |

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| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.<br>The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}^{'})^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$ | |
| In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying 39,35,36 into itself | |
| The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{47})^{(1)}, (T_{47})^{(1)}), ((G_{47})^{(2)}, (T_{47})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \left(\widetilde{(G_{47})}, \widetilde{(T_{47})}\right) = \mathcal{A}^{(9)}\left((G_{47}), (T_{47})\right)$<br>It results<br><br>$\left\|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$<br>$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right) - (a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right| \; e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$<br><br>Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on 45,46,47,28 and 29 it follows | |
| $\left|(G_{47})^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t} \leq$<br>$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)}, (T_{47})^{(1)}; (G_{47})^{(2)}, (T_{47})^{(2)})\right)$ | |

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$

From 99 to 44 it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(9)}t\right)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)}G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t), t\right) < \varepsilon_9, T_{44}\,(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded. The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b''_{46})^{(9)}\left((G_{47})(t),t\right) = (b'_{46})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a'_{13})^{(1)} + (a'_{14})^{(1)} - (a''_{13})^{(1)}(T_{14}, t) + (a''_{14})^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b'_{13})^{(1)} + (b'_{14})^{(1)} - (b''_{13})^{(1)}(G,t) - (b''_{14})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,

and $\boxed{(u_0)^{(1)} = \frac{T^0_{13}}{T^0_{14}}}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$

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| are defined | |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le$<br>$\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a'_{15})^{(1)})}[e^{(S_1)^{(1)}t} - e^{-(a'_{15})^{(1)}t}] + G_{15}^0 e^{-(a'_{15})^{(1)}t})$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b'_{15})^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b'_{15})^{(1)}t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)}t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$<br>$(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ :<br>$(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a'_{16})^{(2)} + (a'_{17})^{(2)} - (a''_{16})^{(2)}(T_{17}, t) + (a''_{17})^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b'_{16})^{(2)} + (b'_{17})^{(2)} - (b''_{16})^{(2)}((G_{19}), t) - (b''_{17})^{(2)}((G_{19}), t) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(v_1)^{(2)}, (v_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |

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| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\left(u^{(2)}\right)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}$,, $(\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\left(u^{(2)}\right)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously $(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$ $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities $G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)}t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le\right.$ | 312 |

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| $\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_2)^{(2)}((S_1)^{(2)}-(a_{18}')^{(2)})}[e^{(S_1)^{(2)}t}-e^{-(a_{18}')^{(2)}t}]+G_{18}^{0}e^{-(a_{18}')^{(2)}t})$ | |
| $\boxed{T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}}T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le\frac{1}{(\mu_2)^{(2)}}T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)}T_{16}^{0}}{(\mu_1)^{(2)}((R_1)^{(2)}-(b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t}-e^{-(b_{18}')^{(2)}t}\right]+T_{18}^{0}e^{-(b_{18}')^{(2)}t}\le T_{18}(t)\le$ $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)},(S_2)^{(2)},(R_1)^{(2)},(R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)}=(a_{16})^{(2)}(m_2)^{(2)}-(a_{16}')^{(2)}$ $(S_2)^{(2)}=(a_{18})^{(2)}-(p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)}=(b_{16})^{(2)}(\mu_2)^{(1)}-(b_{16}')^{(2)}$ $(R_2)^{(2)}=(b_{18}')^{(2)}-(r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$: $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)}\le-(a_{20}')^{(3)}+(a_{21}')^{(3)}-(a_{20}'')^{(3)}(T_{21},t)+(a_{21}'')^{(3)}(T_{21},t)\le-(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)}\le-(b_{20}')^{(3)}+(b_{21}')^{(3)}-(b_{20}'')^{(3)}(G_{23},t)-(b_{21}'')^{(3)}((G_{23}),t)\le-(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)},(v_2)^{(3)},(u_1)^{(3)},(u_2)^{(3)}$: By $(v_1)^{(3)}>0,(v_2)^{(3)}<0$ and respectively $(u_1)^{(3)}>0,(u_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_1)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_1)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ and By $(\bar{v}_1)^{(3)}>0,(\bar{v}_2)^{(3)}<0$ and respectively $(\bar{u}_1)^{(3)}>0,(\bar{u}_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_2)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_2)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ | 320 |
| **Definition of** $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$:- | 321 |

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$, ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,

and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously

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$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$$

$(p_i)^{(3)}$ is defined by equation

$$\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t} \quad (323)$$

$$\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$$

$$\left. \frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right) \quad (324)$$

$$\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}} \quad (325)$$

$$\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t} \quad (326)$$

$$\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$$

$$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t} \quad (327)$$

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- (328)

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$$

$$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$$

$$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}\big((G_{27}), t\big) - (b''_{25})^{(4)}\big((G_{27}), t\big) \le -(\tau_1)^{(4)}$$

**Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : 329

By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{v}_1)^{(4)} > 0$, $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (v_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}, \ \textbf{\textit{if}}\ (v_0)^{(4)} < (v_1)^{(4)}$$

$$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}, \textbf{\textit{if}}\ (v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)},$$

and $\boxed{(v_0)^{(4)} = \frac{G^0_{24}}{G^0_{25}}}$

$$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}, \ \textbf{\textit{if}}\ (\bar{v}_4)^{(4)} < (v_0)^{(4)}$$

and analogously 331

$$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}, \ \textbf{\textit{if}}\ (u_0)^{(4)} < (u_1)^{(4)}$$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}, \textbf{\textit{if}}\ (u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)},$$

and $\boxed{(u_0)^{(4)} = \frac{T^0_{24}}{T^0_{25}}}$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \textbf{\textit{if}}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)} \text{ where } (u_1)^{(4)}, (\bar{u}_1)^{(4)}$$

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| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}')^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}')^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)}t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(\nu_1)^{(5)}, (\nu_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, \nu^{(5)}, u^{(5)}$ :<br>By $(\nu_1)^{(5)} > 0$, $(\nu_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}(\nu^{(5)})^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)} = 0$ | 339 |

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| and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ :<br><br>By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)},(v_0)^{(5)}$ :-<br><br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)}$ by<br><br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br><br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br><br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |

$\frac{(b_{30})^{(5)}T_{28}^{0}}{(\mu_1)^{(5)}((R_1)^{(5)}-(b_{30}^{'})^{(5)})}\left[e^{(R_1)^{(5)}t}-e^{-(b_{30}^{'})^{(5)}t}\right]+T_{30}^{0}e^{-(b_{30}^{'})^{(5)}t}\le T_{30}(t)\le$

$\frac{(a_{30})^{(5)}T_{28}^{0}}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t}-e^{-(R_2)^{(5)}t}\right]+T_{30}^{0}e^{-(R_2)^{(5)}t}$ 347

**Definition of** $(S_1)^{(5)},(S_2)^{(5)},(R_1)^{(5)},(R_2)^{(5)}$:- 348

Where $(S_1)^{(5)}=(a_{28})^{(5)}(m_2)^{(5)}-(a_{28}^{'})^{(5)}$

$(S_2)^{(5)}=(a_{30})^{(5)}-(p_{30})^{(5)}$

$(R_1)^{(5)}=(b_{28})^{(5)}(\mu_2)^{(5)}-(b_{28}^{'})^{(5)}$

$(R_2)^{(5)}=(b_{30}^{'})^{(5)}-(r_{30})^{(5)}$

**Behavior of the solutions of equation** 349

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ four constants satisfying

$-(\sigma_2)^{(6)}\le-(a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}-(a_{32}^{''})^{(6)}(T_{33},t)+(a_{33}^{''})^{(6)}(T_{33},t)\le-(\sigma_1)^{(6)}$

$-(\tau_2)^{(6)}\le-(b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(b_{32}^{''})^{(6)}\left((G_{35}),t\right)-(b_{33}^{''})^{(6)}\left((G_{35}),t\right)\le-(\tau_1)^{(6)}$

**Definition of** $(\nu_1)^{(6)},(\nu_2)^{(6)},(u_1)^{(6)},(u_2)^{(6)},\nu^{(6)},u^{(6)}$ : 350

By $(\nu_1)^{(6)}>0$ , $(\nu_2)^{(6)}<0$ and respectively $(u_1)^{(6)}>0$ , $(u_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_1)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2+(\tau_1)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$ and

**Definition of** $(\bar{\nu}_1)^{(6)},,(\bar{\nu}_2)^{(6)},(\bar{u}_1)^{(6)},(\bar{u}_2)^{(6)}$ : 351

By $(\bar{\nu}_1)^{(6)}>0$ , $(\bar{\nu}_2)^{(6)}<0$ and respectively $(\bar{u}_1)^{(6)}>0$ , $(\bar{u}_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_2)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2+(\tau_2)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$

**Definition of** $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)},(\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)}$ by

$(m_2)^{(6)}=(\nu_0)^{(6)},(m_1)^{(6)}=(\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)}<(\nu_1)^{(6)}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)}<(\nu_0)^{(6)}<(\bar{\nu}_1)^{(6)}$,

and $\boxed{(\nu_0)^{(6)}=\frac{G_{32}^{0}}{G_{33}^{0}}}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)}<(\nu_0)^{(6)}$

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| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},\ \boldsymbol{if}\ (u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}, \boldsymbol{if}\ (u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)},$<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}, \boldsymbol{if}\ (\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le\right.$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b_{34}')^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b_{34}')^{(6)} t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$<br>$(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ : | |

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| $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying<br><br>$-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$<br><br>$-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}((G_{39}), t) - (b''_{37})^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ :<br><br>By $(\nu_1)^{(7)} > 0, (\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ :<br><br>By $(\bar{\nu}_1)^{(7)} > 0, (\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$<br>**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-<br><br>If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by<br><br>$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,<br>and $\boxed{(\nu_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | 362 |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,<br>and $\boxed{(u_0)^{(7)} = \frac{T^0_{36}}{T^0_{37}}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities | 364 |

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|---|---|
| $G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^{0}e^{(S_1)^{(7)}t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | |
| $\frac{1}{(m_7)^{(7)}} G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^{0}e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^{0}e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}^{'})^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}^{'})^{(7)}t}] + G_{38}^{0}e^{-(a_{38}^{'})^{(7)}t})$ | 366 |
| $\boxed{T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)}T_{36}^{0}}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}^{'})^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}^{'})^{(7)}t}\right] + T_{38}^{0}e^{-(b_{38}^{'})^{(7)}t} \le T_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)}T_{36}^{0}}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^{0}e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}^{'})^{(7)}$<br><br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br><br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}^{'})^{(7)}$<br><br>$(R_2)^{(7)} = (b_{38}^{'})^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br><br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br><br>$-(\sigma_2)^{(8)} \le -(a_{40}^{'})^{(8)} + (a_{41}^{'})^{(8)} - (a_{40}^{''})^{(8)}(T_{41}, t) + (a_{41}^{''})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br><br>$-(\tau_2)^{(8)} \le -(b_{40}^{'})^{(8)} + (b_{41}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}), t) - (b_{41}^{''})^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |
| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : | 372 |

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| By $(\nu_1)^{(8)} > 0\,, (\nu_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0\,, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$ and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(8)},, (\bar{\nu}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :<br><br>By $(\bar{\nu}_1)^{(8)} > 0\,, (\bar{\nu}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0\,, (\bar{u}_2)^{(8)} < 0$ the<br><br>roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$<br><br>and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$<br><br>**Definition of** $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (\nu_0)^{(8)}$ :-<br><br>If we define $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by<br><br>$(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)},$ ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$<br><br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)},$<br>and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br><br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)},$ ***if*** $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br><br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},$ ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)},$ ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)}t}$<br><br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$ | 376 |

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| $\left(\frac{(a_{42})^{(8)}G_{40}^{0}}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t}-e^{-(S_2)^{(8)}t}\right]+G_{42}^{0}e^{-(S_2)^{(8)}t}\leq G_{42}(t)\leq \frac{(a_{42})^{(8)}G_{40}^{0}}{(m_2)^{(8)}((S_1)^{(8)}-(a_{42}^{'})^{(8)})}[e^{(S_1)^{(8)}t}-e^{-(a_{42}^{'})^{(8)}t}]+G_{42}^{0}e^{-(a_{42}^{'})^{(8)}t}\right)$ | 377 |
| $\boxed{T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}}T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq \frac{1}{(\mu_2)^{(8)}}T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^{0}}{(\mu_1)^{(8)}((R_1)^{(8)}-(b_{42}^{'})^{(8)})}\left[e^{(R_1)^{(8)}t}-e^{-(b_{42}^{'})^{(8)}t}\right]+T_{42}^{0}e^{-(b_{42}^{'})^{(8)}t}\leq T_{42}(t)\leq$ $\frac{(a_{42})^{(8)}T_{40}^{0}}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t}-e^{-(R_2)^{(8)}t}\right]+T_{42}^{0}e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)},(S_2)^{(8)},(R_1)^{(8)},(R_2)^{(8)}$:-<br>Where $(S_1)^{(8)}=(a_{40})^{(8)}(m_2)^{(8)}-(a_{40}^{'})^{(8)}$<br>$(S_2)^{(8)}=(a_{42})^{(8)}-(p_{42})^{(8)}$<br>$(R_1)^{(8)}=(b_{40})^{(8)}(\mu_2)^{(8)}-(b_{40}^{'})^{(8)}$<br>$(R_2)^{(8)}=(b_{42}^{'})^{(8)}-(r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define | 382 |
| **Definition of** $(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)}\leq -(a_{44}^{'})^{(9)}+(a_{45}^{'})^{(9)}-(a_{44}^{''})^{(9)}(T_{45},t)+(a_{45}^{''})^{(9)}(T_{45},t)\leq -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)}\leq -(b_{44}^{'})^{(9)}+(b_{45}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47}),t)-(b_{45}^{''})^{(9)}((G_{47}),t)\leq -(\tau_1)^{(9)}$ | |
| **Definition of** $(\nu_1)^{(9)},(\nu_2)^{(9)},(u_1)^{(9)},(u_2)^{(9)},\nu^{(9)},u^{(9)}$ :<br>By $(\nu_1)^{(9)}>0$ , $(\nu_2)^{(9)}<0$ and respectively $(u_1)^{(9)}>0$ , $(u_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_1)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$<br>and $(b_{45})^{(9)}(u^{(9)})^2+(\tau_1)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(9)},,(\bar{\nu}_2)^{(9)},(\bar{u}_1)^{(9)},(\bar{u}_2)^{(9)}$ :<br>By $(\bar{\nu}_1)^{(9)}>0$ , $(\bar{\nu}_2)^{(9)}<0$ and respectively $(\bar{u}_1)^{(9)}>0$ , $(\bar{u}_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_2)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$ | |

and $(b_{45})^{(9)}\left(u^{(9)}\right)^2+(\tau_2)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$

**Definition of** $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)},(\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)}$ by

$(m_2)^{(9)}=(\nu_0)^{(9)},(m_1)^{(9)}=(\nu_1)^{(9)},$ ***if*** $(\nu_0)^{(9)}<(\nu_1)^{(9)}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)}<(\nu_0)^{(9)}<(\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)}=\frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\nu_0)^{(9)},$ ***if*** $(\bar{\nu}_1)^{(9)}<(\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)}=(u_0)^{(9)},(\mu_1)^{(9)}=(u_1)^{(9)},$ ***if*** $(u_0)^{(9)}<(u_1)^{(9)}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)}<(u_0)^{(9)}<(\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)}=\frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(u_0)^{(9)},$ ***if*** $(\bar{u}_1)^{(9)}<(u_0)^{(9)}$ where $(u_1)^{(9)},(\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{44}(t)\le G_{44}^0e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{45}(t)\le\frac{1}{(m_2)^{(9)}}G_{44}^0e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)}G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^0e^{-(S_2)^{(9)}t}\le G_{46}(t)\le\right.$$

$$\left.\frac{(a_{46})^{(9)}G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}')^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}')^{(9)}t}]+\ G_{46}^0e^{-(a_{46}')^{(9)}t}\right)$$

$$\boxed{T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le\frac{1}{(\mu_2)^{(9)}}T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}')^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}')^{(9)}t}\right]+T_{46}^0e^{-(b_{46}')^{(9)}t}\le T_{46}(t)\le$$

$$\frac{(a_{46})^{(9)}T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^0e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain (383)

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right) - (a''_{14})^{(1)}(T_{14}, t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get (384)

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{v}_1)^{(1)}$ we find like in the previous case, (385)

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}'')^{(1)} = (a_{14}'')^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)}G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}'')^{(1)} = (b_{14}'')^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)}T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}')^{(2)} - (a_{17}')^{(2)} + (a_{16}'')^{(2)}(T_{17}, t)\right) - (a_{17}'')^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)}+(C)^{(2)}(\nu_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_0)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)}-(\nu_0)^{(2)}}{(\nu_0)^{(2)}-(\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)}-(\nu_0)^{(2)}}{(\nu_0)^{(2)}-(\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)}+(C)^{(2)}(\nu_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$$

If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain 394

$$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(2)}(t)$ :- 395

$$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}, \quad \boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$$

In a completely analogous way, we obtain 396

**Definition of** $u^{(2)}(t)$ :-

$$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}, \quad \boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :** 397

If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$

Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then

$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$ if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$

**Proof** : From global equations we obtain 398

$$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$$

**Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$ 399

It follows

$$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$$

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From which one obtains

For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)} + (C)^{(3)}(\nu_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

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In the same manner , we get

$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)} + (\bar{C})^{(3)}(\bar{\nu}_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, 402

$$(\nu_1)^{(3)} \le \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \le \nu^{(3)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \le (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \le (\bar{\nu}_1)^{(3)} \le (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

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$$(\nu_1)^{(3)} \le \nu^{(3)}(t) \le \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \le (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \le \nu^{(3)}(t) \le (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \le u^{(3)}(t) \le (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right) - (a_{25}^{''})^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\quad \boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \le \frac{d\nu^{(4)}}{dt} \le -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

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$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} =$

$(\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b''_{24})^{(4)} = (b''_{25})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a'_{28})^{(5)} - (a'_{29})^{(5)} + (a''_{28})^{(5)}(T_{29}, t)\right) - (a''_{29})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \le \frac{d\nu^{(5)}}{dt} \le -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \ge \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \le \nu^{(5)}(t) \le (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \le \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \le \nu^{(5)}(t) \le (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \le \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \le \nu^{(5)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \le (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \le (\bar{\nu}_1)^{(5)} \le \boxed{(\nu_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\quad \boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get 413

$$v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{v}_1)^{(6)}-(v_0)^{(6)}}{(v_0)^{(6)}-(\bar{v}_2)^{(6)}}}$$

From which we deduce $(v_0)^{(6)} \le v^{(6)}(t) \le (\bar{v}_1)^{(6)}$

414

If $0 < (v_1)^{(6)} < (v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{v}_1)^{(6)}$ we find like in the previous case,

$$(v_1)^{(6)} \le \frac{(v_1)^{(6)}+(C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}} \le v^{(6)}(t) \le$$

$$\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (\bar{v}_1)^{(6)}$$

415

If $0 < (v_1)^{(6)} \le (\bar{v}_1)^{(6)} \le \boxed{(v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(v_1)^{(6)} \le v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (v_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$(m_2)^{(6)} \le v^{(6)}(t) \le (m_1)^{(6)}$, $\boxed{v^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \le u^{(6)}(t) \le (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(v_1)^{(6)} = (\bar{v}_1)^{(6)}$ if in addition $(v_0)^{(6)} = (v_1)^{(6)}$ then $v^{(6)}(t) = (v_0)^{(6)}$ and as a consequence $G_{32}(t) = (v_0)^{(6)}G_{33}(t)$ **this also defines** $(v_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(v_1)^{(6)}$ and $(\bar{v}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right) - (a_{37}^{''})^{(7)}(T_{37},t)v^{(7)} - (a_{37})^{(7)}v^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

$$\nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \leq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}} \leq \nu^{(7)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\bar{\nu}_1)^{(7)}$$

418

If $0 < (\nu_1)^{(7)} \leq (\bar{\nu}_1)^{(7)} \leq \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain

$$(\nu_1)^{(7)} \leq \nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \leq \nu^{(7)}(t) \leq (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

419

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :- 420

$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)} G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain 421

$$\frac{d\nu^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right) - (a_{41}^{''})^{(8)}(T_{41}, t)\nu^{(8)} - (a_{41})^{(8)}\nu^{(8)}$$

**Definition of** $\nu^{(8)}$ :- $\boxed{\nu^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right) \leq \frac{d\nu^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)}-(\nu_0)^{(8)}}{(\nu_0)^{(8)}-(\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \le \nu^{(8)}(t) \le (\bar{\nu}_8)^{(8)}$

If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case, 423

$$(\nu_1)^{(8)} \le \frac{(\nu_1)^{(8)}+(C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}} \le \nu^{(8)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\bar{\nu}_1)^{(8)}$$

If $0 < (\nu_1)^{(8)} \le (\bar{\nu}_1)^{(8)} \le \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$, we obtain 424

$$(\nu_1)^{(8)} \le \nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have
**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \le \nu^{(8)}(t) \le (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain
**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \le u^{(8)}(t) \le (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)}G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)}T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

**Proof** : From 99,20,44,22,23,44 we obtain 424 A

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right) - (a''_{45})^{(9)}(T_{45},t)v^{(9)} - (a_{45})^{(9)}v^{(9)}$$

**Definition of** $v^{(9)}$ :- $\boxed{v^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)}\right) \le \frac{dv^{(9)}}{dt} \le -\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(9)}, (v_0)^{(9)}$ :-

For $0 < \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (v_1)^{(9)} < (\bar{v}_1)^{(9)}$

$$v^{(9)}(t) \ge \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(v_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (v_2)^{(9)}}}$$

it follows $(v_0)^{(9)} \le v^{(9)}(t) \le (v_9)^{(9)}$

In the same manner , we get

$$v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{v}_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (\bar{v}_2)^{(9)}}}$$

From which we deduce $(v_0)^{(9)} \le v^{(9)}(t) \le (\bar{v}_1)^{(9)}$

If $0 < (v_1)^{(9)} < (v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}} < (\bar{v}_1)^{(9)}$ we find like in the previous case,

$$(v_1)^{(9)} \le \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}} \le v^{(9)}(t) \le$$

$$\frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (\bar{v}_1)^{(9)}$$

If $0 < (v_1)^{(9)} \le (\bar{v}_1)^{(9)} \le \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$ , we obtain

$$(v_1)^{(9)} \le v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (v_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(9)}(t)$ :-

$$(m_2)^{(9)} \le v^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{v^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}$, $\boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ 430

$with\ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations 431

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$

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| $with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | |
| We can prove the following<br><br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br><br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br><br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br><br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$ | 435 |

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| $with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |

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| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27})\big)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\big((G_{27})\big)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\big((G_{27})\big)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |

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| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |

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| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(f) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) +$ | 489 |

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| $(a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that | 496 |

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| there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{32}=\frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}=\frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{36}=\frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}=\frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |
| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*})=0$. With this value , we obtain from the three first equations | 501 |

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| $G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a'_{40})^{(8)}+(a''_{40})^{(8)}(T^*_{41})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a'_{42})^{(8)}+(a''_{42})^{(8)}(T^*_{41})]}$ | |
| **Definition and uniqueness of** $T^*_{45}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T^*_{45}$ for which $f(T^*_{45}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a'_{44})^{(9)}+(a''_{44})^{(9)}(T^*_{45})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a'_{46})^{(9)}+(a''_{46})^{(9)}(T^*_{45})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br>$[(b'_{13})^{(1)}(b''_{14})^{(1)}(G) + (b'_{14})^{(1)}(b''_{13})^{(1)}(G)]+(b''_{13})^{(1)}(G)(b''_{14})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G^*_{14}$ such that $\varphi(G^*) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b'_{16})^{(2)}(b'_{17})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br>$[(b'_{16})^{(2)}(b''_{17})^{(2)}(G_{19}) + (b'_{17})^{(2)}(b''_{16})^{(2)}(G_{19})]+(b''_{16})^{(2)}(G_{19})(b''_{17})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G^*_{14}$ such that $\varphi((G_{19})^*) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b'_{20})^{(3)}(b'_{21})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$[(b'_{20})^{(3)}(b''_{21})^{(3)}(G_{23}) + (b'_{21})^{(3)}(b''_{20})^{(3)}(G_{23})]+(b''_{20})^{(3)}(G_{23})(b''_{21})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G^*_{21}$ such that $\varphi((G_{23})^*) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b'_{24})^{(4)}(b'_{25})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br>$[(b'_{24})^{(4)}(b''_{25})^{(4)}(G_{27}) + (b'_{25})^{(4)}(b''_{24})^{(4)}(G_{27})]+(b''_{24})^{(4)}(G_{27})(b''_{25})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ | 506 |

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| is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution | 511 |

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| $G_{14}^{*}$ given by $\varphi(G^{*})=0$ , $T_{14}^{*}$ given by $f(T_{14}^{*})=0$ and<br>$G_{13}^{*}=\frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*}=\frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br>$T_{13}^{*}=\frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*}=\frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br>Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*})=0$ , $T_{17}^{*}$ given by $f(T_{17}^{*})=0$ and | 512 |
| $G_{16}^{*}=\frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*}=\frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*}=\frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*}=\frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*})=0$ , $T_{21}^{*}$ given by $f(T_{21}^{*})=0$ and<br>$G_{20}^{*}=\frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*}=\frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br>$T_{20}^{*}=\frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*}=\frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^{*}$ given by $\varphi(G_{27})=0$ , $T_{25}^{*}$ given by $f(T_{25}^{*})=0$ and<br>$G_{24}^{*}=\frac{(a_{24})^{(4)}G_{25}^{*}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*}=\frac{(a_{26})^{(4)}G_{25}^{*}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*}=\frac{(b_{24})^{(4)}T_{25}^{*}}{[(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*}=\frac{(b_{26})^{(4)}T_{25}^{*}}{[(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}((G_{27})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*})=0$ , $T_{29}^{*}$ given by $f(T_{29}^{*})=0$ and<br>$G_{28}^{*}=\frac{(a_{28})^{(5)}G_{29}^{*}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*}=\frac{(a_{30})^{(5)}G_{29}^{*}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |

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| $T_{28}^{*} = \frac{(b_{28})^{(5)}T_{29}^{*}}{[(b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)}T_{29}^{*}}{[(b_{30}^{'})^{(5)}-(b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br>$G_{32}^{*} = \frac{(a_{32})^{(6)}G_{33}^{*}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)}G_{33}^{*}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)}T_{33}^{*}}{[(b_{32}^{'})^{(6)}-(b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)}T_{33}^{*}}{[(b_{34}^{'})^{(6)}-(b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br>$G_{36}^{*} = \frac{(a_{36})^{(7)}G_{37}^{*}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)}G_{37}^{*}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br>$T_{36}^{*} = \frac{(b_{36})^{(7)}T_{37}^{*}}{[(b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)}T_{37}^{*}}{[(b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br>$G_{40}^{*} = \frac{(a_{40})^{(8)}G_{41}^{*}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)}G_{41}^{*}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br>$T_{40}^{*} = \frac{(b_{40})^{(8)}T_{41}^{*}}{[(b_{40}^{'})^{(8)}-(b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)}T_{41}^{*}}{[(b_{42}^{'})^{(8)}-(b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)}G_{45}^{*}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)}G_{45}^{*}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)}T_{45}^{*}}{[(b_{44}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)}T_{45}^{*}}{[(b_{46}^{'})^{(9)}-(b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | 523 A |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^*) = (q_{14})^{(1)} \quad , \frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^*) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)}G_{13}^*\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^*\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^*\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^*\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^*\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^*\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)} \quad , \frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |

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| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ $and$ $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)}$ , $\frac{\partial(b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_{j}\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_{j}\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_{j}\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 548 |

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| $\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^{*}) = (q_{25})^{(4)}$ , $\frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^{*}) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^{*}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^{*}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^{*}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^{*}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^{*}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^{*}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^{*}) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^{*}) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^{*}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^{*}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^{*}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^{*}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^{*}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^{*}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 563 |

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| **Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ $and$ $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^*\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^*\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^*\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^*\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ $and$ $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}((G_{39})^{**}) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |

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| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^{*}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^{*}\mathbb{G}_{j}\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^{*}\mathbb{G}_{j}\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^{*}\mathbb{G}_{j}\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^{*}) = (q_{41})^{(8)} \quad , \frac{\partial(b_i^{''})^{(8)}}{\partial G_j}\left((G_{43})^{*}\right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^{*}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^{*}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^{*}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^{*}\mathbb{G}_{j}\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b_{41}^{'})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T_{41}^{*}\mathbb{G}_{j}\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b_{42}^{'})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T_{42}^{*}\mathbb{G}_{j}\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |

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| | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \ \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b_{44}^{'})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T_{44}^*\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b_{45}^{'})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T_{45}^*\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b_{46}^{'})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T_{46}^*\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)$<br>$\left[\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(q_{13})^{(1)}G_{13}^*\right)\Big]$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T_{14}^* + (b_{14})^{(1)}s_{(13),(14)}T_{14}^*\Big)$<br>$+\left(\left((\lambda)^{(1)} + (a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^* + (a_{13})^{(1)}(q_{14})^{(1)}G_{14}^*\right)$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^* + (b_{14})^{(1)}s_{(13),(13)}T_{13}^*\Big)$<br>$\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$<br>$\left((\lambda)^{(1)}\right)^2 + \left((b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$<br>$+\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$<br>$+\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^*\right)$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^* + (b_{14})^{(1)}s_{(13),(15)}T_{13}^*\Big)\} = 0$ | |

$$+$$

$$((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)})\{((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)})$$

$$\left[\left(((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)})(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)\right]$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(17)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^{*}_{17}\right)$$

$$+\left(((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)})(q_{16})^{(2)}\mathrm{G}^{*}_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(16)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^{*}_{16}\right)$$

$$\left(((\lambda)^{(2)})^{2} + (\,(a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)})\,(\lambda)^{(2)}\right)$$

$$\left(((\lambda)^{(2)})^{2} + (\,(b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)})\,(\lambda)^{(2)}\right)$$

$$+\left(((\lambda)^{(2)})^{2} + (\,(a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)})\,(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)})\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(18)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(18)}\mathrm{T}^{*}_{16}\right)\} = 0$$

$$+$$

$$((\lambda)^{(3)} + (b'_{22})^{(3)} - (r_{22})^{(3)})\{((\lambda)^{(3)} + (a'_{22})^{(3)} + (p_{22})^{(3)})$$

$$\left[\left(((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)})(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)\right]$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(21)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(21)}T^{*}_{21}\right)$$

$$+\left(((\lambda)^{(3)} + (a'_{21})^{(3)} + (p_{21})^{(3)})(q_{20})^{(3)}G^{*}_{20} + (a_{20})^{(3)}(q_{21})^{(1)}G^{*}_{21}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(20)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(20)}T^{*}_{20}\right)$$

$$\left(((\lambda)^{(3)})^{2} + (\,(a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)})\,(\lambda)^{(3)}\right)$$

$$\left(((\lambda)^{(3)})^{2} + (\,(b'_{20})^{(3)} + (b'_{21})^{(3)} - (r_{20})^{(3)} + (r_{21})^{(3)})\,(\lambda)^{(3)}\right)$$

$$+\left(((\lambda)^{(3)})^{2} + (\,(a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)})\,(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)})\left((a_{22})^{(3)}(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(22)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(22)}T^{*}_{20}\right)\} = 0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\right)\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\right)\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\right)\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(8)} + (b'_{42})^{(8)} - (r_{42})^{(8)})\{((\lambda)^{(8)} + (a'_{42})^{(8)} + (p_{42})^{(8)})$$

$$\left[\left(((\lambda)^{(8)} + (a'_{40})^{(8)} + (p_{40})^{(8)})(q_{41})^{(8)}G^*_{41} + (a_{41})^{(8)}(q_{40})^{(8)}G^*_{40}\right)\right]$$

$$\left(((\lambda)^{(8)} + (b'_{40})^{(8)} - (r_{40})^{(8)})s_{(41),(41)}T^*_{41} + (b_{41})^{(8)}s_{(40),(41)}T^*_{41}\right)$$

$$+\left(((\lambda)^{(8)} + (a'_{41})^{(8)} + (p_{41})^{(8)})(q_{40})^{(8)}G^*_{40} + (a_{40})^{(8)}(q_{41})^{(8)}G^*_{41}\right)$$

$$\left(((\lambda)^{(8)} + (b'_{40})^{(8)} - (r_{40})^{(8)})s_{(41),(40)}T^*_{41} + (b_{41})^{(8)}s_{(40),(40)}T^*_{40}\right)$$

$$\left(((\lambda)^{(8)})^2 + (\,(a'_{40})^{(8)} + (a'_{41})^{(8)} + (p_{40})^{(8)} + (p_{41})^{(8)})\,(\lambda)^{(8)}\right)$$

$$\left(((\lambda)^{(8)})^2 + (\,(b'_{40})^{(8)} + (b'_{41})^{(8)} - (r_{40})^{(8)} + (r_{41})^{(8)})\,(\lambda)^{(8)}\right)$$

$$+\left(((\lambda)^{(8)})^2 + (\,(a'_{40})^{(8)} + (a'_{41})^{(8)} + (p_{40})^{(8)} + (p_{41})^{(8)})\,(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+((\lambda)^{(8)} + (a'_{40})^{(8)} + (p_{40})^{(8)})\,((a_{42})^{(8)}(q_{41})^{(8)}G^*_{41} + (a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^*_{40})$$

$$\left(((\lambda)^{(8)} + (b'_{40})^{(8)} - (r_{40})^{(8)})s_{(41),(42)}T^*_{41} + (b_{41})^{(8)}s_{(40),(42)}T^*_{40}\right)\} = 0$$

$$+$$

$$(\lambda)^{(9)} + (b'_{46})^{(9)} - (r_{46})^{(9)})\{((\lambda)^{(9)} + (a'_{46})^{(9)} + (p_{46})^{(9)})$$

$$\left[\left(((\lambda)^{(9)} + (a'_{44})^{(9)} + (p_{44})^{(9)})(q_{45})^{(9)}G^*_{45} + (a_{45})^{(9)}(q_{44})^{(9)}G^*_{44}\right)\right]$$

$$\left(((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)})s_{(45),(45)}T^*_{45} + (b_{45})^{(9)}s_{(44),(45)}T^*_{45}\right)$$

$$+\left(((\lambda)^{(9)} + (a'_{45})^{(9)} + (p_{45})^{(9)})(q_{44})^{(9)}G^*_{44} + (a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\right)$$

$$\left(((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)})s_{(45),(44)}T^*_{45} + (b_{45})^{(9)}s_{(44),(44)}T^*_{44}\right)$$

$$\left(((\lambda)^{(9)})^2 + (\,(a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)})\,(\lambda)^{(9)}\right)$$

$$\left(((\lambda)^{(9)})^2 + (\,(b'_{44})^{(9)} + (b'_{45})^{(9)} - (r_{44})^{(9)} + (r_{45})^{(9)})\,(\lambda)^{(9)}\right)$$

$$+\left(((\lambda)^{(9)})^2 + (\,(a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)})\,(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+((\lambda)^{(9)} + (a'_{44})^{(9)} + (p_{44})^{(9)})\,((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45} + (a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44})$$

$\left(\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\right)\} = 0$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i \in R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i\ X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor

bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k \tag{6}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 6. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,...x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone $\{ x_i \epsilon C_i: \prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i(x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \in I, x \in K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\,\delta\,(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C\backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population *i* is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int_0^t (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\to\infty}{limsup}\frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \epsilon R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v=au_i(x)+w$ with $a \epsilon R$ and $w \epsilon F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|\ B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |\ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)|\ w|)$. Hence, $\underset{t\to\infty}{limsup}\frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\underset{t\to\infty}{limsup}\frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \underset{t\to\infty}{\text{limsup}}\frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)|-|B_i(t,x)w|$$

$$\geq |\ B_i(t,x)u_i(x)|(a- \alpha+ exp\ (-\beta t)|\ w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \underset{t\to\infty}{\text{limsup}}\frac{1}{t} \text{In}\ |B_i(t,x)v| \geq \lambda i(x) \geq \underset{t\to\infty}{\text{limsup}}\frac{1}{t} \text{In}\ |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In}\ |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In}\ b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x)$ =0 implies

$$\frac{d}{dt}b_i(t,x) = \lim_{s\to 0}\frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

=

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \le 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \le \gamma$ for all $t \ge 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \operatorname{In} |B_i(t,x)x_i| = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \operatorname{In} |\, (x.t)_i\, |$$

$$\le \operatorname{limsup}_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0. If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\ge 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\operatorname{limsup}_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x) =$ Integral $h(x.t)d\,\mu(x)$for all $t \ge 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure*. $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$ , $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x) \rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1 \le j \le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as $n \to \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |\mathrm{h}(x.t) - h(x^n.t)|d\mu_n(x)| + |\int_c h(x^n.t)|d\mu_n(x) - \int_c \mathrm{h}(x)d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x)d\mu_n(x) - \int_c h(x)d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $\mathrm{h}_i^n(x) = \langle \mathrm{A}_i^n(x)\mathrm{u}_i^n(x), \mathrm{u}_i^n(x)\rangle$ where $\mathrm{u}_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $\mathrm{A}_i^n(x) \to \mathrm{A}_i(x)$ uniformly for $x \in N_1((\mathrm{A}))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $\mathrm{u}_i^n(x) \to \mathrm{u}_i(x)$ converges uniformly for $x \in N_1(\Lambda(\mathrm{A}))$. Given $\in > 0$, choose $n$ sufficiently large so that $|\mathrm{h}_i^n(x) - \mathrm{h}_i(x)| \leq \in$ for all $x \in N_1(\Lambda(\mathrm{A}))$ and $|\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$\lambda_i(\mu) \leq \int_c |\mathrm{h}_i(x) - \mathrm{h}_i^n(x)|d\mu(x) + |\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)|$

$+ \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)$

$\leq \quad 2\in$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\, \mathrm{ds}$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument

implies that $\mu$ is $x.t$ invariant. Define $h_i$: C → R by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

(6) $\min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since $\min\mu$ $\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta -$perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \begin{cases} \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} & if\ i \epsilon I \\ if\ i \notin I \end{cases}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t >$ $0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set $K$ if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$?then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$?and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume x* $\mapsto$ *A(x) is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists j* $\epsilon$ { 1,….,k} *such that $M_i$ ----- { $x \in C : x_l \gg 0$ for $l \neq j$} Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_iM_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----{1,…,k} such that $\mu$(---$_i \in_I C_+$) = 1

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures μ with support in ∂C, $\lambda_i(\mu)>0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $$\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j \qquad i = 1,..,k \qquad j=1,\ldots,m$$

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge

to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j$ ($x^j_i$) = 0 i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product ($E_1^1$ ,…,$E_1^m$), and the matrix $A^2$ ($E_1$) is close to the diagonal matrix diag($f^i_2(E_1{}^j)$)$^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f_2^j (E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j (E^j_1)>0$ and $\max_j f_2^j (E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial$C is given by two equilibria {0,$E_1$}, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j (E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$

$\frac{dx_2^j}{dt} = x_2^j(1 - \alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1 - \beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > 1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$and $\lambda_3(E_1)$are close to $\max_j 1 - a^j$ and $\max_j 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1 - a^j$ and $\max_j 1 - \beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1 - a^j) > -\max_j(1 - \beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i\left(r_i - \sum_s a_{is}x_s\right), \qquad i = 1,\dots,k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$$

It has been shown [12,ch.16.1,16.2] That for *k* =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \ \ with\ \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing

holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & O \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,($K$,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance $N$ and exhibits logistic dynamics f($N$)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left[ \beta(1 - y - z)\frac{P}{\epsilon + P} - r - m(1 - y) - b(N)\, y \right]$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*′ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\Lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\Lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\Lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\,(x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + Kp\, x_i x_i - 1}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords

Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation

of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1$ *(t)* and $P_2$ *(t)* denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial

alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.16) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.17) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.18) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_{1,}$ $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_{i,}$ $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic ($as$ $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 6. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, \text{R}^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1, 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) \quad P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^\infty f_i(a + s, P(t + S)) ds \right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[ - \int_o^x f_1(a + s, p(t + s)) ds \right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0) \exp\left[ - \int_o^t f_1(a - t + s, p_1(s)) ds \right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(\text{a+s}, p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da+\int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$$t> 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \epsilon[-\tau, 0], i = 1,2,$

(2.8) $\beta_i(\text{t}) = \int_0^t \beta_i(\text{a}) b_i(\text{t-a}, P_i(t)) M_i.(a, t, P_i(t))da$

$$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$$

$$i = 1, 2, \text{t} > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t)\, \phi(0) d_i^*]t \quad for\ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T],R^+), \quad B_i \in C([0,T],R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau,T] = \{f \in C([-\tau,T],R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau,T]$. Then for any

$(x, y) \in C^+[-\tau,T] \times C^+[-\tau,T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau,T], \quad C_T^2 = C^+[-\tau,T] \times C^+[-\tau,T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau,T]$ *and* $t \in [0,T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$B: C_T^2 \to C_T^2$ *and* $\Pi C_T^2 \to C_T^2$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = P_t \left\{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right)$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Pi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Pi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Pi(P_t^{(1)}) - \Pi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Pi(P^{(1)}(t)) - \Pi(P^{(2)}(t)| \le \int_0^t B_1^{(1)}(a) \left\{ M_1(a, t, P_t^{(1)}) - M_1(a, t, P_t^{(2)}) \right\} da$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left( \int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_* ds \right) da$$

$$\leq d_1 \beta_1^* \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o) \left[ \int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds \right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \|_t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$=\int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\mathrm{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \left\| P_t^{(1)} - P_t^{(2)} \right\| \text{ for } a \varepsilon [o, t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \leq \beta_1 \Phi_1(0) ||P_t^{(1)} - P_t^{(2)}||_t^* + \Phi_1(0) \beta_1 d_1 ||P_t^{(1)} - P_t^{(2)}||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) e^{\delta 1^a} \beta_1 \left\| P_t^{(1)} - P_t^{(2)} \right\|_{t^*} + \beta_1 d_1 \left\| P_t^{(1)} - P_t^{(2)} \right\|_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) e^{\delta 1^a} \left\| P_t^{(1)} - P_t^{(2)} \right\|_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| B_1^{*d}\sigma$$

$$+ \Phi_1(0) \left\| P_t^{(1)} - P_t^{(2)} \right\|_{t^*} \left\{ B_1 + B_2 d_1 a + B_1^* \left(B_2 a + d_1 a^2/2\right) e^{\delta 1^a} \right\}$$

$$\leq \beta_1^* \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| do + \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \to \beta_1$ as $a \to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{g_1(a) + 0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a)) - b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a)) do \right|$$

$$\left| B_1^* \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi\left(P_t^{(1)} - \pi\left(P_t^{(2)}\right) /\!/ \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^*)$$

As $t^* \to o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp P_2^*(a)}{da} + f_i(a, P_1^*, P_2^*) P_i^*(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^* = \int_o^\infty P_i^*(a) da, i = 1,\ 2,$$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i=1.\ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s,P_1^*,P_2^*)ds\right], i=1,\quad 2, a\geq 0,$

Which together with (3.3) lead to

$$1=\int_o^\infty b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^\infty b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1,P_2) = \int_o^\infty b_i(a,P_1,P_2)\exp\left[-\int_o^\infty f_i(s,P_1,P_2)ds\right]da,$

$i=1,\ 2,\quad P_1\geq 0,\quad P_2\geq 0,$

**And suppose the following hold:**

(3.7) $A_{4.}\quad F_{i.}\ (0,\ 0)>1,\ i=1,2$:

$A_{5.}$ *there exists a positive constant c for which*

$$F_1(C,0)< F_{2.}(C,0),$$

(3.8) $$F_1(0.C)< F_{2.}(0,C),$$

Then there exists a unique pair $(P_1^*,P_2^*)$ of real numbers $P_1^*>o,$

$P_2^*>0$ such that

(3,9) $F_1(P_1^*,P_2^*)=1=F_2(P_1^*,P_2^*)$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a)=[P_2^*\exp[-f_i(s,P_1^*,P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s,P_1^*,P_2^*)ds]da),$

$i=1,2,\quad a\geq 0,$

Proof Consider the elements of the surfaces defined by

$$Z= F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0,$$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

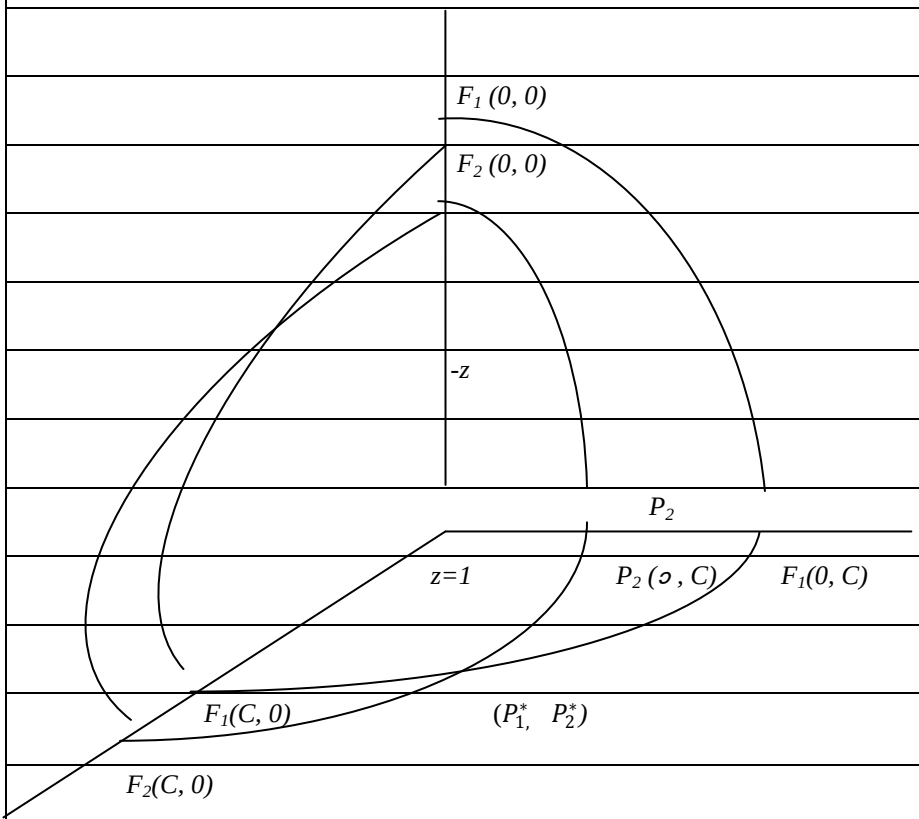
$$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0,$ These two curves are defined by

$$F_1(P_1, P_2 = 1 \; and \; F_2(P_1, P_2) = 1, \; F_1(P_1, P_2 \geq 0,$$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1< F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11)
$$\begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square=1}^{2} \frac{\partial b_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u\square(a)\exp[\lambda t], \quad \square(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty \square(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d\square}{da} + \lambda\square = -f_i\left(a, p_1^*, p_2^*\right)\square(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_{i\square}^*\right\},$$

(3.15)

$$\square(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\square(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*} p_{i\square}^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^{*} = p_{1}^{*}, \quad p_{12}^{*} = p_{2}^{*} \exp\{-\lambda\tau\},$$

$$p_{21}^{*} = p_{1}^{*}, \exp\{-\lambda\tau\}, \ p_{22}^{*} = p_{2}^{*}.$$

If we integrate the first of (3.15) and use the fact that $\mu_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\mu_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\mu_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\mu_i(a) = \{\mu_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \mu_i\right\} e^{\lambda s} ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\mu_i(a)\left[p_i^*(a)/p_i^*(0)c_i \exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \mu_i\right\} da\right.$$

$$\left. - \int_0^\infty b_i(a, p_1^*, p_2^*)\, p_i^*(a)\, e^{-\lambda a}\left\{\int_0^a \left(\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \mu_i\right) e^{\lambda s} ds\right\} da\right]$$

$$- p_i^*(a)\, e^{-\lambda a}\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \mu_i\right\} e^{\lambda s} ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i(a, p_1^*, p_2^*)p_i^*(a)e^{-\lambda a}da, \quad i = 1,\ 2,$$

Now supplying $\mu_1(a)$ and $\mu_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^* \exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^* \exp(-\lambda\tau) + (\lambda - A_{22})\, p_2^* = g_{21}(\lambda)\, p_1^* \exp(-\lambda\tau) + g_{22}(\lambda)\, p_2^*$$

Where

(3.22) $$A_{i} = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_{}^*} - \frac{\partial f_i}{\partial p_{}^*}\right]da,$$

(3.23) $$g_{i}(\lambda) = \left[p_i^*(0)c_i\right]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\, da\right\}$$
$$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_{}^*} - b_i(a, p_1^*, p_2^*)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{}^*}\, e^{\lambda s}\, ds\right)\right]da\right\}$$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{}^*}\, e^{\lambda s}\, ds\right)da, \quad i = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*), \quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})\, e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$$S_2(\lambda, \tau) = \left[A_{21}\, g_{12}(\lambda) + A_{12}\, g_{21}(\lambda) + g_{12}(\lambda)\, g_{21}(\lambda)\right]\exp[-2\lambda\tau]$$

$$-A_{11}\, g_{22}(\lambda) - A_{22}\, g_{11}(\lambda) - g_{11}(\lambda)\, g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6$. $0 < \beta_i^* < d_i^*$ $(i = 1, 2.)$ $(see(A_1) and (A_2))$;

$A_7$. $\{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \text{[illegible]}(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - \left| F_2(\lambda, \tau) \right|$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \,| - |\, A_{11} + A_{22} + \| S_1(0) \| \,|^2 \right\}/2,$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational

ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the

Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have

obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is:

individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

**TITLE**

**INTRODUCTION—VARIABLES USED**

**PARRHESIA NUMBER 1 2006 • 62 – 87 Numerology or Ideology Zizek's Critique of Deleuze Robert Sinner brink I have deleted some sentences and added my own remarks to herein below exposition. For acknowledgements and references kindly see the original article. / Please note that the expatiation is model able and the following presupposition format is preeminently suitable for modelling**

(1) By contrast, Hegelian immanence is marked by (e) an irreducible gap or rupture, namely the inherent negativity or inconsistency marking the immanence of phenomena, which gives rise to (eb) unavoidable illusions of transcendence.

(2) One should note that this in fact parallels Deleuze's argumentation (from Nietzsche and Philosophy to A Thousand Plateaus) that the reactive forces defining consciousness are prey to (e) the illusions of transcendence precisely because of (e) the inherent 'inconsistency' or dynamic flux of (unconscious) active forces comprising (e) the plane of immanence.

(3) So perhaps here, ironically, Deleuze and Zizek are really much closer than Zizek will admit! Zizek then examines various examples of the Hegelian dialectical movement to show the operation of the Deleuzian 'minimal difference' in action: 1) We begin with the real as transcendent hard kernel,

always perceived in a refracted way; 2) we then move through the 'postmodern' reversal in which we reject the transcendent Real, since the play of refracted appearances is all there is. 3) We conclude with the dialectical Aufhebung through which we return to the Real, but now as 'desubstantialized Real,' the Real reduced to a 'minimal difference' between appearances

(4) The Real reveals itself as (=) the cause of refraction, the difference in itself of appearances, not the inner core beyond refracted appearances (OwB, 62). Zizek then attempts to show that Deleuzian difference in itself is in effect "the core of Hegelian dialectics" (OwB, 64), a startling claim given Deleuze's explicit critique of Hegelian dialectic as an 'orgiastic' representation of difference, that is, conceptual difference rather than a concept of difference. The parallel with Hegelian dialectic, I suggest, runs presumably as follows: 1). We begin with difference as an inaccessible transcendent real, unthinkable and unrepresentable in itself. 2) We then move through a rejection of this inaccessible difference, reversing difference such that it is now revealed in the play of appearances. 3)

(5) We conclude with the synthesis of these two aspects, where difference is now construed as (=) minimal difference, difference not between appearances but difference in itself: "the 'pure' difference that differentiates an element not from (e) other particular elements but from itself, from (e) its own place of inscription"; the minimal difference that "explodes this identity into (e&eb) the in (de)finite generative process of differentiation" (OwB, 64).

(6) The most perfect repetition thus reveals difference in itself, which is precisely not a relational difference between articulated identities but a differentiation process (event) that undermines any articulated identity. The problem here is that Hegelian dialectics, even in the most heterodox reading, is always directed towards the moment of reconciliatory unity in which identity and difference, individuality and particularity, are sublated into a concrete universality.

(7) Difference is inscribed within the very heart of universality, the Hegelian 'identity of identity and difference,' no longer as the reflective opposition between identity and difference, but now as one aspect of the threefold conceptual unity of universal, particular, and individual. Speculative dialectic, for Hegel, always recuperates difference, understood as singularity, within conceptual universality. This was precisely the spirit of Deleuze's criticism of Hegel: that Hegel inscribes difference within universality, as conceptual difference within a system of representation, rather than arriving at the thought of a concept of difference, of difference in itself. Zizek, however, flattens this Deleuzian criticism, ignoring the crucial difference between conceptual difference and a concept of difference, and attempts instead to sublate Deleuzian difference into the Hegelian speculative identity of identity and difference. Now Deleuze's complaint on this point, we recall, is that Hegelian difference cannot think the advent of the New, precisely because it integrates difference as an aspect of universality. Zizek, however, proceeds to posit this Deleuzian difference in itself, and its self-differentiation in the productive repetition of the New, as the equivalent to Hegelian dialectical difference; a conceptual difference that reduces the New to the particularity and contingency of phenomena that find their conceptual meaning and ground within speculative thought.

(8) Zizek then goes on to argue that Hegel anticipates Deleuze's contributions to the philosophy of difference, presenting what one might be tempted to call his own 'poststructuralist' (decentred, open-ended, heterogenous) reading of Hegelian dialectic (an ironic twist given Zizek's critical dismissal of the 'poststructuralist' critique of Hegel elsewhere). The Deleuzian concept of difference in itself is thus reinscribed as Hegelian conceptual difference.

(9) The New conceals the Old; Deleuzian difference is a repetition of the Hegelian Same rather than an attempt to think the New, let alone break free of Hegelian dialectics, as Deleuze claimed of his own

work. One of the more interesting implications of Deleuze's attempted Aufhebung of Deleuzian difference is to suggest that there is a parallel between Deleuze's pure difference and Ernesto Laclau's concept of an antagonism (OwB, 65 ff.). Laclau argues, in very Hegelian fashion, that while the logic of difference and logic of equivalence are opposed, pursued to their logical conclusion each converts into its opposite. A system of pure differentiality (defined by purely relational elements) without the possibility of antagonism renders all elements equivalent to each other; pure difference converts into pure equivalence in relation to an absolute outside (OwB, 65).

(10) The logic of difference and the logic of equivalence assume opposing poles, but are in fact mutually reliant since they convert into each other once taken to their respective extremes. Laclau argues that there is therefore no primordial opposition of poles (between difference and equivalence) but rather "only the inherent gap of the One" (OwB, 65). Equivalence is not opposed to difference; equivalence emerges because difference remains incomplete in itself. A system of pure antagonism without mediation, on the other hand, devolves to an intractable conflict between a naturalised Us and Them without any possibility of resolution or transformation

**(11)** Hence the need find a way of rethinking antagonism so as to avoid this conceptual and political dead end. Within antagonisms such as sexual difference or class struggle, Laclau argues, it is not opposition but rather "the fissure" that is "primary" (OwB 65). To take the example of sexual difference, it is not that there is a straightforward natural opposition between the sexes; 'woman,' rather, is the effect of the structural incompleteness of 'man' (understood as symbolic categories that fail to adequately represent the traumatic 'real' of sexual difference). In short, "the tension between the Same and the Other," as Zizek concludes, "is secondary with regard to the noncoincidence of the Same with itself" (OwB, 65). The opposition between difference and identity is a product of the inherent difference fissuring the one itself.

**NOTATION**

**Module One**

By contrast, Hegelian immanence is marked by (e) an irreducible gap or rupture, namely the inherent negativity or inconsistency marking the immanence of phenomena, which gives rise to (eb) unavoidable illusions of transcendence

$G_{13}$ : Category one of irreducible gap or rupture, namely the inherent negativity or inconsistency marking the immanence of phenomena, which gives rise to (eb) unavoidable illusions of transcendence

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of Hegelian immanence is marked

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

By contrast, Hegelian immanence is marked by an irreducible gap or rupture, namely the inherent negativity or inconsistency marking the immanence of phenomena, which gives rise to (eb) unavoidable illusions of transcendence

$G_{16}$ : Category one of Hegelian immanence is marked by an irreducible gap or rupture, namely the inherent negativity or inconsistency marking the immanence of phenomena

| | |
|---|---|
| $G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of unavoidable illusions of transcendence<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>One should note that this in fact parallels Deleuze's argumentation (from Nietzsche and Philosophy to A Thousand Plateaus) that the reactive forces defining consciousness are prey to (e) the illusions of transcendence precisely because of (e) the inherent 'inconsistency' or dynamic flux of (unconscious) active forces comprising (e) the plane of immanence | |
| $G_{20}$ : Category one of illusions of transcendence precisely because of (e) the inherent 'inconsistency' or dynamic flux of (unconscious) active forces comprising (e) the plane of immanence<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of reactive forces defining consciousness are prey<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>One should note that this in fact parallels Deleuze's argumentation (from Nietzsche and Philosophy to A Thousand Plateaus) that the reactive forces defining consciousness are prey to the illusions of transcendence precisely because of (e) the inherent 'inconsistency' or dynamic flux of (unconscious) active forces comprising (e) the plane of immanence | |
| $G_{24}$ : Category one of inherent 'inconsistency' or dynamic flux of (unconscious) active forces comprising (e) the plane of immanence<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of reactive forces defining consciousness are prey to the illusions of transcendence<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>One should note that this in fact parallels Deleuze's argumentation (from Nietzsche and Philosophy to A Thousand Plateaus) that the reactive forces defining consciousness are prey to the illusions of transcendence precisely because of the inherent 'inconsistency' or dynamic flux of (unconscious) active forces comprising | |

| | |
|---|---|
| (e) the plane of immanence | |
| $G_{28}$ : Category one of plane of immanence<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of reactive forces defining consciousness are prey to the illusions of transcendence precisely because of the inherent 'inconsistency' or dynamic flux of (unconscious) active forces<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Real reveals itself as (=) the cause of refraction, the difference in itself of appearances, not the inner core beyond refracted appearances (OwB, 62). | |
| $G_{32}$ : Category one of Real reveals itself<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of cause of refraction, the difference in itself of appearances, not the inner core beyond refracted appearances (OwB, 62).<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>We conclude with the synthesis of these two aspects, where difference is now construed as (=) minimal difference, difference not between appearances but difference in itself: "the 'pure' difference that differentiates an element not from (e) other particular elements but from itself, from (e) its own place of inscription"; the minimal difference that "explodes this identity into (e&eb) the in (de)finite generative process of differentiation" (OwB, 64). | |
| $G_{36}$ : Category one of difference is now construed<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of minimal difference, difference not between appearances but difference in itself:<br><br>$T_{37}$ : Category two of SAS | |

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| $T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>We conclude with the synthesis of these two aspects, where difference is now construed as (=) minimal difference, difference not between appearances but difference in itself: "the 'pure' difference that differentiates an element not from (e) other particular elements but from itself, from (e) its own place of inscription"; the minimal difference that "explodes this identity into (e&eb) the in (de)finite generative process of differentiation" (OwB, 64). | |
| $G_{40}$ : Category one of its own place of inscription"; the minimal difference that "explodes this identity into (e&eb) the in (de)finite generative process of differentiation" (OwB, 64).<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of'pure' difference that differentiates an element not from (e) other particular elements but from itself<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>We conclude with the synthesis of these two aspects, where difference is now construed as minimal difference, difference not between appearances but difference in itself: "the 'pure' difference that differentiates an element not from other particular elements but from itself, from (e) its own place of inscription"; the minimal difference that "explodes this identity into (e&eb) the in (de)finite generative process of differentiation" (OwB, 64). | |
| $G_{44}$ : Category one of**difference is now construed as minimal difference, difference not between appearances but difference in itself: "the 'pure' difference that differentiates an element not from other particular elements but from itself, from (e) its own place of inscription"; the minimal difference that "explodes this identity**; the in (de)finite generative process of differentiation" (OwB, 64).<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of the in (de)finite generative process of differentiation" (OwB, 64).; **difference is now construed as minimal difference, difference not between appearances but difference in itself: "the 'pure' difference that differentiates an element not from other particular elements but from itself, from (e) its own place of inscription"; the minimal difference that "explodes this identity**<br><br>$T_{45}$ : Category two of SAS | |

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| $T_{46}$ : Category three of SAS | |

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|---|---|
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$<br>$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$<br>$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}\ (a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,<br>$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$<br>$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,<br>$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,<br>$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b'_{13})^{(1)} - (b''_{13})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b'_{14})^{(1)} - (b''_{14})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b'_{15})^{(1)} - (b''_{15})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a''_{13})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b''_{13})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |

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|---|---|
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}, t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}, t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21}, t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23}, t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25}, t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}, t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27}), t\right)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27}), t\right)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27}), t\right)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25}, t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}\left((G_{27}), t\right) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29}, t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}, t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}\left((G_{31}), t\right)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}\left((G_{31}), t\right)\right]T_{29}$ | 29 |

| | |
|---|---|
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b'_{30})^{(5)} - (b''_{30})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a''_{28})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b''_{28})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a'_{32})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b'_{32})^{(6)} - (b''_{32})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b'_{33})^{(6)} - (b''_{33})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b'_{34})^{(6)} - (b''_{34})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a''_{32})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b'_{36})^{(7)} - (b''_{36})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b'_{37})^{(7)} - (b''_{37})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b'_{38})^{(7)} - (b''_{38})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a''_{36})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a'_{41})^{(8)} + (a''_{41})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b'_{40})^{(8)} - (b''_{40})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b'_{41})^{(8)} - (b''_{41})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b'_{42})^{(8)} - (b''_{42})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |

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| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}\left((G_{47}),t\right) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}^{'})^{(1)} & +(a_{13}^{''})^{(1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,)}(T_{17},t) & +(a_{20}^{''})^{(3,3,)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t) & \\ +(a_{36}^{''})^{(7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)} & +(a_{14}^{''})^{(1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,)}(T_{17},t) & +(a_{21}^{''})^{(3,3,)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t) & \\ +(a_{37}^{''})^{(7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)} & +(a_{15}^{''})^{(1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,)}(T_{17},t) & +(a_{22}^{''})^{(3,3,)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t) & \\ +(a_{38}^{''})^{(7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{15}$ | *57* |
| Where $(a_{13}^{''})^{(1)}(T_{14},t)$, $(a_{14}^{''})^{(1)}(T_{14},t)$, $(a_{15}^{''})^{(1)}(T_{14},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{38}^{''})^{(7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3<br>$+(a_{40}^{''})^{(8,8)}(T_{41},t)$ $+(a_{41}^{''})^{(8,8)}(T_{41},t)$ $+(a_{42}^{''})^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3<br>$+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)} & -(b_{13}^{''})^{(1)}(G,t) & -(b_{16}^{''})^{(2,2,)}(G_{19},t) & -(b_{20}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t) & \\ -(b_{36}^{''})^{(7,7,)}(G_{39},t) & -(b_{40}^{''})^{(8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{13}$ | 58 |

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| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\boxed{-(b_{15}^{''})^{(1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | *60* |
| Where $\boxed{-(b_{13}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}^{'})^{(2)}\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}^{'})^{(2)}\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{17}$ | *62* |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}^{'})^{(2)}\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{18}$ | *63* |
| Where $\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)}$ are first augmentation coefficients for | |

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| category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$ , $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$ , $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9)}(T_{45},t)$ , $+(a_{45}^{''})^{(9,9)}(T_{45},t)$ , $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)} & -(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t) & \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{16}$ | *64* |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)} & -(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t) & \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{17}$ | *65* |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)} & -(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t) & \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{18}$ | *66* |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$ , $-(b_{17}^{''})^{(2)}(G_{19},t)$ , $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,)}(G,t)$ , $-(b_{14}^{''})^{(1,1,)}(G,t)$ , $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3<br>$-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$ , $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$ , $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3 | |

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| $-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[ \begin{matrix} (a_{20}^{'})^{(3)} & +(a_{20}^{''})^{(3)}(T_{21},t) & +(a_{16}^{''})^{(2,2,2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t) & \\ +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9)}(T_{45},t) & \end{matrix} \right] G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[ \begin{matrix} (a_{21}^{'})^{(3)} & +(a_{21}^{''})^{(3)}(T_{21},t) & +(a_{17}^{''})^{(2,2,2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t) & \\ +(a_{37}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9)}(T_{45},t) & \end{matrix} \right] G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[ \begin{matrix} (a_{22}^{'})^{(3)} & +(a_{22}^{''})^{(3)}(T_{21},t) & +(a_{18}^{''})^{(2,2,2)}(T_{17},t) & +(a_{15}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t) & \\ +(a_{38}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9)}(T_{45},t) & \end{matrix} \right] G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |

$$\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b'_{20})^{(3)} & -(b''_{20})^{(3)}(G_{23},t) & -(b''_{16})^{(2,2,2)}(G_{19},t) & -(b''_{13})^{(1,1,1,)}(G,t) \\ -(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b''_{36})^{(7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9)}(G_{47},t) & \end{bmatrix} T_{20} \quad (70)$$

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b'_{21})^{(3)} & -(b''_{21})^{(3)}(G_{23},t) & -(b''_{17})^{(2,2,2)}(G_{19},t) & -(b''_{14})^{(1,1,1,)}(G,t) \\ -(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b''_{37})^{(7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9)}(G_{47},t) & \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)} & -(b''_{22})^{(3)}(G_{23},t) & -(b''_{18})^{(2,2,2)}(G_{19},t) & -(b''_{15})^{(1,1,1,)}(G,t) \\ -(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b''_{38})^{(7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9)}(G_{47},t) & \end{bmatrix} T_{22} \quad (72)$$

$-(b''_{20})^{(3)}(G_{23},t)$, $-(b''_{21})^{(3)}(G_{23},t)$, $-(b''_{22})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*

$-(b''_{16})^{(2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*

$-(b''_{13})^{(1,1,1,)}(G,t)$, $-(b''_{14})^{(1,1,1,)}(G,t)$, $-(b''_{15})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*

$-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*

$-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*

$-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*

$-(b''_{36})^{(7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7)}(G_{39},t)$, $-(b''_{38})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b''_{40})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b''_{46})^{(9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a'_{24})^{(4)} & +(a''_{24})^{(4)}(T_{25},t) & +(a''_{28})^{(5,5,)}(T_{29},t) & +(a''_{32})^{(6,6,)}(T_{33},t) \\ +(a''_{13})^{(1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3)}(T_{21},t) & \\ +(a''_{36})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a'_{25})^{(4)} & +(a''_{25})^{(4)}(T_{25},t) & +(a''_{29})^{(5,5,)}(T_{29},t) & +(a''_{33})^{(6,6)}(T_{33},t) \\ +(a''_{14})^{(1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3)}(T_{21},t) & \\ +(a''_{37})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a'_{26})^{(4)} & +(a''_{26})^{(4)}(T_{25},t) & +(a''_{30})^{(5,5,)}(T_{29},t) & +(a''_{34})^{(6,6,)}(T_{33},t) \\ & +(a''_{15})^{(1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a''_{24})^{(4)}(T_{25},t)$, $(a''_{25})^{(4)}(T_{25},t)$, $(a''_{26})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a''_{28})^{(5,5,)}(T_{29},t)$, $+(a''_{29})^{(5,5,)}(T_{29},t)$, $+(a''_{30})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3

$+(a''_{32})^{(6,6,)}(T_{33},t)$, $+(a''_{33})^{(6,6,)}(T_{33},t)$, $+(a''_{34})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1)}(T_{14},t)$ *are fourth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{16})^{(2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2)}(T_{17},t)$ *are fifth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{20})^{(3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3)}(T_{21},t)$ *are sixth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{36})^{(7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7)}(T_{37},t)$ *are seventh augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{40})^{(8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8)}(T_{41},t)$ *are eighth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{46})^{(9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9)}(T_{45},t)$, $+(a''_{44})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b'_{24})^{(4)} & -(b''_{24})^{(4)}(G_{27},t) & -(b''_{28})^{(5,5,)}(G_{31},t) & -(b''_{32})^{(6,6,)}(G_{35},t) \\ & -(b''_{13})^{(1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b'_{25})^{(4)} & -(b''_{25})^{(4)}(G_{27},t) & -(b''_{29})^{(5,5,)}(G_{31},t) & -(b''_{33})^{(6,6,)}(G_{35},t) \\ & -(b''_{14})^{(1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b'_{26})^{(4)} & -(b''_{26})^{(4)}(G_{27},t) & -(b''_{30})^{(5,5,)}(G_{31},t) & -(b''_{34})^{(6,6,)}(G_{35},t) \\ & -(b''_{15})^{(1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{26} \quad (78)$$

*Where* $-(b''_{24})^{(4)}(G_{27},t)$, $-(b''_{25})^{(4)}(G_{27},t)$, $-(b''_{26})^{(4)}(G_{27},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{28})^{(5,5,)}(G_{31},t)$, $-(b''_{29})^{(5,5,)}(G_{31},t)$, $-(b''_{30})^{(5,5,)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b''_{32})^{(6,6,)}(G_{35},t)$, $-(b''_{33})^{(6,6,)}(G_{35},t)$, $-(b''_{34})^{(6,6,)}(G_{35},t)$ *are third detrition coefficients for category* 1, 2 *and* 3

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| $-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2 and 3 $-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 $-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3 $-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[\begin{matrix} (a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,)}(T_{25},t) & +(a_{32}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{matrix}\right] G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[\begin{matrix} (a_{29}^{'})^{(5)}+(a_{29}^{''})^{(5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,)}(T_{25},t) & +(a_{33}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{matrix}\right] G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[\begin{matrix} (a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,)}(T_{25},t) & +(a_{34}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{matrix}\right] G_{30}$ | *81* |
| *Where* $+(a_{28}^{''})^{(5)}(T_{29},t)$, $+(a_{29}^{''})^{(5)}(T_{29},t)$, $+(a_{30}^{''})^{(5)}(T_{29},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3 *And* $+(a_{24}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,)}(T_{25},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3 $+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3 $+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3 $+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3 $+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3 $+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3 $+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation | |

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|---|---|
| coefficients for category 1,2, 3<br>$+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b_{28}^{'})^{(5)} & -(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) & -(b_{32}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{28}$ | 82 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b_{29}^{'})^{(5)} & -(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) & -(b_{33}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{29}$ | 83 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b_{30}^{'})^{(5)} & -(b_{30}^{''})^{(5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,)}(G_{27},t) & -(b_{34}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{15}^{''})^{(1,1,1,1,1,)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{30}$ | 84 |
| *where* $-(b_{28}^{''})^{(5)}(G_{31},t)$, $-(b_{29}^{''})^{(5)}(G_{31},t)$, $-(b_{30}^{''})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3<br>$-(b_{24}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3<br>$-(b_{32}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3<br>$-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3<br>$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3<br>$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3<br>$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3<br>$-(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3<br>$-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3 | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a_{32}^{'})^{(6)} & +(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,4,)}(T_{25},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{32}$ | 85 |

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a'_{33})^{(6)} & +(a''_{33})^{(6)}(T_{33},t) & +(a''_{29})^{(5,5,5)}(T_{29},t) & +(a''_{25})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients*

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| $for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{28}^{''})^{(5,5,5)}(G_{31},t)}, \boxed{-(b_{29}^{''})^{(5,5,5)}(G_{31},t)}, \boxed{-(b_{30}^{''})^{(5,5,5)}(G_{31},t)}\ are\ second\ detrition\ coefficients$ $for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)}\ are\ third\ detrition\ coefficients$ $for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2, and 3 | |
| $\frac{dG_{36}}{dt}$ $= (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)}\boxed{+(a_{36}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *91* |
| $\frac{dG_{37}}{dt}$ $= (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)}\boxed{+(a_{37}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *92* |
| $\frac{dG_{38}}{dt}$ $= (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)}\boxed{+(a_{38}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *93* |
| Where $\boxed{(a_{36}^{''})^{(7)}(T_{37},t)}, \boxed{(a_{37}^{''})^{(7)}(T_{37},t)}, \boxed{(a_{38}^{''})^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3 $\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3 | |

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| $+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ , $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ , $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$ , $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$ , $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3<br>$+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ , $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ , $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{36}}{dt} =$<br>$(b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 94 |
| $\frac{dT_{37}}{dt} =$<br>$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | |
| $\frac{dT_{38}}{dt} =$<br>$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | |
| Where $-(b_{36}^{''})^{(7)}(G_{39},t)$ , $-(b_{37}^{''})^{(7)}(G_{39},t)$ , $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ , $-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ , $-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ , $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ , $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 | |

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| $-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ , $-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ , $-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ , $-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ , $-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)$<br>are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ , $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ , $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3<br>$-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$ , $-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$ , $-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{40}}{dt}$<br>$= (a_{40})^{(8)}G_{41}$<br>$-\begin{bmatrix} (a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13}$ | *95* |
| $\frac{dG_{41}}{dt}$<br>$= (a_{41})^{(8)}G_{40}$<br>$-\begin{bmatrix} (a_{41}^{'})^{(8)}+(a_{41}^{''})^{(8)}(T_{41},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$ | |
| $\frac{dG_{42}}{dt}$<br>$= (a_{42})^{(8)}G_{41}$<br>$-\begin{bmatrix} (a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$ | |
| Where $+(a_{40}^{''})^{(8)}(T_{41},t)$ , $+(a_{41}^{''})^{(8)}(T_{41},t)$ , $+(a_{42}^{''})^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ , $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ , $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ , $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ , $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ , $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ , $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 | |

$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ , $+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ , $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ , $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & \left[-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & \left[-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & \left[-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$ , $-(b_{37}^{''})^{(7)}(G_{39},t)$ , $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$ , $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$ , $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$ , $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$ , $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$ , $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$ , $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$ , $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)$ , $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)$ , $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

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| $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a_{44}^{'})^{(9)}\boxed{+(a_{44}^{''})^{(9)}(T_{45},t)}\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13}$ | *96* |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a_{45}^{'})^{(9)}\boxed{+(a_{45}^{''})^{(9)}(T_{45},t)}\boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$ | |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a_{46}^{'})^{(9)}\boxed{+(a_{46}^{''})^{(9)}(T_{37},t)}\boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$ | |
| Where $+(a_{44}^{''})^{(9)}(T_{45},t)$, $+(a_{45}^{''})^{(9)}(T_{45},t)$, $+(a_{46}^{''})^{(9)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ are Seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3<br>$+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$ are ninth | |

augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \begin{bmatrix} (b'_{46})^{(9)}\boxed{-(b''_{46})^{(9)}(G_{47},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{44})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$$(a_i)^{(1)}, (a'_i)^{(1)}, (a''_i)^{(1)}, (b_i)^{(1)}, (b'_i)^{(1)}, (b''_i)^{(1)} > 0, \quad i,j = 13,14,15 \qquad (97)$$

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| The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br><br>$(a_i^{''})^{(1)}(T_{14},t) \leq (p_i)^{(1)} \leq (\hat{A}_{13})^{(1)}$<br><br>$(b_i^{''})^{(1)}(G,t) \leq (r_i)^{(1)} \leq (b_i^{'})^{(1)} \leq (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$<br><br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br><br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'},t) - (a_i^{''})^{(1)}(T_{14},t)\rvert \leq (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\lvert(b_i^{''})^{(1)}(G^{'},t) - (b_i^{''})^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G^{'}\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}$, $(r_i)^{(1)}, i = 13,14,15$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i^{'})^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i^{'})^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |

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| $(a_i)^{(2)}, (a_i^{'})^{(2)}, (a_i^{''})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (b_i^{''})^{(2)} > 0, \quad i,j = 16,17,18$ | |
| The functions $(a_i^{''})^{(2)}, (b_i^{''})^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(\mathrm{p_i})^{(2)}$, $(\mathrm{r_i})^{(2)}$: | |
| $(a_i^{''})^{(2)}(T_{17}, t) \leq (p_i)^{(2)} \leq \left(\hat{A}_{16}\right)^{(2)}$ | 102 |
| $(b_i^{''})^{(2)}(G_{19}, t) \leq \ (r_i)^{(2)} \leq (b_i^{'})^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(2)}(T_{17}, t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty} (b_i^{''})^{(2)}\left((G_{19}), t\right) = \ (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $\boxed{(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, \ (r_i)^{(2)}}$ are positive constants and $\boxed{i = 16,17,18}$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'}, t) - (a_i^{''})^{(2)}(T_{17}, t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}^{'}\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'}, t) - (b_i^{''})^{(2)}\left((G_{19}), t\right)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})^{'}\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'}, t)$ and $(a_i^{''})^{(2)}(T_{17}, t)$ . $(T_{17}^{'}, t)$ and $(T_{17}, t)$ are points belonging to the interval $\left[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}\right]$. It is to be noted that $(a_i^{''})^{(2)}(T_{17}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i^{''})^{(2)}(T_{17}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, \ (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + \ (\hat{A}_{16})^{(2)} + \ (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |

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| $\frac{1}{(\hat{M}_{16})^{(2)}}[\ (b_i)^{(2)} + (b_i^{'})^{(2)} +\ (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)}\ (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23}, t) \leq\ (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(3)}\ (T_{21}, t) = (p_i)^{(3)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(3)}\ (G_{23}, t) =\ (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)},\ (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'}, t) - (a_i^{''})^{(3)}(T_{21}, t)\rvert \leq (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}{}^{'}, t) - (b_i^{''})^{(3)}(G_{23}, t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}{}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'}, t)$ and$(a_i^{''})^{(3)}(T_{21}, t)$ . $(T_{21}^{'}, t)$ And $(T_{21}, t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |

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| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}((G_{27}), t) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(4)}((G_{27}), t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\rvert \leq (\hat{k}_{24})^{(4)} \lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'}, t) - (b_i^{''})^{(4)}((G_{27}), t)\rvert < (\hat{k}_{24})^{(4)} \lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)}t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants | 120 |

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| $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)} + (b_i^{'})^{(4)} + (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}, (r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}((G_{31}), t) \leq (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(5)} (T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(5)} (G_{31}, t) = (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)} \lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)} t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})', t) - (b_i^{''})^{(5)}((G_{31}), t)\rvert < (\hat{k}_{28})^{(5)} \lvert\lvert(G_{31}) - (G_{31})'\rvert\rvert e^{-(\hat{M}_{28})^{(5)} t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ : | 125 |

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| $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$ | 126 |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(6)} (T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(6)} ((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)} \lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)} t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})', t) - (b_i^{''})^{(6)}((G_{35}), t)\rvert < (\hat{k}_{32})^{(6)} \lvert\lvert(G_{35}) - (G_{35})'\rvert\rvert e^{-(\hat{M}_{32})^{(6)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and $(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |

**Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ : 129

$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)},$ are positive constants

$$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$$

**Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ : 130

There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

Where we suppose

(KK) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i, j = 36,37,38$ 131

(LL) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}$, $(r_i)^{(7)}$:

$$(a_i^{''})^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$$

$$(b_i^{''})^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$$

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*(MM)* $\lim_{T_2 \to \infty} (a_i^{''})^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(NN)*

$\lim_{G \to \infty} (b_i^{''})^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)} |T_{37} - T_{37}^{'}| e^{-(\hat{M}_{36})^{(7)} t}$$

$$|(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)} ||(G_{39}) - (G_{39})^{'}|| e^{-(\hat{M}_{36})^{(7)} t}$$

| | |
|---|---|
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'},t)$ and$(a_i^{''})^{(7)}(T_{37},t)$ . $(T_{37}^{'},t)$ and $(T_{37},t)$ are points belonging to the interval $\left[(\hat{k}_{36})^{(7)},(\widehat{M}_{36})^{(7)}\right]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{36})^{(7)}=1$ then the function $(a_i^{''})^{(7)}(T_{37},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)}$ :<br><br>(OO) $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)},$ are positive constants<br><br>$\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}<1$ | 134 |
| **Definition of** $(\hat{P}_{36})^{(7)},(\hat{Q}_{36})^{(7)}$ :<br><br>(PP) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)},(\hat{A}_{36})^{(7)}and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)},(a_i^{'})^{(7)},(b_i)^{(7)},(b_i^{'})^{(7)},(p_i)^{(7)},\ (r_i)^{(7)},i=36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\widehat{M}_{36})^{(7)}}[(a_i)^{(7)}+(a_i^{'})^{(7)}+(\hat{A}_{36})^{(7)}+(\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}]<1$<br><br>$\frac{1}{(\widehat{M}_{36})^{(7)}}[(b_i)^{(7)}+(b_i^{'})^{(7)}+(\hat{B}_{36})^{(7)}+(\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}]<1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)},(a_i^{'})^{(8)},(a_i^{''})^{(8)},(b_i)^{(8)},(b_i^{'})^{(8)},(b_i^{''})^{(8)}>0,\quad i,j=40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)},(b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)},\ (r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41},t)\leq(p_i)^{(8)}\leq(\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}),t)\leq\ (r_i)^{(8)}\leq(b_i^{'})^{(8)}\leq(\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(8)}(T_{41},t)=(p_i)^{(8)}$ | 140 |
| $\lim_{G\to\infty}(b_i^{''})^{(8)}\left((G_{43}),t\right)=\ (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)},(\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)},(\hat{B}_{40})^{(8)},(p_i)^{(8)},\ (r_i)^{(8)}}$ are positive constants and $\boxed{i=40,41,42}$ | |

They satisfy Lipschitz condition:

$$|(a_i'')^{(8)}(T_{41}', t) - (a_i'')^{(8)}(T_{41}, t)| \leq (\hat{k}_{40})^{(8)}|T_{41} - T_{41}'|e^{-(\hat{M}_{40})^{(8)}t} \quad (142)$$

$$|(b_i'')^{(8)}((G_{43})', t) - (b_i'')^{(8)}((G_{43}), t)| < (\hat{k}_{40})^{(8)}||(G_{43}) - (G_{43})'||e^{-(\hat{M}_{40})^{(8)}t} \quad (143)$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(8)}(T_{41}', t)$ and $(a_i'')^{(8)}(T_{41}, t)$ . $(T_{41}', t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i'')^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i'')^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ :

$(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants

$$\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}, \frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1 \quad (144)$$

**Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :

There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$
Satisfy the inequalities

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1 \quad (145)$$

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1 \quad (146)$$

Where we suppose

$$(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i, j = 44,45,46 \quad (146A)$$

The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:

$$(a_i'')^{(9)}(T_{45}, t) \leq (p_i)^{(9)} \leq (\hat{A}_{44})^{(9)}$$

$$(b_i'')^{(9)}(G_{47}, t) \leq (r_i)^{(9)} \leq (b_i')^{(9)} \leq (\hat{B}_{44})^{(9)}$$

$$\lim_{T_2 \to \infty} (a_i'')^{(9)}(T_{45}, t) = (p_i)^{(9)}$$

$$\lim_{G \to \infty} (b_i'')^{(9)}(G_{47}, t) = (r_i)^{(9)}$$

**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :

Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$

They satisfy Lipschitz condition:

$$|(a_i^{''})^{(9)}(T_{45}^{'}, t) - (a_i^{''})^{(9)}(T_{45}, t)| \leq (\hat{k}_{44})^{(9)}|T_{45} - T_{45}^{'}|e^{-(\hat{M}_{44})^{(9)}t}$$

$$|(b_i^{''})^{(9)}((G_{47})^{'}, t) - (b_i^{''})^{(9)}\big((G_{47}), t\big)| < (\hat{k}_{44})^{(9)}||(G_{47}) - (G_{47})^{'}||e^{-(\hat{M}_{44})^{(9)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(9)}(T_{45}^{'}, t)$ and$(a_i^{''})^{(9)}(T_{45}, t)$ . $(T_{45}^{'}, t)$ and $(T_{45}, t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i^{''})^{(9)}(T_{45}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i^{''})^{(9)}(T_{45}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)},$ are positive constants

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}} \ , \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$$

**Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and \ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}, i = 44,45,46,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0), T_i(0)$ :

$$G_i(t) \leq (\hat{P}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t} \ , \ \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \leq (\hat{Q}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t} \ , \ \boxed{T_i(0) = T_i^0 > 0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0), T_i(0)$

$$G_i(t) \leq (\hat{P}_{16})^{(2)}e^{(\hat{M}_{16})^{(2)}t} \ , \ G_i(0) = G_i^0 > 0$$

| | |
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| $T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$ | 153 A |

| | |
|---|---|
| $T_i(t) \leq (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br><br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br><br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |

| | |
|---|---|
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[ (a_{18})^{(2)} G_{17}(s_{(16)}) - \left( (a_{18}')^{(2)} + (a_{18}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{18}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[ (b_{16})^{(2)} T_{17}(s_{(16)}) - \left( (b_{16}')^{(2)} - (b_{16}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{16}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[ (b_{17})^{(2)} T_{16}(s_{(16)}) - \left( (b_{17}')^{(2)} - (b_{17}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[ (b_{18})^{(2)} T_{17}(s_{(16)}) - \left( (b_{18}')^{(2)} - (b_{18}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{18}(s_{(16)}) \right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a_{20}')^{(3)} + a_{20}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a_{21}')^{(3)} + (a_{21}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a_{22}')^{(3)} + (a_{22}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b_{20}')^{(3)} - (b_{20}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b_{21}')^{(3)} - (b_{21}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b_{22}')^{(3)} - (b_{22}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |

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| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{24})^{(4)}$ , $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{28})^{(5)}$ , $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |

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| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b'_{29})^{(5)} - (b''_{29})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{\mathrm{T}}_{30}(\mathrm{t}) = \mathrm{T}_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b'_{30})^{(5)} - (b''_{30})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{32})^{(6)}$ , $T_i^0 \le (\hat{Q}_{32})^{(6)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a'_{32})^{(6)} + a''_{32})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b'_{32})^{(6)} - (b''_{32})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b'_{33})^{(6)} - (b''_{33})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{\mathrm{T}}_{34}(\mathrm{t}) = \mathrm{T}_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b'_{34})^{(6)} - (b''_{34})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{36})^{(7)}$ , $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |

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| By<br>$\bar{G}_{36}(t) = G_{36}^{0} + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a_{36}^{'})^{(7)} + a_{36}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^{0} + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^{0} + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{40})^{(8)}$, $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^{0} + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}^{'})^{(8)} + a_{40}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^{0} + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^{0} + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^{0} + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^{0} + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |

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| $\overline{T}_{42}(t) = T_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br><br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166<br>A |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{44})^{(9)}$, $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br><br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\overline{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br><br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\bar{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br><br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left(e^{(\bar{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br><br>$(G_{13}(t) - G_{13}^0) e^{-(\bar{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br><br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |

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| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)}\left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$ $\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that $(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$ $\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$ | 171 |
| From which it follows that $(G_{20}(t) - G_{20}^0) e^{-(\hat{M}_{20})^{(3)} t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{P}_{20})^{(3)} + G_{21}^0\right) e^{\left(-\frac{(\hat{P}_{20})^{(3)} + G_{21}^0}{G_{21}^0}\right)} + (\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that $G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0 + (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} s_{(24)}}\right)\right] ds_{(24)} =$ $\left(1 + (a_{24})^{(4)} t\right) G_{25}^0 + \frac{(a_{24})^{(4)} (\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)} t} - 1\right)$ | 173 |
| From which it follows that $(G_{24}(t) - G_{24}^0) e^{-(\hat{M}_{24})^{(4)} t} \le \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[\left((\hat{P}_{24})^{(4)} + G_{25}^0\right) e^{\left(-\frac{(\hat{P}_{24})^{(4)} + G_{25}^0}{G_{25}^0}\right)} + (\hat{P}_{24})^{(4)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0 + (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} s_{(28)}}\right)\right] ds_{(28)} =$ $\left(1 + (a_{28})^{(5)} t\right) G_{29}^0 + \frac{(a_{28})^{(5)} (\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left(e^{(\hat{M}_{28})^{(5)} t} - 1\right)$ | |

From which it follows that

$$(G_{28}(t)-G_{28}^{0})e^{-(\hat{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^{0})e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^{0}}{G_{29}^{0}}\right)}+(\hat{P}_{28})^{(5)}\right] \quad (175)$$

$(G_i^0)$ is as defined in the statement of theorem 5

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{32}(t) \le G_{32}^{0}+\int_0^t\left[(a_{32})^{(6)}\left(G_{33}^{0}+(\hat{P}_{32})^{(6)}e^{(\hat{M}_{32})^{(6)}s_{(32)}}\right)\right]ds_{(32)} = \quad (176)$$

$$\left(1+(a_{32})^{(6)}t\right)G_{33}^{0}+\frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)}t}-1\right)$$

From which it follows that

$$(G_{32}(t)-G_{32}^{0})e^{-(\hat{M}_{32})^{(6)}t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^{0})e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^{0}}{G_{33}^{0}}\right)}+(\hat{P}_{32})^{(6)}\right] \quad (177)$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

(g) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that (178)

$$G_{36}(t) \le G_{36}^{0}+\int_0^t\left[(a_{36})^{(7)}\left(G_{37}^{0}+(\hat{P}_{36})^{(7)}e^{(\hat{M}_{36})^{(7)}s_{(36)}}\right)\right]ds_{(36)} =$$

$$\left(1+(a_{36})^{(7)}t\right)G_{37}^{0}+\frac{(a_{36})^{(7)}(\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)}t}-1\right)$$

From which it follows that

$$(G_{36}(t)-G_{36}^{0})e^{-(\hat{M}_{36})^{(7)}t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)}+G_{37}^{0})e^{\left(-\frac{(\hat{P}_{36})^{(7)}+G_{37}^{0}}{G_{37}^{0}}\right)}+(\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \le G_{40}^{0}+\int_0^t\left[(a_{40})^{(8)}\left(G_{41}^{0}+(\hat{P}_{40})^{(8)}e^{(\hat{M}_{40})^{(8)}s_{(40)}}\right)\right]ds_{(40)} = \quad (180)$$

$$\left(1+(a_{40})^{(8)}t\right)G_{41}^{0}+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)}t}-1\right)$$

From which it follows that

$$(G_{40}(t)-G_{40}^{0})e^{-(\hat{M}_{40})^{(8)}t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^{0})e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^{0}}{G_{41}^{0}}\right)}+(\hat{P}_{40})^{(8)}\right] \quad (181)$$

$(G_i^0)$ is as defined in the statement of theorem 8
Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$

The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)}\left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$$

$$\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)} t} - 1\right)$$

From which it follows that

$$(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{P}_{44})^{(9)} + G_{45}^0\right) e^{\left(-\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0}\right)} + (\hat{P}_{44})^{(9)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 9

Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$

It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ and to choose

$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have

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$$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)} + \left((\hat{P}_{13})^{(1)} + G_j^0\right) e^{-\left(\frac{(\hat{P}_{13})^{(1)} + G_j^0}{G_j^0}\right)}\right] \le (\hat{P}_{13})^{(1)} \quad (183)$$

$$\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)} + T_j^0\right) e^{-\left(\frac{(\hat{Q}_{13})^{(1)} + T_j^0}{T_j^0}\right)} + (\hat{Q}_{13})^{(1)}\right] \le (\hat{Q}_{13})^{(1)} \quad (184)$$

In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric (185)

$$d\left(\left(G^{(1)}, T^{(1)}\right), \left(G^{(2)}, T^{(2)}\right)\right) =$$

$$\sup_i \{\max_{t \in \mathbb{R}_+} \left|G_i^{(1)}(t) - G_i^{(2)}(t)\right| e^{-(\hat{M}_{13})^{(1)} t}, \max_{t \in \mathbb{R}_+} \left|T_i^{(1)}(t) - T_i^{(2)}(t)\right| e^{-(\hat{M}_{13})^{(1)} t}\}$$

Indeed if we denote

**Definition of** $\tilde{G}, \tilde{T}$ : $\left(\tilde{G}, \tilde{T}\right) = \mathcal{A}^{(1)}(G, T)$

It results

$$\left|\tilde{G}_{13}^{(1)}-\tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{13})^{(1)} \left|G_{14}^{(1)}-G_{14}^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)} s_{(13)}} e^{(\widehat{M}_{13})^{(1)} s_{(13)}}\, ds_{(13)} +$$

$$\int_0^t \{(a_{13}')^{(1)} \left|G_{13}^{(1)}-G_{13}^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)} s_{(13)}} e^{-(\widehat{M}_{13})^{(1)} s_{(13)}} +$$

$$(a_{13}'')^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right)\left|G_{13}^{(1)}-G_{13}^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)} s_{(13)}} e^{(\widehat{M}_{13})^{(1)} s_{(13)}} +$$

$$G_{13}^{(2)} \left|(a_{13}'')^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right) - (a_{13}'')^{(1)}\left(T_{14}^{(2)}, s_{(13)}\right)\right|\ e^{-(\widehat{M}_{13})^{(1)} s_{(13)}} e^{(\widehat{M}_{13})^{(1)} s_{(13)}}\} ds_{(13)}$$

Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses it follows

$$\left|G^{(1)}-G^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)} t} \leq$$
$$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a_{13}')^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)} (\widehat{k}_{13})^{(1)}\right) d\left(\left(G^{(1)}, T^{(1)};\ G^{(2)}, T^{(2)}\right)\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

(186)

**Remark 1:** The fact that we supposed $(a_{13}'')^{(1)}$ and $(b_{13}'')^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)} t}$ *and* $(\widehat{Q}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(1)}$ and $(b_i'')^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 2:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$

From 19 to 24 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(1)} - (a_i'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(1)} t\right)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :

(187)

**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if

$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a_{14}')^{(1)} G_{14}$ and by integrating

$$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)} \left((\widehat{M}_{13})^{(1)}\right)_1 / (a_{14}')^{(1)}$$

In the same way , one can obtain

$$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)} \left((\widehat{M}_{13})^{(1)}\right)_2 / (a_{15}')^{(1)}$$

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| If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t),t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br><br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote | 195 |

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| **Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right) d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$ | 201 |

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| In the same way , one can obtain<br>$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | |
| **Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br>Indeed let $t_2$ be so that for $t > t_2$<br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$ | 211 |

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| $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}},\widetilde{T_{23}}:(\widetilde{(G_{23})},\widetilde{(T_{23})})=\mathcal{A}^{(3)}((G_{23}),(T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{20})^{(3)}\left\|G_{21}^{(1)}-G_{21}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)}+$<br>$\int_0^t\{(a_{20}^{'})^{(3)}\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$G_{20}^{(2)}\left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)-(a_{20}^{''})^{(3)}\left(T_{21}^{(2)},s_{(20)}\right)\right\|\ e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)}-G_{23}{}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}t}\le$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)}+(a_{20}^{'})^{(3)}+(\widehat{A}_{20})^{(3)}+(\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)},(T_{23})^{(1)};\ (G_{23})^{(2)},(T_{23})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ *and* $(\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i=20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(3)}-(a_i^{''})^{(3)}(T_{21}(s_{(20)}),s_{(20)})\}ds_{(20)}\right]}\ge 0$<br>$T_i(t)\ge T_i^0e^{(-(b_i^{'})^{(3)}t)}>0$ for t > 0 | 216 |

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| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |

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| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{24})^{(4)}\left\|G_{25}^{(1)}-G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$<br>$\int_0^t\{(a_{24}^{'})^{(4)}\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$G_{24}^{(2)}\left\|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)}-(G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t}\le$<br>$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 226 |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24{,}25{,}26$ depend only on $\mathrm{T}_{25}$ and respectively on $(G_{27})$*(and not on* $t$*)* and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results | 228 |

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| $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(4)} - (a_i'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(4)} t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ :<br><br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if<br><br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}')^{(4)} G_{25}$ and by integrating<br><br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)} \left((\widehat{M}_{24})^{(4)}\right)_1 / (a_{25}')^{(4)}$<br><br>In the same way , one can obtain<br><br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)} \left((\widehat{M}_{24})^{(4)}\right)_2 / (a_{26}')^{(4)}$<br><br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t), t)) = (b_{25}')^{(4)}$ then $T_{25} \to \infty$.<br><br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br><br>Indeed let $t_4$ be so that for $t > t_4$<br><br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t), t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}\left((G_{27})(t), t\right) = (b_{26}')^{(4)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br><br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br><br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |

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| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)}+\left((\hat{P}_{28})^{(5)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{28})^{(5)}\right]\leq(\hat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}ds_{(28)}+$<br>$\int_0^t\{(a_{28}')^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}'')^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\leq$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}')^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}'')^{(5)}$ and $(b_{28}'')^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}\ and\ (\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it | 238 |

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| suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 22:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(5)} - (a_i^{''})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(5)} t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2\ and\ \left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}\ and\ G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}^{'})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a_{29}^{'})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a_{30}^{'})^{(5)}$<br><br>If $G_{29}\ or\ G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}\ and\ G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t), t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t), t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t), t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |

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| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\widehat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{32})^{(6)}\right] \leq (\widehat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)}, (T_{35})^{(1)}), ((G_{35})^{(2)}, (T_{35})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $\widetilde{(G_{35})}, \widetilde{(T_{35})}$ : $\left(\widetilde{(G_{35})}, \widetilde{(T_{35})}\right) = \mathcal{A}^{(6)}((G_{35}), (T_{35}))$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{32})^{(6)}\left\|G_{33}^{(1)} - G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\,ds_{(32)} +$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)} - G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)\left\|G_{32}^{(1)} - G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right) - (a''_{32})^{(6)}\left(T_{33}^{(2)}, s_{(32)}\right)\right\| \; e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)} - (G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t} \leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)} + (a'_{32})^{(6)} + (\widehat{A}_{32})^{(6)} + (\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)}, (T_{35})^{(1)}; (G_{35})^{(2)}, (T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | 249 |

condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i = 32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 27:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$ 250

it results

$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(6)} - (a_i^{''})^{(6)}(T_{33}(s_{(32)}),s_{(32)})\}ds_{(32)}\right]} \geq 0$

$T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(6)}t)} > 0$ for t > 0

**Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2\ and\ \left((\widehat{M}_{32})^{(6)}\right)_3$ : 251

**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}\ and\ G_{34}$ . indeed if

$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}^{'})^{(6)}G_{33}$ and by integrating

$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1/(a_{33}^{'})^{(6)}$

In the same way , one can obtain

$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2/(a_{34}^{'})^{(6)}$

If $G_{33}\ or\ G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively.

**Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}\ and\ G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. 252

**Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}\ ((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$. 253

**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :

Indeed let $t_6$ be so that for $t > t_6$

$(b_{33})^{(6)} - (b_i^{''})^{(6)}\left((G_{35})(t),t\right) < \varepsilon_6, T_{32}\,(t) > (m)^{(6)}$

Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to 254

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right),\ \ t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.
The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$

It is now sufficient to take $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ and to choose
$(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ large to have

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$$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[(\hat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)} \quad (256)$$

$$\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)} \quad (257)$$

In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric

$$d\left(((G_{39})^{(1)}, (T_{39})^{(1)}), ((G_{39})^{(2)}, (T_{39})^{(2)})\right) =$$
$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{36})^{(7)}t}\} \quad (258)$$

Indeed if we denote

**Definition of** $\widetilde{(G_{39})}, \widetilde{(T_{39})} : \left(\widetilde{(G_{39})}, \widetilde{(T_{39})}\right) = \mathcal{A}^{(7)}((G_{39}), (T_{39}))$
It results

$$\left|\tilde{G}_{36}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{36})^{(7)}\left|G_{37}^{(1)} - G_{37}^{(2)}\right|e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{(\hat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)} +$$
$$\int_0^t\{(a'_{36})^{(7)}\left|G_{36}^{(1)} - G_{36}^{(2)}\right|e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{-(\hat{M}_{36})^{(7)}s_{(36)}} +$$
$$(a''_{36})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right)\left|G_{36}^{(1)} - G_{36}^{(2)}\right|e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{(\hat{M}_{36})^{(7)}s_{(36)}} +$$
$$G_{36}^{(2)}\left|(a''_{36})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) - (a''_{36})^{(7)}\left(T_{37}^{(2)}, s_{(36)}\right)\right|\ e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{(\hat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$$

Where $s_{(36)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on it follows

$$\left|(G_{39})^{(1)} - (G_{39})^{(2)}\right|e^{-(\hat{M}_{36})^{(7)}t} \leq$$
$$\frac{1}{(\hat{M}_{36})^{(7)}}\left((a_{36})^{(7)} + (a'_{36})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)}, (T_{39})^{(1)};\ (G_{39})^{(2)}, (T_{39})^{(2)})\right) \quad (259)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

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| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ $and$ $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\}ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)}t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)}G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1/(a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ $is$ bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}\,((G_{39})(t),t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |

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| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right), \quad t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t), t\right) = (b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\bar{M}_{40})^{(8)}}, \frac{(b_i)^{(8)}}{(\bar{M}_{40})^{(8)}} < 1$ and to choose<br>$(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\bar{M}_{40})^{(8)}}\left[(\hat{P}_{40})^{(8)} + \left((\hat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\bar{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{40})^{(8)}\right] \leq (\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)}, (T_{43})^{(1)}), ((G_{43})^{(2)}, (T_{43})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\bar{M}_{40})^{(8)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\bar{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})}, \widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})}, \widetilde{(T_{43})}\right) = \mathcal{A}^{(8)}((G_{43}), (T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{40})^{(8)}\left\|G_{41}^{(1)} - G_{41}^{(2)}\right\|e^{-(\bar{M}_{40})^{(8)}s_{(40)}}e^{(\bar{M}_{40})^{(8)}s_{(40)}}\,ds_{(40)} +$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\bar{M}_{40})^{(8)}s_{(40)}}e^{-(\bar{M}_{40})^{(8)}s_{(40)}} +$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\bar{M}_{40})^{(8)}s_{(40)}}e^{(\bar{M}_{40})^{(8)}s_{(40)}} +$<br>$G_{40}^{(2)}\left\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right) - (a_{40}^{''})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)\right\| e^{-(\bar{M}_{40})^{(8)}s_{(40)}}e^{(\bar{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval [0, t] | 272 |

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|---|---|
| From the hypotheses it follows | |
| $\left\|(G_{43})^{(1)} - (G_{43})^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)}t} \leq$ $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)} + (a'_{40})^{(8)} + (\widehat{A}_{40})^{(8)} + (\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right) d\left(((G_{43})^{(1)}, (T_{43})^{(1)};\ (G_{43})^{(2)}, (T_{43})^{(2)})\right)$ And analogous inequalities for $G_i$ $and$ $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a''_{40})^{(8)}$ and $(b''_{40})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(8)}$ and $(b''_i)^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t) = 0$ $and$ $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(8)} - (a''_i)^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{(-(b'_i)^{(8)}t)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ $and$ $\left((\widehat{M}_{40})^{(8)}\right)_3$ : **Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ $and$ $G_{42}$ . indeed if $G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a'_{41})^{(8)} G_{41}$ and by integrating $G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a'_{41})^{(8)}$ In the same way , one can obtain $G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a'_{42})^{(8)}$ If $G_{41}$ $or$ $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}$ $and$ $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(8)}((G_{43})(t), t)) = (b'_{41})^{(8)}$ then $T_{41} \to \infty$. | 278 |

**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :

Indeed let $t_8$ be so that for $t > t_8$

$(b_{41})^{(8)} - (b_i^{''})^{(8)}\big((G_{43})(t), t\big) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$

Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.
The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\big((G_{43})(t), t(t), t\big) = (b_{42}^{'})^{(8)}$

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It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have

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$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \big((\hat{P}_{44})^{(9)} + G_j^0\big)e^{-\left(\frac{(\hat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\big((\hat{Q}_{44})^{(9)} + T_j^0\big)e^{-\left(\frac{(\hat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{44})^{(9)}\right] \leq (\hat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\Big(\big((G_{47})^{(1)}, (T_{47})^{(1)}\big), \big((G_{47})^{(2)}, (T_{47})^{(2)}\big)\Big) =$$
$$sup_i\{max_{t\in\mathbb{R}_+}\big|G_i^{(1)}(t) - G_i^{(2)}(t)\big|e^{-(\widehat{M}_{44})^{(9)}t}, max_{t\in\mathbb{R}_+}\big|T_i^{(1)}(t) - T_i^{(2)}(t)\big|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \big(\widetilde{(G_{47})}, \widetilde{(T_{47})}\big) = \mathcal{A}^{(9)}\big((G_{47}), (T_{47})\big)$
It results

$$\big|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\big| \leq \int_0^t (a_{44})^{(9)}\big|G_{45}^{(1)} - G_{45}^{(2)}\big|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\big|G_{44}^{(1)} - G_{44}^{(2)}\big|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$(a_{44}^{''})^{(9)}\big(T_{45}^{(1)}, s_{(44)}\big)\big|G_{44}^{(1)} - G_{44}^{(2)}\big|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$G_{44}^{(2)}\big|(a_{44}^{''})^{(9)}\big(T_{45}^{(1)}, s_{(44)}\big) - (a_{44}^{''})^{(9)}\big(T_{45}^{(2)}, s_{(44)}\big)\big|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)} - G^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)}t} \leq \frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a'_{44})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right) d\left(((G_{47})^{(1)}, (T_{47})^{(1)};\ (G_{47})^{(2)}, (T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a''_{44})^{(9)}$ and $(b''_{44})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)} e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)} e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(9)}$ and $(b_i'')^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$

From 99 to 44 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(9)} - (a_i'')^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b_i')^{(9)}t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2$ *and* $\left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}$ *and* $G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a'_{45})^{(9)} G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1 / (a'_{45})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2 / (a'_{46})^{(9)}$$

If $G_{45}$ *or* $G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}$ *and* $G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(9)}\left((G_{47})(t), t\right)) = (b'_{45})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\big((G_{47})(t),t\big) < \varepsilon_9, T_{44}\,(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\big((G_{47})(t),t\big) = (b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} - (a_{13}^{''})^{(1)}(T_{14}, t) + (a_{14}^{''})^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\big(\nu^{(1)}\big)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\big(u^{(1)}\big)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\big(\nu^{(1)}\big)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\big(u^{(1)}\big)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

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| $(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)},$<br>and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$<br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$<br>are defined | |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)} t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)} t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)} t}\right] + G_{15}^0 e^{-(S_2)^{(1)} t} \le G_{15}(t) \le \right.$<br>$\left.\frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a'_{15})^{(1)})}[e^{(S_1)^{(1)} t} - e^{-(a'_{15})^{(1)} t}] + G_{15}^0 e^{-(a'_{15})^{(1)} t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)} T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)} - (b'_{15})^{(1)})}\left[e^{(R_1)^{(1)} t} - e^{-(b'_{15})^{(1)} t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)} t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)} T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)} + (r_{13})^{(1)} + (R_2)^{(1)})}\left[e^{((R_1)^{(1)} + (r_{13})^{(1)})t} - e^{-(R_2)^{(1)} t}\right] + T_{15}^0 e^{-(R_2)^{(1)} t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$<br>$(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ :<br>$(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |

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| $-(\sigma_2)^{(2)} \le -(a_{16}^{'})^{(2)} + (a_{17}^{'})^{(2)} - (a_{16}^{''})^{(2)}(\mathrm{T}_{17}, t) + (a_{17}^{''})^{(2)}(\mathrm{T}_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b_{16}^{'})^{(2)} + (b_{17}^{'})^{(2)} - (b_{16}^{''})^{(2)}\big((G_{19}), t\big) - (b_{17}^{''})^{(2)}\big((G_{19}), t\big) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}$,, $(\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{\mathrm{G}_{16}^0}{\mathrm{G}_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{\mathrm{T}_{16}^0}{\mathrm{T}_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$\mathrm{G}_{16}^0 \mathrm{e}^{((\mathrm{S}_1)^{(2)} - (p_{16})^{(2)})\mathrm{t}} \le G_{16}(t) \le \mathrm{G}_{16}^0 \mathrm{e}^{(\mathrm{S}_1)^{(2)}\mathrm{t}}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |

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| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)}-(p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)}-(p_{16})^{(2)}-(S_2)^{(2)})}\left[e^{((S_1)^{(2)}-(p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le \right.$ $\left.\frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)}-(a_{18}')^{(2)})}\left[e^{(S_1)^{(2)}t} - e^{-(a_{18}')^{(2)}t}\right] + G_{18}^0 e^{-(a_{18}')^{(2)}t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)}t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)}t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)}-(b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t} - e^{-(b_{18}')^{(2)}t}\right] + T_{18}^0 e^{-(b_{18}')^{(2)}t} \le T_{18}(t) \le$ $\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t} - e^{-(R_2)^{(2)}t}\right] + T_{18}^0 e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a_{16}')^{(2)}$ $(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b_{16}')^{(2)}$ $(R_2)^{(2)} = (b_{18}')^{(2)} - (r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$: $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)} \le -(a_{20}')^{(3)} + (a_{21}')^{(3)} - (a_{20}'')^{(3)}(T_{21}, t) + (a_{21}'')^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)} \le -(b_{20}')^{(3)} + (b_{21}')^{(3)} - (b_{20}'')^{(3)}(G_{23}, t) - (b_{21}'')^{(3)}((G_{23}), t) \le -(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)}, (v_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$: By $(v_1)^{(3)} > 0$, $(v_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$, $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$ and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and By $(\bar{v}_1)^{(3)} > 0$, $(\bar{v}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$, $(\bar{u}_2)^{(3)} < 0$ the | 320 |

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| roots of the equations $(a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$<br>and $(b_{21})^{(3)}\left(u^{(3)}\right)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ | |
| **Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :-<br>If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by<br>$(m_2)^{(3)} = (v_0)^{(3)}, (m_1)^{(3)} = (v_1)^{(3)}$, ***if*** $(v_0)^{(3)} < (v_1)^{(3)}$<br>$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (\bar{v}_1)^{(3)}$, ***if*** $(v_1)^{(3)} < (v_0)^{(3)} < (\bar{v}_1)^{(3)}$,<br>and $\boxed{(v_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$<br>$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (v_0)^{(3)}$, ***if*** $(\bar{v}_1)^{(3)} < (v_0)^{(3)}$ | 321 |
| and analogously<br>$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$<br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$<br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$<br>Then the solution of global equations satisfies the inequalities<br>$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)}t}$<br>$(p_i)^{(3)}$ is defined by equation | 322 |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \le G_{22}(t) \le \right.$<br>$\left.\frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a'_{22})^{(3)})}\left[e^{(S_1)^{(3)}t} - e^{-(a'_{22})^{(3)}t}\right] + G_{22}^0 e^{-(a'_{22})^{(3)}t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b'_{22})^{(3)})}\left[e^{(R_1)^{(3)}t} - e^{-(b'_{22})^{(3)}t}\right] + T_{22}^0 e^{-(b'_{22})^{(3)}t} \le T_{22}(t) \le$<br>$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$ | 327 |

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- 328

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$

$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$

$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$

**Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : 329

By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$
and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{v}_1)^{(4)} > 0$, $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$
and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (v_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}$, ***if*** $(v_0)^{(4)} < (v_1)^{(4)}$

$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}$, ***if*** $(v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)}$,
and $\boxed{(v_0)^{(4)} = \frac{G^0_{24}}{G^0_{25}}}$

$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)} < (v_0)^{(4)}$

and analogously 331

$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$

$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$, ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,

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| and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \boldsymbol{if}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)} t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)} t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\right)\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)} t}\right] + G_{26}^0 e^{-(S_2)^{(4)} t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b_{26}')^{(4)})}\left[e^{(R_1)^{(4)} t} - e^{-(b_{26}')^{(4)} t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)} t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)} + (r_{24})^{(4)} + (R_2)^{(4)})}\left[e^{((R_1)^{(4)} + (r_{24})^{(4)})t} - e^{-(R_2)^{(4)} t}\right] + T_{26}^0 e^{-(R_2)^{(4)} t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |

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| **Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ :<br><br>By $(v_1)^{(5)} > 0$ , $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$ , $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | 339 |
| **Definition of** $(\bar{v}_1)^{(5)},, (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ :<br><br>By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :-<br><br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}, (\mu_2)^{(5)}$ by<br><br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br><br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br><br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |

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| $\frac{1}{(\mu_1)^{(5)}}T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}}T_{28}^0 e^{((R_1)^{(5)}+(r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)}T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)}-(b_{30}')^{(5)})}\left[e^{(R_1)^{(5)}t} - e^{-(b_{30}')^{(5)}t}\right] + T_{30}^0 e^{-(b_{30}')^{(5)}t} \le T_{30}(t) \le$<br>$\frac{(a_{30})^{(5)}T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t} - e^{-(R_2)^{(5)}t}\right] + T_{30}^0 e^{-(R_2)^{(5)}t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a_{28}')^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b_{28}')^{(5)}$<br>$(R_2)^{(5)} = (b_{30}')^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :<br>$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying<br>$-(\sigma_2)^{(6)} \le -(a_{32}')^{(6)} + (a_{33}')^{(6)} - (a_{32}'')^{(6)}(T_{33}, t) + (a_{33}'')^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$<br>$-(\tau_2)^{(6)} \le -(b_{32}')^{(6)} + (b_{33}')^{(6)} - (b_{32}'')^{(6)}((G_{35}), t) - (b_{33}'')^{(6)}((G_{35}), t) \le -(\tau_1)^{(6)}$ | 349 |
| **Definition of** $(v_1)^{(6)}, (v_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, v^{(6)}, u^{(6)}$ :<br>By $(v_1)^{(6)} > 0$, $(v_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$, $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(v^{(6)})^2 + (\sigma_1)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{v}_1)^{(6)},, (\bar{v}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br>By $(\bar{v}_1)^{(6)} > 0$, $(\bar{v}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$, $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(v^{(6)})^2 + (\sigma_2)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (v_0)^{(6)}$ :-<br>If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br>$(m_2)^{(6)} = (v_0)^{(6)}, (m_1)^{(6)} = (v_1)^{(6)}$, ***if*** $(v_0)^{(6)} < (v_1)^{(6)}$<br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (\bar{v}_6)^{(6)}$, ***if*** $(v_1)^{(6)} < (v_0)^{(6)} < (\bar{v}_1)^{(6)}$, | 351 |

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| and $\boxed{(v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$<br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (v_0)^{(6)},\ \boldsymbol{if}\ (\bar{v}_1)^{(6)} < (v_0)^{(6)}$ | |
| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},\ \boldsymbol{if}\ (u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}, \boldsymbol{if}\ (u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)},$<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}, \boldsymbol{if}\ (\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\right)\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b'_{34})^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b'_{34})^{(6)} t}\right] + T_{34}^0 e^{-(b'_{34})^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a'_{32})^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b'_{32})^{(6)}$<br>$(R_2)^{(6)} = (b'_{34})^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** | |

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ :

$(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying

$$-(\sigma_2)^{(7)} \le -(a_{36}^{'})^{(7)} + (a_{37}^{'})^{(7)} - (a_{36}^{''})^{(7)}(T_{37}, t) + (a_{37}^{''})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$$

$$-(\tau_2)^{(7)} \le -(b_{36}^{'})^{(7)} + (b_{37}^{'})^{(7)} - (b_{36}^{''})^{(7)}\big((G_{39}), t\big) - (b_{37}^{''})^{(7)}\big((G_{39}), t\big) \le -(\tau_1)^{(7)}$$

**Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ : 361

By $(\nu_1)^{(7)} > 0, (\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ : 362

By $(\bar{\nu}_1)^{(7)} > 0, (\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$

**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,
and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$

and analogously 363

$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,
and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

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| Then the solution of global equations satisfies the inequalities<br>$G_{36}^0 e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)}t}$<br>where $(p_i)^{(7)}$ is defined by equation | 364 |
| $\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^0 e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}')^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}')^{(7)}t}] + G_{38}^0 e^{-(a_{38}')^{(7)}t})$ | 366 |
| $\boxed{T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}')^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}')^{(7)}t}\right] + T_{38}^0 e^{-(b_{38}')^{(7)}t} \le T_{38}(t) \le$<br>$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^0 e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}')^{(7)}$<br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}')^{(7)}$<br>$(R_2)^{(7)} = (b_{38}')^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br>$-(\sigma_2)^{(8)} \le -(a_{40}')^{(8)} + (a_{41}')^{(8)} - (a_{40}'')^{(8)}(T_{41}, t) + (a_{41}'')^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br>$-(\tau_2)^{(8)} \le -(b_{40}')^{(8)} + (b_{41}')^{(8)} - (b_{40}'')^{(8)}((G_{43}), t) - (b_{41}'')^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |

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| **Definition of** $(\nu_1)^{(8)}, (\nu_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, \nu^{(8)}, u^{(8)}$ : <br><br> By $(\nu_1)^{(8)} > 0$ , $(\nu_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0$ , $(u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$ <br> and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | 372 |
| **Definition of** $(\bar{\nu}_1)^{(8)},, (\bar{\nu}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ : <br><br> By $(\bar{\nu}_1)^{(8)} > 0$ , $(\bar{\nu}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0$ , $(\bar{u}_2)^{(8)} < 0$ the <br><br> roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$ <br><br> and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ <br><br> **Definition of** $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)}, (\mu_2)^{(8)}, (\nu_0)^{(8)}$ :- <br><br> If we define $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)}, (\mu_2)^{(8)}$ by <br><br> $(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)}$, ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$ <br><br> $(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)}$, <br> and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ <br><br> $(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)}$, ***if*** $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously <br><br> $(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)}$, ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$ <br><br> $(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)}$, <br> and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$ <br><br> $(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}$, ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities <br><br> $G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{40}(t) \leq G_{40}^0 e^{(S_1)^{(8)}t}$ <br><br> where $(p_i)^{(8)}$ is defined by equation | 375 |

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| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)}-(p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)}G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t} - e^{-(S_2)^{(8)}t}\right] + G_{42}^0 e^{-(S_2)^{(8)}t} \le G_{42}(t) \le \right.$ $\left.\frac{(a_{42})^{(8)}G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)}-(a'_{42})^{(8)})}[e^{(S_1)^{(8)}t} - e^{-(a'_{42})^{(8)}t}] + G_{42}^0 e^{-(a'_{42})^{(8)}t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)}-(b'_{42})^{(8)})}\left[e^{(R_1)^{(8)}t} - e^{-(b'_{42})^{(8)}t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)}t} \le T_{42}(t) \le$ $\frac{(a_{42})^{(8)}T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t} - e^{-(R_2)^{(8)}t}\right] + T_{42}^0 e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$<br>$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}((G_{47}), t) - (b''_{45})^{(9)}((G_{47}), t) \le -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :<br>By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$<br>and $(b_{45})^{(9)}(u^{(9)})^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(9)},, (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ : | |

By $(\bar{v}_1)^{(9)} > 0$ , $(\bar{v}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\left(u^{(9)}\right)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}$ , $(m_2)^{(9)}$ , $(\mu_1)^{(9)}$, $(\mu_2)^{(9)}$, $(v_0)^{(9)}$ :-

If we define $(m_1)^{(9)}$ , $(m_2)^{(9)}$ , $(\mu_1)^{(9)}$, $(\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (v_0)^{(9)}, (m_1)^{(9)} = (v_1)^{(9)}$, ***if*** $(v_0)^{(9)} < (v_1)^{(9)}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (\bar{v}_1)^{(9)}$ , ***if*** $(v_1)^{(9)} < (v_0)^{(9)} < (\bar{v}_1)^{(9)}$,

and $\boxed{(v_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (v_0)^{(9)}$, ***if*** $(\bar{v}_1)^{(9)} < (v_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^0 e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}} G_{44}^0 e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)} G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)} - (p_{44})^{(9)} - (S_2)^{(9)})}\left[e^{((S_1)^{(9)} - (p_{44})^{(9)})t} - e^{-(S_2)^{(9)}t}\right] + G_{46}^0 e^{-(S_2)^{(9)}t} \le G_{46}(t) \le \right.$$

$$\left. \frac{(a_{46})^{(9)} G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)} - (a'_{46})^{(9)})}\left[e^{(S_1)^{(9)}t} - e^{-(a'_{46})^{(9)}t}\right] + G_{46}^0 e^{-(a'_{46})^{(9)}t}\right)$$

$$\boxed{T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}} T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}} T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)} T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)} - (b'_{46})^{(9)})}\left[e^{(R_1)^{(9)}t} - e^{-(b'_{46})^{(9)}t}\right] + T_{46}^0 e^{-(b'_{46})^{(9)}t} \le T_{46}(t) \le$$

$$\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}\left((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)}\right)}\left[e^{\left((R_1)^{(9)}+(r_{44})^{(9)}\right)t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain (383)

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14},t)\right) - (a''_{14})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(1)}, (\nu_0)^{(1)}$ :-

For $0 < \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (\nu_1)^{(1)} < (\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t) \ge \frac{(\nu_1)^{(1)}+(C)^{(1)}(\nu_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_0)^{(1)}\right)t\right]}}{1+(C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(\nu_1)^{(1)}-(\nu_0)^{(1)}}{(\nu_0)^{(1)}-(\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)} \le \nu^{(1)}(t) \le (\nu_1)^{(1)}$

In the same manner , we get (384)

$$\nu^{(1)}(t) \le \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{\nu}_1)^{(1)}-(\nu_0)^{(1)}}{(\nu_0)^{(1)}-(\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)} \le \nu^{(1)}(t) \le (\bar{\nu}_1)^{(1)}$

If $0 < (\nu_1)^{(1)} < (\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{\nu}_1)^{(1)}$ we find like in the previous case, (385)

$$(\nu_1)^{(1)} \le \frac{(\nu_1)^{(1)}+(C)^{(1)}(\nu_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_2)^{(1)}\right)t\right]}}{1+(C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_2)^{(1)}\right)t\right]}} \le \nu^{(1)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \le (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \le (\bar{\nu}_1)^{(1)} \le \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \le \nu^{(1)}(t) \le \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \le (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \le \nu^{(1)}(t) \le (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \le u^{(1)}(t) \le (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)}G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)}T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}')^{(2)} - (a_{17}')^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

| | |
|---|---|
| $-\left((a_{17})^{(2)}\left(v^{(2)}\right)^2+(\sigma_2)^{(2)}v^{(2)}-(a_{16})^{(2)}\right)\le\frac{\mathrm{d}v^{(2)}}{\mathrm{dt}}\le-\left((a_{17})^{(2)}\left(v^{(2)}\right)^2+(\sigma_1)^{(2)}v^{(2)}-(a_{16})^{(2)}\right)$ | |
| From which one obtains<br><br>**Definition of** $(\bar{v}_1)^{(2)},(v_0)^{(2)}$ :-<br><br>For $0<(v_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}<(v_1)^{(2)}<(\bar{v}_1)^{(2)}$<br><br>$v^{(2)}(t)\ge\frac{(v_1)^{(2)}+(C)^{(2)}(v_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_0)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_0)^{(2)}\right)t\right]}}$ , $\boxed{(C)^{(2)}=\frac{(v_1)^{(2)}-(v_0)^{(2)}}{(v_0)^{(2)}-(v_2)^{(2)}}}$<br><br>it follows $(v_0)^{(2)}\le v^{(2)}(t)\le(v_1)^{(2)}$ | 390 |
| In the same manner , we get<br><br>$v^{(2)}(t)\le\frac{(\bar{v}_1)^{(2)}+(\bar{C})^{(2)}(\bar{v}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(2)}=\frac{(\bar{v}_1)^{(2)}-(v_0)^{(2)}}{(v_0)^{(2)}-(\bar{v}_2)^{(2)}}}$ | 391 |
| From which we deduce $(v_0)^{(2)}\le v^{(2)}(t)\le(\bar{v}_1)^{(2)}$ | 392 |
| If $0<(v_1)^{(2)}<(v_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}<(\bar{v}_1)^{(2)}$ we find like in the previous case,<br><br>$(v_1)^{(2)}\le\frac{(v_1)^{(2)}+(C)^{(2)}(v_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_2)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_2)^{(2)}\right)t\right]}}\le v^{(2)}(t)\le$<br><br>$\frac{(\bar{v}_1)^{(2)}+(\bar{C})^{(2)}(\bar{v}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}\le(\bar{v}_1)^{(2)}$ | 393 |
| If $0<(v_1)^{(2)}\le(\bar{v}_1)^{(2)}\le(v_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(v_1)^{(2)}\le v^{(2)}(t)\le\frac{(\bar{v}_1)^{(2)}+(\bar{C})^{(2)}(\bar{v}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}\le(v_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $v^{(2)}(t)$ :-<br><br>$(m_2)^{(2)}\le v^{(2)}(t)\le(m_1)^{(2)}$, $\boxed{v^{(2)}(t)=\frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :- | 396 |

| | |
|---|---|
| $(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| From which one obtains<br><br>For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$<br><br>$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)} + (C)^{(3)}(\nu_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}$ , $\boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\nu_2)^{(3)}}}$<br><br>it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$ | 400 |
| In the same manner , we get<br><br>$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)} + (C)^{(3)}(\bar{\nu}_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\bar{\nu}_2)^{(3)}}}$<br><br>**Definition of** $(\bar{\nu}_1)^{(3)}$ :-<br><br>From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$ | 401 |

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If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case,

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

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If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}'')^{(3)} = (a_{21}'')^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b_{20}'')^{(3)} = (b_{21}'')^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}')^{(4)} - (a_{25}')^{(4)} + (a_{24}'')^{(4)}(T_{25},t)\right) - (a_{25}'')^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_2)^{(4)}v^{(4)}-(a_{24})^{(4)}\right)\le\frac{dv^{(4)}}{dt}\le-\left((a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_4)^{(4)}v^{(4)}-(a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(4)},(v_0)^{(4)}$ :-

For $0<\boxed{(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}<(v_1)^{(4)}<(\bar{v}_1)^{(4)}$

$$v^{(4)}(t)\ge\frac{(v_1)^{(4)}+(C)^{(4)}(v_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_0)^{(4)}\right)t\right]}}{4+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_0)^{(4)}\right)t\right]}}\quad,\quad\boxed{(C)^{(4)}=\frac{(v_1)^{(4)}-(v_0)^{(4)}}{(v_0)^{(4)}-(v_2)^{(4)}}}$$

it follows $(v_0)^{(4)}\le v^{(4)}(t)\le(v_1)^{(4)}$

In the same manner , we get

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$$v^{(4)}(t)\le\frac{(\bar{v}_1)^{(4)}+(\bar{C})^{(4)}(\bar{v}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}{4+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(4)}=\frac{(\bar{v}_1)^{(4)}-(v_0)^{(4)}}{(v_0)^{(4)}-(\bar{v}_2)^{(4)}}}$$

From which we deduce $(v_0)^{(4)}\le v^{(4)}(t)\le(\bar{v}_1)^{(4)}$

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If $0<(v_1)^{(4)}<(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}<(\bar{v}_1)^{(4)}$ we find like in the previous case,

$$(v_1)^{(4)}\le\frac{(v_1)^{(4)}+(C)^{(4)}(v_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_2)^{(4)}\right)t\right]}}{1+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_2)^{(4)}\right)t\right]}}\le v^{(4)}(t)\le$$

$$\frac{(\bar{v}_1)^{(4)}+(\bar{C})^{(4)}(\bar{v}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}\le(\bar{v}_1)^{(4)}$$

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If $0<(v_1)^{(4)}\le(\bar{v}_1)^{(4)}\le\boxed{(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(v_1)^{(4)}\le v^{(4)}(t)\le\frac{(\bar{v}_1)^{(4)}+(\bar{C})^{(4)}(\bar{v}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}\le(v_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(4)}(t)$ :-

$$(m_2)^{(4)}\le v^{(4)}(t)\le(m_1)^{(4)},\quad\boxed{v^{(4)}(t)=\frac{G_{24}(t)}{G_{25}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$$(\mu_2)^{(4)}\le u^{(4)}(t)\le(\mu_1)^{(4)},\quad\boxed{u^{(4)}(t)=\frac{T_{24}(t)}{T_{25}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}'')^{(5)} = (b_{29}'')^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}')^{(6)} - (a_{33}')^{(6)} + (a_{32}'')^{(6)}(T_{33}, t)\right) - (a_{33}'')^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

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In the same manner , we get

$$\nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\bar{\nu}_1)^{(6)}$

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If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

$$(\nu_1)^{(6)} \leq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_2)^{(6)}\right)t\right]}} \leq \nu^{(6)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\bar{\nu}_1)^{(6)}$$

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If $0 < (\nu_1)^{(6)} \leq (\bar{\nu}_1)^{(6)} \leq \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(\nu_1)^{(6)} \leq \nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

$$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}, \quad \boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}, \quad \boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)}G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a'_{36})^{(7)} - (a'_{37})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right) - (a''_{37})^{(7)}(T_{37},t)v^{(7)} - (a_{37})^{(7)}v^{(7)}$$

**Definition of** $v^{(7)}$ :- $\boxed{v^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)}\right) \le \frac{dv^{(7)}}{dt} \le -\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(7)}, (v_0)^{(7)}$ :-

For $0 < \boxed{(v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}} < (v_1)^{(7)} < (\bar{v}_1)^{(7)}$

$$v^{(7)}(t) \ge \frac{(v_1)^{(7)} + (C)^{(7)}(v_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(v_1)^{(7)} - (v_0)^{(7)}}{(v_0)^{(7)} - (v_2)^{(7)}}}$$

it follows $(v_0)^{(7)} \le v^{(7)}(t) \le (v_1)^{(7)}$

In the same manner , we get 417

$$v^{(7)}(t) \le \frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{v}_1)^{(7)} - (v_0)^{(7)}}{(v_0)^{(7)} - (\bar{v}_2)^{(7)}}}$$

From which we deduce $(v_0)^{(7)} \le v^{(7)}(t) \le (\bar{v}_1)^{(7)}$

If $0 < (v_1)^{(7)} < (v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}} < (\bar{v}_1)^{(7)}$ we find like in the previous case, 418

$$(v_1)^{(7)} \le \frac{(v_1)^{(7)} + (C)^{(7)}(v_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_2)^{(7)}\right)t\right]}} \le v^{(7)}(t) \le$$

$$\frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \le (\bar{v}_1)^{(7)}$$

If $0 < (v_1)^{(7)} \le (\bar{v}_1)^{(7)} \le \boxed{(v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$ , we obtain 419

$$(v_1)^{(7)} \le v^{(7)}(t) \le \frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \le (v_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(7)}(t)$ :-

$(m_2)^{(7)} \leq v^{(7)}(t) \leq (m_1)^{(7)}$, $\boxed{v^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :-

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$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $v^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)} G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

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$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right) - (a_{41}^{''})^{(8)}(T_{41}, t) v^{(8)} - (a_{41})^{(8)} v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)} v^{(8)} - (a_{40})^{(8)}\right) \leq \frac{dv^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)} v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$v^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)}-(\nu_0)^{(8)}}{(\nu_0)^{(8)}-(\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\bar{\nu}_8)^{(8)}$

If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case, 423

$$(\nu_1)^{(8)} \leq \frac{(\nu_1)^{(8)}+(C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}} \leq \nu^{(8)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\bar{\nu}_1)^{(8)}$$

If $0 < (\nu_1)^{(8)} \leq (\bar{\nu}_1)^{(8)} \leq \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain 424

$$(\nu_1)^{(8)} \leq \nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)}+(C)^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \leq \nu^{(8)}(t) \leq (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)}G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)}T_{41}(t)$ This is an important

consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{d\nu^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right) - (a''_{45})^{(9)}(T_{45},t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \leq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}} \leq \nu^{(9)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \leq (\bar{\nu}_1)^{(9)} \leq \boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$ , we obtain

$$(\nu_1)^{(9)} \leq \nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$(m_2)^{(9)} \leq \nu^{(9)}(t) \leq (m_1)^{(9)}$, $\boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}$, $\boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ 430

$with\ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations 431

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$

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| $(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$ | 435 |

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| $(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |

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| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |

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| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |

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| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(g) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) +$ | 488 |

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| $(a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first | 494 |

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| equations | |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |

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| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{40}=\frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}=\frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 501 |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{44}=\frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}=\frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G)=(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)}-(b_{13})^{(1)}(b_{14})^{(1)}-$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G)+(b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G)=0$<br><br>Where in $G(G_{13},G_{14},G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*})=0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19})=(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)}-(b_{16})^{(2)}(b_{17})^{(2)}-$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19})+(b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19})=0$ | 503 |
| Where in $(G_{19})(G_{16},G_{17},G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*})=0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23})=(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)}-(b_{20})^{(3)}(b_{21})^{(3)}-$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23})+(b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23})=0$<br><br>Where in $G_{23}(G_{20},G_{21},G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*})=0$ | 505 |

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| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right] + (b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br><br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br><br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br><br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br><br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$ | |

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| $\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right]+(b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0$ , $T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0$ , $T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*}) = 0$ , $T_{21}^{*}$ given by $f(T_{21}^{*}) = 0$ and<br><br>$G_{20}^{*} = \frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*} = \frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br><br>$T_{20}^{*} = \frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*} = \frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br><br>$G_{25}^{*}$ given by $\varphi(G_{27}) = 0$ , $T_{25}^{*}$ given by $f(T_{25}^{*}) = 0$ and<br><br>$G_{24}^{*} = \frac{(a_{24})^{(4)}G_{25}^{*}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*} = \frac{(a_{26})^{(4)}G_{25}^{*}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*} = \frac{(b_{24})^{(4)}T_{25}^{*}}{[(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*} = \frac{(b_{26})^{(4)}T_{25}^{*}}{[(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}((G_{27})^{*})]}$ | 517 |

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| Obviously, these values represent an equilibrium solution of global equations | |
| Finally we obtain the unique solution<br><br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*}) = 0$ , $T_{29}^{*}$ given by $f(T_{29}^{*}) = 0$ and<br><br>$G_{28}^{*} = \frac{(a_{28})^{(5)} G_{29}^{*}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*} = \frac{(a_{30})^{(5)} G_{29}^{*}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |
| $T_{28}^{*} = \frac{(b_{28})^{(5)} T_{29}^{*}}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)} T_{29}^{*}}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br><br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br><br>$G_{32}^{*} = \frac{(a_{32})^{(6)} G_{33}^{*}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)} G_{33}^{*}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)} T_{33}^{*}}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)} T_{33}^{*}}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br><br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br><br>$G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99 | 523 A |

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| $G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof:** Denote<br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*}\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*}\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*}\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*}\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$ | 532 |

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| $\frac{\partial(a_{17}^{''})^{(2)}}{\partial \mathrm{T}_{17}}(\mathrm{T}_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial(b_i^{''})^{(2)}}{\partial \mathrm{G}_j}((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{\mathrm{d}\mathbb{G}_{16}}{\mathrm{dt}} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}\mathrm{G}_{16}^{*}\mathbb{T}_{17}$ | 534 |
| $\frac{\mathrm{d}\mathbb{G}_{17}}{\mathrm{dt}} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}\mathrm{G}_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{\mathrm{d}\mathbb{G}_{18}}{\mathrm{dt}} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}\mathrm{G}_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{\mathrm{d}\mathbb{T}_{16}}{\mathrm{dt}} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}\mathrm{T}_{16}^{*}\mathbb{G}_j\right)$ | 537 |
| $\frac{\mathrm{d}\mathbb{T}_{17}}{\mathrm{dt}} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}\mathrm{T}_{17}^{*}\mathbb{G}_j\right)$ | 538 |
| $\frac{\mathrm{d}\mathbb{T}_{18}}{\mathrm{dt}} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}\mathrm{T}_{18}^{*}\mathbb{G}_j\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)}$ , $\frac{\partial(b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_j\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 547 |

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| **Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)}$ , $\frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^*) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ $and$ $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^*) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |

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| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^{*}\mathbb{G}_{j}\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^{*}\mathbb{G}_{j}\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^{*}\mathbb{G}_{j}\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^{*}) = (q_{33})^{(6)}$ , $\frac{\partial(b_i^{''})^{(6)}}{\partial G_j}((G_{35})^{*}) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^{*}\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^{*}\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^{*}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^{*}\mathbb{G}_{j}\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^{*}\mathbb{G}_{j}\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^{*}\mathbb{G}_{j}\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 572 |

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| $\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^{*}) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^{*}\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^{*}\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^{*}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^{*}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^{*}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^{*}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^{*}) = (q_{41})^{(8)}$ , $\frac{\partial (b_i^{''})^{(8)}}{\partial G_j}\left((G_{43})^{*}\right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^{*}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^{*}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^{*}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^{*}\mathbb{G}_j\right)$ | 584 |

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| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{45}^{''})^{(9)}}{\partial T_{45}}(T^*_{45}) = (q_{45})^{(9)} \;,\; \frac{\partial(b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a'_{44})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G^*_{44}\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a'_{45})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G^*_{45}\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a'_{46})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G^*_{46}\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$<br><br>$\left[\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\Big]$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\Big)$<br><br>$+\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\Big)$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\Big)$<br><br>$\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$ | |

$$\left((\lambda)^{(1)}\right)^2+\left((b'_{13})^{(1)}+(b'_{14})^{(1)}-(r_{13})^{(1)}+(r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$$

$$+\left(\left((\lambda)^{(1)}\right)^2+\left((a'_{13})^{(1)}+(a'_{14})^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$$

$$+\left((\lambda)^{(1)}+(a'_{13})^{(1)}+(p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G^*_{14}+(a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G^*_{13}\right)$$

$$\left((\lambda)^{(1)}+(b'_{13})^{(1)}-(r_{13})^{(1)}\right)s_{(14),(15)}T^*_{14}+(b_{14})^{(1)}s_{(13),(15)}T^*_{13}\Big)\}=0$$

$$+$$

$$(\lambda)^{(2)}+(b'_{18})^{(2)}-(r_{18})^{(2)}\}\{\left((\lambda)^{(2)}+(a'_{18})^{(2)}+(p_{18})^{(2)}\right)$$

$$\left[\left((\lambda)^{(2)}+(a'_{16})^{(2)}+(p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}^*_{17}+(a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^*_{16}\right)\Big]$$

$$\left((\lambda)^{(2)}+(b'_{16})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}^*_{17}+(b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^*_{17}\Big)$$

$$+\left(\left((\lambda)^{(2)}+(a'_{17})^{(2)}+(p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}^*_{16}+(a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^*_{17}\right)$$

$$\left((\lambda)^{(2)}+(b'_{16})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}^*_{17}+(b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^*_{16}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((a'_{16})^{(2)}+(a'_{17})^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((b'_{16})^{(2)}+(b'_{17})^{(2)}-(r_{16})^{(2)}+(r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left(\left((\lambda)^{(2)}\right)^2+\left((a'_{16})^{(2)}+(a'_{17})^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a'_{16})^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}^*_{17}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}^*_{16}\right)$$

$$\left((\lambda)^{(2)}+(b'_{16})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}^*_{17}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}^*_{16}\Big)\}=0$$

$$+$$

$$(\lambda)^{(3)}+(b'_{22})^{(3)}-(r_{22})^{(3)}\}\{\left((\lambda)^{(3)}+(a'_{22})^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left((\lambda)^{(3)}+(a'_{20})^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G^*_{21}+(a_{21})^{(3)}(q_{20})^{(3)}G^*_{20}\right)\Big]$$

$$\left((\lambda)^{(3)}+(b'_{20})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T^*_{21}+(b_{21})^{(3)}s_{(20),(21)}T^*_{21}\Big)$$

$$+\left(\left((\lambda)^{(3)}+(a'_{21})^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G^*_{20}+(a_{20})^{(3)}(q_{21})^{(1)}G^*_{21}\right)$$

$$\left((\lambda)^{(3)}+(b'_{20})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T^*_{21}+(b_{21})^{(3)}s_{(20),(20)}T^*_{20}\Big)$$

$$\left((\lambda)^{(3)}\right)^2+\left((a'_{20})^{(3)}+(a'_{21})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((b_{20}')^{(3)}+(b_{21}')^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(4)}+(b_{26}')^{(4)}-(r_{26})^{(4)}\{\left((\lambda)^{(4)}+(a_{26}')^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}')^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}')^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}')^{(4)}+(a_{25}')^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((b_{24}')^{(4)}+(b_{25}')^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}')^{(4)}+(a_{25}')^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}')^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}')^{(5)}-(r_{30})^{(5)}\{\left((\lambda)^{(5)}+(a_{30}')^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}')^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}')^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}')^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}')^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}')^{(5)}+(a_{29}')^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left((\lambda)^{(5)}\right)^2+\left((b_{28}')^{(5)}+(b_{29}')^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$+\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}')^{(5)}+(a_{29}')^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}')^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}')^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}')^{(6)}-(r_{34})^{(6)}\{\left((\lambda)^{(6)}+(a_{34}')^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left((\lambda)^{(6)}+(a_{32}')^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\Big]$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\Big)$$

$$+\left((\lambda)^{(6)}+(a_{33}')^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\Big)$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((a_{32}')^{(6)}+(a_{33}')^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((b_{32}')^{(6)}+(b_{33}')^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$+\left(\left((\lambda)^{(6)}\right)^2+\left((a_{32}')^{(6)}+(a_{33}')^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}')^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b_{38}')^{(7)}-(r_{38})^{(7)}\{\left((\lambda)^{(7)}+(a_{38}')^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left((\lambda)^{(7)}+(a_{36}')^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\Big]$$

$$\left((\lambda)^{(7)}+(b_{36}')^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\Big)$$

$$+\left((\lambda)^{(7)}+(a_{37}')^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\Big)$$

$$\left((\lambda)^{(7)}+(b_{36}')^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a_{36}')^{(7)}+(a_{37}')^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b_{36}')^{(7)}+(b_{37}')^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left(\left((\lambda)^{(7)}\right)^2+\left((a_{36}')^{(7)}+(a_{37}')^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}')^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left((\lambda)^{(7)}+(b_{36}')^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b_{42}')^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a_{42}')^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a_{40}')^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(q_{40})^{(8)}G_{40}^{*}\right)\right]$$

$$\left((\lambda)^{(8)}+(b_{40}')^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(41)}T_{41}^{*}\Big)$$

$$+\left(\left((\lambda)^{(8)}+(a_{41}')^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G_{40}^{*}+(a_{40})^{(8)}(q_{41})^{(8)}G_{41}^{*}\right)$$

$$\left((\lambda)^{(8)}+(b_{40}')^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(40)}T_{40}^{*}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((a_{40}')^{(8)}+(a_{41}')^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b_{40}')^{(8)}+(b_{41}')^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left(\left((\lambda)^{(8)}\right)^2+\left((a_{40}')^{(8)}+(a_{41}')^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a_{40}')^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G_{40}^{*}\right)$$

$$\left((\lambda)^{(8)}+(b_{40}')^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(42)}T_{40}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b_{46}')^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a_{46}')^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a_{44}')^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G_{45}^{*}+(a_{45})^{(9)}(q_{44})^{(9)}G_{44}^{*}\right)\right]$$

$$\left((\lambda)^{(9)}+(b_{44}')^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T_{45}^{*}+(b_{45})^{(9)}s_{(44),(45)}T_{45}^{*}\Big)$$

$$+\left(\left((\lambda)^{(9)}+(a_{45}')^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G_{44}^{*}+(a_{44})^{(9)}(q_{45})^{(9)}G_{45}^{*}\right)$$

$$\left((\lambda)^{(9)}+(b_{44}')^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T_{45}^{*}+(b_{45})^{(9)}s_{(44),(44)}T_{44}^{*}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45}+(b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt}= A_i\ (x)\ xi$ where $x_i\ \epsilon\ R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ \ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i\ \ X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions

are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$(7) \qquad \frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17].

In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 7. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i(x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \epsilon C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population $i$. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility

assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C\backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^n_i$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\in \Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two

propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population $i$ persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population $i$ is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) In \ |B_i \ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^t_0 (A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = limsup_{t\to\infty} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni}= E_{i(x)} \ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \ \leq \ a| \ B_i(t,x)u_i(x)|+ \ |B_i(t,x)w| \ \ \leq \ | \ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)| \ w|)$. Hence, $limsup_{t\to\infty} \frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$limsup_{t\to\infty} \frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \ \geq \ a|B_i(t,x)u_i(x)|-|B_i(t,x)w|$$

$$\geq | \ B_i(t,x)u_i(x)|(a- \alpha+ exp\ (-\beta t)| \ w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \ \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \ \geq \ \lambda i(x) \geq \ \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) \ = \ In|B_i(t+s,x)\ u_i(x)|= In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

| |
|---|
| $= \text{In } \lvert B_i(s,x.t)\, u_i(x.t) \rvert \lvert B_i(t,x)\, u_i(x) \rvert$ |
| $= \text{In } b_i(s,t.x)\, b_i(t,x)$ |
| |
| Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies |
| $\frac{d}{dt} b_i(t,x) = \lim_{s \to 0} \frac{bi(t+s,x-bi(t,x)}{S-}$ |
| $= \lim_{s \to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}\lvert_{s=0}\, b_i(s,x.t)$ |
| = |
| $=(A_i(x.t)u_i(x.s),u_i(x.s),$ |
| The Fundamental Theorem of Calculus implies |
| $B_i(t,x) = \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$ |
| and the second assertion follows. |
| **Proposition 2:** *For the solutions of (2) we have:* if $x_i > 0$, then $\lambda i(x) \leq 0$. If $\lambda i(x) < 0$, then $\lim_{t \to \infty} (x.t)_i = 0$ If $\limsup_{t \to \infty}$ $(x.t)_i \rvert > 0$, then $\lambda i(x) = 0$. |
| **Proof:** First, assume that x satisfies $x_i > 0$. Since the semi flow is dissipative, there exists $\gamma > 0$ such that $\lvert x.t \rvert \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply |
| $\lambda i(x) = \limsup_{t \to \infty} \frac{1}{t} \text{In } \lvert B_i(t,x)x_i \rvert = \limsup_{t \to \infty} \frac{1}{t} \text{In } \lvert (x.t)_i \rvert$ |
| $\leq \limsup_{t \to \infty} \frac{In\, \gamma}{t} = 0$ |
| To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i = 0$, then the invariance of the faces of C imply that $(x.t)_i = 0$ for all $t \geq 0$. Alternatively, if $x_i > 0$, then Proposition I and the skew product flow imply |
| $\limsup_{t \to \infty} \frac{1}{t} In \lvert (x.t)_i \rvert = \lambda i(x) < 0.$ |
| The final assertion follows from the first two assertions. 5 |
| |

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on $\boldsymbol{C}$, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:*$\lambda_i(\mathrm{x}) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle \mathrm{A}_i(x.s)\mathrm{u}_i(x.s), \mathrm{u}_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(\mathrm{x}) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $\mathrm{h}_i(x) = \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I}\boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I}\boldsymbol{C}_i^+$such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following

equivalent conditions hold For all invariant measures $\mu$ supported by $K$ (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by $K$.

then $K$ is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and $K$ is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by $K$.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$. Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large

so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \leq \epsilon$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \leq \epsilon$, and $|h(x.t) - h(x^n.t)| \leq \epsilon$ for all $x \in N_1(\Gamma(A))$. Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$$\leq |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$$

$$\leq \epsilon + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$$

$$\leq 2\epsilon + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$$

$$\leq 3\epsilon$$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\epsilon > 0$ and $t > 0$ are arbitrary, we have $\int_c h(x.t) d\mu(x) = \int_c h(x) d\mu(x)$ for all $t > 0$ and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x) u_i^n(x), u_i^n(x) \rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop. 3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop. 3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\epsilon > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \epsilon$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)| \leq \epsilon$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)| d\mu(x) + |\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\, d\mu_n(x)$$

$$\leq 2\epsilon$$

Since $\epsilon > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm. 2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}$

$\int_0^t \delta_{y.s}$ ds where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$ . Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}_{n=1}^{\infty}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

| |
|---|
| Min max $\lambda i$ ($\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_{i} Pi\lambda i(\mu)$ |
| |
| Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that |
| (6) min max $\lambda i(\mu) = \min_{\mu} \max_{p\epsilon\Delta} \sum_{i} Pi\lambda i(\mu)$ |
| |
| Where the minimum is taken over invariant probability measures $\mu$ with support in *K.* Since min$\mu$ |
| $\sum_{i} P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established. |
| |
| To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta -$perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ [0,1]$be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by |
| |
| $\tilde{A}_i(x) = \{\frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i\epsilon I}{if\ i \notin I}$ |
| |
| Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t >$ 0. $Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in Kand\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of |

$\dot{x} = \tilde{A}(x)x$ *and* $\lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\} \backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). *Since* $E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently *K* is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent*. Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume x* $\mapsto$ *A(x) is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely*

*ergodic, or There exists j € { 1,….,k} such that* $M_i$ ----- *{* $x \in C : x_l \gg 0$ *for* $l \neq j$*} Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by* $U_i M_i$.

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----{1,…,k} such that $\mu(---_i \in_I C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures* $\mu$ *with support in* $\partial$C, $\lambda_i(\mu) > 0$ *for all* $I \in \{1,\ldots,k\} \setminus supp(\mu)$ *(2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1, \ldots, x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda$ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species,* e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0 < e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j \ (x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1{}^1, \ldots, E_1{}^m)$, and the matrix $A^2$ ($E_1$) is close to the diagonal matrix $\text{diag}(f^i{}_2(E_1{}^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f_2^j (E^j_1) > 0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j (E^j_1) > 0$ and $\max_j f_2^j (E^j_2) > 0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0, E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is

sufficiently slow (i.e. $0< e^{j}{}_{i}\ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j (E^j{}_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1-x_1^j - \beta^j\ x_2^j\ - \alpha^j\ x_3^j\ ) + \sum d_1^{jk}\ \ x_1^k\ - e_1^j\ x_1^j$$

$$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j\ x_1^j\ - x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_2^{jk}\ \ x_2^k\ - e_2^j\ x_2^j$$

$$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j\ x_1^j\ - \alpha^j x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_3^{jk}\ \ x_3^k\ - e_3^j\ x_3^j$$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all *j*. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j{}_i = \sum_l d^{l,j}_i$ for all *i, j*) , the maximal invariant set in $\partial$C consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0< e^j{}_i\ll 1$ for all *i*, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2\ (E_1)$and $\lambda_3\ (E_1)$are close to $\max_j 1-a^j$ and $\max_j\ 1\ -\ \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j\ 1\ -\ \beta^j$. Consider the Morse decomposition of $\partial$C given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1> 0$ for *i* = 1, 2, 3, 0 is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j\ (1\ -\ \beta^j\ )$$

Equivalently,

$$2>\min_j a^j\ +\ \min_j\ \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j (r_i - \sum_s a_{is} x_s^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j , \qquad i = 1, \dots, k;\ j = 1, \dots, m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$ . We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set H= $\{x \in C: x_i^j = x_i^l \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \qquad i = 1, \dots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \dots, k$$

It has been shown [12,ch.16.1,16.2] That for $k$ =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there

is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\, \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \rightarrow \mathbf{E_2} \rightarrow \mathbf{E_3} \rightarrow \mathbf{E_1}$ then the following inequality holds

12) $\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as

well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that

there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has

abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left( \beta(1 - y - z)\frac{P}{\epsilon + P} - r - m(1 - y) - b(N)\ y \right)$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1 = N$, $x_2 = p$, $x_3 = (y,z)'$ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} - y_i)$$

With

$$(16)\quad F(x)_i = x_i g\,(x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\quad \begin{pmatrix} x_i \\ y_i\ dot \end{pmatrix} = \begin{pmatrix} -\beta & \beta \\ ag\,(x_i, x_i) - 1 & \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon

plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational

consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical

propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$ to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.19) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.20) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.21) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the

model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1$, $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as* $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 7. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^{+}\text{x}\ k_{\ 2}\text{x}\ k_{\ 1}, R^{-})$,

$A_3$. $\varphi_i\ (\cdot\ , s) \in L^{\ 2}\ (R^{+}, R^{+})\ \cap C(R^{+}, R^{+})$ for each $s \in\ [-\tau, 0]$,

$i,\ = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i$: $R^{+}$ ▯$[-\tau, T] \rightarrow R^{+}$ $(i,\ = 1,\ 2,)$ With the following properties:

$S_1$. $P_i(\cdot\ , t) \in L^{\ 1}\ (R^{+}, R^{+})$ for each $t\ \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot\ ) \in C\ ([-\tau, T], R^{+})$ for each $a\ \in R^{+}$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^{+}$ ▯$\{R^{+} \cap\ [0, T]\}$ And is continuous $(i,\ = 1,\ 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,)\ \in R^{+}$ ▯$[-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$P_i(a + x, t + x) =\ \ P_i(x), (i,\ = 1,\ 2,)$ ,

Then (1.3) considered at $(1.3)$ considered at $(a + x,\ t + x)$ becomes

$\frac{dp_{\ i}(x)}{dx} + f_i(a + x,\ P(t + x))\ \ P_i(x) = 0$

Which has a solution of the form?

$$P_i(x) = P_i(0)\ \exp\left[-\int_0^{\infty} f_{\ i}(a + s, P(t + S))ds\right]$$

And hence

(2.1) $p_{\ 1}\ (a + x, t + x) = p_{\ 1}\ (a, t)\ exp\left[-\int_o^{x} f_{\ 1}\ (a + s, p(t + s))ds\right]$

For $a \geq t$ *we choose* $(a,t) = (a - t, 0)$ in (1.3) and $x = t$ in (2.1) so that

(2.2) $p_{\ 1}\ (a, t) =\ \varphi_{\ 1}\ (a - t, 0)\exp$▯$\left[-\int_o^{t} f_{\ 1}\left(a - t + s, p_{\ 1}\ (s)\right) ds\right]$

$O \leq t \leq a, i = 1,2,$

With

$P_1(s)=(p_1(s),p_2(s-t)),$

$P_2(s)=(p_1(s-t),p_2(s)),$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$t > a \geq, i = 1,2,$

Where

(2.4) $B_i(t) = p_i(0, t), i = 1,2, t > 0.$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$

(2.5) $L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(a+s, p_i(s))ds]$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da + \int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$t > 0, i = 1,2,$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.(a, t, P_i(t))da$

$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$

$i = 1, 2, t > 0$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$

(2.5), (2.6) and (2.10) lead to

(2.11) $|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i a - d_i^*(t-a)]\, da$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t) \left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \beta(0) d_i^* t$$

$$\le [\beta_i^* \beta(0)\, exp(\beta_i^* t) \beta(0) d_i^*] t \quad for \ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T],R^+), \quad B_i \in C([0,T],R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau,T] = \{f \in C([-\tau,T],R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau,T]$. Then for any

$(x, y) \in C^+[-\tau,T] \times C^+[-\tau,T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau,T], \quad C_T^2 = C^+[-\tau,T] \times C^+[-\tau,T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau,T]$ *and* $t \in [0,T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Phi C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = P_t \left\{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on [$-\tau$, $t^*$]

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Phi(P_t) - \Phi_t\|_{t*} \le \left\{ \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right\}$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ and $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for t $\le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \leq \int_0^t B_1^{(1)}(a)\left\{M_1\left(a,t,P_t^{(1)}\right) - M_1\left(a,t,P_t^{(2)}\right)\right\}da$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_t ds\right)da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o)[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds]da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \|_t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$=\int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\mathrm{X}[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \, /\!/ P_t^{(1)} - P_t^{(2)} /\!/ \text{ for } a \, \varepsilon[o, t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \leq \beta_1 \Phi_1(0) || P_t^{(1)} - P_t^{(2)} ||_t^* + \Phi_1(0) \beta_1 d_1 || P_t^{(1)} - P_t^{(2)} ||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+b_1(a,-\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) \, \mathrm{e}^{\delta 1^a} \beta_1 \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 \mathrm{d}_1 \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) \, \mathrm{e}^{\delta 1^a} \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| \mathrm{B}_1^{(1)}(\sigma) - \mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{B}_1^{*\mathrm{d}} \sigma$$

$$+ \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \left\{ B_1 + B_2 d_1 a + B_1^* \left( B_2 a + d_1 a^2/2 \right) e^{\delta 1^a} \right\}$$

$$\leq \beta_1^* \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| do + \left( r^* + /\!/ \Phi_t /\!/_{t*} \right) \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \rightarrow \beta_1$ as $a \rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq \left( r^* + /\!/ \Phi_t /\!/_{t*} \right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{ g_1(a) + 0(a) \}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a)) - b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a)) do \right|$$

$$\left| B_1^* \left( r^* + /\!/ \Phi_t /\!/_{t*} \right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} 0(a) as\ a \rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1(P_t^{(2)}) \right| \leq ((r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2(P_t^{(2)}) \right|$ leads to

$$\| \pi(P_t^{(1)} - \pi(P_t^{(2)}) /\!/ \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^*)$$

As $t^* \rightarrow o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

(3.1) $\frac{dp\,P_2^*(a)}{da}+f_i(\mathrm{a}, P_1^*, P_2^*)\,P_i^*(a) = 0,\ \ i = 1,2, a > 0,$

(3.2) $P_i^* = \int_o^\infty P_i^*(a)da, i = 1,\ 2,$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i = 1.\ \ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s, P_1^*, P_2^*)ds\right], i = 1,\quad 2, a \geq 0,$

Which together with (3.3) lead to

$1=\int_o^\infty b_1(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_1(s, P_1^*, P_2^*)ds\right]da,$

(3.5) $1=\int_o^\infty b_2(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_2(s, P_1^*, P_2^*)ds\right]da$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1, P_2) = \int_o^\infty b_i(a, P_1, P_2)\exp\left[-\int_o^\infty f_i(s, P_1, P_2)ds\right]da,$

$i$ =1, 2, $P_1 \geq 0$, $P_2 \geq 0$,

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_5$. *there exists a positive constant c for which*

$F_1$(C,0)< $F_{2.}$(C,0),

(3.8) $F_1$(0.C)< $F_{2.}$(0,C),

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $F_1(P_1^*, P_2^*) = 1 = F_2(P_1^*, P_2^*)$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a) = [P_2^* \exp[-f_i(s, P_1^*, P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s, P_1^*, P_2^*)ds]da)$,

$i = 1, 2, \quad a \geq 0,$

Proof Consider the elements of the surfaces defined by

Z= $F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0,$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

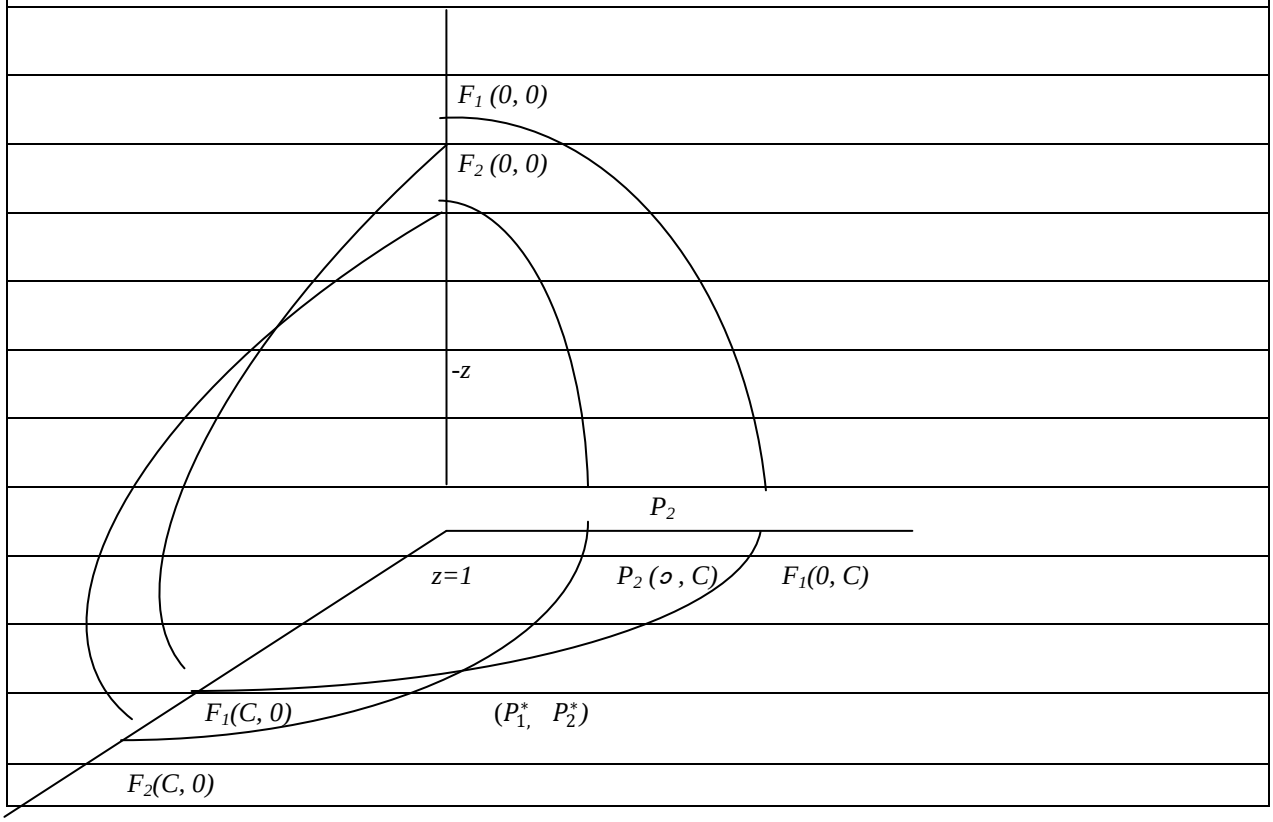
$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, F_1(P_1, P_2 \geq 0,$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.



(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11)
$$\begin{cases} p_i(a,t) = p^*_i(a) + u_i(a,t) \\ p_i(t) = p^*_i + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_i} p_i(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial b_i}{\partial p^*_i} p_i(t)\right\} da \end{cases}$$

Where $i = 1,2$ and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_i(a) \exp[\lambda t], \quad \pi_i(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty \pi_i(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d\pi_i}{da} + \lambda\pi_i = -f_i\left(a, p^*_1, p^*_2\right)\pi_i(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j} p^*_i\right\},$$

(3.15)

$$\mu_i(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\mu_i(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial b_i}{\partial p_j^*}\ p_i^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \ p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\mu_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1\left(a, p_1^*, p_2^*\right) - f_1\left(a, p_1^*, p_2^*\right)\right]\mu_i(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2\left(a, p_1^*, p_2^*\right) - f_2\left(a, p_1^*, p_2^*\right)\right]\mu_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\mu_i(a) = \left\{\mu_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\right\} - p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\mu_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\,e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)\,e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\varphi_1(a)$ and $\varphi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$\left(\lambda - A_{11}\right) p_1^* - A_{12} p_2^* \exp\left(-\lambda\tau\right) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21} p_1^* \exp\left(-\lambda\tau\right) + \left(\lambda - A_{22}\right) p_2^* = g_{21}(\lambda) p_1^* \exp\left(-\lambda\tau\right) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $$A_{i\varphi} = \int_0^\infty p_i^*(a) \left[\frac{\partial b_i}{\partial p_\varphi^*} - \frac{\partial f_i}{\partial p_\varphi^*}\right] da,$$

(3.23) $$g_{i\varphi}(\lambda) = \left[p_i^*(0) c_i\right]^{-1} \left\{\int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} da\right\} \varphi$$
$$\left\{\int_0^\infty \left[p_i^*(a) \frac{\partial b_i}{\partial p_\varphi^*} - b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_\varphi^*} e^{\lambda s} ds\right)\right] da\right\}$$

$$- \int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_\varphi^*} e^{\lambda s} ds\right) da, \quad i, \varphi = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i\left(a, p_1^*, p_2^*\right) - f_i\left(a, p_1^*, p_2^*\right), \quad i, = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $\left(p_1^*, p_2^*\right)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et \begin{bmatrix} \lambda - \left(A_{11} + g_{11}(\lambda)\right) & -\left(A_{12} + g_{12}\right) e^{-\lambda\tau} \\ -\left(A_{21} + g_{21}\right) e^{-\lambda\tau} & \lambda - \left(A_{22} + g_{22}(\lambda)\right) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda\left(A_{11} + A_{22}\right) + A_{11}A_{22} - A_{12}A_{21} \exp\left[-2\lambda\tau\right] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = \left[A_{21} g_{12}(\lambda) + A_{12} g_{21}(\lambda) + g_{12}(\lambda) g_{21}(\lambda)\right] \exp\left[-2\lambda\tau\right]$

$$-A_{11} g_{22}(\lambda) - A_{22} g_{11}(\lambda) - g_{11}(\lambda) g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ =1,\ 2.)\ \ (see(A_1)\ and\ (A_2))$;

$A_7.\ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = |A_{11}\, g_{12}(0)| + |A_{12}\, g_{21}(0)| + |g_{12}(0)\, g_{21}(0)|$$

$$+ |A_{11}\, g_{22}(0)| + |A_{22}\, g_{11}(0)| + |g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau]$;

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$, $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) | - F_2(\lambda, \tau) \right|$$

$$\geq |\lambda|^2 - |\lambda|\{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[(|\lambda| - P)^2 + (q)\right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$,

$q = \left\{4 | (A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| | - | A_{11} + A_{22} + \| S_1(0) \| |^2\right\}/2$,

It will now follow, from (A) and (3.26), $F_1(\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions

of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends,

assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner

wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the

controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

| **TITLE** | |
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| **INTRODUCTION—VARIABLES USED**<br><br>**PARRHESIA NUMBER 1 2006 • 62 – 87 Numerology or Ideology Zizek's Critique of Deleuze Robert Sinner brink I have deleted some sentences and added my own remarks to herein below exposition. For acknowledgements and references kindly see the original article. / Please note that the expatiation is model able and the following presupposition format is preeminently suitable for modelling**<br><br>(1) Zizek thus draws the very interesting parallel between the Deleuzian minimal difference between an element and its place and Laclau's inherent difference constitutive of an antagonism.<br>(2) In doing so, however, Zizek proceeds to reinterpret Deleuzian difference as though it were readily translatable into (e&eb) the Hegelian triad of universal, particular, and individual: | |

(3) The 'minimal difference' is (=) the difference between the universal and the 'surnumerary' particular which directly stands in for this universal, the paradoxical particular that, precisely insofar as it has no place within the structure of the universal, directly gives body to universality (OwB, 66). Zizek thus reinterprets Deleuzian difference as the "singular universal" or radical antagonism that is inherent to the One itself.

(4) It names the rift between the organic Whole that allocates (eb) each of its parts to its proper place, and the Whole embodied in (eb)the "singular universal," the paradoxical part without (e) any proper place within the Whole, yet which embodies (e) the antagonism at the very heart of universality (OwB, 66).

(5) As we shall see, moreover, this fissure within the ontological structure of the whole is not merely conceptual or logical but also has (e) direct political significance: "It is the rift between Society as the hierarchical structure and (e&eb) Society for which the excluded (demos, the third estate, dissidents) stand" (OwB, 66). At this point, however, a difficulty arises.

(6) This interpretation of difference as "singular universal" is quite different from Zizek's previous interpretation of Deleuzian difference as precisely not a difference between particulars or even between particular and the universal. The Deleuzian difference in itself was previously grasped as the difference between an element and its place; now Zizek interprets it, following Badiou, as a pure difference inscribed within the virtual One or background void of pure multiplicity. In accordance with Badiou's reading, Deleuze is transformed into a thinker of the immanent One rather than of difference in itself. This is where we find, I suggest, the ontological background to Zizek's reading of Deleuze, which, as he remarks, is heavily indebted to Badiou's The Clamour of Being (OwB, 20). Here Zizek reverts from a conception of Deleuzian difference that does not presuppose a virtual One to one that does presuppose such an immanent One, fissured by difference as inherent to this virtual multiplicity. The theoretical advantages of doing so, for Zizek's critique of Deleuze, are clear: the Badiouan reading of Deleuzian difference (as inscribed within the virtual One) make Deleuze's ontology more readily assimilable to Zizek's own Lacanian-Hegelian model of subjectivity as the fissure of negativity inherent within the (fractured) totality, the self differentiation of the singular within the concrete universal. Such a move is perhaps not surprising given Zizek's attempt to subsume, indeed sublate, Deleuzian difference into his own Hegelio-Lacanian schema.

(7) What is interesting is that Deleuze may have already anticipated, and thereby thwarted, this move in his own development of a concept of difference! In the Logic of Sense, Deleuze drew a parallel with Lacan and the radicality of pure difference as the difference between an element and its place, an inherent difference at the heart of the decentred subject in the Lacanian schema. Moreover, the radical potential of such a conception of subjectivity as singular universal has been amply developed, as we know, by Zizek himself.

(8) Yet in Difference and Repetition, Deleuze presents a more independent conception of the concept of difference, as distinct from conceptual difference, namely difference that cannot be subordinated to forms of representation—to identity with respect to concepts, opposition in the determination of concepts, analogy in regard to judgment, and resemblance in relation to objects —which also means a difference that cannot be incorporated into the universality of the self-differentiating concept, embracing opposition and contradiction, of Hegelian speculative dialectics. So in this respect, it is not clear that Zizek's attempt to sublate Deleuzian difference—mediated via Badiou—into the Hegelian conceptual logical schema quite hits the mark. In any case, Zizek's shift from one reading of Deleuzian difference to the other is left without justification, while the theoretical advantages of this move for his critique of Deleuze are not as obvious as first appears.

(9) Zizek draws another important implication from this re-reading of Deleuzian difference, namely that Deleuze badly misunderstands the Hegelian concept of the subject. Indeed, the subject, according to Zizek, just is the name given to this void or radical negativity that fissures substance from within; with the emergence of subjectivity, the void of the One becomes posited as For-Itself; virtuality irrupts into the order of actuality.

(10) Zizek is now in a position to claim that Deleuze's failure to conceive of difference as concrete universality, the fissuring of the immanent One, prompts his reductionist reading of the Hegelian subject and subsequent dismissal of Hegelianism as incapable of thinking difference as singularity and multiplicity. According to Zizek, Deleuze still conceives of the Hegelian subject as substance—"the self-identical foundation/ground of a process" (OwB, 72)—and not yet as the empty, non-substantial subject of radical negativity or pure virtuality.

(11) In other words, we have a (productive!) repetition cum resolution of the Hegel-Spinoza conflict over substance and subject; but instead of a Hegelian Aufhebung, Zizek interprets this conceptual conflict as demonstrating Deleuze's failure to proceed from substance to subject, in other words, to grasp the radical dimension of Hegel's theory of subjectivity as negativity.

(12) Indeed, the Hegelian subject, Zizek argues, "eludes the opposition of impersonal intensities and positive self-identical entities" that is constitutive of the Deleuzian conception of the "cracked" I or dispersed subject (OwB, 72). To remain within this fixed opposition, which Zizek implies Deleuze ultimately does, means that one is committed to either uncritically affirming a positive order of actuality (affirming the flux of virtual becomings that parallels the flux of global capital), or else negating actuality by means of regressive relapse into de-subjectified delirium (the regressive, neoromantic embrace of the schizoid dissolved subject as a form of 'critical resistance' to the established order of meaning, normativity, and practice).

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**(1)** It is this unhappy opposition that distinguishes Deleuzian Nomadology, for Zizek, as an ideological reflection of today's 'digital' capitalism. The question is whether Zizek's criticism is justified on this score, or whether it is an overstatement concealing a theoretical-political difficulty. Deleuze, Zizek, and the Politics of the 'New' Once again, this critique of Deleuze's account of the Hegelian subject has direct political implications, which Zizek explores in the final section of the book: "Politics: A Plea for Cultural Revolution". Zizek begins by reversing Lecercle's amusing anecdote about the Yuppie reading: What is Philosophy? In the Paris Metro.

**NOTATION**

**Module One**

In doing so, however, Zizek proceeds to reinterpret Deleuzian difference as though it were readily translatable into (e&eb) the Hegelian triad of universal, particular, and individual:

$G_{13}$ : Category one of **Deleuzian difference as though it were readily translatable;** Hegelian triad of universal, particular, and individual

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

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| $T_{13}$ : Category one of Hegelian triad of universal, particular, and individual **;Deleuzian difference as though it were readily translatable**<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>The 'minimal difference' is (=) the difference between the universal and the 'surnumerary' particular which directly stands in for this universal, the paradoxical particular that, precisely insofar as it has no place within the structure of the universal, directly gives body to universality (OwB, 66). Zizek thus reinterprets Deleuzian difference as the "singular universal" or radical antagonism that is inherent to the One itself. | |
| $G_{16}$ : Category one of minimal difference'<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of difference between the universal and the 'surnumerary' particular which directly stands in for this universal, the paradoxical particular that, precisely insofar as it has no place within the structure of the universal, directly gives body to universality (OwB, 66). Zizek thus reinterprets Deleuzian difference as the "singular universal" or radical antagonism that is inherent to the One itself.<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>The 'minimal difference' is the difference between the universal and the 'surnumerary' particular which directly stands in for this universal, the paradoxical particular that, precisely insofar as it has no place within the structure of the universal, directly gives body to universality (OwB, 66). Zizek thus reinterprets Deleuzian difference as the "singular universal" or radical antagonism that is inherent to the One itself. | |
| $G_{20}$ : Category one of universality (OwB, 66). Zizek thus reinterprets Deleuzian difference as the "singular universal" or radical antagonism that is inherent to the One itself<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of'minimal difference' is the difference between the universal and the 'surnumerary' particular which directly stands in for this universal, the paradoxical particular that, precisely insofar as it has no place within the structure of the universal, directly gives body<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>The 'minimal difference' is the difference between the universal and the 'surnumerary' particular which directly stands in for this universal, the paradoxical particular that, precisely insofar as it has no place within | |

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| the structure of the universal, directly gives body to universality (OwB, 66). Zizek thus reinterprets Deleuzian difference as the "singular universal" or radical antagonism that is inherent to the One itself | |
| $G_{24}$ : Category one of One itself<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of Deleuzian difference as the "singular universal" or radical antagonism that is inherent<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>Deleuzian difference as the "singular universal" or radical antagonism that is inherent to the One itself | |
| $G_{28}$ : Category one of Deleuzian difference<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of"singular universal" or radical antagonism that is inherent to the One itself<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>It names the rift between the organic Whole that allocates (eb) each of its parts to its proper place, and the Whole embodied in (eb)the "singular universal," the paradoxical part without (e) any proper place within the Whole, yet which embodies (e) the antagonism at the very heart of universality (OwB, 66) | |
| $G_{32}$ : Category one of organic Whole that allocates<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of each of its parts to its proper place, and the Whole embodied in (eb)the "singular universal," the paradoxical part without (e) any proper place within the Whole, yet which embodies (e) the antagonism at the very heart of universality (OwB, 66)<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven** | |

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| It names the rift between the organic Whole that allocates each of its parts to its proper place, and the Whole embodied in (eb)the "singular universal," the paradoxical part without (e) any proper place within the Whole, yet which embodies (e) the antagonism at the very heart of universality (OwB, 66) | |
| $G_{36}$ : Category one of organic Whole that allocates each of its parts to its proper place, and the Whole embodied<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of"singular universal," the paradoxical part without (e) any proper place within the Whole, yet which embodies (e) the antagonism at the very heart of universality (OwB, 66)<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>It names the rift between the organic Whole that allocates each of its parts to its proper place, and the Whole embodied in the "singular universal," the paradoxical part without (e) any proper place within the Whole, yet which embodies (e) the antagonism at the very heart of universality (OwB, 66) | |
| $G_{40}$ : Category one of any proper place within the Whole, yet which embodies (e) the antagonism at the very heart of universality (OwB, 66)<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of rift between the organic Whole that allocates each of its parts to its proper place, and the Whole embodied in the "singular universal," the paradoxical part<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>It names the rift between the organic Whole that allocates each of its parts to its proper place, and the Whole embodied in the "singular universal," the paradoxical part without any proper place within the Whole, yet which embodies (e) the antagonism at the very heart of universality (OwB, 66) | |
| $G_{44}$ : Category one of antagonism at the very heart of universality (OwB, 66)<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |

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| $T_{44}$ : Category one of organic Whole that allocates each of its parts to its proper place, and the Whole embodied in the "singular universal," the paradoxical part without any proper place within the Whole, yet which embodies<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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meets a similar situation: many internal microstates of a system are all compatible with the one observed (macro)state. Thermodynamic entropy quantifies the said multiplicity. Thus by analogy one needs to associate entropy with a black hole . By blocking all signal travel through it, the event horizon prevents an external observer from receiving information about the black hole (save for the mentioned few parameters; see Figure 2). Thus a black hole can be said to hide information. In ordinary physics entropy is a measure of missing information. Hence it makes sense to attribute entropy to a black hole.

**Formula for black hole entropy**

How to express the black hole entropy in a concrete formula? It is clear at the outset that black hole entropy should only depend on the observable properties of the black hole: mass, electric charge and angular momentum. It turns out that these three parameters enter only in the same combination as that which represents the surface area of the black hole. One way to understand why is to recall the "area theorem" (Hawking 1971, Misner, Thorne and Wheeler 1973): the event horizon area of a black hole cannot decrease; it increases in most transformations of the black hole. This increasing behavior is reminiscent of thermodynamic entropy of closed systems. Thus it is reasonable that the black hole entropy should be a monotonic function of area, and it turns out to be simplest such function.

If A stands for the surface area of a black hole (area of the event horizon), then the black hole entropy, in dimensionless form, is given by SBH=A4L2P=c3A4Gℏ,(1)

where LP stands for the Planck length Gℏ/c3 while G,ℏ and c denote, respectively, Newton's gravity constant, the Planck-Dirac constant (h/(2π)) and the speed of light. Of course, if the entropy in the usual (chemist's) form is required, the above should be multiplied by Boltzmann's constant k .

For the spherically symmetric and stationary, or Schwarzschild, black hole (see Schwarzschild metric), the only parameter is the black hole's massM , the horizon's radius is rh=2GM/c2 , and its area is naturally given by 4πr2h , or A=16π(GM/c2)2 .(2)

Note that a one-solar mass Schwarzschild black hole has an horizon area of the same order as the municipal area of Atlanta or Chicago. Its entropy is about 4×1077 , which is about twenty orders of magnitude larger than the thermodynamic entropy of the sun. This observation underscores the fact that one should not think of black hole entropy as the entropy that fell into the black hole when it was formed.

For the most general type of stationary black hole, the Kerr-Newman black hole (rotating black hole), the hole's parameters are mass M , electric charge Q and angular momentum J , and the horizon is no longer spherical. Nevertheless, in the popular Boyer-Lindquist coordinates {t,r,θ,φ} (see Misner, Thorne and Wheeler 1973 or Kerr-Newman metric) it lies at the fixed radial coordinate r=rh≡GM/c2+(GM/c2)2−(G1/2Q/c2)2−(J/Mc)2−−−−−−−−−−−−−−−−−−−−−−−−−−−−√.(3)

Consequently the horizon area is given by A=∫π0dθ∫2π0dφgθθgφφ−−−−−√=4π(r2h+(J/Mc)2).(4)

**The first law of black hole thermodynamics**

When near to equilibrium a thermodynamic system at temperature T changes its state, the consequent increments of its energy E and entropy S are related by the first law of thermodynamics: TdS=dE−dW(5)

Here dW is the work done on the system by exterior agents. When the system is one rotating with angular frequency Ω and charged up to electric potential Φ , the changes in its angular momentum J and charge Q contribute the work dW=ΩdJ+ΦdQ(6)

to the above formula.

A stationary black hole admits a similar relation (Bekenstein 1973). The differential dA from equation (4), when multiplied by a suitable factor, takes the form ΘdA=d(Mc2)−ΩBHdJ−ΦBHdQ(7)

where Θ≡c4(2GA)−1(rh−GM/c2);(8)

ΩBH≡(J/Mc)(r2h+(J/Mc)2)−1;(9)

ΦBH≡Qrh(r2h+(J/Mc)2)−1.(10)

Just now this is just a relation between increments in mechanical and geometrical properties. It turns out that ΩBH is precisely the angular rotation frequency of the black hole in the sense that any test body dropped into it, as it approaches the horizon no matter where, ends up circumnavigating it at just this frequency. And ΦBH turns out to be black hole's electric potential in the sense that it equals the line integral of the hole's electric field from infinity to any location on the horizon.

Because Mc2 is the hole's energy, equation (7) obviously looks like the first law for an ordinary thermodynamic system. It will be the first law if black hole entropy is required to be a function of A and of nothing else, so that dSBH∝dA (Gour and Mayo 2001). With the choice in equation (1) the black hole temperature TBH must be TBH=4L2PΘ=ħc2π(GM/c2)2−(G1/2Q2/c2)2−(J/Mc)2−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−√r2h+(J/Mc)2(11)

The reality of black hole temperature was brought home when Hawking showed (Hawking 1974, 1975) that a non-eternal black hole must spontaneously emit thermal radiation (Hawking radiation) with precisely this temperature (the original calculation was for J=0,Q=0 , but it is now clear that equation (11) is valid for all J and Q). This discovery provided the calibration of the numerical factor in equation (1).

**The generalized second law of thermodynamics**

In ordinary thermodynamics the second law requires that the entropy of a closed system shall never decrease, and shall typically increase as a consequence of generic transformations. While this law may hold good for a system including a black hole, it is not informative in its original form. For example, if an ordinary system falls into a black hole, the ordinary entropy becomes invisible to an exterior observer, so from her viewpoint, saying that ordinary entropy increases does not provide any insight: the ordinary second law is transcended.

Including the black hole entropy in the entropy ledger gives a more useful law, the generalized second law of thermodynamics (GSL) (Bekenstein 1972, 1973, 1974): the sum of ordinary entropy So outside black holes and the total black hole entropy never decreases and typically increases as a consequence of generic transformations of the black hole. In equations ΔSo+ΔSBH≥0.(12)

The GSL extends the reach of the area theorem in both classical and quantum physics as follows:

When matter entropy flows into a black hole, the GSL demands that the increase in black hole entropy shall more than compensate for the disappearance of ordinary entropy from sight. This has been verified by examples (Bekenstein 1973).

During the process of Hawking radiation the black hole's area decreases (basically because of the decrement of black hole mass), in violation of the area theorem. This is known to reflect a failure of the energy condition (assumed by the theorem) as a result of the very quantum fluctuations which engender the radiation. The GSL predicts that the emergent Hawking radiation entropy shall more than compensate for the drop in black hole entropy. This has been

verified amply (Bekenstein 1975, Hawking 1976), and stands as testament to the predictive power of the GSL which was formulated two years before the Hawking radiation was put in evidence.

Varied theoretical arguments have been given in support of the GSL (Frolov and Page 1993, Bombelli et al 1986)

Status of the third law of black hole thermodynamics

In ordinary thermodynamics the third law may be stated in two ways:

Nernst-Simon statement: The entropy of a system at absolute zero temperature either vanishes or becomes independent of the intensive thermodynamic parameters, e.g. pressure, magnetic field, electric potential, etc.

Unattainability statement: To bring a system to absolute zero temperature involves an infinite number of processes or steps.

From formula (11) it is clear that the black hole temperature vanishes when $(GM/c2)2-(G1/2Q2/c2)2-(J/Mc)2=0$.(13)

Kerr-Newman black holes satisfying this condition are called extreme. From equations (1) and (3)-(4) it is clear that for TBH=0 the black hole entropy is not only nonvanishing, but depends on (J/Mc) . Now this last quantity is an analog of a thermodynamic intensive parameter. For example, it is directly related to the hole's angular velocity in equation (9), and angular velocity of a thermodynamic system is an intensive parameter. Thus the Nernst-Simon statement of the third law fails for black holes.

There is some evidence that the unattainability statement of the third law is satisfied by black holes. For example, in an astrophysical setting the process of spinning up a Q=0 black hole gets "hung up" at $(J/Mc)\approx 0.998GM/c2$ , before the extremality condition can be satisfied (Thorne 1973).

**What is behind the black hole entropy**

In ordinary statistical mechanics, the entropy S is a measure of the multiplicity of microstates that hide behind one particular macrostate. A special case of this is Boltzmann's famous formula S=lnW where W stands for the number of equally probable microstates of a particular macrostate. Since black hole entropy plays a role quite analogous to that of ordinary entropy, e.g. it participates in the second law, many have wondered what the microstates that are counted by black hole entropy are. A partial list of interpretations of black hole entropy is the following:

**Black hole entropy counts the number of internal states of matter and gravity**.

As mentioned earlier, the perception that a particular black hole (specific M,J and Q) can be formed in many ways originally suggested the notion of black hole entropy; in this approach the internal states of matter and/or gravity are the sought for microstates. Examples of this viewpoint are provided by Frolov and Novikov (1993) and by Mukhanov (2003).

**Black hole entropy is the entropy of entanglement between degrees of freedom inside and outside the horizon**.

For a black hole formed by collapse, the quantum field degrees of freedom external to the horizon should be entangled with those inside it. To an external observer the last are not accessible, so the meaningful state would be the one from which internal degrees of freedom have been traced out (removed). Even if the global state was pure

and thus entropy free, this so reduced state will be a mixed one and have entropy associated with it. Calculation (Bombelli et al 1986, Srednicki 1993) shows that this entanglement entropy is proportional to the horizon area, just as required to explain black hole entropy, but the coefficient is ultraviolet divergent and thus requires renormalization by physical arguments.

**Black hole entropy counts the number of horizon gravitational states**.

It has been suggested that the sought for states are the states of the gravitational degrees of freedom residing on the black hole's horizon. An example of this approach is a calculation by Carlip (1999) based on the group of symmetries at the horizon. It reproduces formula (1).

**Black hole entropy is a conserved quantity connected with coordinate invariance of the gravitational action**.

This abstract approach has been championed by Wald (1993) as a road to black hole entropy in more general theories of gravity than general relativity. Technically, black hole entropy is the Noether charge of the diffeomorphism symmetry. This notion reproduces formula (1) when the gravitational action is of first order in the curvature, but gives a modified formula for higher order gravity theories. Black hole entropy is thermal entropy of the gas of quanta constituting the thermal atmosphere of the black hole. The atmosphere concept comes from Thorne and Zurek (1985) and 't Hooft (1985,1996); the last introduced the concept of a "brick wall" to keep the said atmosphere from contacting the horizon, and thus making the entropy infinite. This approach recovers the proportionality of entropy to horizon area, but the coefficient has to be chosen by hand.

**Black hole entropy counts the number of states or excitations of a fundamental string**.

Strings in string theory have a variety of excitations, so there is a multitude of string states. Therefore, a string has entropy, which turns out to be proportional to its mass. This is quite in contrast with black hole entropy. However, an argument by Bowick, Smolin and Wijewardhana (1987) suggests that by adiabatically (i.e. sufficiently slowly) reducing the string coupling constant g , it is possible to shrink a black hole's size as well as to reduce its mass (while keeping its entropy constant) until eventually it gets to be the size of the string length scale ls when the black hole should not be distinguishable from a string. At the corresponding value of g , string and black hole entropy are quite similar (see e.g. Zwiebach 2004). This has been taken to mean that there is a one-to-one correspondence between black hole and string states, where both entities have the same entropy (Susskind 1993). This picture has been corroborated in the context of five-dimensional extreme black holes (Strominger and Vafa 1996). Hence black hole entropy can be understood in terms of string entropy.

**Black hole entropy is equivalent to the thermal entropy of the radiation residing on the boundary of the spacetime containing the black hole.**

The AdS/CFT correspondence is a mapping between gravitational degrees of freedom of a certain spacetime and the matter (or field) degrees of freedom residing on its boundary. In particular, certain string theories in five dimensional Anti-deSitter (AdS) spacetime are so mapped to conformal field theories on the corresponding spacetime's four-dimensional boundary which bears some resemblance to Minkowski spacetime (Maldacena et al. 1997). Witten (1998) has shown that the entropy of a black hole residing in the bulk Anti-deSitter spacetime equals that of thermal radiation of the fields residing on its boundary.

Source: Wikipedia: Yang Mills Theory:

We give here some comments about the physical dimensions of the coupling. We note that, in D dimensions, the field scales as $[A] = [L^{\frac{2-D}{2}}]$ and so the coupling must scale as $[g^2] = [L^{D-4}]$. This implies that Yang–Mills theory is not renormalizable for dimensions greater than four. Further, we note that, for D = 4, the coupling is dimensionless and both the field and the square of the coupling have the same dimensions of the field and the coupling of a massless quartic scalar field theory. So, these theories share the scale invariance at the classical level.

**Quantization of Yang–Mills theory**

A method of quantizing the Yang–Mills theory is by functional methods, i.e. path integrals. One introduces a generating functional for n-point functions as

$$Z[j] = \int [dA] \exp\left[-\frac{i}{2}\int d^4x \,\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) + i\int d^4x\, j^a_\mu(x)A^{a\mu}(x)\right],$$

but this integral has no meaning as it is because the potential vector can be arbitrarily chosen due to the gauge freedom. This problem was already known for quantum electrodynamics but here becomes more severe due to non-abelian properties of the gauge group. A way out has been given by Ludvig Faddeev and Victor Popov with the introduction of a ghost field (seeFaddeev–Popov ghost) that has the property of being unphysical since, although it agrees with Fermi–Dirac statistics, it is a complex scalar field, which violates the spin-statistics theorem. So, we can write the generating functional as

$$Z[j,\bar{\varepsilon},\varepsilon] = \int [dA][d\bar{c}][dc] \exp\left\{iS_F[\partial A, A] + iS_{gf}[\partial A] + iS_g[\partial c, \partial\bar{c}, c, \bar{c}, A]\right\}$$
$$\exp\left\{i\int d^4x j^a_\mu(x)A^{a\mu}(x) + i\int d^4x[\bar{c}^a(x)\varepsilon^a(x) + \bar{\varepsilon}^a(x)c^a(x)]\right\}$$

being

$$S_F = -\frac{1}{2}\,\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu})$$

for the field,

$$S_{gf} = -\frac{1}{2\xi}(\partial\cdot A)^2$$

for the gauge fixing and

$$S_g = -(\bar{c}^a\partial_\mu\partial^\mu c^a + g\bar{c}^a f^{abc}\partial_\mu A^{b\mu}c^c)$$

for the ghost. This is the expression commonly used to derive Feynman's rules (see Feynman diagram). Here we have ca for the ghost field while α fixes the gauge's choice for the quantization. Feynman's rules obtained from this functional are the following

These rules for Feynman diagrams can be obtained when the generating functional given above is rewritten as

$$Z[j,\bar{\varepsilon},\varepsilon] = \exp\left(-ig\int d^4x \frac{\delta}{i\delta\bar{\varepsilon}^a(x)} f^{abc}\partial_\mu \frac{i\delta}{\delta j_\mu^b(x)}\frac{i\delta}{\delta\varepsilon^c(x)}\right)$$
$$\times \exp\left(-ig\int d^4x f^{abc}\partial_\mu \frac{i\delta}{\delta j_\nu^a(x)}\frac{i\delta}{\delta j_\mu^b(x)}\frac{i\delta}{\delta j^{c\nu}(x)}\right)$$
$$\times \exp\left(-i\frac{g^2}{4}\int d^4x f^{abc} f^{ars}\frac{i\delta}{\delta j_\mu^b(x)}\frac{i\delta}{\delta j_\nu^c(x)}\frac{i\delta}{\delta j^{r\mu}(x)}\frac{i\delta}{\delta j^{s\nu}(x)}\right)$$
$$\times Z_0[j,\bar{\varepsilon},\varepsilon]$$

with

$$Z_0[j,\bar{\varepsilon},\varepsilon] = \exp\left(-\int d^4x d^4y \bar{\varepsilon}^a(x) C^{ab}(x-y)\varepsilon^b(y)\right)\exp\left(\frac{1}{2}\int d^4x d^4y j_\mu^a(x) D^{ab\mu\nu}(x-y) j_\nu^b(y)\right)$$

being the generating functional of the free theory. Expanding in g and computing the functional derivatives, we are able to obtain all the n-point functions with perturbation theory. Using LSZ reduction formula we get from the n-point functions the corresponding process amplitudes, cross sections and decay rates.

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(1) In this case it is seen that the theory becomes (=) free (trivial ultraviolet fixed point for renormalization group) and both the gluon and ghost propagators are (=) those of a free massless particle.
(2) The asymptotic states of the theory are represented by (e) massless gluons that carry the interaction. The coupling runs to zero as we will see in the next section.
(3) At low momenta (infrared limit) the question has been more involved to (e) settle.
(4) The reason is that the theory becomes (=) strongly coupled in this case and perturbation theory cannot be applied.
(5) The only reliable approach to get an answer is (=) performing lattice computation on a computer powerful enough to afford large volumes.
(6) An answer to this question is a fundamental one as it would provide (eb) an understanding to the problem of confinement.
(7) On the other side, it should not be forgotten that propagators are (=) gauge-dependent quantities and so, they must be managed carefully when (e) one wants to get meaningful physical results.
(8) On the other side, theoretical approaches were conceived to get an understanding of the theory in this case. Pioneering works were due to Vladimir Gribov and Daniel Zwanziger. Gribov uncovered the question of the gauge-fixing in (eb) a Yang–Mills theory: He showed that, even once a gauge is fixed, (eb) a freedom is left yet (Gribov ambiguity).[10] Besides, he was able to provide (eb) a functional form for (e) the gluon propagator in the Landau gauge

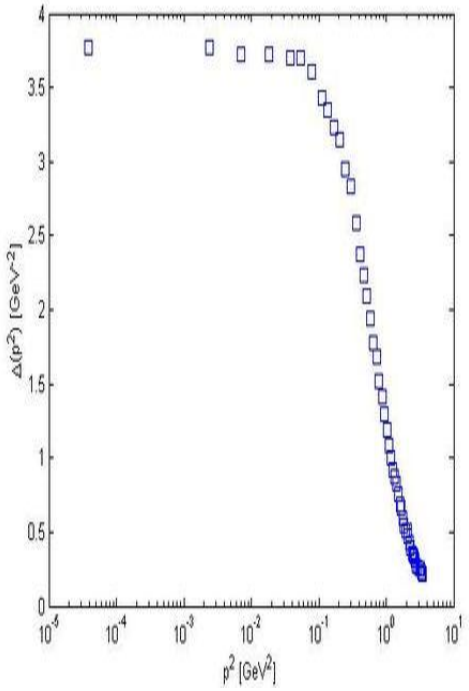
$$D_{\mu\nu}^{ab}(p) = \delta^{ab}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{p^2}{p^4 + M^4}.$$

(9) This propagator cannot be (=) correct in this way as it would violate (e) causality.

(10) On the other side, it provides a linear rising potential, $V(r) \propto r$, that would give reason (eb) to quark confinement.
(11) An important aspect of this functional form is that the gluon propagator appears to go to zero with (e&eb) momenta.
(12) This will become a crucial point in the following. From these studies by Gribov, Zwanziger extended his approach.[11][12] The inescapable conclusion was that the gluon propagator should go to (e*eb)zero with (e&eb) momenta while the ghost propagator should be enhanced with respect to (e&eb) the free case running to infinity.[13][14]
(13) This became known in literature as (=) the Gribov-Zwanziger scenario.
(14) When this scenario was proposed, computational resources were insufficient to decide whether (e&eb) it was correct or not.
(15) Rather, people pursued a different approach using (e) the Dyson-Schwinger equations.
(16) This is a set of coupled equations for (e) the n-point functions of the theory forming a hierarchy.
(17) This means that the equation for (e) the n-point function will depend on (=) the (n+1)-point function.
(18) So, to solve them one needs (e) a proper truncation.
(19) On the other side, these equations are (=) non-perturbative and could permit to obtain (eb) the behavior of the n-point functions in any regime.
(20) A solution to this hierarchy through (e&eb) truncation was proposed by Reinhard Alkofer, Andreas Hauck and Lorenz von Smekal.[15]
(21) This paper and the following publications from this group, the German group, set the agenda for the determination of the behavior of (e) the propagators in the Landau gauge in (eb) the subsequent years.
(22) The main conclusions these authors arrived to were to confirm the Gribov-Zwanziger scenario and that the running coupling should reach (eb) a finite non-null fixed point when (e) momenta run to zero.
(23) This paper represents the birth of the so-called scaling solution as (=) the propagators are seen to obey (e&eb_) scaling laws with given exponents.
(24) A proposal in the eighties by John Cornwall was in contrast with this scenario rather showing that the gluons get massive when (e) momenta goes to zero and the propagator should be (=) finite and non-null there[16] but went ignored at (eb) that time because the theoretical evidence appeared overwhelming for(e) the Gribov-Zwanziger scenario.
(25) Attempts to solve the Dyson-Schwinger equations numerically seemed to provide (eb) a different scenario [17][18] but this could have been due to (e) the way truncation and approximations were applied.
(26) The significant improvement in the computational resources made possible to (e) unveils the proper behavior of (e) the propagators in the Landau gauge. These results where firstly announced in Regensburg at the Lattice 2007 Conference. The results were somewhat unexpected and an example is given in the following figure for the gluon propagator [19]




that was obtained for the SU(2) case with a lattice of $128^4 \sim 10^8$ points reaching momenta in the very deep infrared. This result from a huge lattice shows that the gluon propagator never goes to zero with momenta but rather reaches a plateau with a finite value at zero momenta. This went called the decoupling solution in literature. Similarly, the ghost propagator is seen to behave as that of a free particle. The ghost field just decouples from the gauge field and becomes free in the deep infrared. Other groups at the same conference confirmed similar results.[20][21]

The decoupling scenario is consistent with a Yukawa-like propagator in the very deep infrared

$$D^{ab}_{\mu\nu}(p) \overset{p\to 0}{=} \delta^{ab} \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \frac{Z}{p^2 - M^2 + i0},$$

with $Z$ a constant. The gluon field develops a mass gap parameterized by $M$ in the above formula, while the BRST symmetry appears to be dynamically broken. These results hold in dimensions greater than 2 while for two dimensions the scaling solution holds.[22] Today, this scenario is generally accepted as the correct one for Yang-Mills theories in the infrared limit having such a strong support from lattice computations. Researchers are ongoing for a deeper theoretical understanding of these results and eventual phenomenological applications.

**Beta function and running coupling**

(27) One of the key properties of a quantum field theory is (=) the behavior over all the energy range of (e) the running coupling.
(28) Such a behavior can be obtained from (e) a theory once its beta function is known.
(29) Our ability to extract results from (e) a quantum field theory relies on (e) perturbation theory.

(30) Once the beta function is known, (eb) the behavior at all energy scales of the running coupling is obtained through the equation $\mu^2 \frac{d\alpha_s}{d\mu^2} = \beta(\alpha_s).$ being $\alpha_s = g^2/4\pi$. Yang–Mills theory has the property of being asymptotically free in the large energy limit (ultraviolet limit).

(31) This means that, in this limit, the beta function has (e) a minus sign driving (eb)the behavior of the running coupling toward even smaller values as (=) the energy increases.

(32) Perturbation theory permits (eb) to evaluate beta function in this limit producing (eb)the following result or SU(N) $\beta(\alpha_s) = -\frac{11N}{12\pi}\alpha_s^2 - \frac{17N^2}{24\pi^2}\alpha_s^3 + O\left(\alpha_s^4\right).$

(33) In the opposite limit of low energies (infrared limit), the beta function is not known. It is note the exact one for a supersymmetric Yang–Mills theory. This has been obtained by Novikov, Shifman, Vainshtein and Zakharov [23] and can be written as

$$\beta(\alpha_s) = -\frac{\alpha_s^2}{4\pi}\frac{3N}{1-\frac{N\alpha_s}{2\pi}}.$$

With this starting point, Thomas Ryttov and Francesco Sannino proposed a non-supersymmetric version of it writing down[24]

$$\beta(\alpha_s) = -\alpha_s^2\frac{11N}{12\pi}\frac{1}{1-\frac{17N}{11}\frac{\alpha_s}{2\pi}}.$$

As can be seen from the beta function of the supersymmetric theory, the limit of a large coupling (infrared limit) implies

$$\beta(\alpha_s) \approx \frac{3}{2}\alpha_s.$$

and so the running coupling in the deep infrared limit goes to zero making this theory trivial. This implies that the coupling reaches a maximum at some value of the energy turning again to zero as the energy is lowered. Then, if Ryttov and Sannino hypothesis is correct, the same should be true for ordinary Yang–Mills theory. This would be in agreement with recent lattice computations.[25]

**Open problems**

(1) Yang–Mills theories met with general acceptance in the physics community after Gerard't Hooft, in 1972, worked out their renormalization, relying on (e) a formulation of the problem worked out by (e) his advisor Martinus Veltman. (Their work [26] was recognized by the 1999 Nobel prize in physics.)

**(2)** Renormalizability is obtained even if (e) the gauge bosons described by (e) this theory are massive, as in the electroweak theory, provided (eb) the mass is only an "acquired" one, generated by (e) the Higgs mechanism.

**(3)** Concerning the mathematics, it should be noted that presently, i.e. in 2014, the Yang–Mills theory is a very active field of research, yielding e.g. invariants of differentiable structures on (e&eb) four-dimensional manifolds via (e&eb) work of Simon Donaldson.

**(4)** Furthermore, the field of Yang–Mills theories was included in the Clay Mathematics Institute's list of "Millennium Prize Problems". Here the prize-problem consists, especially, in a proof of the conjecture that the **lowest excitations of (e) a pure Yang–Mills theory (i.e. without (e) matter fields) have (e) a finite mass-gap with regard to (e&eb) the vacuum state.**

**(5) Another open problem, connected with this conjecture, is a proof of the confinement property in the presence of (e) additional Fermion particles.**

In physics the survey of Yang–Mills theories does not usually start from perturbation analysis or analytical methods, but more recently from systematic application of numerical methods tolattice gauge theories.

Faddeev–Popov ghost

| **TITLE** | |
|---|---|
| **INTRODUCTION—VARIABLES USED**<br><br>**Open problems: We assume the proposition to be true and give models for the same. Since we donot find any contradiction in the proof, we can assume that statements are correct.**<br><br>(1) Yang–Mills theories met with general acceptance in the physics community after Gerard't Hooft, in 1972, worked out their renormalization, relying on (e) a formulation of the problem worked out by (e) his advisor Martinus Veltman. (Their work [26] was recognized by the 1999 Nobel prize in physics.)<br>**(2)** Renormalizability is obtained even if (e) the gauge bosons described by (e) this theory are massive, as in the electroweak theory, provided (eb) the mass is only an "acquired" one, generated by (e) the Higgs mechanism.<br>**(3)** Concerning the mathematics, it should be noted that presently, i.e. in 2014, the Yang–Mills theory is a very active field of research, yielding e.g. invariants of differentiable structures on (e&eb) four-dimensional manifolds via (e&eb) work of Simon Donaldson.<br>**(4)** Furthermore, the field of Yang–Mills theories was included in the Clay Mathematics Institute's list of "Millennium Prize Problems". Here the prize-problem consists, especially, in a proof of the conjecture that the **lowest excitations of (e) a pure Yang–Mills theory (i.e. without (e) matter fields) have (e) a finite mass-gap with regard to (e&eb) the vacuum state.**<br>**(5) Another open problem, connected with this conjecture, is a proof of the confinement property in the presence of (e) additional Fermion particles** | |
| **NOTATION** | |
| **Module One**<br>Renormalizability is obtained even if (e) the gauge bosons described by (e) this theory are massive, as in the electroweak theory, provided (eb) the mass is only an "acquired" one, generated by (e) the Higgs mechanism | |
| $G_{13}$ : Category one of gauge bosons described by (e) this theory are massive, as in the electroweak theory, provided (eb) the mass is only an "acquired" one, generated by (e) the Higgs mechanism<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |

| | |
|---|---|
| $T_{13}$ : Category one of Renormalizability is obtained<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>Renormalizability is obtained even if the gauge bosons described by (e) this theory are massive, as in the electroweak theory, provided (e) the mass is only an "acquired" one, generated by (e) the Higgs mechanism | |
| $G_{16}$ : Category one of this theory are massive, as in the electroweak theory, provided (eb) the mass is only an "acquired" one, generated by (e) the Higgs mechanism<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of Renormalizability is obtained even if the gauge bosons described<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>Renormalizability is obtained even if the gauge bosons described by this theory are massive, as in the electroweak theory, provided (e) the mass is only an "acquired" one, generated by (e) the Higgs mechanism | |
| $G_{20}$ : Category one of mass is only an "acquired" one, generated by (e) the Higgs mechanism<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of Renormalizability is obtained even if the gauge bosons described by this theory are massive, as in the electroweak theory<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>Renormalizability is obtained even if the gauge bosons described by this theory are massive, as in the electroweak theory, provided the mass is only an "acquired" one, generated by (e) the **Higgs mechanism** | |
| $G_{24}$ : Category one of Renormalizability is obtained even if the gauge bosons described by this theory are massive, as in the electroweak theory, provided the mass is only an "acquired" one, generated; **Higgs mechanism**<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |

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| $T_{24}$ : Category one of **Higgs mechanism** ;Renormalizability is obtained even if the gauge bosons described by this theory are massive, as in the electroweak theory, provided the mass is only an "acquired" one, generated<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Concerning the mathematics, it should be noted that presently, i.e. in 2014, the Yang–Mills theory is a very active field of research, yielding e.g. invariants of differentiable structures on (e&eb) four-dimensional manifolds via (e&eb) work of Simon Donaldson | |
| $G_{28}$ : Category one of **invariants of differentiable structures;** four-dimensional manifolds via (e&eb) work of Simon Donaldson<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of four-dimensional manifolds via (e&eb) work of Simon Donaldson **;invariants of differentiable structures**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Concerning the mathematics, it should be noted that presently, i.e. in 2014, the **Yang–Mills theory is a very active field of research, yielding e.g. invariants of differentiable structures on four-dimensional manifolds** via (e&eb) work of Simon Donaldson | |
| $G_{32}$ : Category one of **Yang–Mills theory is a very active field of research, yielding e.g. invariants of differentiable structures on four-dimensional manifolds;** work of Simon Donaldson<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of work of Simon Donaldson; **Yang–Mills theory is a very active field of research, yielding e.g. invariants of differentiable structures on four-dimensional manifolds**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Furthermore, the field of Yang–Mills theories was included in the Clay Mathematics Institute's list of | |

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| "Millennium Prize Problems".<br><br>Here the prize-problem consists, especially, in a proof of the conjecture that the **lowest excitations of (e) a pure Yang–Mills theory (i.e. without (e) matter fields) have (e) a finite mass-gap with regard to (e&eb) the vacuum state** | |
| $G_{36}$ : Category one of **pure Yang–Mills theory (i.e. without (e) matter fields) have (e) a finite mass-gap with regard to (e&eb) the vacuum state**<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of **lowest excitations**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Here the prize-problem consists, especially, in a proof of the conjecture that the **lowest excitations of a pure Yang–Mills theory (i.e. without (e) matter fields) have (e) a finite mass-gap with regard to (e&eb) the vacuum state** | |
| $G_{40}$ : Category one of **lowest excitations of a pure Yang–Mills theory (i.e. without (e) matter fields) have (e) a finite mass-gap; vacuum state**<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of **vacuum state ;lowest excitations of a pure Yang–Mills theory (i.e. without (e) matter fields) have (e) a finite mass-gap**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>**another open problem, connected with this conjecture, is a proof of the confinement property in the presence of (e) additional Fermion particles** | |
| $G_{44}$ : Category one of **confinement property; additional Fermion particles**<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |

$T_{44}$ : Category one of **additional Fermion particles** **;confinement property**

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

**The Coefficients:**

$(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:
$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$
$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,
$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$
$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$
$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$
$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$

are Accentuation coefficients
$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$
$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$
$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}\ (a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,
$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$
$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,
$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,
$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,
are Dissipation coefficients

**Module Numbered One**

**The differential system of this model is now (Module Numbered one)**

$$\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right]G_{13} \quad (1)$$

$$\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14}, t)\right]G_{14} \quad (2)$$

$$\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14}, t)\right]G_{15} \quad (3)$$

$$\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b'_{13})^{(1)} - (b''_{13})^{(1)}(G, t)\right]T_{13} \quad (4)$$

$$\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b'_{14})^{(1)} - (b''_{14})^{(1)}(G, t)\right]T_{14} \quad (5)$$

$$\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b'_{15})^{(1)} - (b''_{15})^{(1)}(G, t)\right]T_{15} \quad (6)$$

$+(a''_{13})^{(1)}(T_{14}, t) =$ First augmentation factor

$-(b''_{13})^{(1)}(G, t) =$ First detritions factor

**Module Numbered Two**

**The differential system of this model is now ( Module numbered two)**

$$\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17}, t)\right]G_{16} \quad (7)$$

$$\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17}, t)\right]G_{17} \quad (8)$$

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| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19}), t)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}((G_{19}), t)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19}), t)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}((G_{19}), t) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}, t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}, t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21}, t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23}, t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25}, t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}, t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}), t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}), t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}), t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25}, t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}), t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29}, t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}, t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}), t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}), t)\right]T_{29}$ | 29 |

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| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b'_{30})^{(5)} - (b''_{30})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a''_{28})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b''_{28})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a'_{32})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b'_{32})^{(6)} - (b''_{32})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b'_{33})^{(6)} - (b''_{33})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b'_{34})^{(6)} - (b''_{34})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a''_{32})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b'_{36})^{(7)} - (b''_{36})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b'_{37})^{(7)} - (b''_{37})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b'_{38})^{(7)} - (b''_{38})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a''_{36})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a'_{41})^{(8)} + (a''_{41})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b'_{40})^{(8)} - (b''_{40})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b'_{41})^{(8)} - (b''_{41})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b'_{42})^{(8)} - (b''_{42})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |

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| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}\left((G_{47}),t\right) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}^{'})^{(1)} & +(a_{13}^{''})^{(1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,)}(T_{17},t) & +(a_{20}^{''})^{(3,3,)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t) & & +(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)} & +(a_{14}^{''})^{(1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,)}(T_{17},t) & +(a_{21}^{''})^{(3,3,)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t) & & +(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)} & +(a_{15}^{''})^{(1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,)}(T_{17},t) & +(a_{22}^{''})^{(3,3,)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t) & & +(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{15}$ | *57* |
| Where $(a_{13}^{''})^{(1)}(T_{14},t)$, $(a_{14}^{''})^{(1)}(T_{14},t)$, $(a_{15}^{''})^{(1)}(T_{14},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{38}^{''})^{(7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3<br>$+(a_{40}^{''})^{(8,8)}(T_{41},t)$ $+(a_{41}^{''})^{(8,8)}(T_{41},t)$ $+(a_{42}^{''})^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3<br>$+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)} & -(b_{13}^{''})^{(1)}(G,t) & -(b_{16}^{''})^{(2,2,)}(G_{19},t) & -(b_{20}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t) & & -(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,)}(G_{39},t) & -(b_{40}^{''})^{(8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{13}$ | 58 |

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| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\boxed{-(b_{15}^{''})^{(1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | *60* |
| Where $\boxed{-(b_{13}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}^{'})^{(2)}\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}^{'})^{(2)}\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{17}$ | *62* |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}^{'})^{(2)}\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{18}$ | *63* |
| Where $\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)}$ are first augmentation coefficients for | |

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| category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$ , $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$ , $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9)}(T_{45},t)$ , $+(a_{45}^{''})^{(9,9)}(T_{45},t)$ , $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)} & -(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{16}$ | *64* |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)} & -(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{17}$ | *65* |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)} & -(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{18}$ | *66* |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$ , $-(b_{17}^{''})^{(2)}(G_{19},t)$ , $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,)}(G,t)$ , $-(b_{14}^{''})^{(1,1,)}(G,t)$ , $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3<br>$-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$ , $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$ , $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3 | |

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| $-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[ \begin{matrix} (a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t) + (a_{16}^{''})^{(2,2,2)}(T_{17},t) + (a_{13}^{''})^{(1,1,1,)}(T_{14},t) \\ + (a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t) + (a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t) + (a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ + (a_{36}^{''})^{(7,7,7,7)}(T_{37},t) + (a_{40}^{''})^{(8,8,8,8)}(T_{41},t) + (a_{44}^{''})^{(9,9,9)}(T_{45},t) \end{matrix} \right] G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[ \begin{matrix} (a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t) + (a_{17}^{''})^{(2,2,2)}(T_{17},t) + (a_{14}^{''})^{(1,1,1,)}(T_{14},t) \\ + (a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t) + (a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t) + (a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ + (a_{37}^{''})^{(7,7,7,7)}(T_{37},t) + (a_{41}^{''})^{(8,8,8,8)}(T_{41},t) + (a_{45}^{''})^{(9,9,9)}(T_{45},t) \end{matrix} \right] G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[ \begin{matrix} (a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t) + (a_{18}^{''})^{(2,2,2)}(T_{17},t) + (a_{15}^{''})^{(1,1,1,)}(T_{14},t) \\ + (a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t) + (a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t) + (a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ + (a_{38}^{''})^{(7,7,7,7)}(T_{37},t) + (a_{42}^{''})^{(8,8,8,8)}(T_{41},t) + (a_{46}^{''})^{(9,9,9)}(T_{45},t) \end{matrix} \right] G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |

$$\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b'_{20})^{(3)} & -(b''_{20})^{(3)}(G_{23},t) & -(b''_{16})^{(2,2,2)}(G_{19},t) & -(b''_{13})^{(1,1,1,)}(G,t) \\ -(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b''_{36})^{(7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9)}(G_{47},t) & \end{bmatrix} T_{20} \quad (70)$$

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b'_{21})^{(3)} & -(b''_{21})^{(3)}(G_{23},t) & -(b''_{17})^{(2,2,2)}(G_{19},t) & -(b''_{14})^{(1,1,1,)}(G,t) \\ -(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b''_{37})^{(7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9)}(G_{47},t) & \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)} & -(b''_{22})^{(3)}(G_{23},t) & -(b''_{18})^{(2,2,2)}(G_{19},t) & -(b''_{15})^{(1,1,1,)}(G,t) \\ -(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b''_{38})^{(7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9)}(G_{47},t) & \end{bmatrix} T_{22} \quad (72)$$

$-(b''_{20})^{(3)}(G_{23},t)$, $-(b''_{21})^{(3)}(G_{23},t)$, $-(b''_{22})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*

$-(b''_{16})^{(2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*

$-(b''_{13})^{(1,1,1,)}(G,t)$, $-(b''_{14})^{(1,1,1,)}(G,t)$, $-(b''_{15})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*

$-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*

$-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*

$-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*

$-(b''_{36})^{(7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7)}(G_{39},t)$ $-(b''_{38})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b''_{40})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b''_{46})^{(9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a'_{24})^{(4)} & +(a''_{24})^{(4)}(T_{25},t) & +(a''_{28})^{(5,5,)}(T_{29},t) & +(a''_{32})^{(6,6,)}(T_{33},t) \\ +(a''_{13})^{(1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3)}(T_{21},t) & \\ +(a''_{36})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a'_{25})^{(4)} & +(a''_{25})^{(4)}(T_{25},t) & +(a''_{29})^{(5,5,)}(T_{29},t) & +(a''_{33})^{(6,6)}(T_{33},t) \\ +(a''_{14})^{(1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3)}(T_{21},t) & \\ +(a''_{37})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)} G_{25} - \begin{bmatrix} (a'_{26})^{(4)} & +(a''_{26})^{(4)}(T_{25},t) & +(a''_{30})^{(5,5,)}(T_{29},t) & +(a''_{34})^{(6,6,)}(T_{33},t) \\ & +(a''_{15})^{(1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a''_{24})^{(4)}(T_{25},t)$, $(a''_{25})^{(4)}(T_{25},t)$, $(a''_{26})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a''_{28})^{(5,5,)}(T_{29},t)$, $+(a''_{29})^{(5,5,)}(T_{29},t)$, $+(a''_{30})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3

$+(a''_{32})^{(6,6,)}(T_{33},t)$, $+(a''_{33})^{(6,6,)}(T_{33},t)$, $+(a''_{34})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1)}(T_{14},t)$ *are fourth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{16})^{(2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2)}(T_{17},t)$ *are fifth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{20})^{(3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3)}(T_{21},t)$ *are sixth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{36})^{(7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7)}(T_{37},t)$ *are seventh augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{40})^{(8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8)}(T_{41},t)$ *are eighth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{46})^{(9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9)}(T_{45},t)$, $+(a''_{44})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt} = (b_{24})^{(4)} T_{25} - \begin{bmatrix} (b'_{24})^{(4)} & -(b''_{24})^{(4)}(G_{27},t) & -(b''_{28})^{(5,5,)}(G_{31},t) & -(b''_{32})^{(6,6,)}(G_{35},t) \\ & -(b''_{13})^{(1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)} T_{24} - \begin{bmatrix} (b'_{25})^{(4)} & -(b''_{25})^{(4)}(G_{27},t) & -(b''_{29})^{(5,5,)}(G_{31},t) & -(b''_{33})^{(6,6,)}(G_{35},t) \\ & -(b''_{14})^{(1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)} T_{25} - \begin{bmatrix} (b'_{26})^{(4)} & -(b''_{26})^{(4)}(G_{27},t) & -(b''_{30})^{(5,5,)}(G_{31},t) & -(b''_{34})^{(6,6,)}(G_{35},t) \\ & -(b''_{15})^{(1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{26} \quad (78)$$

*Where* $-(b''_{24})^{(4)}(G_{27},t)$, $-(b''_{25})^{(4)}(G_{27},t)$, $-(b''_{26})^{(4)}(G_{27},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{28})^{(5,5,)}(G_{31},t)$, $-(b''_{29})^{(5,5,)}(G_{31},t)$, $-(b''_{30})^{(5,5,)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b''_{32})^{(6,6,)}(G_{35},t)$, $-(b''_{33})^{(6,6,)}(G_{35},t)$, $-(b''_{34})^{(6,6,)}(G_{35},t)$ *are third detrition coefficients for category* 1, 2 *and* 3

| | |
|---|---|
| $-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2 and 3 $-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 $-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3 $-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[\begin{matrix} (a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,)}(T_{25},t) & +(a_{32}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{matrix}\right] G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[\begin{matrix} (a_{29}^{'})^{(5)}+(a_{29}^{''})^{(5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,)}(T_{25},t) & +(a_{33}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{matrix}\right] G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[\begin{matrix} (a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,)}(T_{25},t) & +(a_{34}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{matrix}\right] G_{30}$ | *81* |
| *Where* $+(a_{28}^{''})^{(5)}(T_{29},t)$, $+(a_{29}^{''})^{(5)}(T_{29},t)$, $+(a_{30}^{''})^{(5)}(T_{29},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3 *And* $+(a_{24}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,)}(T_{25},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3 $+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3 $+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3 $+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3 $+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3 $+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3 $+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation | |

coefficients for category 1,2, 3
$+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3

$$\frac{dT_{28}}{dt}=(b_{28})^{(5)}T_{29}-\begin{bmatrix}(b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) & -(b_{32}^{''})^{(6,6,6)}(G_{35},t)\\ -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)\\ -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt}=(b_{29})^{(5)}T_{28}-\begin{bmatrix}(b_{29}^{'})^{(5)}-(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) & -(b_{33}^{''})^{(6,6,6)}(G_{35},t)\\ -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)\\ -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt}=(b_{30})^{(5)}T_{29}-\begin{bmatrix}(b_{30}^{'})^{(5)}-(b_{30}^{''})^{(5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,)}(G_{27},t) & -(b_{34}^{''})^{(6,6,6)}(G_{35},t)\\ -(b_{15}^{''})^{(1,1,1,1,1,)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)\\ -(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{30} \quad (84)$$

*where* $-(b_{28}^{''})^{(5)}(G_{31},t)$, $-(b_{29}^{''})^{(5)}(G_{31},t)$, $-(b_{30}^{''})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3

$-(b_{32}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3

$-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt}=(a_{32})^{(6)}G_{33}-\begin{bmatrix}(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a'_{33})^{(6)} & +(a''_{33})^{(6)}(T_{33},t) & +(a''_{29})^{(5,5,5)}(T_{29},t) & +(a''_{25})^{(4,4,4,)}(T_{25},t) \\ +(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients*

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| $for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{28}^{''})^{(5,5,5)}(G_{31},t)}, \boxed{-(b_{29}^{''})^{(5,5,5)}(G_{31},t)}, \boxed{-(b_{30}^{''})^{(5,5,5)}(G_{31},t)}\ are\ second\ detrition\ coefficients$ $for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)}\ are\ third\ detrition\ coefficients$ $for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2, and 3 | |
| $\frac{dG_{36}}{dt}$ $= (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)}\boxed{+(a_{36}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *91* |
| $\frac{dG_{37}}{dt}$ $= (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)}\boxed{+(a_{37}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *92* |
| $\frac{dG_{38}}{dt}$ $= (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)}\boxed{+(a_{38}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *93* |
| Where $\boxed{(a_{36}^{''})^{(7)}(T_{37},t)}, \boxed{(a_{37}^{''})^{(7)}(T_{37},t)}, \boxed{(a_{38}^{''})^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3 $\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3 | |

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| $+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ , $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ , $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$ , $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$ , $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3<br>$+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ , $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ , $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{36}}{dt} =$<br>$(b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 94 |
| $\frac{dT_{37}}{dt} =$<br>$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | |
| $\frac{dT_{38}}{dt} =$<br>$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | |
| Where $-(b_{36}^{''})^{(7)}(G_{39},t)$ , $-(b_{37}^{''})^{(7)}(G_{39},t)$ , $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ , $-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ , $-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ , $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ , $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 | |

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| $-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ , $-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ , $-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ , $-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ , $-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)$<br>are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ , $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ , $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3<br>$-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ , $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ , $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13}$ | *95* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}^{'})^{(8)}+(a_{41}^{''})^{(8)}(T_{41},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$ | |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$ | |
| Where $+(a_{40}^{''})^{(8)}(T_{41},t)$ , $+(a_{41}^{''})^{(8)}(T_{41},t)$ , $+(a_{42}^{''})^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ , $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ , $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ , $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ , $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ , $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ , $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 | |

$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & \left[-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & \left[-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & \left[-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

| | |
|---|---|
| $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a_{44}^{'})^{(9)}\boxed{+(a_{44}^{''})^{(9)}(T_{45},t)}\ \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\ \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13}$ | *96* |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a_{45}^{'})^{(9)}\boxed{+(a_{45}^{''})^{(9)}(T_{45},t)}\ \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\ \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$ | |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a_{46}^{'})^{(9)}\boxed{+(a_{46}^{''})^{(9)}(T_{37},t)}\ \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\ \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$ | |
| Where $\boxed{+(a_{44}^{''})^{(9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3<br>$\boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth | |

augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \begin{bmatrix} (b'_{46})^{(9)}\boxed{-(b''_{46})^{(9)}(G_{47},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{44})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$$(a_i)^{(1)}, (a'_i)^{(1)}, (a''_i)^{(1)}, (b_i)^{(1)}, (b'_i)^{(1)}, (b''_i)^{(1)} > 0, \quad i,j = 13,14,15 \qquad (97)$$

| | |
|---|---|
| The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br><br>$(a_i^{''})^{(1)}(T_{14}, t) \leq (p_i)^{(1)} \leq (\hat{A}_{13})^{(1)}$<br><br>$(b_i^{''})^{(1)}(G, t) \leq (r_i)^{(1)} \leq (b_i^{'})^{(1)} \leq (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2 \to \infty}(a_i^{''})^{(1)}(T_{14}, t) = (p_i)^{(1)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(1)}(G, t) = (r_i)^{(1)}$<br><br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br><br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'}, t) - (a_i^{''})^{(1)}(T_{14}, t)\rvert \leq (\hat{k}_{13})^{(1)} \lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)} t}$<br><br>$\lvert(b_i^{''})^{(1)}(G^{'}, t) - (b_i^{''})^{(1)}(G, t)\rvert < (\hat{k}_{13})^{(1)} \lVert G - G^{'}\rVert e^{-(\hat{M}_{13})^{(1)} t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'}, t)$ and $(a_i^{''})^{(1)}(T_{14}, t)$ . $(T_{14}^{'}, t)$ and $(T_{14}, t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i^{'})^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)} (\hat{k}_{13})^{(1)}] < 1$<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i^{'})^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)} (\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |

| | |
|---|---|
| $(a_i)^{(2)}, (a_i^{'})^{(2)}, (a_i^{''})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (b_i^{''})^{(2)} > 0, \quad i,j = 16,17,18$ | |
| The functions $(a_i^{''})^{(2)}, (b_i^{''})^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(\mathrm{p_i})^{(2)}$, $(\mathrm{r_i})^{(2)}$: | |
| $(a_i^{''})^{(2)}(T_{17}, t) \leq (p_i)^{(2)} \leq \left(\hat{A}_{16}\right)^{(2)}$ | 102 |
| $(b_i^{''})^{(2)}(G_{19}, t) \leq \ (r_i)^{(2)} \leq (b_i^{'})^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(2)}(T_{17}, t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty} (b_i^{''})^{(2)}\left((G_{19}), t\right) = \ (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $\boxed{(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, \ (r_i)^{(2)}}$ are positive constants and $\boxed{i = 16,17,18}$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'}, t) - (a_i^{''})^{(2)}(T_{17}, t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}^{'}\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'}, t) - (b_i^{''})^{(2)}\left((G_{19}), t\right)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})^{'}\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'}, t)$ and $(a_i^{''})^{(2)}(T_{17}, t)$ . $(T_{17}^{'}, t)$ and $(T_{17}, t)$ are points belonging to the interval $\left[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}\right]$. It is to be noted that $(a_i^{''})^{(2)}(T_{17}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i^{''})^{(2)}(T_{17}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, \ (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + \ (\hat{A}_{16})^{(2)} + \ (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |

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| $\frac{1}{(\hat{M}_{16})^{(2)}}[\ (b_i)^{(2)} + (b_i^{'})^{(2)} +\ (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)}\ (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23}, t) \leq\ (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(3)}\ (T_{21}, t) = (p_i)^{(3)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(3)}\ (G_{23}, t) =\ (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)},\ (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'}, t) - (a_i^{''})^{(3)}(T_{21}, t)\rvert \leq (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}{}^{'}, t) - (b_i^{''})^{(3)}(G_{23}, t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}{}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'}, t)$ and$(a_i^{''})^{(3)}(T_{21}, t)$ . $(T_{21}^{'}, t)$ And $(T_{21}, t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |

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| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}((G_{27}), t) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(4)}((G_{27}), t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\|(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\| \leq (\hat{k}_{24})^{(4)} \|T_{25} - T_{25}^{'}\| e^{-(\hat{M}_{24})^{(4)} t}$<br><br>$\|(b_i^{''})^{(4)}((G_{27})', t) - (b_i^{''})^{(4)}((G_{27}), t)\| < (\hat{k}_{24})^{(4)} \|(G_{27}) - (G_{27})'\| e^{-(\hat{M}_{24})^{(4)} t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants | 120 |

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| $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)} + (b_i^{'})^{(4)} + (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}((G_{31}), t) \leq (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(5)} (T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(5)} (G_{31}, t) = (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)} \lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)} t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})', t) - (b_i^{''})^{(5)}((G_{31}), t)\rvert < (\hat{k}_{28})^{(5)} \lvert\lvert(G_{31}) - (G_{31})'\rvert\rvert e^{-(\hat{M}_{28})^{(5)} t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ : | 125 |

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| $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$ | 126 |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(6)} (T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(6)} ((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)} \lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)} t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})', t) - (b_i^{''})^{(6)}((G_{35}), t)\rvert < (\hat{k}_{32})^{(6)} \lvert\lvert(G_{35}) - (G_{35})'\rvert\rvert e^{-(\hat{M}_{32})^{(6)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and $(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |

**Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ : 129

$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)},$ are positive constants

$$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$$

**Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ : 130

There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

Where we suppose

(QQ) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i, j = 36,37,38$ 131

(RR) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}$, $(r_i)^{(7)}$:

$$(a_i^{''})^{(7)}(T_{37}, t) \le (p_i)^{(7)} \le (\hat{A}_{36})^{(7)}$$

$$(b_i^{''})^{(7)}(G_{39}, t) \le (r_i)^{(7)} \le (b_i^{'})^{(7)} \le (\hat{B}_{36})^{(7)}$$

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*(SS)* $\lim_{T_2 \to \infty} (a_i^{''})^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(TT)*

$\lim_{G \to \infty} (b_i^{''})^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)| \le (\hat{k}_{36})^{(7)} |T_{37} - T_{37}^{'}| e^{-(\hat{M}_{36})^{(7)} t}$$

$$|(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)} ||(G_{39}) - (G_{39})^{'}|| e^{-(\hat{M}_{36})^{(7)} t}$$

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| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'},t)$ and$(a_i^{''})^{(7)}(T_{37},t)$ . $(T_{37}^{'},t)$ and $(T_{37},t)$ are points belonging to the interval $\left[(\hat{k}_{36})^{(7)},(\widehat{M}_{36})^{(7)}\right]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{36})^{(7)}=1$ then the function $(a_i^{''})^{(7)}(T_{37},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)}$ : <br><br>(UU) $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)},$ are positive constants <br><br>$\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}<1$ | 134 |
| **Definition of** $(\hat{P}_{36})^{(7)},(\hat{Q}_{36})^{(7)}$ : <br><br>(VV) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)},(\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)},(a_i^{'})^{(7)},(b_i)^{(7)},(b_i^{'})^{(7)},(p_i)^{(7)},\ (r_i)^{(7)},i=36,37,38$, satisfy the inequalities <br><br>$\frac{1}{(\widehat{M}_{36})^{(7)}}[(a_i)^{(7)}+(a_i^{'})^{(7)}+(\hat{A}_{36})^{(7)}+(\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}]<1$ <br><br>$\frac{1}{(\widehat{M}_{36})^{(7)}}[(b_i)^{(7)}+(b_i^{'})^{(7)}+(\hat{B}_{36})^{(7)}+(\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}]<1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)},(a_i^{'})^{(8)},(a_i^{''})^{(8)},(b_i)^{(8)},(b_i^{'})^{(8)},(b_i^{''})^{(8)}>0,\quad i,j=40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)},(b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)},\ (r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41},t)\leq(p_i)^{(8)}\leq(\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}),t)\leq\ (r_i)^{(8)}\leq(b_i^{'})^{(8)}\leq(\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(8)}(T_{41},t)=(p_i)^{(8)}$ | 140 |
| $\lim_{G\to\infty}(b_i^{''})^{(8)}\left((G_{43}),t\right)=\ (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)},(\hat{B}_{40})^{(8)}$ : <br><br>Where $\boxed{(\hat{A}_{40})^{(8)},(\hat{B}_{40})^{(8)},(p_i)^{(8)},\ (r_i)^{(8)}}$ are positive constants and $\boxed{i=40,41,42}$ | |

They satisfy Lipschitz condition:

$$|(a_i'')^{(8)}(T_{41}', t) - (a_i'')^{(8)}(T_{41}, t)| \leq (\hat{k}_{40})^{(8)}|T_{41} - T_{41}'|e^{-(\hat{M}_{40})^{(8)}t} \quad (142)$$

$$|(b_i'')^{(8)}((G_{43})', t) - (b_i'')^{(8)}((G_{43}), t)| < (\hat{k}_{40})^{(8)}||(G_{43}) - (G_{43})'||e^{-(\hat{M}_{40})^{(8)}t} \quad (143)$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(8)}(T_{41}', t)$ and $(a_i'')^{(8)}(T_{41}, t)$ . $(T_{41}', t)$ and $(T_{41}, t)$ are points belonging to the interval $\left[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}\right]$. It is to be noted that $(a_i'')^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i'')^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ :

$(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants

$$\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}, \frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1 \quad (144)$$

**Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :

There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$
Satisfy the inequalities

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1 \quad (145)$$

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1 \quad (146)$$

Where we suppose

$$(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i, j = 44,45,46 \quad (146A)$$

The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:

$$(a_i'')^{(9)}(T_{45}, t) \leq (p_i)^{(9)} \leq (\hat{A}_{44})^{(9)}$$

$$(b_i'')^{(9)}(G_{47}, t) \leq (r_i)^{(9)} \leq (b_i')^{(9)} \leq (\hat{B}_{44})^{(9)}$$

$$\lim_{T_2 \to \infty} (a_i'')^{(9)}(T_{45}, t) = (p_i)^{(9)}$$

$$\lim_{G \to \infty} (b_i'')^{(9)}(G_{47}, t) = (r_i)^{(9)}$$

**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :

Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44{,}45{,}46}$

They satisfy Lipschitz condition:

$$|(a_i^{''})^{(9)}(T_{45}^{'}, t) - (a_i^{''})^{(9)}(T_{45}, t)| \leq (\hat{k}_{44})^{(9)}|T_{45} - T_{45}^{'}|e^{-(\hat{M}_{44})^{(9)}t}$$

$$|(b_i^{''})^{(9)}((G_{47})^{'}, t) - (b_i^{''})^{(9)}\big((G_{47}), t\big)| < (\hat{k}_{44})^{(9)}||(G_{47}) - (G_{47})^{'}||e^{-(\hat{M}_{44})^{(9)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(9)}(T_{45}^{'}, t)$ and $(a_i^{''})^{(9)}(T_{45}, t)$ . $(T_{45}^{'}, t)$ and $(T_{45}, t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i^{''})^{(9)}(T_{45}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i^{''})^{(9)}(T_{45}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}} \ , \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$$

**Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and \ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}, i = 44{,}45{,}46,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0), T_i(0)$ :

$$G_i(t) \leq (\hat{P}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t} \ , \ \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \leq (\hat{Q}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t} \ , \ \boxed{T_i(0) = T_i^0 > 0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0), T_i(0)$

$$G_i(t) \leq (\hat{P}_{16})^{(2)}e^{(\hat{M}_{16})^{(2)}t} \ , \ G_i(0) = G_i^0 > 0$$

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| $T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$ | 153 A |

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| $T_i(t) \leq (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br><br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br><br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[ (a_{18})^{(2)} G_{17}(s_{(16)}) - \left( (a_{18}')^{(2)} + (a_{18}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{18}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[ (b_{16})^{(2)} T_{17}(s_{(16)}) - \left( (b_{16}')^{(2)} - (b_{16}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{16}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[ (b_{17})^{(2)} T_{16}(s_{(16)}) - \left( (b_{17}')^{(2)} - (b_{17}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[ (b_{18})^{(2)} T_{17}(s_{(16)}) - \left( (b_{18}')^{(2)} - (b_{18}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{18}(s_{(16)}) \right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a_{20}')^{(3)} + a_{20}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a_{21}')^{(3)} + (a_{21}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a_{22}')^{(3)} + (a_{22}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b_{20}')^{(3)} - (b_{20}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b_{21}')^{(3)} - (b_{21}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b_{22}')^{(3)} - (b_{22}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |

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| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{24})^{(4)}$ , $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{28})^{(5)}$ , $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |

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| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b'_{29})^{(5)} - (b''_{29})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{\mathrm{T}}_{30}(\mathrm{t}) = \mathrm{T}_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b'_{30})^{(5)} - (b''_{30})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{32})^{(6)}$ , $T_i^0 \le (\hat{Q}_{32})^{(6)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a'_{32})^{(6)} + a''_{32})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b'_{32})^{(6)} - (b''_{32})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b'_{33})^{(6)} - (b''_{33})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{\mathrm{T}}_{34}(\mathrm{t}) = \mathrm{T}_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b'_{34})^{(6)} - (b''_{34})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{36})^{(7)}$ , $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |

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| By<br>$\bar{G}_{36}(t) = G_{36}^{0} + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a_{36}^{'})^{(7)} + a_{36}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^{0} + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^{0} + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^{0} + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}^{'})^{(8)} + a_{40}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^{0} + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^{0} + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^{0} + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^{0} + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |

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| $\overline{T}_{42}(t) = T_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br><br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166<br>A |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{44})^{(9)}$, $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br><br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\overline{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br><br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\bar{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br><br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left(e^{(\bar{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br><br>$(G_{13}(t) - G_{13}^0) e^{-(\bar{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br><br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |

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| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \leq G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\bar{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$ $\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\bar{M}_{16})^{(2)}} \left(e^{(\bar{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that $(G_{16}(t) - G_{16}^0) e^{-(\bar{M}_{16})^{(2)} t} \leq \frac{(a_{16})^{(2)}}{(\bar{M}_{16})^{(2)}} \left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{20}(t) \leq G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} \left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$ $\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\bar{M}_{20})^{(3)}} \left(e^{(\bar{M}_{20})^{(3)} t} - 1\right)$ | 171 |
| From which it follows that $(G_{20}(t) - G_{20}^0) e^{-(\bar{M}_{20})^{(3)} t} \leq \frac{(a_{20})^{(3)}}{(\bar{M}_{20})^{(3)}} \left[\left((\hat{P}_{20})^{(3)} + G_{21}^0\right) e^{\left(-\frac{(\hat{P}_{20})^{(3)} + G_{21}^0}{G_{21}^0}\right)} + (\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that $G_{24}(t) \leq G_{24}^0 + \int_0^t \left[(a_{24})^{(4)} \left(G_{25}^0 + (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} s_{(24)}}\right)\right] ds_{(24)} =$ $\left(1 + (a_{24})^{(4)} t\right) G_{25}^0 + \frac{(a_{24})^{(4)} (\hat{P}_{24})^{(4)}}{(\bar{M}_{24})^{(4)}} \left(e^{(\bar{M}_{24})^{(4)} t} - 1\right)$ | 173 |
| From which it follows that $(G_{24}(t) - G_{24}^0) e^{-(\bar{M}_{24})^{(4)} t} \leq \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}} \left[\left((\hat{P}_{24})^{(4)} + G_{25}^0\right) e^{\left(-\frac{(\hat{P}_{24})^{(4)} + G_{25}^0}{G_{25}^0}\right)} + (\hat{P}_{24})^{(4)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{28}(t) \leq G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} \left(G_{29}^0 + (\hat{P}_{28})^{(5)} e^{(\bar{M}_{28})^{(5)} s_{(28)}}\right)\right] ds_{(28)} =$ $\left(1 + (a_{28})^{(5)} t\right) G_{29}^0 + \frac{(a_{28})^{(5)} (\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}} \left(e^{(\bar{M}_{28})^{(5)} t} - 1\right)$ | |

From which it follows that

$$(G_{28}(t)-G_{28}^{0})e^{-(\bar{M}_{28})^{(5)}t}\leq\frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^{0})e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^{0}}{G_{29}^{0}}\right)}+(\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5

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The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{32}(t)\leq G_{32}^{0}+\int_{0}^{t}\left[(a_{32})^{(6)}\left(G_{33}^{0}+(\hat{P}_{32})^{(6)}e^{(\bar{M}_{32})^{(6)}s_{(32)}}\right)\right]ds_{(32)}=$$

$$\left(1+(a_{32})^{(6)}t\right)G_{33}^{0}+\frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left(e^{(\bar{M}_{32})^{(6)}t}-1\right)$$

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From which it follows that

$$(G_{32}(t)-G_{32}^{0})e^{-(\bar{M}_{32})^{(6)}t}\leq\frac{(a_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^{0})e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^{0}}{G_{33}^{0}}\right)}+(\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

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(h) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{36}(t)\leq G_{36}^{0}+\int_{0}^{t}\left[(a_{36})^{(7)}\left(G_{37}^{0}+(\hat{P}_{36})^{(7)}e^{(\bar{M}_{36})^{(7)}s_{(36)}}\right)\right]ds_{(36)}=$$

$$\left(1+(a_{36})^{(7)}t\right)G_{37}^{0}+\frac{(a_{36})^{(7)}(\hat{P}_{36})^{(7)}}{(\bar{M}_{36})^{(7)}}\left(e^{(\bar{M}_{36})^{(7)}t}-1\right)$$

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From which it follows that

$$(G_{36}(t)-G_{36}^{0})e^{-(\bar{M}_{36})^{(7)}t}\leq\frac{(a_{36})^{(7)}}{(\bar{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)}+G_{37}^{0})e^{\left(-\frac{(\hat{P}_{36})^{(7)}+G_{37}^{0}}{G_{37}^{0}}\right)}+(\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t)\leq G_{40}^{0}+\int_{0}^{t}\left[(a_{40})^{(8)}\left(G_{41}^{0}+(\hat{P}_{40})^{(8)}e^{(\bar{M}_{40})^{(8)}s_{(40)}}\right)\right]ds_{(40)}=$$

$$\left(1+(a_{40})^{(8)}t\right)G_{41}^{0}+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\bar{M}_{40})^{(8)}}\left(e^{(\bar{M}_{40})^{(8)}t}-1\right)$$

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From which it follows that

$$(G_{40}(t)-G_{40}^{0})e^{-(\bar{M}_{40})^{(8)}t}\leq\frac{(a_{40})^{(8)}}{(\bar{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^{0})e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^{0}}{G_{41}^{0}}\right)}+(\hat{P}_{40})^{(8)}\right]$$

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| $(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)}\left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$<br>$\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)} t} - 1\right)$ | |
| From which it follows that<br>$(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{P}_{44})^{(9)} + G_{45}^0\right) e^{\left(-\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0}\right)} + (\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ and to choose<br>$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)} + \left((\hat{P}_{13})^{(1)} + G_j^0\right) e^{-\left(\frac{(\hat{P}_{13})^{(1)} + G_j^0}{G_j^0}\right)}\right] \le (\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)} + T_j^0\right) e^{-\left(\frac{(\hat{Q}_{13})^{(1)} + T_j^0}{T_j^0}\right)} + (\hat{Q}_{13})^{(1)}\right] \le (\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)}, T^{(1)}\right), \left(G^{(2)}, T^{(2)}\right)\right) =$<br>$\sup_i \{\max_{t \in \mathbb{R}_+} \left|G_i^{(1)}(t) - G_i^{(2)}(t)\right| e^{-(\hat{M}_{13})^{(1)} t}, \max_{t \in \mathbb{R}_+} \left|T_i^{(1)}(t) - T_i^{(2)}(t)\right| e^{-(\hat{M}_{13})^{(1)} t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G}, \tilde{T}$ : $\left(\tilde{G}, \tilde{T}\right) = \mathcal{A}^{(1)}(G, T)$<br>It results | |

$$\left|\tilde{G}_{13}^{(1)}-\tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{13})^{(1)} \left|G_{14}^{(1)}-G_{14}^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\, ds_{(13)} +$$

$$\int_0^t \{(a'_{13})^{(1)} \left|G_{13}^{(1)}-G_{13}^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} +$$

$$(a''_{13})^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right)\left|G_{13}^{(1)}-G_{13}^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} e^{(\widehat{M}_{13})^{(1)}s_{(13)}} +$$

$$G_{13}^{(2)}\left|(a''_{13})^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right)-(a''_{13})^{(1)}\left(T_{14}^{(2)}, s_{(13)}\right)\right|\ e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\} ds_{(13)}$$

Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses it follows

$$\left|G^{(1)}-G^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)}t} \leq$$
$$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right) d\left(\left(G^{(1)}, T^{(1)};\ G^{(2)}, T^{(2)}\right)\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

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**Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(1)}$ and $(b''_i)^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 2:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$

From 19 to 24 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(1)} - (a''_i)^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(1)}t\right)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :

**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if

$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating

$$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1 / (a'_{14})^{(1)}$$

In the same way , one can obtain

$$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2 / (a'_{15})^{(1)}$$

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| If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t),t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br><br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote | 195 |

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| **Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right) d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$ | 201 |

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| In the same way , one can obtain<br>$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | |
| **Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br>Indeed let $t_2$ be so that for $t > t_2$<br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$ | 211 |

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| $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}},\widetilde{T_{23}}:(\widetilde{(G_{23})},\widetilde{(T_{23})})=\mathcal{A}^{(3)}((G_{23}),(T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{20})^{(3)}\left\|G_{21}^{(1)}-G_{21}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)}+$<br>$\int_0^t\{(a_{20}^{'})^{(3)}\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$G_{20}^{(2)}\left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)-(a_{20}^{''})^{(3)}\left(T_{21}^{(2)},s_{(20)}\right)\right\|\ e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)}-G_{23}{}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}t}\le$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)}+(a_{20}^{'})^{(3)}+(\widehat{A}_{20})^{(3)}+(\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)},(T_{23})^{(1)};\ (G_{23})^{(2)},(T_{23})^{(2)})\right)$<br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ and\ (\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i=20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(3)}-(a_i^{''})^{(3)}(T_{21}(s_{(20)}),s_{(20)})\}ds_{(20)}\right]}\ge 0$<br>$T_i(t)\ge T_i^0e^{(-(b_i^{'})^{(3)}t)}>0$ for t > 0 | 216 |

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| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |

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| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{24})^{(4)}\left\|G_{25}^{(1)}-G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$<br>$\int_0^t\{(a_{24}^{'})^{(4)}\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$G_{24}^{(2)}\left\|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)}-(G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t}\le$<br>$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 226 |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24{,}25{,}26$ depend only on $\mathrm{T}_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i\,(t)=0\ and\ T_i\,(t)=0$<br>it results | 228 |

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| $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(4)} - (a_i'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(4)} t)} > 0$ for t > 0 | |
| **Definition of** $((\widehat{M}_{24})^{(4)})_1, ((\widehat{M}_{24})^{(4)})_2$ *and* $((\widehat{M}_{24})^{(4)})_3$ :<br><br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if<br><br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq ((\widehat{M}_{24})^{(4)})_1 - (a_{25}')^{(4)} G_{25}$ and by integrating<br><br>$G_{25} \leq ((\widehat{M}_{24})^{(4)})_2 = G_{25}^0 + 2(a_{25})^{(4)} ((\widehat{M}_{24})^{(4)})_1 / (a_{25}')^{(4)}$<br><br>In the same way , one can obtain<br><br>$G_{26} \leq ((\widehat{M}_{24})^{(4)})_3 = G_{26}^0 + 2(a_{26})^{(4)} ((\widehat{M}_{24})^{(4)})_2 / (a_{26}')^{(4)}$<br><br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t), t)) = (b_{25}')^{(4)}$ then $T_{25} \to \infty$.<br><br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br><br>Indeed let $t_4$ be so that for $t > t_4$<br><br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t), t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}((G_{27})(t), t) = (b_{26}')^{(4)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br><br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br><br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |

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| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)}+\left((\hat{P}_{28})^{(5)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{28})^{(5)}\right]\leq(\hat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}ds_{(28)}+$<br>$\int_0^t\{(a_{28}')^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}'')^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\leq$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}')^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}'')^{(5)}$ and $(b_{28}'')^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it | 238 |

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| suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 22:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(5)} - (a_i^{''})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(5)} t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2\ and\ \left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}\ and\ G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}^{'})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a_{29}^{'})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a_{30}^{'})^{(5)}$<br><br>If $G_{29}\ or\ G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}\ and\ G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t), t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t), t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t), t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |

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| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\widehat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{32})^{(6)}\right] \leq (\widehat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)}, (T_{35})^{(1)}), ((G_{35})^{(2)}, (T_{35})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $\widetilde{(G_{35})}, \widetilde{(T_{35})}$ : $\left(\widetilde{(G_{35})}, \widetilde{(T_{35})}\right) = \mathcal{A}^{(6)}((G_{35}), (T_{35}))$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{32})^{(6)}\left\|G_{33}^{(1)} - G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\,ds_{(32)} +$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)} - G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)\left\|G_{32}^{(1)} - G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right) - (a''_{32})^{(6)}\left(T_{33}^{(2)}, s_{(32)}\right)\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)} - (G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t} \leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)} + (a'_{32})^{(6)} + (\widehat{A}_{32})^{(6)} + (\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)}, (T_{35})^{(1)}; (G_{35})^{(2)}, (T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | 249 |

condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i = 32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 27:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$ 250

it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(6)} - (a_i^{''})^{(6)}(T_{33}(s_{(32)}),s_{(32)})\}ds_{(32)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(6)}t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2\ and\ \left((\widehat{M}_{32})^{(6)}\right)_3$ : 251

**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}\ and\ G_{34}$ . indeed if

$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}^{'})^{(6)}G_{33}$ and by integrating

$$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1/(a_{33}^{'})^{(6)}$$

In the same way , one can obtain

$$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2/(a_{34}^{'})^{(6)}$$

If $G_{33}\ or\ G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively.

**Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}\ and\ G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. 252

**Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}\ ((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$. 253

**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :

Indeed let $t_6$ be so that for $t > t_6$

$$(b_{33})^{(6)} - (b_i^{''})^{(6)}\left((G_{35})(t),t\right) < \varepsilon_6, T_{32}\,(t) > (m)^{(6)}$$

Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to 254

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right), \quad t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.
The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$

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| We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ large to have | 255 |
| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[(\hat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)}, (T_{39})^{(1)}), ((G_{39})^{(2)}, (T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $(\widetilde{G_{39}}), (\widetilde{T_{39}}) : \left((\widetilde{G_{39}}), (\widetilde{T_{39}})\right) = \mathcal{A}^{(7)}((G_{39}), (T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{36})^{(7)}\left\|G_{37}^{(1)} - G_{37}^{(2)}\right\|e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{(\hat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)} +$<br>$\int_0^t\{(a'_{36})^{(7)}\left\|G_{36}^{(1)} - G_{36}^{(2)}\right\|e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{-(\hat{M}_{36})^{(7)}s_{(36)}} +$<br>$(a''_{36})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right)\left\|G_{36}^{(1)} - G_{36}^{(2)}\right\|e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{(\hat{M}_{36})^{(7)}s_{(36)}} +$<br>$G_{36}^{(2)}\left\|(a''_{36})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) - (a''_{36})^{(7)}\left(T_{37}^{(2)}, s_{(36)}\right)\right\|\ e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{(\hat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)} - (G_{39})^{(2)}\right\|e^{-(\hat{M}_{36})^{(7)}t} \leq$<br>$\frac{1}{(\hat{M}_{36})^{(7)}}\left((a_{36})^{(7)} + (a'_{36})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)}, (T_{39})^{(1)};\ (G_{39})^{(2)}, (T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |

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| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ $and$ $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\}ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)}t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)}G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1/(a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ $is$ bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}\,((G_{39})(t),t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |

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| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right), \quad t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t), t\right) = (b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ and to choose<br>$(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\left[(\hat{P}_{40})^{(8)} + \left((\hat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{40})^{(8)}\right] \leq (\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)}, (T_{43})^{(1)}), ((G_{43})^{(2)}, (T_{43})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{40})^{(8)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})}, \widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})}, \widetilde{(T_{43})}\right) = \mathcal{A}^{(8)}((G_{43}), (T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{40})^{(8)}\left\|G_{41}^{(1)} - G_{41}^{(2)}\right\|e^{-(\hat{M}_{40})^{(8)}s_{(40)}}e^{(\hat{M}_{40})^{(8)}s_{(40)}}\,ds_{(40)} +$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\hat{M}_{40})^{(8)}s_{(40)}}e^{-(\hat{M}_{40})^{(8)}s_{(40)}} +$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\hat{M}_{40})^{(8)}s_{(40)}}e^{(\hat{M}_{40})^{(8)}s_{(40)}} +$<br>$G_{40}^{(2)}\left\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right) - (a_{40}^{''})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)\right\| e^{-(\hat{M}_{40})^{(8)}s_{(40)}}e^{(\hat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval [0, t] | 272 |

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| From the hypotheses it follows | |
| $\left\|(G_{43})^{(1)} - (G_{43})^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)}t} \leq$ $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)} + (a'_{40})^{(8)} + (\widehat{A}_{40})^{(8)} + (\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right) d\left(((G_{43})^{(1)}, (T_{43})^{(1)};\ (G_{43})^{(2)}, (T_{43})^{(2)})\right)$ And analogous inequalities for $G_i$ $and$ $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a''_{40})^{(8)}$ and $(b''_{40})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(8)}$ and $(b''_i)^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t) = 0$ $and$ $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(8)} - (a''_i)^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ $and$ $\left((\widehat{M}_{40})^{(8)}\right)_3$ : **Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ $and$ $G_{42}$ . indeed if $G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a'_{41})^{(8)} G_{41}$ and by integrating $G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a'_{41})^{(8)}$ In the same way , one can obtain $G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a'_{42})^{(8)}$ If $G_{41}$ $or$ $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}$ $and$ $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(8)}((G_{43})(t), t)) = (b'_{41})^{(8)}$ then $T_{41} \to \infty$. | 278 |

**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :

Indeed let $t_8$ be so that for $t > t_8$

$(b_{41})^{(8)} - (b_i^{''})^{(8)}\big((G_{43})(t), t\big) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$

Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.
The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\big((G_{43})(t), t(t), t\big) = (b_{42}^{'})^{(8)}$

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It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have

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$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \big((\hat{P}_{44})^{(9)} + G_j^0\big)e^{-\left(\frac{(\hat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\big((\hat{Q}_{44})^{(9)} + T_j^0\big)e^{-\left(\frac{(\hat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{44})^{(9)}\right] \leq (\hat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\Big(\big((G_{47})^{(1)}, (T_{47})^{(1)}\big), \big((G_{47})^{(2)}, (T_{47})^{(2)}\big)\Big) =$$
$$\sup_i\{\max_{t\in\mathbb{R}_+}\big|G_i^{(1)}(t) - G_i^{(2)}(t)\big|e^{-(\widehat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\big|T_i^{(1)}(t) - T_i^{(2)}(t)\big|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \big(\widetilde{(G_{47})}, \widetilde{(T_{47})}\big) = \mathcal{A}^{(9)}\big((G_{47}), (T_{47})\big)$
It results

$$\big|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\big| \leq \int_0^t (a_{44})^{(9)}\big|G_{45}^{(1)} - G_{45}^{(2)}\big|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\big|G_{44}^{(1)} - G_{44}^{(2)}\big|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$(a_{44}^{''})^{(9)}\big(T_{45}^{(1)}, s_{(44)}\big)\big|G_{44}^{(1)} - G_{44}^{(2)}\big|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$G_{44}^{(2)}\big|(a_{44}^{''})^{(9)}\big(T_{45}^{(1)}, s_{(44)}\big) - (a_{44}^{''})^{(9)}\big(T_{45}^{(2)}, s_{(44)}\big)\big|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)} - G^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)}t} \leq \frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a'_{44})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right) d\left(((G_{47})^{(1)}, (T_{47})^{(1)};\ (G_{47})^{(2)}, (T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a''_{44})^{(9)}$ and $(b''_{44})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)} e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)} e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(9)}$ and $(b''_i)^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$

From 99 to 44 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(9)} - (a''_i)^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b'_i)^{(9)}t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2$ *and* $\left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a'_{45})^{(9)} G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1 / (a'_{45})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2 / (a'_{46})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(9)}\left((G_{47})(t), t\right)) = (b'_{45})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\big((G_{47})(t),t\big) < \varepsilon_9, T_{44}\,(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\big((G_{47})(t),t\big) = (b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} - (a_{13}^{''})^{(1)}(T_{14}, t) + (a_{14}^{''})^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\big(\nu^{(1)}\big)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\big(u^{(1)}\big)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\big(\nu^{(1)}\big)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\big(u^{(1)}\big)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

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| $(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)},$ and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$ $(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$ are defined | |
| Then the solution of global equations satisfies the inequalities $G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)} t}$ where $(p_i)^{(1)}$ is defined by equation $\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)} t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)} t}\right] + G_{15}^0 e^{-(S_2)^{(1)} t} \le G_{15}(t) \le \frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a'_{15})^{(1)})}[e^{(S_1)^{(1)} t} - e^{-(a'_{15})^{(1)} t}] + G_{15}^0 e^{-(a'_{15})^{(1)} t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)} T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)} - (b'_{15})^{(1)})}\left[e^{(R_1)^{(1)} t} - e^{-(b'_{15})^{(1)} t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)} t} \le T_{15}(t) \le \frac{(a_{15})^{(1)} T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)} + (r_{13})^{(1)} + (R_2)^{(1)})}\left[e^{((R_1)^{(1)} + (r_{13})^{(1)})t} - e^{-(R_2)^{(1)} t}\right] + T_{15}^0 e^{-(R_2)^{(1)} t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:- Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$ $(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$ $(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$ $(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ : $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |

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| $-(\sigma_2)^{(2)} \leq -(a_{16}^{'})^{(2)} + (a_{17}^{'})^{(2)} - (a_{16}^{''})^{(2)}(\mathrm{T}_{17}, t) + (a_{17}^{''})^{(2)}(\mathrm{T}_{17}, t) \leq -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \leq -(b_{16}^{'})^{(2)} + (b_{17}^{'})^{(2)} - (b_{16}^{''})^{(2)}\big((G_{19}), t\big) - (b_{17}^{''})^{(2)}\big((G_{19}), t\big) \leq -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}$,, $(\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{\mathrm{G}_{16}^0}{\mathrm{G}_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{\mathrm{T}_{16}^0}{\mathrm{T}_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$\mathrm{G}_{16}^0 \mathrm{e}^{((\mathrm{S}_1)^{(2)} - (p_{16})^{(2)})\mathrm{t}} \leq G_{16}(t) \leq \mathrm{G}_{16}^0 \mathrm{e}^{(\mathrm{S}_1)^{(2)}\mathrm{t}}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |

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| $\frac{1}{(m_1)^{(2)}} \mathrm{G}_{16}^{0} \mathrm{e}^{((\mathrm{S}_1)^{(2)}-(p_{16})^{(2)})\mathrm{t}} \leq G_{17}(t) \leq \frac{1}{(m_2)^{(2)}} \mathrm{G}_{16}^{0} \mathrm{e}^{(\mathrm{S}_1)^{(2)}\mathrm{t}}$ | 311 |
| $\left(\frac{(a_{18})^{(2)}\mathrm{G}_{16}^{0}}{(m_1)^{(2)}((\mathrm{S}_1)^{(2)}-(p_{16})^{(2)}-(\mathrm{S}_2)^{(2)})}\left[\mathrm{e}^{((\mathrm{S}_1)^{(2)}-(p_{16})^{(2)})\mathrm{t}} - \mathrm{e}^{-(\mathrm{S}_2)^{(2)}\mathrm{t}}\right] + \mathrm{G}_{18}^{0}\mathrm{e}^{-(\mathrm{S}_2)^{(2)}\mathrm{t}} \leq \mathrm{G}_{18}(t) \leq \right.$ $\left.\frac{(a_{18})^{(2)}\mathrm{G}_{16}^{0}}{(m_2)^{(2)}((\mathrm{S}_1)^{(2)}-(a_{18}^{'})^{(2)})}\left[\mathrm{e}^{(\mathrm{S}_1)^{(2)}\mathrm{t}} - \mathrm{e}^{-(a_{18}^{'})^{(2)}\mathrm{t}}\right] + \mathrm{G}_{18}^{0}\mathrm{e}^{-(a_{18}^{'})^{(2)}\mathrm{t}}\right)$ | 312 |
| $\boxed{\mathrm{T}_{16}^{0}\mathrm{e}^{(\mathrm{R}_1)^{(2)}t} \leq T_{16}(t) \leq \mathrm{T}_{16}^{0}\mathrm{e}^{((\mathrm{R}_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} \mathrm{T}_{16}^{0}\mathrm{e}^{(\mathrm{R}_1)^{(2)}t} \leq T_{16}(t) \leq \frac{1}{(\mu_2)^{(2)}} \mathrm{T}_{16}^{0}\mathrm{e}^{((\mathrm{R}_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)}\mathrm{T}_{16}^{0}}{(\mu_1)^{(2)}((\mathrm{R}_1)^{(2)}-(b_{18}^{'})^{(2)})}\left[\mathrm{e}^{(\mathrm{R}_1)^{(2)}t} - \mathrm{e}^{-(b_{18}^{'})^{(2)}t}\right] + \mathrm{T}_{18}^{0}\mathrm{e}^{-(b_{18}^{'})^{(2)}t} \leq T_{18}(t) \leq$ $\frac{(a_{18})^{(2)}\mathrm{T}_{16}^{0}}{(\mu_2)^{(2)}((\mathrm{R}_1)^{(2)}+(r_{16})^{(2)}+(\mathrm{R}_2)^{(2)})}\left[\mathrm{e}^{((\mathrm{R}_1)^{(2)}+(r_{16})^{(2)})t} - \mathrm{e}^{-(\mathrm{R}_2)^{(2)}t}\right] + \mathrm{T}_{18}^{0}\mathrm{e}^{-(\mathrm{R}_2)^{(2)}t}$ | 315 |
| **Definition of** $(\mathrm{S}_1)^{(2)}, (\mathrm{S}_2)^{(2)}, (\mathrm{R}_1)^{(2)}, (\mathrm{R}_2)^{(2)}$:- | 316 |
| Where $(\mathrm{S}_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a_{16}^{'})^{(2)}$ $(\mathrm{S}_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b_{16}^{'})^{(2)}$ $(\mathrm{R}_2)^{(2)} = (b_{18}^{'})^{(2)} - (r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ : $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)} \leq -(a_{20}^{'})^{(3)} + (a_{21}^{'})^{(3)} - (a_{20}^{''})^{(3)}(T_{21}, t) + (a_{21}^{''})^{(3)}(T_{21}, t) \leq -(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)} \leq -(b_{20}^{'})^{(3)} + (b_{21}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t) - (b_{21}^{''})^{(3)}((G_{23}), t) \leq -(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(\nu_1)^{(3)}, (\nu_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ : By $(\nu_1)^{(3)} > 0$, $(\nu_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$, $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}(\nu^{(3)})^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$ and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and By $(\bar{\nu}_1)^{(3)} > 0$, $(\bar{\nu}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$, $(\bar{u}_2)^{(3)} < 0$ the | 320 |

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| roots of the equations $(a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$<br><br>and $(b_{21})^{(3)}\left(u^{(3)}\right)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ | |
| **Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :-<br><br>If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by<br>$(m_2)^{(3)} = (v_0)^{(3)}, (m_1)^{(3)} = (v_1)^{(3)}$, ***if*** $(v_0)^{(3)} < (v_1)^{(3)}$<br><br>$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (\bar{v}_1)^{(3)}$, ***if*** $(v_1)^{(3)} < (v_0)^{(3)} < (\bar{v}_1)^{(3)}$,<br><br>and $\boxed{(v_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$<br><br>$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (v_0)^{(3)}$, ***if*** $(\bar{v}_1)^{(3)} < (v_0)^{(3)}$ | 321 |
| and analogously<br><br>$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$<br><br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$<br><br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$<br><br>Then the solution of global equations satisfies the inequalities<br><br>$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)}t}$<br><br>$(p_i)^{(3)}$ is defined by equation | 322 |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \le G_{22}(t) \le \right.$<br>$\left.\frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a'_{22})^{(3)})}\left[e^{(S_1)^{(3)}t} - e^{-(a'_{22})^{(3)}t}\right] + G_{22}^0 e^{-(a'_{22})^{(3)}t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b'_{22})^{(3)})}\left[e^{(R_1)^{(3)}t} - e^{-(b'_{22})^{(3)}t}\right] + T_{22}^0 e^{-(b'_{22})^{(3)}t} \le T_{22}(t) \le$<br><br>$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$ | 327 |

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- 328

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$

$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$

$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$

**Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : 329

By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{v}_1)^{(4)} > 0$, $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (v_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}$, ***if*** $(v_0)^{(4)} < (v_1)^{(4)}$

$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}$, ***if*** $(v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)}$,

and $\boxed{(v_0)^{(4)} = \frac{G^0_{24}}{G^0_{25}}}$

$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)} < (v_0)^{(4)}$

and analogously 331

$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$

$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$, ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,

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| and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \boldsymbol{if}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)} t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)} t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\right)\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)} t}\right] + G_{26}^0 e^{-(S_2)^{(4)} t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b_{26}')^{(4)})}\left[e^{(R_1)^{(4)} t} - e^{-(b_{26}')^{(4)} t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)} t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)} + (r_{24})^{(4)} + (R_2)^{(4)})}\left[e^{((R_1)^{(4)} + (r_{24})^{(4)})t} - e^{-(R_2)^{(4)} t}\right] + T_{26}^0 e^{-(R_2)^{(4)} t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |

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| **Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ :<br><br>By $(v_1)^{(5)} > 0$ , $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$ , $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | 339 |
| **Definition of** $(\bar{v}_1)^{(5)},, (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ :<br><br>By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :-<br><br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}, (\mu_2)^{(5)}$ by<br><br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br><br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br><br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |

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| $\frac{1}{(\mu_1)^{(5)}}T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}}T_{28}^0 e^{((R_1)^{(5)}+(r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)}T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)}-(b_{30}')^{(5)})}\left[e^{(R_1)^{(5)}t} - e^{-(b_{30}')^{(5)}t}\right] + T_{30}^0 e^{-(b_{30}')^{(5)}t} \le T_{30}(t) \le$<br>$\frac{(a_{30})^{(5)}T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t} - e^{-(R_2)^{(5)}t}\right] + T_{30}^0 e^{-(R_2)^{(5)}t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a_{28}')^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b_{28}')^{(5)}$<br>$(R_2)^{(5)} = (b_{30}')^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :<br>$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying<br>$-(\sigma_2)^{(6)} \le -(a_{32}')^{(6)} + (a_{33}')^{(6)} - (a_{32}'')^{(6)}(T_{33}, t) + (a_{33}'')^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$<br>$-(\tau_2)^{(6)} \le -(b_{32}')^{(6)} + (b_{33}')^{(6)} - (b_{32}'')^{(6)}((G_{35}), t) - (b_{33}'')^{(6)}((G_{35}), t) \le -(\tau_1)^{(6)}$ | 349 |
| **Definition of** $(v_1)^{(6)}, (v_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, v^{(6)}, u^{(6)}$ :<br>By $(v_1)^{(6)} > 0$, $(v_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$, $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(v^{(6)})^2 + (\sigma_1)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{v}_1)^{(6)},, (\bar{v}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br>By $(\bar{v}_1)^{(6)} > 0$, $(\bar{v}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$, $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(v^{(6)})^2 + (\sigma_2)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (v_0)^{(6)}$ :-<br>If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br>$(m_2)^{(6)} = (v_0)^{(6)}, (m_1)^{(6)} = (v_1)^{(6)}$, ***if*** $(v_0)^{(6)} < (v_1)^{(6)}$<br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (\bar{v}_6)^{(6)}$, ***if*** $(v_1)^{(6)} < (v_0)^{(6)} < (\bar{v}_1)^{(6)}$, | 351 |

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| and $\boxed{(v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$<br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (v_0)^{(6)},$ ***if*** $(\bar{v}_1)^{(6)} < (v_0)^{(6)}$ | |
| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},$ ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)},$ ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \leq G_{32}(t) \leq G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \leq G_{33}(t) \leq \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\right)\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \leq G_{34}(t) \leq$<br>*(a34)6G320(m2)6(S1)6−(a34')6e(S1)6t−e−(a34')6t+ G340e−(a34')6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \leq T_{32}(t) \leq T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \leq T_{32}(t) \leq \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b_{34}')^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b_{34}')^{(6)} t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)} t} \leq T_{34}(t) \leq$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$<br>$(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** | |

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ :

$(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying

$$-(\sigma_2)^{(7)} \le -(a_{36}^{'})^{(7)} + (a_{37}^{'})^{(7)} - (a_{36}^{''})^{(7)}(T_{37}, t) + (a_{37}^{''})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$$

$$-(\tau_2)^{(7)} \le -(b_{36}^{'})^{(7)} + (b_{37}^{'})^{(7)} - (b_{36}^{''})^{(7)}\big((G_{39}), t\big) - (b_{37}^{''})^{(7)}\big((G_{39}), t\big) \le -(\tau_1)^{(7)}$$

**Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ : 361

By $(\nu_1)^{(7)} > 0, (\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ : 362

By $(\bar{\nu}_1)^{(7)} > 0, (\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$

**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,
and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$

and analogously 363

$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,
and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

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| Then the solution of global equations satisfies the inequalities<br>$G_{36}^0 e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)}t}$<br>where $(p_i)^{(7)}$ is defined by equation | 364 |
| $\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^0 e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}')^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}')^{(7)}t}] + G_{38}^0 e^{-(a_{38}')^{(7)}t})$ | 366 |
| $\boxed{T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}')^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}')^{(7)}t}\right] + T_{38}^0 e^{-(b_{38}')^{(7)}t} \le T_{38}(t) \le$<br>$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^0 e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}')^{(7)}$<br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}')^{(7)}$<br>$(R_2)^{(7)} = (b_{38}')^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br>$-(\sigma_2)^{(8)} \le -(a_{40}')^{(8)} + (a_{41}')^{(8)} - (a_{40}'')^{(8)}(T_{41}, t) + (a_{41}'')^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br>$-(\tau_2)^{(8)} \le -(b_{40}')^{(8)} + (b_{41}')^{(8)} - (b_{40}'')^{(8)}((G_{43}), t) - (b_{41}'')^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |

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|---|---|
| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : <br><br> By $(v_1)^{(8)} > 0$ , $(v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0$ , $(u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$ <br> and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | 372 |
| **Definition of** $(\bar{v}_1)^{(8)}$,, $(\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ : <br><br> By $(\bar{v}_1)^{(8)} > 0$ , $(\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0$ , $(\bar{u}_2)^{(8)} < 0$ the <br><br> roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$ <br><br> and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ <br><br> **Definition of** $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :- <br><br> If we define $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)}, (\mu_2)^{(8)}$ by <br><br> $(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)}$, ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$ <br><br> $(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}$ , ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)}$, <br> and $\boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ <br><br> $(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (v_0)^{(8)}$, ***if*** $(\bar{v}_1)^{(8)} < (v_0)^{(8)}$ | |
| and analogously <br><br> $(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)}$, ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$ <br><br> $(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)}$, <br> and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$ <br><br> $(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}$, ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities <br><br> $G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{40}(t) \leq G_{40}^0 e^{(S_1)^{(8)}t}$ <br><br> where $(p_i)^{(8)}$ is defined by equation | 375 |

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| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)}-(p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)}G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t} - e^{-(S_2)^{(8)}t}\right] + G_{42}^0 e^{-(S_2)^{(8)}t} \le G_{42}(t) \le \right.$ $\left.\frac{(a_{42})^{(8)}G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)}-(a'_{42})^{(8)})}[e^{(S_1)^{(8)}t} - e^{-(a'_{42})^{(8)}t}] + G_{42}^0 e^{-(a'_{42})^{(8)}t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)}-(b'_{42})^{(8)})}\left[e^{(R_1)^{(8)}t} - e^{-(b'_{42})^{(8)}t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)}t} \le T_{42}(t) \le$ $\frac{(a_{42})^{(8)}T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t} - e^{-(R_2)^{(8)}t}\right] + T_{42}^0 e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$<br>$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}((G_{47}), t) - (b''_{45})^{(9)}((G_{47}), t) \le -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :<br>By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$<br>and $(b_{45})^{(9)}(u^{(9)})^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(9)},, (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ : | |

By $(\bar{v}_1)^{(9)} > 0$ , $(\bar{v}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\left(u^{(9)}\right)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}$ , $(m_2)^{(9)}$ , $(\mu_1)^{(9)}$, $(\mu_2)^{(9)}$, $(v_0)^{(9)}$ :-

If we define $(m_1)^{(9)}$ , $(m_2)^{(9)}$ , $(\mu_1)^{(9)}$, $(\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (v_0)^{(9)}, (m_1)^{(9)} = (v_1)^{(9)}$, ***if*** $(v_0)^{(9)} < (v_1)^{(9)}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (\bar{v}_1)^{(9)}$ , ***if*** $(v_1)^{(9)} < (v_0)^{(9)} < (\bar{v}_1)^{(9)}$,

and $\boxed{(v_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (v_0)^{(9)}$, ***if*** $(\bar{v}_1)^{(9)} < (v_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^0 e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}} G_{44}^0 e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)} G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)} - (p_{44})^{(9)} - (S_2)^{(9)})}\left[e^{((S_1)^{(9)} - (p_{44})^{(9)})t} - e^{-(S_2)^{(9)}t}\right] + G_{46}^0 e^{-(S_2)^{(9)}t} \le G_{46}(t) \le \right.$$

$$\left. \frac{(a_{46})^{(9)} G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)} - (a_{46}')^{(9)})}\left[e^{(S_1)^{(9)}t} - e^{-(a_{46}')^{(9)}t}\right] + G_{46}^0 e^{-(a_{46}')^{(9)}t}\right)$$

$$\boxed{T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}} T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}} T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)} T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)} - (b_{46}')^{(9)})}\left[e^{(R_1)^{(9)}t} - e^{-(b_{46}')^{(9)}t}\right] + T_{46}^0 e^{-(b_{46}')^{(9)}t} \le T_{46}(t) \le$$

$$\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)},(S_2)^{(9)},(R_1)^{(9)},(R_2)^{(9)}$:-

Where $(S_1)^{(9)}=(a_{44})^{(9)}(m_2)^{(9)}-(a_{44}^{'})^{(9)}$

$(S_2)^{(9)}=(a_{46})^{(9)}-(p_{46})^{(9)}$

$(R_1)^{(9)}=(b_{44})^{(9)}(\mu_2)^{(9)}-(b_{44}^{'})^{(9)}$

$(R_2)^{(9)}=(b_{46}^{'})^{(9)}-(r_{46})^{(9)}$

**Proof :** From global equations we obtain 383

$$\frac{dv^{(1)}}{dt}=(a_{13})^{(1)}-\left((a_{13}^{'})^{(1)}-(a_{14}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14},t)\right)-(a_{14}^{''})^{(1)}(T_{14},t)v^{(1)}-(a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)}=\frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_2)^{(1)}v^{(1)}-(a_{13})^{(1)}\right)\leq\frac{dv^{(1)}}{dt}\leq-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_1)^{(1)}v^{(1)}-(a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(1)},(\nu_0)^{(1)}$ :-

For $0<\boxed{(\nu_0)^{(1)}=\frac{G_{13}^{0}}{G_{14}^{0}}}<(\nu_1)^{(1)}<(\bar{\nu}_1)^{(1)}$

$$v^{(1)}(t)\geq\frac{(\nu_1)^{(1)}+(C)^{(1)}(\nu_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_0)^{(1)}\right)t\right]}}{1+(C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_0)^{(1)}\right)t\right]}}\quad,\quad\boxed{(C)^{(1)}=\frac{(\nu_1)^{(1)}-(\nu_0)^{(1)}}{(\nu_0)^{(1)}-(\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)}\leq v^{(1)}(t)\leq(\nu_1)^{(1)}$

In the same manner , we get 384

$$v^{(1)}(t)\leq\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(1)}=\frac{(\bar{\nu}_1)^{(1)}-(\nu_0)^{(1)}}{(\nu_0)^{(1)}-(\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)}\leq v^{(1)}(t)\leq(\bar{\nu}_1)^{(1)}$

If $0<(\nu_1)^{(1)}<(\nu_0)^{(1)}=\frac{G_{13}^{0}}{G_{14}^{0}}<(\bar{\nu}_1)^{(1)}$ we find like in the previous case, 385

$$(\nu_1)^{(1)} \le \frac{(\nu_1)^{(1)}+(C)^{(1)}(\nu_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_2)^{(1)}\right)t\right]}}{1+(C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_2)^{(1)}\right)t\right]}} \le \nu^{(1)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \le (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \le (\bar{\nu}_1)^{(1)} \le \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \le \nu^{(1)}(t) \le \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \le (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \le \nu^{(1)}(t) \le (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \le u^{(1)}(t) \le (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)}G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)}T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}')^{(2)} - (a_{17}')^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

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| $-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2+(\sigma_2)^{(2)}\nu^{(2)}-(a_{16})^{(2)}\right)\le\frac{d\nu^{(2)}}{dt}\le-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2+(\sigma_1)^{(2)}\nu^{(2)}-(a_{16})^{(2)}\right)$ | |
| From which one obtains<br><br>**Definition of** $(\bar{\nu}_1)^{(2)},(\nu_0)^{(2)}$ :-<br><br>For $0<(\nu_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}<(\nu_1)^{(2)}<(\bar{\nu}_1)^{(2)}$<br><br>$\nu^{(2)}(t)\ge\frac{(\nu_1)^{(2)}+(C)^{(2)}(\nu_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_0)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_0)^{(2)}\right)t\right]}}$ , $\boxed{(C)^{(2)}=\frac{(\nu_1)^{(2)}-(\nu_0)^{(2)}}{(\nu_0)^{(2)}-(\nu_2)^{(2)}}}$<br><br>it follows $(\nu_0)^{(2)}\le\nu^{(2)}(t)\le(\nu_1)^{(2)}$ | 390 |
| In the same manner , we get<br><br>$\nu^{(2)}(t)\le\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(2)}=\frac{(\bar{\nu}_1)^{(2)}-(\nu_0)^{(2)}}{(\nu_0)^{(2)}-(\bar{\nu}_2)^{(2)}}}$ | 391 |
| From which we deduce $(\nu_0)^{(2)}\le\nu^{(2)}(t)\le(\bar{\nu}_1)^{(2)}$ | 392 |
| If $0<(\nu_1)^{(2)}<(\nu_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}<(\bar{\nu}_1)^{(2)}$ we find like in the previous case,<br><br>$(\nu_1)^{(2)}\le\frac{(\nu_1)^{(2)}+(C)^{(2)}(\nu_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}}\le\nu^{(2)}(t)\le$<br><br>$\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}\le(\bar{\nu}_1)^{(2)}$ | 393 |
| If $0<(\nu_1)^{(2)}\le(\bar{\nu}_1)^{(2)}\le(\nu_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(\nu_1)^{(2)}\le\nu^{(2)}(t)\le\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}\le(\nu_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br><br>$(m_2)^{(2)}\le\nu^{(2)}(t)\le(m_1)^{(2)}$, $\boxed{\nu^{(2)}(t)=\frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :- | 396 |

| | |
|---|---|
| $(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| From which one obtains<br><br>For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$<br><br>$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)} + (C)^{(3)}(\nu_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}$ , $\boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\nu_2)^{(3)}}}$<br><br>it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$ | 400 |
| In the same manner , we get<br><br>$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)} + (C)^{(3)}(\bar{\nu}_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\bar{\nu}_2)^{(3)}}}$<br><br>**Definition of** $(\bar{\nu}_1)^{(3)}$ :-<br><br>From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$ | 401 |

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If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case,

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

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If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}'')^{(3)} = (a_{21}'')^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b_{20}'')^{(3)} = (b_{21}'')^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}')^{(4)} - (a_{25}')^{(4)} + (a_{24}'')^{(4)}(T_{25},t)\right) - (a_{25}'')^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_2)^{(4)}v^{(4)}-(a_{24})^{(4)}\right)\le\frac{dv^{(4)}}{dt}\le-\left((a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_4)^{(4)}v^{(4)}-(a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(4)},(v_0)^{(4)}$ :-

For $0<\boxed{(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}<(v_1)^{(4)}<(\bar{v}_1)^{(4)}$

$$v^{(4)}(t)\ge\frac{(v_1)^{(4)}+(C)^{(4)}(v_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_0)^{(4)}\right)t\right]}}{4+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_0)^{(4)}\right)t\right]}}\quad,\quad\boxed{(C)^{(4)}=\frac{(v_1)^{(4)}-(v_0)^{(4)}}{(v_0)^{(4)}-(v_2)^{(4)}}}$$

it follows $(v_0)^{(4)}\le v^{(4)}(t)\le(v_1)^{(4)}$

In the same manner , we get

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$$v^{(4)}(t)\le\frac{(\bar{v}_1)^{(4)}+(\bar{C})^{(4)}(\bar{v}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}{4+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(4)}=\frac{(\bar{v}_1)^{(4)}-(v_0)^{(4)}}{(v_0)^{(4)}-(\bar{v}_2)^{(4)}}}$$

From which we deduce $(v_0)^{(4)}\le v^{(4)}(t)\le(\bar{v}_1)^{(4)}$

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If $0<(v_1)^{(4)}<(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}<(\bar{v}_1)^{(4)}$ we find like in the previous case,

$$(v_1)^{(4)}\le\frac{(v_1)^{(4)}+(C)^{(4)}(v_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_2)^{(4)}\right)t\right]}}{1+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_2)^{(4)}\right)t\right]}}\le v^{(4)}(t)\le$$

$$\frac{(\bar{v}_1)^{(4)}+(\bar{C})^{(4)}(\bar{v}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}\le(\bar{v}_1)^{(4)}$$

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If $0<(v_1)^{(4)}\le(\bar{v}_1)^{(4)}\le\boxed{(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(v_1)^{(4)}\le v^{(4)}(t)\le\frac{(\bar{v}_1)^{(4)}+(\bar{C})^{(4)}(\bar{v}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}\le(v_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(4)}(t)$ :-

$$(m_2)^{(4)}\le v^{(4)}(t)\le(m_1)^{(4)},\quad\boxed{v^{(4)}(t)=\frac{G_{24}(t)}{G_{25}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$$(\mu_2)^{(4)}\le u^{(4)}(t)\le(\mu_1)^{(4)},\quad\boxed{u^{(4)}(t)=\frac{T_{24}(t)}{T_{25}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}'')^{(5)} = (b_{29}'')^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}')^{(6)} - (a_{33}')^{(6)} + (a_{32}'')^{(6)}(T_{33}, t)\right) - (a_{33}'')^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get 413

$$\nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\bar{\nu}_1)^{(6)}$

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If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

$$(\nu_1)^{(6)} \leq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_2)^{(6)}\right)t\right]}} \leq \nu^{(6)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\bar{\nu}_1)^{(6)}$$

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If $0 < (\nu_1)^{(6)} \leq (\bar{\nu}_1)^{(6)} \leq \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(\nu_1)^{(6)} \leq \nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

$$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}, \quad \boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}, \quad \boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)}G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a'_{36})^{(7)} - (a'_{37})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right) - (a''_{37})^{(7)}(T_{37},t)v^{(7)} - (a_{37})^{(7)}v^{(7)}$$

**Definition of** $v^{(7)}$ :- $\boxed{v^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)}\right) \le \frac{dv^{(7)}}{dt} \le -\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(7)}, (v_0)^{(7)}$ :-

For $0 < \boxed{(v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}} < (v_1)^{(7)} < (\bar{v}_1)^{(7)}$

$$v^{(7)}(t) \ge \frac{(v_1)^{(7)} + (C)^{(7)}(v_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(v_1)^{(7)} - (v_0)^{(7)}}{(v_0)^{(7)} - (v_2)^{(7)}}}$$

it follows $(v_0)^{(7)} \le v^{(7)}(t) \le (v_1)^{(7)}$

In the same manner , we get

$$v^{(7)}(t) \le \frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{v}_1)^{(7)} - (v_0)^{(7)}}{(v_0)^{(7)} - (\bar{v}_2)^{(7)}}}$$

From which we deduce $(v_0)^{(7)} \le v^{(7)}(t) \le (\bar{v}_1)^{(7)}$

If $0 < (v_1)^{(7)} < (v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}} < (\bar{v}_1)^{(7)}$ we find like in the previous case,

$$(v_1)^{(7)} \le \frac{(v_1)^{(7)} + (C)^{(7)}(v_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_2)^{(7)}\right)t\right]}} \le v^{(7)}(t) \le$$

$$\frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \le (\bar{v}_1)^{(7)}$$

If $0 < (v_1)^{(7)} \le (\bar{v}_1)^{(7)} \le \boxed{(v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$ , we obtain

$$(v_1)^{(7)} \le v^{(7)}(t) \le \frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \le (v_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

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**Definition of** $v^{(7)}(t)$ :-

$(m_2)^{(7)} \leq v^{(7)}(t) \leq (m_1)^{(7)}$, $\boxed{v^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :-

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$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $v^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)} G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

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$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right) - (a_{41}^{''})^{(8)}(T_{41}, t) v^{(8)} - (a_{41})^{(8)} v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)} v^{(8)} - (a_{40})^{(8)}\right) \leq \frac{dv^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)} v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$v^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)}-(\nu_0)^{(8)}}{(\nu_0)^{(8)}-(\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\bar{\nu}_8)^{(8)}$

If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case, 423

$$(\nu_1)^{(8)} \leq \frac{(\nu_1)^{(8)}+(C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}} \leq \nu^{(8)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\bar{\nu}_1)^{(8)}$$

If $0 < (\nu_1)^{(8)} \leq (\bar{\nu}_1)^{(8)} \leq \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain 424

$$(\nu_1)^{(8)} \leq \nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)}+(C)^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \leq \nu^{(8)}(t) \leq (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)}G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)}T_{41}(t)$ This is an important

consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{d\nu^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right) - (a''_{45})^{(9)}(T_{45},t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \le \frac{d\nu^{(9)}}{dt} \le -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \ge \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \le \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}} \le \nu^{(9)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \le (\bar{\nu}_1)^{(9)} \le \boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$ , we obtain

$$(\nu_1)^{(9)} \le \nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$(m_2)^{(9)} \leq \nu^{(9)}(t) \leq (m_1)^{(9)}$, $\boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}$, $\boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ 430

$with\ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations 431

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$

| | |
|---|---|
| $(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$ | 435 |

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| $(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |

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| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |

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| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |

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| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(h) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) +$ | 488 |

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| $(a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first | 494 |

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| equations | |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |

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| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{40}=\frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}=\frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 501 |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{44}=\frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}=\frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G)=(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)}-(b_{13})^{(1)}(b_{14})^{(1)}-$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G)+(b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G)=0$<br><br>Where in $G(G_{13},G_{14},G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*})=0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19})=(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)}-(b_{16})^{(2)}(b_{17})^{(2)}-$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19})+(b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19})=0$ | 503 |
| Where in $(G_{19})(G_{16},G_{17},G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*})=0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23})=(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)}-(b_{20})^{(3)}(b_{21})^{(3)}-$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23})+(b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23})=0$<br><br>Where in $G_{23}(G_{20},G_{21},G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*})=0$ | 505 |

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| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right] + (b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br><br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br><br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br><br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br><br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$ | |

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| $\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47})+(b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right]+(b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47})=0$<br><br>Where in $(G_{47})(G_{44},G_{45},G_{46}),G_{44},G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*})=0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*})=0$ , $T_{14}^{*}$ given by $f(T_{14}^{*})=0$ and<br><br>$G_{13}^{*}=\frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*}=\frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*}=\frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*}=\frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*})=0$ , $T_{17}^{*}$ given by $f(T_{17}^{*})=0$ and | 512 |
| $G_{16}^{*}=\frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*}=\frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*}=\frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*}=\frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*})=0$ , $T_{21}^{*}$ given by $f(T_{21}^{*})=0$ and<br><br>$G_{20}^{*}=\frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*}=\frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br><br>$T_{20}^{*}=\frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*}=\frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br><br>$G_{25}^{*}$ given by $\varphi(G_{27})=0$ , $T_{25}^{*}$ given by $f(T_{25}^{*})=0$ and<br><br>$G_{24}^{*}=\frac{(a_{24})^{(4)}G_{25}^{*}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*}=\frac{(a_{26})^{(4)}G_{25}^{*}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*}=\frac{(b_{24})^{(4)}T_{25}^{*}}{[(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*}=\frac{(b_{26})^{(4)}T_{25}^{*}}{[(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}((G_{27})^{*})]}$ | 517 |

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| Obviously, these values represent an equilibrium solution of global equations | |
| Finally we obtain the unique solution<br><br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*}) = 0$ , $T_{29}^{*}$ given by $f(T_{29}^{*}) = 0$ and<br><br>$G_{28}^{*} = \frac{(a_{28})^{(5)} G_{29}^{*}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*} = \frac{(a_{30})^{(5)} G_{29}^{*}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |
| $T_{28}^{*} = \frac{(b_{28})^{(5)} T_{29}^{*}}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)} T_{29}^{*}}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br><br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br><br>$G_{32}^{*} = \frac{(a_{32})^{(6)} G_{33}^{*}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)} G_{33}^{*}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)} T_{33}^{*}}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)} T_{33}^{*}}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br><br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br><br>$G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99 | 523 A |

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| $G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof:** Denote<br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*}\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*}\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*}\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*}\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$ | 532 |

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| $\frac{\partial(a_{17}^{''})^{(2)}}{\partial \mathrm{T}_{17}}(\mathrm{T}_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial(b_i^{''})^{(2)}}{\partial \mathrm{G}_j}((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{\mathrm{d}\mathbb{G}_{16}}{\mathrm{dt}} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}\mathrm{G}_{16}^{*}\mathbb{T}_{17}$ | 534 |
| $\frac{\mathrm{d}\mathbb{G}_{17}}{\mathrm{dt}} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}\mathrm{G}_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{\mathrm{d}\mathbb{G}_{18}}{\mathrm{dt}} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}\mathrm{G}_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{\mathrm{d}\mathbb{T}_{16}}{\mathrm{dt}} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}\mathrm{T}_{16}^{*}\mathbb{G}_j\right)$ | 537 |
| $\frac{\mathrm{d}\mathbb{T}_{17}}{\mathrm{dt}} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}\mathrm{T}_{17}^{*}\mathbb{G}_j\right)$ | 538 |
| $\frac{\mathrm{d}\mathbb{T}_{18}}{\mathrm{dt}} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}\mathrm{T}_{18}^{*}\mathbb{G}_j\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_{+})$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)}$ , $\frac{\partial(b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_j\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 547 |

**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-

$$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)} \quad , \frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^* ) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25} \quad (549)$$

$$\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25} \quad (550)$$

$$\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25} \quad (551)$$

$$\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right) \quad (552)$$

$$\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right) \quad (553)$$

$$\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right) \quad (554)$$

**ASYMPTOTIC STABILITY ANALYSIS** (555)

**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ $and$ $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- (556)

$$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)} \quad , \frac{\partial (b_i^{''})^{(5)}}{\partial G_j}(\,(G_{31})^*\,) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29} \quad (557)$$

$$\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29} \quad (558)$$

$$\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29} \quad (559)$$

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| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^{*}\mathbb{G}_{j}\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^{*}\mathbb{G}_{j}\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^{*}\mathbb{G}_{j}\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^{*}) = (q_{33})^{(6)}$ , $\frac{\partial(b_i^{''})^{(6)}}{\partial G_j}((G_{35})^{*}) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^{*}\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^{*}\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^{*}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^{*}\mathbb{G}_{j}\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^{*}\mathbb{G}_{j}\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^{*}\mathbb{G}_{j}\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 572 |

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|---|---|
| $\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^{*}) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^{*}\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^{*}\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^{*}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^{*}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^{*}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^{*}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^{*}) = (q_{41})^{(8)}$ , $\frac{\partial (b_i^{''})^{(8)}}{\partial G_j}\left((G_{43})^{*}\right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^{*}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^{*}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^{*}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^{*}\mathbb{G}_j\right)$ | 584 |

| | |
|---|---|
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a''_i)^{(9)}$ *and* $(b''_i)^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{45})^{(9)}}{\partial T_{45}}(T^*_{45}) = (q_{45})^{(9)} \ , \ \frac{\partial(b''_i)^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a'_{44})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G^*_{44}\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a'_{45})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G^*_{45}\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a'_{46})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G^*_{46}\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$<br><br>$\left[\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\Big]$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\Big)$<br><br>$+\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\Big)$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\Big)$<br><br>$\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$ | |

$$\left((\lambda)^{(1)}\right)^2+\left((b_{13}')^{(1)}+(b_{14}')^{(1)}-(r_{13})^{(1)}+(r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$$

$$+\left(\left((\lambda)^{(1)}\right)^2+\left((a_{13}')^{(1)}+(a_{14}')^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$$

$$+\left((\lambda)^{(1)}+(a_{13}')^{(1)}+(p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^{*}+(a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^{*}\right)$$

$$\left((\lambda)^{(1)}+(b_{13}')^{(1)}-(r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^{*}+(b_{14})^{(1)}s_{(13),(15)}T_{13}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(2)}+(b_{18}')^{(2)}-(r_{18})^{(2)}\}\{\left((\lambda)^{(2)}+(a_{18}')^{(2)}+(p_{18})^{(2)}\right)$$

$$\left[\left(\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(17)}\mathrm{T}_{17}^{*}\right)$$

$$+\left(\left((\lambda)^{(2)}+(a_{17}')^{(2)}+(p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}_{16}^{*}+(a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(16)}\mathrm{T}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)$$

$$\left(\left((\lambda)^{(2)}\right)^2+\left((b_{16}')^{(2)}+(b_{17}')^{(2)}-(r_{16})^{(2)}+(r_{17})^{(2)}\right)(\lambda)^{(2)}\right)$$

$$+\left(\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\}\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left(\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\right)$$

$$+\left(\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((b_{20}')^{(3)}+(b_{21}')^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(4)}+(b_{26}')^{(4)}-(r_{26})^{(4)}\{\left((\lambda)^{(4)}+(a_{26}')^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}')^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}')^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}')^{(4)}+(a_{25}')^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((b_{24}')^{(4)}+(b_{25}')^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}')^{(4)}+(a_{25}')^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}')^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}')^{(5)}-(r_{30})^{(5)}\{\left((\lambda)^{(5)}+(a_{30}')^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}')^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}')^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}')^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}')^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}')^{(5)}+(a_{29}')^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left((\lambda)^{(5)}\right)^2+\left((b_{28}')^{(5)}+(b_{29}')^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$+\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}')^{(5)}+(a_{29}')^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}')^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}')^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}')^{(6)}-(r_{34})^{(6)}\{\left((\lambda)^{(6)}+(a_{34}')^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left((\lambda)^{(6)}+(a_{32}')^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\Big]$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\Big)$$

$$+\left((\lambda)^{(6)}+(a_{33}')^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\Big)$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((a_{32}')^{(6)}+(a_{33}')^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((b_{32}')^{(6)}+(b_{33}')^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$+\left(\left((\lambda)^{(6)}\right)^2+\left((a_{32}')^{(6)}+(a_{33}')^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}')^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b_{38}')^{(7)}-(r_{38})^{(7)}\{\left((\lambda)^{(7)}+(a_{38}')^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left((\lambda)^{(7)}+(a_{36}')^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\Big]$$

$$\left((\lambda)^{(7)}+(b_{36}')^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\Big)$$

$$+\left((\lambda)^{(7)}+(a_{37}')^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\Big)$$

$$\left((\lambda)^{(7)}+(b_{36}')^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a_{36}')^{(7)}+(a_{37}')^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b_{36}')^{(7)}+(b_{37}')^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left(\left((\lambda)^{(7)}\right)^2+\left((a_{36}')^{(7)}+(a_{37}')^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}')^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left((\lambda)^{(7)}+(b_{36}')^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b_{42}')^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a_{42}')^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a_{40}')^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(q_{40})^{(8)}G_{40}^{*}\right)\right]$$

$$\left((\lambda)^{(8)}+(b_{40}')^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(41)}T_{41}^{*}\Big)$$

$$+\left(\left((\lambda)^{(8)}+(a_{41}')^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G_{40}^{*}+(a_{40})^{(8)}(q_{41})^{(8)}G_{41}^{*}\right)$$

$$\left((\lambda)^{(8)}+(b_{40}')^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(40)}T_{40}^{*}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((a_{40}')^{(8)}+(a_{41}')^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b_{40}')^{(8)}+(b_{41}')^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left(\left((\lambda)^{(8)}\right)^2+\left((a_{40}')^{(8)}+(a_{41}')^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a_{40}')^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G_{40}^{*}\right)$$

$$\left((\lambda)^{(8)}+(b_{40}')^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(42)}T_{40}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b_{46}')^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a_{46}')^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a_{44}')^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G_{45}^{*}+(a_{45})^{(9)}(q_{44})^{(9)}G_{44}^{*}\right)\right]$$

$$\left((\lambda)^{(9)}+(b_{44}')^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T_{45}^{*}+(b_{45})^{(9)}s_{(44),(45)}T_{45}^{*}\Big)$$

$$+\left(\left((\lambda)^{(9)}+(a_{45}')^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G_{44}^{*}+(a_{44})^{(9)}(q_{45})^{(9)}G_{45}^{*}\right)$$

$$\left((\lambda)^{(9)}+(b_{44}')^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T_{45}^{*}+(b_{45})^{(9)}s_{(44),(44)}T_{44}^{*}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45}+(b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt}= A_i\ (x)\ xi$ where $x_i \in R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i \ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i \ X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions

are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$(8) \qquad \frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17].

In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 8. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i(x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \epsilon C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population $i$. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility

assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C\backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^n_i$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\in \Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two

propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population $i$ persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population $i$ is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) In \ |B_i \ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^t_0 (A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = limsup_{t\to\infty} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni}= E_{i(x)} \ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \ \leq \ a| \ B_i(t,x)u_i(x)|+ \ |B_i(t,x)w| \ \ \leq \ | \ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)| \ w|)$. Hence, $limsup_{t\to\infty} \frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$limsup_{t\to\infty} \frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \ \geq \ a|B_i(t,x)u_i(x)|-|B_i(t,x)w|$$

$$\geq | \ B_i(t,x)u_i(x)|(a- \alpha+ exp\ (-\beta t)| \ w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \ \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \ \geq \ \lambda i(x) \geq \ \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) \ = \ In|B_i(t+s,x)\ u_i(x)|= In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

| |
|---|
| $= \text{In } \lvert B_i(s,x.t)\, u_i(x.t) \rvert \lvert B_i(t,x)\, u_i(x) \rvert$ |
| $= \text{In } b_i(s,t.x)\, b_i(t,x)$ |
| |
| Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies |
| $\frac{d}{dt} b_i(t,x) = \lim_{s \to 0} \frac{bi(t+s,x-bi(t,x)}{S-}$ |
| $= \lim_{s \to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}\lvert_{s=0}\, b_i(s,x.t)$ |
| = |
| $=(A_i(x.t)u_i(x.s),u_i(x.s),$ |
| The Fundamental Theorem of Calculus implies |
| $B_i(t,x) = \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$ |
| and the second assertion follows. |
| **Proposition 2:** *For the solutions of (2) we have:* if $x_i > 0$, then $\lambda i(x) \leq 0$. If $\lambda i(x) < 0$, then $\lim_{t \to \infty} (x.t)_i = 0$ If $\limsup_{t \to \infty}$ $(x.t)_i \rvert > 0$, then $\lambda i(x) = 0$. |
| **Proof:** First, assume that x satisfies $x_i > 0$. Since the semi flow is dissipative, there exists $\gamma > 0$ such that $\lvert x.t \rvert \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply |
| $\lambda i(x) = \limsup_{t \to \infty} \frac{1}{t} \text{In } \lvert B_i(t,x)x_i \rvert = \limsup_{t \to \infty} \frac{1}{t} \text{In } \lvert (x.t)_i \rvert$ |
| $\leq \limsup_{t \to \infty} \frac{In\, \gamma}{t} = 0$ |
| To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i = 0$, then the invariance of the faces of C imply that $(x.t)_i = 0$ for all $t \geq 0$. Alternatively, if $x_i > 0$, then Proposition I and the skew product flow imply |
| $\limsup_{t \to \infty} \frac{1}{t} In \lvert (x.t)_i \rvert = \lambda i(x) < 0.$ |
| The final assertion follows from the first two assertions. 5 |
| |

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on $\boldsymbol{C}$, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:*$\lambda_i(\mathrm{x}) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle \mathrm{A}_i(x.s)\mathrm{u}_i(x.s), \mathrm{u}_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(\mathrm{x}) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $\mathrm{h}_i(x) = \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I}\boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I}\boldsymbol{C}_i^+$such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following

equivalent conditions hold For all invariant measures $\mu$ supported by $K$ (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by $K$.

then $K$ is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and $K$ is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by $K$.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$. Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large

so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \leq \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \leq \in$, and $|h(x.t) - h(x^n.t)| \leq \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$$\leq \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$$

$$\leq \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$$

$$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$$

$$\leq \quad 3 \in$$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t) d\mu(x) = \int_c h(x) d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x) u_i^n(x), u_i^n(x) \rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop. 3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop. 3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)| d\mu(x) + |\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\, d\mu_n(x)$$

$$\leq 2 \in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm. 2.1]$ implies that either there exists $\mathcal{Y} \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}$

$\int_0^t \delta_{y.s}$ ds where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$ . Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i$: C → R by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}_{n=1}^{\infty}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

| |
|---|
| Min max $\lambda i$ ($\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_{i} Pi\lambda i(\mu)$ |
| |
| Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that |
| (6) min max $\lambda i(\mu) = \min_{\mu} \max_{p\epsilon\Delta} \sum_{i} Pi\lambda i(\mu)$ |
| |
| Where the minimum is taken over invariant probability measures $\mu$ with support in *K.* Since min$\mu$ |
| $\sum_{i} P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established. |
| |
| To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta -$perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ [0,1]$be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by |
| |
| $\tilde{A}_i(x) = \{ \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i \epsilon I}{if\ i \notin I}$ |
| |
| Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t >$ 0. $Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in Kand\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of |

$\dot{x} = \tilde{A}(x)x$ *and* $\lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\} \backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). *Since* $E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently *K* is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent*. Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume x* $\mapsto$ *A(x) is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely*

*ergodic, or There exists j € { 1,….,k} such that $M_i$ ----- { x ∈ C :$x_l$ ≫ 0 for l ≠ j} Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_i M_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset *I* -----{1,…,k} such that $\mu(---_i \in_I C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures μ with support in ∂*C, $\lambda_i(\mu)>0$ *for all I ∈{ 1,…,k} \ supp(μ) (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i,..., x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species,* e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0 < e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j \ (x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1{}^1, \ldots, E_1{}^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i_2(E_1{}^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f^j_2(E^j_1) > 0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f^j_2(E^j_1) > 0$ and $\max_j f^j_2(E^j_2) > 0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0, E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is

sufficiently slow (i.e. $0< e^{j}{}_{i}\ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j (E^j{}_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1-x_1^j - \beta^j\ x_2^j\ - \alpha^j\ x_3^j\ ) + \sum d_1^{jk}\ \ x_1^k\ - e_1^j\ x_1^j$$

$$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j\ x_1^j\ - x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_2^{jk}\ \ x_2^k\ - e_2^j\ x_2^j$$

$$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j\ x_1^j\ - \alpha^j x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_3^{jk}\ \ x_3^k\ - e_3^j\ x_3^j$$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all *j*. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j{}_i = \sum_l d^{l,j}_i$ for all *i, j*) , the maximal invariant set in $\partial$C consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0< e^j{}_i\ll 1$ for all *i*, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2$ $(E_1)$and $\lambda_3$ $(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1 - \beta^j$. Consider the Morse decomposition of $\partial$C given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1> 0$ for *i* = 1, 2, 3, 0 is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j (1 - \beta^j)$$

Equivalently,

$$2>\min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j (r_i - \sum_s a_{is} x_s^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j , \qquad i = 1, \dots, k; \; j = 1, \dots, m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$ . We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set H= $\{x \in C: x_i^j = x_i^l \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \qquad i = 1, \dots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \dots, k$$

It has been shown [12,ch.16.1,16.2] That for $k$ =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there

is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\, \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \rightarrow \mathbf{E_2} \rightarrow \mathbf{E_3} \rightarrow \mathbf{E_1}$ then the following inequality holds

12) $\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as

well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that

there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has

abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left( \beta(1 - y - z)\frac{P}{\epsilon + P} - r - m(1 - y) - b(N)\ y \right)$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1 = N$, $x_2 = p$, $x_3 = (y,z)'$ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} - y_i)$$

With

$$(16)\quad F(x)_i = x_i g\,(x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\quad \begin{pmatrix} x_i \\ y_i\ dot \end{pmatrix} = \begin{pmatrix} -\beta & \beta \\ ag\,(x_i, x_i) - 1 & \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon

plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational

consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical

propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.22) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.23) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.24) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the

model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1$, $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as* $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 8. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^{+}\text{x}\ k_{\ 2}\text{x}\ k_{\ 1}, R^{-})$,

$A_3$. $\varphi_i\ (\cdot\ , s) \in L^{\ 2}\ (R^{+}, R^{+})\ \cap C(R^{+}, R^{+})$ for each $s \in\ [-\tau, 0]$,

$i,\ = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i$: $R^{+}$ $[-\tau, T] \rightarrow R^{+}$ $(i,\ = 1,\ 2,)$ With the following properties:

$S_1$. $P_i(\cdot\ , t) \in L^{\ 1}\ (R^{+}, R^{+})$ for each $t\ \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot\ ) \in C\ ([-\tau, T], R^{+})$ for each $a\ \in R^{+}$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^{+}$ $\{R^{+} \cap\ [0, T]\}$ And is continuous $(i,\ = 1,\ 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,)\ \in R^{+}$ $[-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$P_i(a + x, t + x) =\ \ P_i(x), (i,\ = 1,\ 2,)$ ,

Then (1.3) considered at $(1.3)$ considered at $(a + x,\ t + x)$ becomes

$$\frac{dp_{\ i}(x)}{dx} + f_i(a + x,\ P(t + x))\ \ P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0)\ \exp\left[-\int_0^{\infty} f_{\ i}(a + s, P(t + S))ds\right]$$

And hence

(2.1) $p_{\ 1}\ (a + x, t + x) = p_{\ 1}\ (a, t)\ exp\left[-\int_o^{x} f_{\ 1}\ (a + s, p(t + s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_{\ 1}\ (a, t) =\ \varphi_{\ 1}\ (a - t, 0)\exp\left[-\int_o^{t} f_{\ 1}\left(a - t + s, p_{\ 1}\ (s)\right) ds\right]$

$O \leq t \leq a, i = 1,2,$

With

$P_1(s)=(p_1(s),p_2(s-t)),$

$P_2(s)=(p_1(s-t),p_2(s)),$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$t > a \geq, i = 1,2,$

Where

(2.4) $B_i(t) = p_i(0, t), i = 1,2, t > 0.$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$

(2.5) $L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(a+s, p_i(s))ds]$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da + \int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$t > 0, i = 1,2,$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.(a, t, P_i(t))da$

$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$

$i = 1, 2, t > 0$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$

(2.5), (2.6) and (2.10) lead to

(2.11) $|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i a - d_i^*(t-a)]\, da$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \beta(0) d_i^* t$$

$$\le [\beta_i^* \beta(0)\, exp(\beta_i^* t) \beta(0) d_i^*] t \quad for \ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T],R^+), \quad B_i \in C([0,T],R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau,T] = \{f \in C([-\tau,T],R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau,T]$. Then for any

$(x, y) \in C^+[-\tau,T] \times C^+[-\tau,T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau,T], \quad C_T^2 = C^+[-\tau,T] \times C^+[-\tau,T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau,T]$ *and* $t \in [0,T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Phi: C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = P_t \left\{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Phi(P_t) - \Phi_t\|_{t*} \le \left\{ \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right\}$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ and $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \leq \int_0^t B_1^{(1)}(a)\left\{M_1\left(a,t,P_t^{(1)}\right) - M_1\left(a,t,P_t^{(2)}\right)\right\}da$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_t ds\right)da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o)\left[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds\right]da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \|_t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$=\int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\mathrm{X}[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \, /\!/ P_t^{(1)} - P_t^{(2)} /\!/ \text{ for } a \, \varepsilon[o, t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \leq \beta_1 \Phi_1(0) ||P_t^{(1)} - P_t^{(2)}||_t^* + \Phi_1(0) \beta_1 d_1 ||P_t^{(1)} - P_t^{(2)}||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) \, \mathrm{e}^{\delta 1^a} \beta_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 \mathrm{d}_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) \, \mathrm{e}^{\delta 1^a} /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| \mathrm{B}_1^{(1)}(\sigma) - \mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{B}_1^{*\mathrm{d}} \sigma$$

$$+ \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \left\{ B_1 + B_2 d_1 a + B_1^* \left( B_2 a + d_1 a^2/2 \right) e^{\delta 1^a} \right\}$$

$$\leq \beta_1^* \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| do + \left( r^* + /\!/ \Phi_t /\!/_{t*} \right) \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \rightarrow \beta_1$ as $a \rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq \left( r^* + /\!/ \Phi_t /\!/_{t*} \right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{ g_1(a) + 0(a) \}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a)) - b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a)) do \right|$$

$$\left| B_1^* \left( r^* + /\!/ \Phi_t /\!/_{t*} \right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} 0(a) as\ a \rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1(P_t^{(2)}) \right| \leq ((r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2(P_t^{(2)}) \right|$ leads to

$$\| \pi (P_t^{(1)} - \pi (P_t^{(2)}) /\!/ \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^*)$$

As $t^* \rightarrow o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

(3.1) $\frac{dp\,P_2^*(a)}{da}+f_i(\mathrm{a}, P_1^*, P_2^*)\,P_i^*(a)=0,\ \ i=1,2, a>0,$

(3.2) $P_i^*=\int_o^\infty P_i^*(a)da, i=1,\ 2,$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i=1.\ \ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a)=P_i^*(o)\exp\left[-\int_o^\infty f_i(s,P_1^*,P_2^*)ds\right], i=1,\quad 2, a\geq 0,$

Which together with (3.3) lead to

$1=\int_o^\infty b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$

(3.5) $1=\int_o^\infty b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1,P_2)=\ \int_o^\infty b_i(a,P_1,P_2)\exp\left[-\int_o^\infty f_i(s,P_1,P_2)ds\right]da,$

$i=1,\ 2,\quad P_1\geq 0,\quad P_2\geq 0,$

**And suppose the following hold:**

(3.7) $A_4.\quad F_{i.}\ (0,\ 0)>1,\ i=1,2$:

$A_5.$ *there exists a positive constant c for which*

$F_1(\mathrm{C},0)< F_{2.}(\mathrm{C},0),$

(3.8) $F_1(0.\mathrm{C})< F_{2.}(0,\mathrm{C}),$

Then there exists a unique pair $(P_1^*,P_2^*)$ of real numbers $P_1^*>o,$

$P_2^*>0$ such that

(3,9) $F_1(P_1^*,P_2^*)=1=F_2(P_1^*,P_2^*)$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a) = [P_2^* \exp[-f_i(s, P_1^*, P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s, P_1^*, P_2^*)ds]da)$,

$i$ =1, 2, $a \geq 0$,

Proof Consider the elements of the surfaces defined by

Z= $F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0$,

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

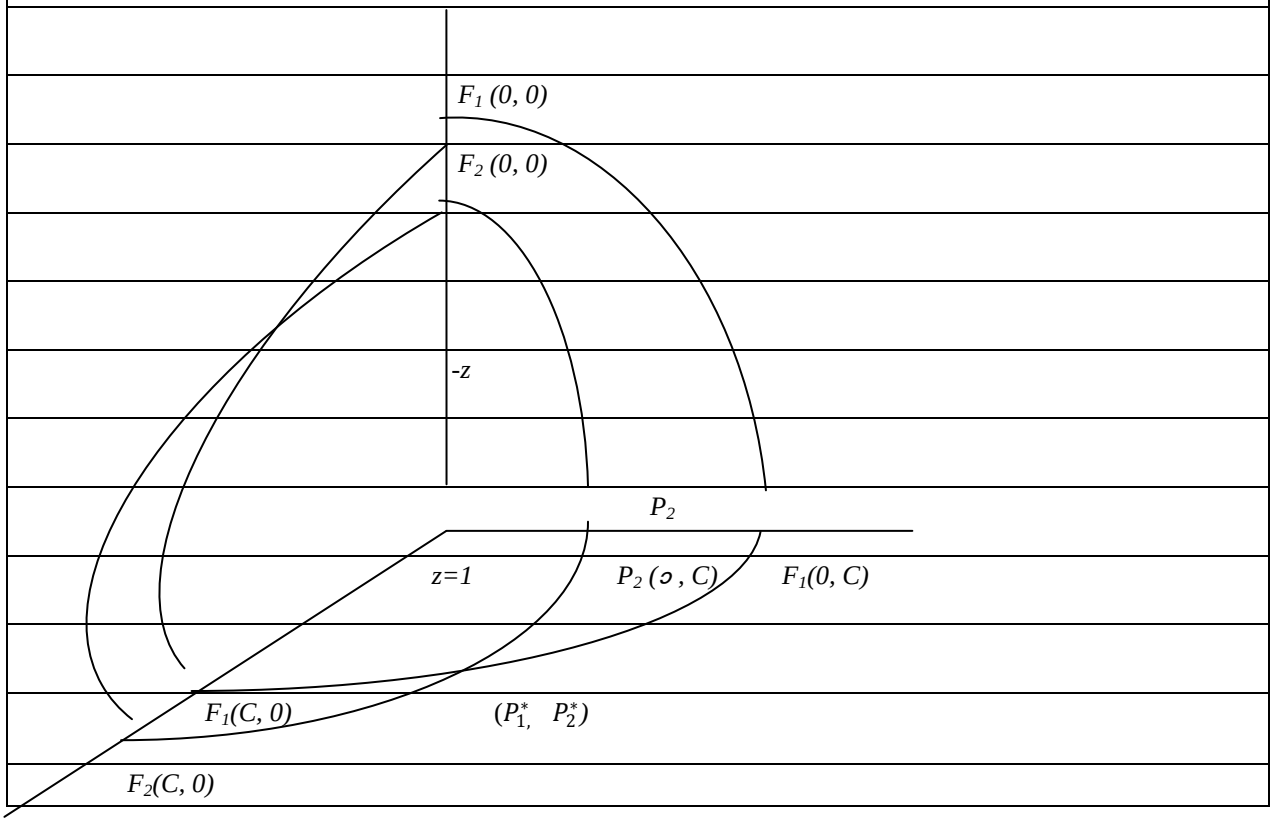
$P_1 \geq 0, P_2 \geq 0$, $i$ =1, 2,

$j$ =1, 2, Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, F_1(P_1, P_2 \geq 0$,

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.



(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11)
$$\begin{cases} p_i(a,t) = p^*_i(a) + u_i(a,t) \\ p_i(t) = p^*_i + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_i} p_i(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial b_i}{\partial p^*_i} p_i(t)\right\} da \end{cases}$$

Where $i = 1,2$ and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_i(a)\exp[\lambda t], \quad u_i(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty u_i(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{du_i}{da} + \lambda u_i = -f_i\left(a, p^*_1, p^*_2\right) u_i(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j} p^*_i\right\},$$

(3.15)

$$\mu_i(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\mu_i(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial b_i}{\partial p_j^*}\ p_i^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*,\ \exp\{-\lambda\tau\},\ p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\mu_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1\left(a, p_1^*, p_2^*\right) - f_1\left(a, p_1^*, p_2^*\right)\right]\mu_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2\left(a, p_1^*, p_2^*\right) - f_2\left(a, p_1^*, p_2^*\right)\right]\mu_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\mu_i(a) = \left\{\mu_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\right\} - p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\mu_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\,e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)\,e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\varphi_1(a)$ and $\varphi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$\left(\lambda - A_{11}\right) p_1^* - A_{12} p_2^* \exp\left(-\lambda\tau\right) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21} p_1^* \exp\left(-\lambda\tau\right) + \left(\lambda - A_{22}\right) p_2^* = g_{21}(\lambda) p_1^* \exp\left(-\lambda\tau\right) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $$A_{i\varphi} = \int_0^\infty p_i^*(a) \left[\frac{\partial b_i}{\partial p_\varphi^*} - \frac{\partial f_i}{\partial p_\varphi^*}\right] da,$$

(3.23) $$g_{i\varphi}(\lambda) = \left[p_i^*(0) c_i\right]^{-1} \left\{\int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} da\right\} \varphi \left\{\int_0^\infty \left[p_i^*(a) \frac{\partial b_i}{\partial p_\varphi^*} - b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_\varphi^*} e^{\lambda s} ds\right)\right] da\right\}$$

$$- \int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_\varphi^*} e^{\lambda s} ds\right) da, \quad i, \varphi = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i\left(a, p_1^*, p_2^*\right) - f_i\left(a, p_1^*, p_2^*\right), \quad i, = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $\left(p_1^*, p_2^*\right)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et \begin{bmatrix} \lambda - \left(A_{11} + g_{11}(\lambda)\right) & -\left(A_{12} + g_{12}\right) e^{-\lambda\tau} \\ -\left(A_{21} + g_{21}\right) e^{-\lambda\tau} & \lambda - \left(A_{22} + g_{22}(\lambda)\right) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda\left(A_{11} + A_{22}\right) + A_{11}A_{22} - A_{12}A_{21} \exp\left[-2\lambda\tau\right] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = \left[A_{21} g_{12}(\lambda) + A_{12} g_{21}(\lambda) + g_{12}(\lambda) g_{21}(\lambda)\right] \exp\left[-2\lambda\tau\right]$

$$-A_{11} g_{22}(\lambda) - A_{22} g_{11}(\lambda) - g_{11}(\lambda) g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ =1,\ 2.)\ \ (see(A_1)\ and\ (A_2));$

$A_7.\ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\},$

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = |A_{11}\, g_{12}(0)| + |A_{12}\, g_{21}(0)| + |g_{12}(0)\, g_{21}(0)|$$

$$+ |A_{11}\, g_{22}(0)| + |A_{22}\, g_{11}(0)| + |g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau),$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$, $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) | - F_2(\lambda, \tau) \right|$$

$$\geq |\lambda|^2 - |\lambda|\{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[(|\lambda| - P)^2 + (q)\right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2,$

$q = \left\{4 | (A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| | - | A_{11} + A_{22} + \| S_1(0) \| |^2\right\}/2,$

It will now follow, from (A) and (3.26), $F_1(\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions

of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends,

assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the “Grand design “is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner

wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the

controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## SECTION A(1)

## Quantum Query Without Unitary Operators

### INTRODUCTION—VARIABLES USED

**Quantum query without unitary operators S. A. Grillo, F. L. Marquezino (Submitted on 24 Feb 2016)**

(1) The Quantum Query Model is a framework that allows us (eb) to express most known quantum algorithms. Algorithms represented by this model consist on (e&eb) a set of unitary operators acting over (e&eb) a finite Hilbert space and a final measurement step consisting on (e&eb) a set of

projectors.

(2) In this work, **S. A. Grillo, F. L. Marquezino** prove that the application of these unitary operators before the measurement step is equivalent to (=) decomposing (e&eb) a unit vector into (e&eb) a sum of vectors and then inverting some of their relative phases.

(3) Authors also prove that the vectors of (e) that sum must fulfill a list of properties and authors call such vectors a Block Set.

(4) If we take the same measurement step on the Quantum Query Model and (e&eb) on the Block Set Formulation, then authors prove that both formulations give (eb) the same Gram matrix of output states, although the Block Set Formulation allows (eb) a much more explicit form.

(5) Therefore, the Block Set reformulation of (e) the Quantum Query Model has (e) advantages for the design of exact quantum algorithms.

**(6)** Finally, authors test these advantages of Block Sets by applying approach to (e&eb) the construction of quantum exact algorithms. Subjects: Quantum Physics (quant-ph); Computational Complexity (cs.CC) Cite as: arXiv: 1602.07716 [quant-ph] (or arXiv: 1602.07716v1 [quant-ph] for this version) **Quantum query without unitary operators S. A. Grillo, F. L. Marquezino (Submitted on 24 Feb 2016)**

**NOTATION**

**Module One**

The Quantum Query Model is a framework that allows us (eb) to express most known quantum algorithms. Algorithms represented by this model consist on (e&eb) a set of unitary operators acting over (e&eb) a finite Hilbert space and a final measurement step consisting on (e&eb) a set of projectors

$G_{13}$ : Category one of Quantum Query Model is a framework

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of express most known quantum algorithms. Algorithms represented by this model consist on (e&eb) a set of unitary operators acting over (e&eb) a finite Hilbert space and a final measurement step consisting on (e&eb) a set of projectors

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

The Quantum Query Model is a framework that allows us to express most known quantum algorithms.

Algorithms represented by this model consist on (e&eb) a set of unitary operators acting over (e&eb) a finite Hilbert space and a final measurement step consisting on (e&eb) a set of projectors

$G_{16}$ : Category one of **Algorithms represented by this model;** set of unitary operators acting over (e&eb) a finite Hilbert space and a final measurement step consisting on (e&eb) a set of projectors

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

| | |
|---|---|
| $T_{16}$ : Category one of set of unitary operators acting over (e&eb) a finite Hilbert space and a final measurement step consisting on (e&eb) a set of projectors **;Algorithms represented by this model**<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>Algorithms represented by this model consist on a set of unitary operators acting over (e&eb) a finite Hilbert space and a final measurement step consisting on (e&eb) a set of projectors | |
| $G_{20}$ : Category one of **Algorithms represented by this model consist on a set of unitary operators acting over;** finite Hilbert space and a final measurement step consisting on (e&eb) a set of projectors<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of finite Hilbert space and a final measurement step consisting on (e&eb) a set of projectors **;Algorithms represented by this model consist on a set of unitary operators acting over**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Algorithms represented by this model consist on a set of unitary operators acting over a finite Hilbert space and a final measurement step consisting on (e&eb) a set of projectors | |
| $G_{24}$ : Category one of **Algorithms represented by this model consist on a set of unitary operators acting over a finite Hilbert space and a final measurement step consisting;** set of projectors<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of set of projectors **;Algorithms represented by this model consist on a set of unitary operators acting over a finite Hilbert space and a final measurement step consisting**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>In this work, **S. A. Grillo, F. L. Marquezino** prove that the application of these unitary operators before the measurement step is equivalent to (=) decomposing (e&eb) a unit vector into (e&eb) a sum of vectors and then inverting some of their relative phases | |
| $G_{28}$ : Category one of application of these unitary operators before the measurement step<br><br>$G_{29}$ : Category two of SAS | |

| | |
|---|---|
| $G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of decomposing (e&eb) a unit vector into (e&eb) a sum of vectors and then inverting some of their relative phases<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>In this work, **S. A. Grillo, F. L. Marquezino** prove that the application of these unitary operators before the measurement step is equivalent to decomposing a unit vector into (e&eb) a sum of vectors and then inverting some of their relative phases | |
| $G_{32}$ : Category one of **unitary operators before the measurement step is equivalent to decomposing a unit vector;** sum of vectors and then inverting some of their relative phases<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of sum of vectors and then inverting some of their relative phases **;unitary operators before the measurement step is equivalent to decomposing a unit vector**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Authors also prove that the vectors of that sum must fulfill a list of properties and authors call such vectors a Block Set | |
| $G_{36}$ : Category one of vectors of that sum must fulfill<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of list of properties and authors call such vectors a Block Set<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>If we take the same measurement step on the Quantum Query Model and (e&eb) on the Block Set Formulation, then authors prove that both formulations give (eb) the same Gram matrix of output states, although the Block Set Formulation allows (eb) a much more explicit form | |

| | |
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| $G_{40}$ : Category one of **same measurement step on the Quantum Query Model;** Block Set Formulation,<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of Block Set Formulation,; **same measurement step on the Quantum Query Model**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>If we take the same measurement step on the Quantum Query Model and on the Block Set Formulation, then authors prove that both formulations give (eb) the same Gram matrix of output states, although the Block Set Formulation allows (eb) a much more explicit form | |
| $G_{44}$ : Category one of same measurement step on the Quantum Query Model and on the Block Set Formulation, then authors prove that both formulations<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of same Gram matrix of output states, although the Block Set Formulation allows (eb) a much more explicit form<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

| | |
|---|---|
| **SECTION A(1)-2**<br><br>**Block Set Reformulation Of The Quantum Query Model** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Quantum query without unitary operators S. A. Grillo, F. L. Marquezino (Submitted on 24 Feb 2016)**<br><br>**(1)** Therefore, the Block Set reformulation of (e) the Quantum Query Model has (e) advantages for the design of exact quantum algorithms.<br>**(2)** Finally, authors test these advantages of Block Sets by applying approach to (e&eb) the construction of quantum exact algorithms. Subjects: Quantum Physics (quant-ph); Computational Complexity (cs.CC) Cite as: arXiv: 1602.07716 [quant-ph] (or arXiv: 1602.07716v1 [quant-ph] for this version) **Quantum query without unitary operators S. A. Grillo, F. L. Marquezino (Submitted on 24 Feb 2016)**<br><br>**SIAM J. Comput., 37(2), 413–424. (12 pages) Quantum Algorithms for the Triangle Problem** | |

**Published online: 16 May 2007 Keywords 05C85, 68R10, 68W99 AMS Subject Headings quantum algorithm, query complexity, triangle problem, quantum walk Publication Data ISSN (print): 0097-5397 ISSN (online): 1095-7111 Publisher: Society for Industrial and Applied Mathematics Frédéric Magniez, Miklos Santha, and Mario SzegedyDOI:10.1137/050643684**

**(6)** Authors present two new quantum algorithms that either find a triangle (a copy of $K_{3}$) in (eb) an undirected graph G on n nodes, or reject if (e) G is triangle free.

**(7)** The first algorithm uses (e) combinatorial ideas with (e&eb) Grover Search and makes (eb) $\tilde{O}(n^{10/7})$ queries.

**(8)** The second algorithm uses (e) $\tilde{O}(n^{13/10})$ queries and is based on (e) a design concept of Ambainis [in Proceedings of the $45$th IEEE Symposium on Foundations of Computer Science, 2004, pp. 22–31] that incorporates (e) the benefits of quantum walks into Grover Search [L. Grover, in Proceedings of the Twenty-Eighth ACM Symposium on Theory of Computing, 1996, pp. 212–219].

**(9)** The first algorithm uses (e) only $O(\log n)$ qubits in its quantum subroutines, whereas (e) the second one uses (e) $O(n)$ qubits. The Triangle Problem was first treated in [H. Buhrman et al., SIAM J. Comput., 34 (2005), pp. 1324–1330], where (e) an algorithm with $O(n+\sqrt{nm})$ query complexity was presented, where (e) m is the number of edges of G.Copyright © 2007 Society for Industrial and Applied Mathematics

**NOTATION**

**Module One**

Therefore, the Block Set reformulation of (e) the Quantum Query Model has (e) advantages for the design of exact quantum algorithms

$G_{13}$ : Category one of Quantum Query Model has (e) advantages for the design of exact quantum algorithms

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of Block Set reformulation

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

Therefore, the Block Set reformulation of the Quantum Query Model has (e) advantages for the design of exact quantum algorithms

$G_{16}$ : Category one of advantages for the design of exact quantum algorithms

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

$T_{16}$ : Category one of Block Set reformulation of the Quantum Query Model

$T_{17}$ : Category two of SAS

| | |
|---|---|
| $T_{18}$ : Category three of SAS | |
| **Module three**<br><br>Finally, authors test these advantages of Block Sets by applying approach to (e&eb) the construction of quantum exact algorithms.<br><br>Subjects: Quantum Physics (quant-ph); Computational Complexity (cs.CC) Cite as: arXiv: 1602.07716 [quant-ph] (or arXiv: 1602.07716v1 [quant-ph] for this version) **Quantum query without unitary operators S. A. Grillo, F. L. Marquezino (Submitted on 24 Feb 2016)** | |
| $G_{20}$ : Category one of **advantages of Block Sets by applying approach;** construction of quantum exact algorithms<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of construction of quantum exact algorithms; **advantages of Block Sets by applying approach**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Authors present two new quantum algorithms that either find a triangle (a copy of $K_{3}$) in (eb) an undirected graph G on n nodes, or reject if (e) G is triangle free | |
| $G_{24}$ : Category one of new quantum algorithms that either find a triangle (a copy of $K_{3}$)<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of undirected graph G on n nodes, or reject if (e) G is triangle free<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Authors present two new quantum algorithms that either find a triangle (a copy of $K_{3}$) in an undirected graph G on n nodes, or reject if (e) G is triangle free | |
| $G_{28}$ : Category one of G is triangle free<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |

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|---|---|
| $T_{28}$ : Category one of new quantum algorithms that either find a triangle (a copy of $K_{3}$) in an undirected graph G on n nodes, or reject<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>The first algorithm uses (e) combinatorial ideas with (e&eb) Grover Search and makes (eb) $\tilde{O}(n^{10/7})$ queries | |
| $G_{32}$ : Category one of combinatorial ideas with (e&eb) Grover Search and makes (eb) $\tilde{O}(n^{10/7})$ queries<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of first algorithm<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>The first algorithm uses combinatorial ideas with (e&eb) Grover Search and makes (eb) $\tilde{O}(n^{10/7})$ queries | |
| $G_{36}$ : Category one of **first algorithm uses combinatorial ideas;** Grover Search and makes (eb) $\tilde{O}(n^{10/7})$ queries<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of Grover Search and makes (eb) $\tilde{O}(n^{10/7})$ queries **;first algorithm uses combinatorial ideas**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The first algorithm uses combinatorial ideas with Grover Search and makes (eb) $\tilde{O}(n^{10/7})$ queries | |

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| $G_{40}$ : Category one of first algorithm uses combinatorial ideas with Grover Search and makes<br>$G_{41}$ : Category two of SAS<br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of$\tilde{O}(n^{10/7})$ queries<br>$T_{41}$ : Category two of SAS<br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br>The second algorithm uses (e) $\tilde{O}(n^{13/10})$ queries and is based on (e) a design concept of Ambainis [in Proceedings of the $45$th IEEE Symposium on Foundations of Computer Science, 2004, pp. 22–31] that incorporates (e) the benefits of quantum walks into Grover Search [L. Grover, in Proceedings of the Twenty-Eighth ACM Symposium on Theory of Computing, 1996, pp. 212–219]. | |
| $G_{44}$ : Category one of$\tilde{O}(n^{13/10})$ queries and is based on (e) a design concept of Ambainis<br>$G_{45}$ : Category two of SAS<br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of second algorithm<br>$T_{45}$ : Category two of SAS<br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$<br>$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$<br>$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}\ (a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,<br>$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$<br>$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,<br>$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,<br>$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$, | |

| | |
|---|---|
| are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14},t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14},t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14},t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G,t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G,t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14},t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G,t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17},t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17},t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17},t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17},t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}),t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |

| | |
|---|---|
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25}, t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}, t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}), t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}), t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}), t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25}, t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}), t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29}, t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}, t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}), t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}), t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}), t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29}, t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}), t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33}, t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}, t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}), t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}), t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}), t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33}, t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}, t)\right]G_{37}$ | *38* |

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| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)}\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)}\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)}\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for | |

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| category 1, 2 and 3<br>$+(a_{16}'')^{(2,2,)}(T_{17},t)$, $+(a_{17}'')^{(2,2,)}(T_{17},t)$, $+(a_{18}'')^{(2,2,)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}'')^{(3,3,)}(T_{21},t)$, $+(a_{21}'')^{(3,3,)}(T_{21},t)$, $+(a_{22}'')^{(3,3,)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}'')^{(4,4,4,4,)}(T_{25},t)$, $+(a_{25}'')^{(4,4,4,4,)}(T_{25},t)$, $+(a_{26}'')^{(4,4,4,4,)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}'')^{(5,5,5,5,)}(T_{29},t)$, $+(a_{29}'')^{(5,5,5,5,)}(T_{29},t)$, $+(a_{30}'')^{(5,5,5,5,)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}'')^{(6,6,6,6,)}(T_{33},t)$, $+(a_{33}'')^{(6,6,6,6,)}(T_{33},t)$, $+(a_{34}'')^{(6,6,6,6,)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{38}'')^{(7,7)}(T_{37},t)$ $+(a_{37}'')^{(7,7)}(T_{37},t)$ $+(a_{36}'')^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3<br>$+(a_{40}'')^{(8,8)}(T_{41},t)$ $+(a_{41}'')^{(8,8)}(T_{41},t)$ $+(a_{42}'')^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3<br>$+(a_{44}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}')^{(1)}\left[-(b_{13}'')^{(1)}(G,t)\right] & \left[-(b_{16}'')^{(2,2,)}(G_{19},t)\right] & \left[-(b_{20}'')^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{24}'')^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{28}'')^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{32}'')^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{36}'')^{(7,7,)}(G_{39},t)\right] & \left[-(b_{40}'')^{(8,8)}(G_{43},t)\right] & \left[-(b_{44}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}')^{(1)}\left[-(b_{14}'')^{(1)}(G,t)\right] & \left[-(b_{17}'')^{(2,2,)}(G_{19},t)\right] & \left[-(b_{21}'')^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{25}'')^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{29}'')^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{33}'')^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{37}'')^{(7,7,)}(G_{39},t)\right] & \left[-(b_{41}'')^{(8,8)}(G_{43},t)\right] & \left[-(b_{45}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}')^{(1)}\left[-(b_{15}'')^{(1)}(G,t)\right] & \left[-(b_{18}'')^{(2,2,)}(G_{19},t)\right] & \left[-(b_{22}'')^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{26}'')^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{30}'')^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{34}'')^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{38}'')^{(7,7,)}(G_{39},t)\right] & \left[-(b_{42}'')^{(8,8)}(G_{43},t)\right] & \left[-(b_{46}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{15}$ | *60* |
| Where $-(b_{13}'')^{(1)}(G,t)$, $-(b_{14}'')^{(1)}(G,t)$, $-(b_{15}'')^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{16}'')^{(2,2,)}(G_{19},t)$, $-(b_{17}'')^{(2,2,)}(G_{19},t)$, $-(b_{18}'')^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{20}'')^{(3,3,)}(G_{23},t)$, $-(b_{21}'')^{(3,3,)}(G_{23},t)$, $-(b_{22}'')^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}'')^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}'')^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}'')^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}'')^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}'')^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}'')^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}'')^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}'')^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}'')^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for | |

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| category 1, 2 and 3 $\boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[\begin{array}{ccc} (a_{16}^{'})^{(2)} & \boxed{+(a_{16}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)} \quad \boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)} \end{array}\right] G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[\begin{array}{ccc} (a_{17}^{'})^{(2)} & \boxed{+(a_{17}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)} \quad \boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)} \end{array}\right] G_{17}$ | *62* |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[\begin{array}{ccc} (a_{18}^{'})^{(2)} & \boxed{+(a_{18}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)} \quad \boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)} \end{array}\right] G_{18}$ | *63* |
| Where $\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)}$ are first augmentation coefficients for category 1, 2 and 3 $\boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)}$ are second augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)}$ are seventh augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)}$ are eight augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)}$ are ninth augmentation coefficient for category 1, 2 and 3 | |

$$\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b'_{16})^{(2)}\boxed{-(b''_{16})^{(2)}(G_{19},t)} & \boxed{-(b''_{13})^{(1,1,)}(G,t)} & \boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{16} \quad (64)$$

$$\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b'_{17})^{(2)}\boxed{-(b''_{17})^{(2)}(G_{19},t)} & \boxed{-(b''_{14})^{(1,1,)}(G,t)} & \boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{17} \quad (65)$$

$$\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b'_{18})^{(2)}\boxed{-(b''_{18})^{(2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,)}(G,t)} & \boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{18} \quad (66)$$

where $\boxed{-(b''_{16})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{17})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{18})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,)}(G,t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{44})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2 and 3

$$\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a'_{20})^{(3)}\boxed{+(a''_{20})^{(3)}(T_{21},t)} & \boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{20} \quad (67)$$

$$\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a'_{21})^{(3)}\boxed{+(a''_{21})^{(3)}(T_{21},t)} & \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{21} \quad (68)$$

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a'_{22})^{(3)}\boxed{+(a''_{22})^{(3)}(T_{21},t)} & \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{22} \quad (69)$$

$\boxed{+(a''_{20})^{(3)}(T_{21},t)}, \boxed{+(a''_{21})^{(3)}(T_{21},t)}, \boxed{+(a''_{22})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)}, \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)}, \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)}, \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)}, \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)}$ are third augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)}, \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)}, \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)}, \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)}, \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)}, \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)}, \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)}, \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)}, \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)}, \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)}, \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)}$ are eight augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)}, \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)}, \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1, 2 and 3

$$\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b'_{20})^{(3)}\boxed{-(b''_{20})^{(3)}(G_{23},t)} & \boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{13})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{20} \quad (70)$$

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b'_{21})^{(3)}\boxed{-(b''_{21})^{(3)}(G_{23},t)} & \boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{14})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)}\boxed{-(b''_{22})^{(3)}(G_{23},t)} & \boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{22} \quad (72)$$

$\boxed{-(b''_{20})^{(3)}(G_{23},t)}, \boxed{-(b''_{21})^{(3)}(G_{23},t)}, \boxed{-(b''_{22})^{(3)}(G_{23},t)}$ *are first detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)}, \boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)}, \boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)}$ *are second detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{13})^{(1,1,1,)}(G,t)}, \boxed{-(b''_{14})^{(1,1,1,)}(G,t)}, \boxed{-(b''_{15})^{(1,1,1,)}(G,t)}$ *are third detrition coefficients for category 1,2 and 3*

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|---|---|
| $-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*<br>$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$ $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a_{24}^{'})^{(4)} & +(a_{24}^{''})^{(4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,)}(T_{29},t) & +(a_{32}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24}$ | *73* |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a_{25}^{'})^{(4)} & +(a_{25}^{''})^{(4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,)}(T_{29},t) & +(a_{33}^{''})^{(6,6)}(T_{33},t) \\ & +(a_{14}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25}$ | *74* |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a_{26}^{'})^{(4)} & +(a_{26}^{''})^{(4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,)}(T_{29},t) & +(a_{34}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26}$ | *75* |

$(a_{24}^{''})^{(4)}(T_{25},t)$, $(a_{25}^{''})^{(4)}(T_{25},t)$, $(a_{26}^{''})^{(4)}(T_{25},t)$ are first augmentation coefficients category 1, 2 3

$+(a_{28}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,)}(T_{29},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,)}(T_{33},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1, 2 and 3

$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3

$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1, 2 and 3

$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b_{24}^{'})^{(4)} & -(b_{24}^{''})^{(4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,)}(G_{35},t) \\ & -(b_{13}^{''})^{(1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3)}(G_{23},t) \\ & -(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b_{25}^{'})^{(4)} & -(b_{25}^{''})^{(4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,)}(G_{35},t) \\ & -(b_{14}^{''})^{(1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3)}(G_{23},t) \\ & -(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b_{26}^{'})^{(4)} & -(b_{26}^{''})^{(4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,)}(G_{35},t) \\ & -(b_{15}^{''})^{(1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3)}(G_{23},t) \\ & -(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{26} \quad (78)$$

Where $-(b_{24}^{''})^{(4)}(G_{27},t)$, $-(b_{25}^{''})^{(4)}(G_{27},t)$, $-(b_{26}^{''})^{(4)}(G_{27},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,)}(G_{31},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,)}(G_{35},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2 and 3

| | |
|---|---|
| $-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 $-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3 $-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,)}(T_{25},t) & +(a_{32}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{28}$ | 79 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a_{29}^{'})^{(5)}+(a_{29}^{''})^{(5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,)}(T_{25},t) & +(a_{33}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{29}$ | 80 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,)}(T_{25},t) & +(a_{34}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{30}$ | 81 |
| Where $+(a_{28}^{''})^{(5)}(T_{29},t)$, $+(a_{29}^{''})^{(5)}(T_{29},t)$, $+(a_{30}^{''})^{(5)}(T_{29},t)$ are first augmentation coefficients for category 1, 2 and 3 And $+(a_{24}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,)}(T_{25},t)$ are second augmentation coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ are third augmentation coefficient for category 1, 2 and 3 $+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3 $+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3 $+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3 $+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3 $+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3 $+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b'_{28})^{(5)} & -(b''_{28})^{(5)}(G_{31},t) & -(b''_{24})^{(4,4,)}(G_{27},t) & -(b''_{32})^{(6,6,6)}(G_{35},t) \\ -(b''_{13})^{(1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b'_{29})^{(5)} & -(b''_{29})^{(5)}(G_{31},t) & -(b''_{25})^{(4,4,)}(G_{27},t) & -(b''_{33})^{(6,6,6)}(G_{35},t) \\ -(b''_{14})^{(1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b'_{30})^{(5)} & -(b''_{30})^{(5)}(G_{31},t) & -(b''_{26})^{(4,4,)}(G_{27},t) & -(b''_{34})^{(6,6,6)}(G_{35},t) \\ -(b''_{15})^{(1,1,1,1,1,)}(G,t) & -(b''_{18})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{30} \quad (84)$$

*where* $-(b''_{28})^{(5)}(G_{31},t)$ , $-(b''_{29})^{(5)}(G_{31},t)$ , $-(b''_{30})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{24})^{(4,4,)}(G_{27},t)$, $-(b''_{25})^{(4,4,)}(G_{27},t)$, $-(b''_{26})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3

$-(b''_{32})^{(6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b''_{13})^{(1,1,1,1,1)}(G,t)$, $-(b''_{14})^{(1,1,1,1,1)}(G,t)$ , $-(b''_{15})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3

$-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3

$-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)$, $-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)$, $-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3

$-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3

$-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3

$-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a'_{32})^{(6)} & +(a''_{32})^{(6)}(T_{33},t) & +(a''_{28})^{(5,5,5)}(T_{29},t) & +(a''_{24})^{(4,4,4,)}(T_{25},t) \\ +(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a'_{33})^{(6)} & +(a''_{33})^{(6)}(T_{33},t) & +(a''_{29})^{(5,5,5)}(T_{29},t) & +(a''_{25})^{(4,4,4,)}(T_{25},t) \\ +(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)} G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)} T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)} T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)} T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{28})^{(5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)} & +(a_{36}^{''})^{(7)}(T_{37},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)} & +(a_{37}^{''})^{(7)}(T_{37},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)} & +(a_{38}^{''})^{(7)}(T_{37},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15} \quad (93)$$

Where $(a_{36}^{''})^{(7)}(T_{37},t)$, $(a_{37}^{''})^{(7)}(T_{37},t)$, $(a_{38}^{''})^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation

coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3 $+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}(G_{39},t) & -(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$ 94

$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}-(b_{37}^{''})^{(7)}(G_{39},t) & -(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}(G_{39},t) & -(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3 $-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3 $-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3 $-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

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$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}')^{(8)}+(a_{40}^{''})^{(8)}(T_{41},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13}$$

$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}')^{(8)}+(a_{41}^{''})^{(8)}(T_{41},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}')^{(8)}+(a_{42}^{''})^{(8)}(T_{41},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$$

Where $+(a_{40}^{''})^{(8)}(T_{41},t)$, $+(a_{41}^{''})^{(8)}(T_{41},t)$, $+(a_{42}^{''})^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & -(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & -(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & -(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{13}$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[\begin{array}{ccc} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$\frac{dT_{44}}{dt} =$

$$(b_{44})^{(9)}T_{45} - \left[\begin{array}{ccc} (b_{44}^{'})^{(9)}\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[\begin{array}{ccc} (b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[\begin{array}{ccc} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$$

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

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| **Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br>$(a_i'')^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$<br>$(b_i'')^{(1)}(G,t) \le (r_i)^{(1)} \le (b_i')^{(1)} \le (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2\to\infty}(a_i'')^{(1)}(T_{14},t) = (p_i)^{(1)}$<br>$\lim_{G\to\infty}(b_i'')^{(1)}(G,t) = (r_i)^{(1)}$<br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(1)}(T_{14}',t) - (a_i'')^{(1)}(T_{14},t)\rvert \le (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}'\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br>$\lvert(b_i'')^{(1)}(G',t) - (b_i'')^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G'\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(1)}(T_{14}',t)$ and $(a_i'')^{(1)}(T_{14},t)$ . $(T_{14}',t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i'')^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i'')^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i')^{(1)}, (b_i)^{(1)}, (b_i')^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i,j = 16,17,18$ | |

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| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17},t) \le (p_i)^{(2)} \le (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19},t) \le (r_i)^{(2)} \le (b_i')^{(2)} \le (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2\to\infty} (a_i'')^{(2)}(T_{17},t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty} (b_i'')^{(2)}((G_{19}),t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}$ are positive constants and $i = 16,17,18$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(2)}(T_{17}',t) - (a_i'')^{(2)}(T_{17},t)\rvert \le (\hat{k}_{16})^{(2)}\lvert T_{17} - T_{17}'\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i'')^{(2)}((G_{19})',t) - (b_i'')^{(2)}((G_{19}),t)\rvert < (\hat{k}_{16})^{(2)}\lvert\lvert(G_{19}) - (G_{19})'\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}',t)$ and $(a_i'')^{(2)}(T_{17},t)$ . $(T_{17}',t)$ and $(T_{17},t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}]$ . It is to be noted that $(a_i'')^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i'')^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i')^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ | 111 |

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| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(3)} (T_{21}, t) = (p_i)^{(3)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(3)} (G_{23}, t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'}, t) - (a_i^{''})^{(3)}(T_{21}, t)\rvert \leq (\hat{k}_{20})^{(3)} \lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)} t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'}, t) - (b_i^{''})^{(3)}(G_{23}, t)\rvert < (\hat{k}_{20})^{(3)} \lvert\lvert G_{23} - G_{23}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)} t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'}, t)$ and$(a_i^{''})^{(3)}(T_{21}, t)$ . $(T_{21}^{'}, t)$ And $(T_{21}, t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$<br>satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$ | 116 |

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| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24{,}25{,}26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}\big((G_{27}), t\big) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(4)}\big((G_{27}), t\big) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24{,}25{,}26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\rvert \leq (\hat{k}_{24})^{(4)} \lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)} t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'}, t) - (b_i^{''})^{(4)}\big((G_{27}), t\big)\rvert < (\hat{k}_{24})^{(4)} \lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)} t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $\big[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\big]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24{,}25{,}26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |

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| $\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i^{'})^{(4)} +\ (\hat{B}_{24})^{(4)} +\ (\hat{Q}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}\big((G_{31}), t\big) \leq\ (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(5)}\,(T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(5)}\,(G_{31}, t) =\ (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'}, t) - (b_i^{''})^{(5)}\big((G_{31}), t\big)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $\big[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\big]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br><br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\ , \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities | 126 |

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| $\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(6)}(T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(6)}((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)}\lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})^{'}, t) - (b_i^{''})^{(6)}((G_{35}), t)\rvert < (\hat{k}_{32})^{(6)}\lvert\lvert(G_{35}) - (G_{35})^{'}\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and$(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants | 130 |

$(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$
satisfy the inequalities

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

Where we suppose

(WW) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i, j = 36,37,38$ 131

(XX) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$:

$$(a_i^{''})^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$$

$$(b_i^{''})^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$$

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*(YY)* $\lim_{T_2 \to \infty} (a_i^{''})^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(ZZ)*

$\lim_{G \to \infty} (b_i^{''})^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, \ (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)} |T_{37} - T_{37}^{'}| e^{-(\hat{M}_{36})^{(7)} t}$$

$$|(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)} ||(G_{39}) - (G_{39})^{'}|| e^{-(\hat{M}_{36})^{(7)} t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'}, t)$ and $(a_i^{''})^{(7)}(T_{37}, t)$ . $(T_{37}^{'}, t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$ . It is to be noted that $(a_i^{''})^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ : 134

(AAA) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants

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| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :<br><br>(BBB) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}, (r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})', t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})'\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would | |

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| be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i, j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45}, t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47}, t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2 \to \infty} (a_i'')^{(9)}(T_{45}, t) = (p_i)^{(9)}$<br>$\lim_{G \to \infty} (b_i'')^{(9)}(G_{47}, t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}', t) - (a_i'')^{(9)}(T_{45}, t)\rvert \le (\hat{k}_{44})^{(9)} \lvert T_{45} - T_{45}' \rvert e^{-(\hat{M}_{44})^{(9)} t}$<br>$\lvert(b_i'')^{(9)}((G_{47})', t) - (b_i'')^{(9)}((G_{47}), t)\rvert < (\hat{k}_{44})^{(9)} \lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}', t)$ | |

and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)},(\widehat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{44})^{(9)}=1$ then the function $(a_i'')^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$ :

$(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},$ are positive constants

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\ ,\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}<1$$

**Definition of** $(\hat{P}_{44})^{(9)},(\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},(\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)},(a_i')^{(9)},(b_i)^{(9)},(b_i')^{(9)},(p_i)^{(9)},\ (r_i)^{(9)},i=44,45,46,$ satisfy the inequalities

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[(a_i)^{(9)}+(a_i')^{(9)}+\ (\hat{A}_{44})^{(9)}+\ (\hat{P}_{44})^{(9)}\ (\hat{k}_{44})^{(9)}]<1$$

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[\ (b_i)^{(9)}+(b_i')^{(9)}+\ (\hat{B}_{44})^{(9)}+\ (\hat{Q}_{44})^{(9)}\ \ (\hat{k}_{44})^{(9)}]<1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0)\ ,T_i(0)$ :

$$G_i(t)\le\ \left(\hat{P}_{13}\right)^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{G_i(0)=G_i^0>0}$$

$$T_i(t)\le\ (\hat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{T_i(0)=T_i^0>0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0)\ ,T_i(0)$

$$G_i(t)\le\ (\hat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t)\le\ (\hat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

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| **Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$ | 153 B |

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| $T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |

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| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} G_{21}(s_{(20)}) - \left((a_{20}')^{(3)} + a_{20}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{20}(s_{(20)})\right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[(a_{21})^{(3)} G_{20}(s_{(20)}) - \left((a_{21}')^{(3)} + (a_{21}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}')^{(3)} - (b_{21}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{28})^{(5)}$, $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[ (b_{29})^{(5)} T_{28}(s_{(28)}) - \left( (b'_{29})^{(5)} - (b''_{29})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[ (b_{30})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{30})^{(5)} - (b''_{30})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{30}(s_{(28)}) \right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** | |

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| Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[ (a_{32})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{32}(s_{(32)}) \right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[ (a_{33})^{(6)} G_{32}(s_{(32)}) - \left( (a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[ (a_{34})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{34}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[ (b_{32})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{32}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[ (b_{33})^{(6)} T_{32}(s_{(32)}) - \left( (b'_{33})^{(6)} - (b''_{33})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[ (b_{34})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{34})^{(6)} - (b''_{34})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{34}(s_{(32)}) \right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |

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| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}^{'})^{(8)} + a_{40}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^{0} + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^{0} + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^{0} + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{13}(t) \leq G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \leq \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}$ , $G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \leq G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that | 170 |

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| $(G_{16}(t)-G_{16}^0)e^{-(\bar{M}_{16})^{(2)}t} \le \frac{(a_{16})^{(2)}}{(\bar{M}_{16})^{(2)}}\left[((\hat{P}_{16})^{(2)}+G_{17}^0)e^{\left(-\frac{(\hat{P}_{16})^{(2)}+G_{17}^0}{G_{17}^0}\right)}+(\hat{P}_{16})^{(2)}\right]$ | |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0+(\hat{P}_{20})^{(3)}e^{(\bar{M}_{20})^{(3)}s_{(20)}}\right)\right] ds_{(20)} =$<br>$(1+(a_{20})^{(3)}t)G_{21}^0 + \frac{(a_{20})^{(3)}(\hat{P}_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left(e^{(\bar{M}_{20})^{(3)}t}-1\right)$ | 171 |
| From which it follows that<br>$(G_{20}(t)-G_{20}^0)e^{-(\bar{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that<br>$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\bar{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$<br>$(1+(a_{24})^{(4)}t)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left(e^{(\bar{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that<br>$(G_{24}(t)-G_{24}^0)e^{-(\bar{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$<br>$(1+(a_{28})^{(5)}t)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)}t}-1\right)$ | |
| From which it follows that<br>$(G_{28}(t)-G_{28}^0)e^{-(\bar{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 5 | 175 |
| The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious | 176 |

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| that<br>$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} \left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$<br>$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}} \left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$ | |
| From which it follows that<br>$(G_{32}(t) - G_{32}^0) e^{-(\hat{M}_{32})^{(6)} t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}} \left[((\hat{P}_{32})^{(6)} + G_{33}^0) e^{\left(-\frac{(\hat{P}_{32})^{(6)} + G_{33}^0}{G_{33}^0}\right)} + (\hat{P}_{32})^{(6)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 6<br>Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$ | 177 |
| (i) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} \left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$<br>$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$ | 178 |
| From which it follows that<br>$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left[((\hat{P}_{36})^{(7)} + G_{37}^0) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 7 | |
| The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} \left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$<br>$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$ | 180 |
| From which it follows that<br>$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left[((\hat{P}_{40})^{(8)} + G_{41}^0) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} \left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$ | |

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| $\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$ | |
| From which it follows that<br>$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t}\leq\frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{P}_{44})^{(9)}+G_{45}^{0}\right)e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ and to choose<br>$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{13})^{(1)}\left\|G_{14}^{(1)}-G_{14}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$<br>$\int_0^t\{(a_{13}')^{(1)}\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{-(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$G_{13}^{(2)}\left\\|(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}'')^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right\\|\ e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$ | |

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| Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | |
| $\left\|G^{(1)} - G^{(2)}\right\| e^{-(\widehat{M}_{13})^{(1)}t} \leq$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right) d\left(\left(G^{(1)}, T^{(1)};\ G^{(2)}, T^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(1)}$ and $(b''_i)^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(1)} - (a''_i)^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1 / (a'_{14})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2 / (a'_{15})^{(1)}$<br><br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(1)}(G(t), t)) = (b'_{14})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ : | 189 |

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| Indeed let $t_1$ be so that for $t > t_1$<br>$(b_{14})^{(1)} - (b_i'')^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}'')^{(1)}(G(t),t) = (b_{15}')^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ $and$ $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |

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| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$(a_{16}^{''})^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)} \left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}} \left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)} (\widehat{k}_{16})^{(2)}\right) d\left(\left((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)}\right)\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)} (T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0 \quad \text{for } t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)} \left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$<br>In the same way , one can obtain | 201 |

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| $G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$, $G_{18}$ and $G_{16}$, $G_{17}$ respectively. | |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$. The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}$, $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}} < 1$ and to choose<br><br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$, $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{20})^{(3)}t}\}$ | 211 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}}$ :$\left( \widetilde{(G_{23})}, \widetilde{(T_{23})} \right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)} \left\|G_{21}^{(1)} - G_{21}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\, ds_{(20)} +$<br><br>$\int_0^t \{(a_{20}^{'})^{(3)} \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right)\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$G_{20}^{(2)} \left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}^{''})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\} ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} t} \leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a_{20}^{'})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)} (\widehat{k}_{20})^{(3)}\right) d\left(\left((G_{23})^{(1)}, (T_{23})^{(1)}; (G_{23})^{(2)}, (T_{23})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ *and* $(\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)} t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating | 217 |

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| $G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}')^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}')^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(3)}\left((G_{23})(t),t\right) = (b_{21}')^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i'')^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}'')^{(3)}\left((G_{23})(t),t\right) = (b_{22}')^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}\left[(\hat{P}_{24})^{(4)} + \left((\hat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}}\left[\left((\hat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{24})^{(4)}\right] \leq (\hat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$ | 225 |

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{24})^{(4)}\left|G_{25}^{(1)}-G_{25}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$$

$$\int_0^t\{(a_{24}^{'})^{(4)}\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$G_{24}^{(2)}\left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)}-(G_{27})^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}t}\le$$
$$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows

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**Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}\ and\ (\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

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**Remark 17:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

it results

$$G_i\ (t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]}\ge 0$$

$$T_i\ (t)\ge T_i^0e^{\left(-(b_i^{'})^{(4)}t\right)}>0\quad \text{for t} > 0$$

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**Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1,\left((\widehat{M}_{24})^{(4)}\right)_2\ and\ \left((\widehat{M}_{24})^{(4)}\right)_3$ :

**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if

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| $G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a'_{25})^{(4)} G_{25}$ and by integrating<br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a'_{25})^{(4)}$<br>In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a'_{26})^{(4)}$<br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t),t)) = (b_{25}^{'})^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i^{''})^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + \left((\widehat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\widehat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{28})^{(5)}\right] \leq (\widehat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying | |

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| Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}((G_{31}),(T_{31}))$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)}+$<br>$\int_0^t\{(a_{28}^{'})^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\le$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}^{'})^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}^{''})^{(5)}$ and $(b_{28}^{''})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(5)}-(a_i^{''})^{(5)}(T_{29}(s_{(28)}),s_{(28)})\}ds_{(28)}\right]}\ge 0$ | 239 |

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| $T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(5)}t)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}^{'})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1/(a_{29}^{'})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2/(a_{30}^{'})^{(5)}$<br><br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t),t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\,\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |

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| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{32})^{(6)}\right]\leq(\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}ds_{(32)}+$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a''_{32})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a'_{32})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(6)}$ and $(b_i'')^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 249 |

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| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(6)} - (a_i'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(6)} t)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}')^{(6)} G_{33}$ and by integrating<br><br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}')^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}')^{(6)}$<br><br>If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(6)}((G_{35})(t), t)) = (b_{33}')^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i'')^{(6)}((G_{35})(t), t) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty} (b_{34}'')^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}')^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have | 255 |

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| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)}+\left((\hat{P}_{36})^{(7)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{36})^{(7)}\right]\leq(\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}')^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}'')^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}')^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}'')^{(7)}$ and $(b_{36}'')^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$. | 260 |

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| If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2 / (a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t), t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t), t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right),\ \ t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded. | 265 |

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| The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t),t\right)=(b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}<1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)}+\left((\hat{P}_{40})^{(8)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{40})^{(8)}\right]\leq(\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}ds_{(40)}+$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\leq$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |

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| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(8)} - (a_i^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2\ and\ \left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}^{'})^{(8)}$<br><br>If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t), t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t), t\right) < \varepsilon_8, T_{40}\,(t) > (m)^{(8)}$ | 278 |

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| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.<br>The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}^{'})^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$ | |
| In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying 39,35,36 into itself | |
| The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{47})^{(1)}, (T_{47})^{(1)}), ((G_{47})^{(2)}, (T_{47})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \left(\widetilde{(G_{47})}, \widetilde{(T_{47})}\right) = \mathcal{A}^{(9)}\left((G_{47}), (T_{47})\right)$<br>It results<br><br>$\left\|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$<br>$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right) - (a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right| \; e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$<br><br>Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on 45,46,47,28 and 29 it follows | |
| $\left|(G_{47})^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t} \leq$<br>$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)}, (T_{47})^{(1)}; (G_{47})^{(2)}, (T_{47})^{(2)})\right)$ | |

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$

From 99 to 44 it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(9)}t\right)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)}G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t), t\right) < \varepsilon_9, T_{44}\,(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded. The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b''_{46})^{(9)}\left((G_{47})(t),t\right) = (b'_{46})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a'_{13})^{(1)} + (a'_{14})^{(1)} - (a''_{13})^{(1)}(T_{14}, t) + (a''_{14})^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b'_{13})^{(1)} + (b'_{14})^{(1)} - (b''_{13})^{(1)}(G,t) - (b''_{14})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,

and $\boxed{(u_0)^{(1)} = \frac{T^0_{13}}{T^0_{14}}}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$

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| are defined | |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le$<br>$\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a'_{15})^{(1)})}[e^{(S_1)^{(1)}t} - e^{-(a'_{15})^{(1)}t}] + G_{15}^0 e^{-(a'_{15})^{(1)}t})$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b'_{15})^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b'_{15})^{(1)}t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)}t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$<br>$(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ :<br>$(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a'_{16})^{(2)} + (a'_{17})^{(2)} - (a''_{16})^{(2)}(T_{17}, t) + (a''_{17})^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b'_{16})^{(2)} + (b'_{17})^{(2)} - (b''_{16})^{(2)}((G_{19}), t) - (b''_{17})^{(2)}((G_{19}), t) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(v_1)^{(2)}, (v_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |

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| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\left(u^{(2)}\right)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}$,, $(\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\left(u^{(2)}\right)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously $(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$ $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities $G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)}t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le\right.$ | 312 |

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| $\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_2)^{(2)}((S_1)^{(2)}-(a_{18}')^{(2)})}[e^{(S_1)^{(2)}t}-e^{-(a_{18}')^{(2)}t}]+G_{18}^{0}e^{-(a_{18}')^{(2)}t})$ | |
| $\boxed{T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}}T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le\frac{1}{(\mu_2)^{(2)}}T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)}T_{16}^{0}}{(\mu_1)^{(2)}((R_1)^{(2)}-(b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t}-e^{-(b_{18}')^{(2)}t}\right]+T_{18}^{0}e^{-(b_{18}')^{(2)}t}\le T_{18}(t)\le$ $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)},(S_2)^{(2)},(R_1)^{(2)},(R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)}=(a_{16})^{(2)}(m_2)^{(2)}-(a_{16}')^{(2)}$ $(S_2)^{(2)}=(a_{18})^{(2)}-(p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)}=(b_{16})^{(2)}(\mu_2)^{(1)}-(b_{16}')^{(2)}$ $(R_2)^{(2)}=(b_{18}')^{(2)}-(r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$: $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)}\le-(a_{20}')^{(3)}+(a_{21}')^{(3)}-(a_{20}'')^{(3)}(T_{21},t)+(a_{21}'')^{(3)}(T_{21},t)\le-(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)}\le-(b_{20}')^{(3)}+(b_{21}')^{(3)}-(b_{20}'')^{(3)}(G_{23},t)-(b_{21}'')^{(3)}((G_{23}),t)\le-(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)},(v_2)^{(3)},(u_1)^{(3)},(u_2)^{(3)}$: By $(v_1)^{(3)}>0,(v_2)^{(3)}<0$ and respectively $(u_1)^{(3)}>0,(u_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_1)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_1)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ and By $(\bar{v}_1)^{(3)}>0,(\bar{v}_2)^{(3)}<0$ and respectively $(\bar{u}_1)^{(3)}>0,(\bar{u}_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_2)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_2)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ | 320 |
| **Definition of** $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$:- | 321 |

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$, ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,

and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously

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$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$$

$(p_i)^{(3)}$ is defined by equation

$$\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t} \quad (323)$$

$$\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$$

$$\left. \frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right) \quad (324)$$

$$\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}} \quad (325)$$

$$\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t} \quad (326)$$

$$\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$$

$$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t} \quad (327)$$

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- (328)

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$$

$$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$$

$$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}\big((G_{27}), t\big) - (b''_{25})^{(4)}\big((G_{27}), t\big) \le -(\tau_1)^{(4)}$$

**Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : 329

By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{v}_1)^{(4)} > 0$, $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (v_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}, \ \textbf{\textit{if}}\ (v_0)^{(4)} < (v_1)^{(4)}$$

$$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}, \textbf{\textit{if}}\ (v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)},$$

and $\boxed{(v_0)^{(4)} = \frac{G^0_{24}}{G^0_{25}}}$

$$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}, \ \textbf{\textit{if}}\ (\bar{v}_4)^{(4)} < (v_0)^{(4)}$$

and analogously 331

$$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}, \ \textbf{\textit{if}}\ (u_0)^{(4)} < (u_1)^{(4)}$$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}, \textbf{\textit{if}}\ (u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)},$$

and $\boxed{(u_0)^{(4)} = \frac{T^0_{24}}{T^0_{25}}}$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \textbf{\textit{if}}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)} \text{ where } (u_1)^{(4)}, (\bar{u}_1)^{(4)}$$

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| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le\right.$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}')^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}')^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)}t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(\nu_1)^{(5)}, (\nu_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, \nu^{(5)}, u^{(5)}$ :<br>By $(\nu_1)^{(5)} > 0$, $(\nu_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}(\nu^{(5)})^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)} = 0$ | 339 |

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| and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ :<br><br>By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)},(v_0)^{(5)}$ :-<br><br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)}$ by<br><br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br><br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br><br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |

$\frac{(b_{30})^{(5)}T_{28}^{0}}{(\mu_1)^{(5)}((R_1)^{(5)}-(b_{30}^{'})^{(5)})}\left[e^{(R_1)^{(5)}t}-e^{-(b_{30}^{'})^{(5)}t}\right]+T_{30}^{0}e^{-(b_{30}^{'})^{(5)}t}\le T_{30}(t)\le$

$\frac{(a_{30})^{(5)}T_{28}^{0}}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t}-e^{-(R_2)^{(5)}t}\right]+T_{30}^{0}e^{-(R_2)^{(5)}t}$ 347

**Definition of** $(S_1)^{(5)},(S_2)^{(5)},(R_1)^{(5)},(R_2)^{(5)}$:- 348

Where $(S_1)^{(5)}=(a_{28})^{(5)}(m_2)^{(5)}-(a_{28}^{'})^{(5)}$

$(S_2)^{(5)}=(a_{30})^{(5)}-(p_{30})^{(5)}$

$(R_1)^{(5)}=(b_{28})^{(5)}(\mu_2)^{(5)}-(b_{28}^{'})^{(5)}$

$(R_2)^{(5)}=(b_{30}^{'})^{(5)}-(r_{30})^{(5)}$

**Behavior of the solutions of equation** 349

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ four constants satisfying

$-(\sigma_2)^{(6)}\le-(a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}-(a_{32}^{''})^{(6)}(T_{33},t)+(a_{33}^{''})^{(6)}(T_{33},t)\le-(\sigma_1)^{(6)}$

$-(\tau_2)^{(6)}\le-(b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(b_{32}^{''})^{(6)}\left((G_{35}),t\right)-(b_{33}^{''})^{(6)}\left((G_{35}),t\right)\le-(\tau_1)^{(6)}$

**Definition of** $(\nu_1)^{(6)},(\nu_2)^{(6)},(u_1)^{(6)},(u_2)^{(6)},\nu^{(6)},u^{(6)}$ : 350

By $(\nu_1)^{(6)}>0$ , $(\nu_2)^{(6)}<0$ and respectively $(u_1)^{(6)}>0$ , $(u_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_1)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2+(\tau_1)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$ and

**Definition of** $(\bar{\nu}_1)^{(6)},,(\bar{\nu}_2)^{(6)},(\bar{u}_1)^{(6)},(\bar{u}_2)^{(6)}$ : 351

By $(\bar{\nu}_1)^{(6)}>0$ , $(\bar{\nu}_2)^{(6)}<0$ and respectively $(\bar{u}_1)^{(6)}>0$ , $(\bar{u}_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_2)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2+(\tau_2)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$

**Definition of** $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)},(\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)}$ by

$(m_2)^{(6)}=(\nu_0)^{(6)},(m_1)^{(6)}=(\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)}<(\nu_1)^{(6)}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)}<(\nu_0)^{(6)}<(\bar{\nu}_1)^{(6)}$,

and $\boxed{(\nu_0)^{(6)}=\frac{G_{32}^{0}}{G_{33}^{0}}}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)}<(\nu_0)^{(6)}$

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| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},\ \boldsymbol{if}\ (u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}, \boldsymbol{if}\ (u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)},$<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}, \boldsymbol{if}\ (\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le\right.$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b_{34}')^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b_{34}')^{(6)} t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$<br>$(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ : | |

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| $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying<br><br>$-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$<br><br>$-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}((G_{39}), t) - (b''_{37})^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ :<br><br>By $(\nu_1)^{(7)} > 0, (\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ :<br><br>By $(\bar{\nu}_1)^{(7)} > 0, (\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$<br>**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-<br><br>If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by<br><br>$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,<br>and $\boxed{(\nu_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | 362 |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,<br>and $\boxed{(u_0)^{(7)} = \frac{T^0_{36}}{T^0_{37}}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities | 364 |

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| $G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^{0}e^{(S_1)^{(7)}t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | |
| $\frac{1}{(m_7)^{(7)}} G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^{0}e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^{0}e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}^{'})^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}^{'})^{(7)}t}] + G_{38}^{0}e^{-(a_{38}^{'})^{(7)}t})$ | 366 |
| $\boxed{T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)}T_{36}^{0}}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}^{'})^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}^{'})^{(7)}t}\right] + T_{38}^{0}e^{-(b_{38}^{'})^{(7)}t} \le T_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)}T_{36}^{0}}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^{0}e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}^{'})^{(7)}$<br><br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br><br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}^{'})^{(7)}$<br><br>$(R_2)^{(7)} = (b_{38}^{'})^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br><br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br><br>$-(\sigma_2)^{(8)} \le -(a_{40}^{'})^{(8)} + (a_{41}^{'})^{(8)} - (a_{40}^{''})^{(8)}(T_{41}, t) + (a_{41}^{''})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br><br>$-(\tau_2)^{(8)} \le -(b_{40}^{'})^{(8)} + (b_{41}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}), t) - (b_{41}^{''})^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |
| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : | 372 |

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| By $(\nu_1)^{(8)} > 0\,, (\nu_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0\,, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$ and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(8)},, (\bar{\nu}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :<br><br>By $(\bar{\nu}_1)^{(8)} > 0\,, (\bar{\nu}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0\,, (\bar{u}_2)^{(8)} < 0$ the<br><br>roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$<br><br>and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$<br><br>**Definition of** $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (\nu_0)^{(8)}$ :-<br><br>If we define $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by<br><br>$(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)},$ ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$<br><br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)},$<br>and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br><br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)},$ ***if*** $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br><br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},$ ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)},$ ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)}t}$<br><br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$ | 376 |

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| $\left(\frac{(a_{42})^{(8)}G_{40}^{0}}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t}-e^{-(S_2)^{(8)}t}\right]+G_{42}^{0}e^{-(S_2)^{(8)}t}\leq G_{42}(t)\leq \frac{(a_{42})^{(8)}G_{40}^{0}}{(m_2)^{(8)}((S_1)^{(8)}-(a_{42}^{'})^{(8)})}[e^{(S_1)^{(8)}t}-e^{-(a_{42}^{'})^{(8)}t}]+G_{42}^{0}e^{-(a_{42}^{'})^{(8)}t}\right)$ | 377 |
| $\boxed{T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}}T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq \frac{1}{(\mu_2)^{(8)}}T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^{0}}{(\mu_1)^{(8)}((R_1)^{(8)}-(b_{42}^{'})^{(8)})}\left[e^{(R_1)^{(8)}t}-e^{-(b_{42}^{'})^{(8)}t}\right]+T_{42}^{0}e^{-(b_{42}^{'})^{(8)}t}\leq T_{42}(t)\leq$ $\frac{(a_{42})^{(8)}T_{40}^{0}}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t}-e^{-(R_2)^{(8)}t}\right]+T_{42}^{0}e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)},(S_2)^{(8)},(R_1)^{(8)},(R_2)^{(8)}$:-<br>Where $(S_1)^{(8)}=(a_{40})^{(8)}(m_2)^{(8)}-(a_{40}^{'})^{(8)}$<br>$(S_2)^{(8)}=(a_{42})^{(8)}-(p_{42})^{(8)}$<br>$(R_1)^{(8)}=(b_{40})^{(8)}(\mu_2)^{(8)}-(b_{40}^{'})^{(8)}$<br>$(R_2)^{(8)}=(b_{42}^{'})^{(8)}-(r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)}\leq -(a_{44}^{'})^{(9)}+(a_{45}^{'})^{(9)}-(a_{44}^{''})^{(9)}(T_{45},t)+(a_{45}^{''})^{(9)}(T_{45},t)\leq -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)}\leq -(b_{44}^{'})^{(9)}+(b_{45}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47}),t)-(b_{45}^{''})^{(9)}((G_{47}),t)\leq -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(\nu_1)^{(9)},(\nu_2)^{(9)},(u_1)^{(9)},(u_2)^{(9)},\nu^{(9)},u^{(9)}$ :<br>By $(\nu_1)^{(9)}>0$ , $(\nu_2)^{(9)}<0$ and respectively $(u_1)^{(9)}>0$ , $(u_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_1)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$<br>and $(b_{45})^{(9)}(u^{(9)})^2+(\tau_1)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(9)},,(\bar{\nu}_2)^{(9)},(\bar{u}_1)^{(9)},(\bar{u}_2)^{(9)}$ :<br>By $(\bar{\nu}_1)^{(9)}>0$ , $(\bar{\nu}_2)^{(9)}<0$ and respectively $(\bar{u}_1)^{(9)}>0$ , $(\bar{u}_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_2)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$ | |

and $(b_{45})^{(9)}\left(u^{(9)}\right)^2+(\tau_2)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$

**Definition of** $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)},(\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)}$ by

$(m_2)^{(9)}=(\nu_0)^{(9)},(m_1)^{(9)}=(\nu_1)^{(9)},$ ***if*** $(\nu_0)^{(9)}<(\nu_1)^{(9)}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)}<(\nu_0)^{(9)}<(\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)}=\frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\nu_0)^{(9)},$ ***if*** $(\bar{\nu}_1)^{(9)}<(\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)}=(u_0)^{(9)},(\mu_1)^{(9)}=(u_1)^{(9)},$ ***if*** $(u_0)^{(9)}<(u_1)^{(9)}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)}<(u_0)^{(9)}<(\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)}=\frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(u_0)^{(9)},$ ***if*** $(\bar{u}_1)^{(9)}<(u_0)^{(9)}$ where $(u_1)^{(9)},(\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{44}(t)\le G_{44}^0e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{45}(t)\le\frac{1}{(m_2)^{(9)}}G_{44}^0e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)}G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^0e^{-(S_2)^{(9)}t}\le G_{46}(t)\le\right.$$

$$\left.\frac{(a_{46})^{(9)}G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}')^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}')^{(9)}t}]+\ G_{46}^0e^{-(a_{46}')^{(9)}t}\right)$$

$$\boxed{T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le\frac{1}{(\mu_2)^{(9)}}T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}')^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}')^{(9)}t}\right]+T_{46}^0e^{-(b_{46}')^{(9)}t}\le T_{46}(t)\le$$

$$\frac{(a_{46})^{(9)}T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^0e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain (383)

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right) - (a''_{14})^{(1)}(T_{14}, t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get (384)

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{v}_1)^{(1)}$ we find like in the previous case, (385)

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)}G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)}T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$$

If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain 394

$$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(2)}(t)$ :- 395

$$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}, \quad \boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$$

In a completely analogous way, we obtain 396

**Definition of** $u^{(2)}(t)$ :-

$$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}, \quad \boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :** 397

If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$

Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then

$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$ if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$

**Proof** : From global equations we obtain 398

$$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$$

**Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$ 399

It follows

$$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$$

From which one obtains 400

For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)} + (C)^{(3)}(\nu_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

In the same manner , we get 401

$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)} + (\bar{C})^{(3)}(\bar{\nu}_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, 402

$$(\nu_1)^{(3)} \le \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \le \nu^{(3)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \le (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \le (\bar{\nu}_1)^{(3)} \le (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

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$$(\nu_1)^{(3)} \le \nu^{(3)}(t) \le \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \le (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \le \nu^{(3)}(t) \le (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \le u^{(3)}(t) \le (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right) - (a_{25}^{''})^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\quad \boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \le \frac{d\nu^{(4)}}{dt} \le -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

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$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} =$

$(\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b''_{24})^{(4)} = (b''_{25})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a'_{28})^{(5)} - (a'_{29})^{(5)} + (a''_{28})^{(5)}(T_{29}, t)\right) - (a''_{29})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\quad \boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get 413

$$v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{v}_1)^{(6)}-(v_0)^{(6)}}{(v_0)^{(6)}-(\bar{v}_2)^{(6)}}}$$

From which we deduce $(v_0)^{(6)} \le v^{(6)}(t) \le (\bar{v}_1)^{(6)}$

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If $0 < (v_1)^{(6)} < (v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{v}_1)^{(6)}$ we find like in the previous case,

$$(v_1)^{(6)} \le \frac{(v_1)^{(6)}+(C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}} \le v^{(6)}(t) \le$$

$$\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (\bar{v}_1)^{(6)}$$

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If $0 < (v_1)^{(6)} \le (\bar{v}_1)^{(6)} \le \boxed{(v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(v_1)^{(6)} \le v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (v_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$$(m_2)^{(6)} \le v^{(6)}(t) \le (m_1)^{(6)}, \quad \boxed{v^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)} \le u^{(6)}(t) \le (\mu_1)^{(6)}, \quad \boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(v_1)^{(6)} = (\bar{v}_1)^{(6)}$ if in addition $(v_0)^{(6)} = (v_1)^{(6)}$ then $v^{(6)}(t) = (v_0)^{(6)}$ and as a consequence $G_{32}(t) = (v_0)^{(6)}G_{33}(t)$ **this also defines** $(v_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(v_1)^{(6)}$ and $(\bar{v}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right) - (a_{37}^{''})^{(7)}(T_{37},t)v^{(7)} - (a_{37})^{(7)}v^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get 417

$$\nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\bar{\nu}_1)^{(7)}$

If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case, 418

$$(\nu_1)^{(7)} \leq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}} \leq \nu^{(7)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\bar{\nu}_1)^{(7)}$$

If $0 < (\nu_1)^{(7)} \leq (\bar{\nu}_1)^{(7)} \leq \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain 419

$$(\nu_1)^{(7)} \leq \nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \leq \nu^{(7)}(t) \leq (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :- 420

$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)} G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain 421

$$\frac{d\nu^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right) - (a_{41}^{''})^{(8)}(T_{41}, t)\nu^{(8)} - (a_{41})^{(8)}\nu^{(8)}$$

**Definition of** $\nu^{(8)}$ :- $\boxed{\nu^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right) \leq \frac{d\nu^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)}-(\nu_0)^{(8)}}{(\nu_0)^{(8)}-(\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \le \nu^{(8)}(t) \le (\bar{\nu}_8)^{(8)}$

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If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case,

$$(\nu_1)^{(8)} \le \frac{(\nu_1)^{(8)}+(C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}} \le \nu^{(8)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\bar{\nu}_1)^{(8)}$$

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If $0 < (\nu_1)^{(8)} \le (\bar{\nu}_1)^{(8)} \le \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$, we obtain

$$(\nu_1)^{(8)} \le \nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \le \nu^{(8)}(t) \le (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \le u^{(8)}(t) \le (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)}G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)}T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right) - (a''_{45})^{(9)}(T_{45},t)v^{(9)} - (a_{45})^{(9)}v^{(9)}$$

**Definition of** $v^{(9)}$ :- $\boxed{v^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)}\right) \le \frac{dv^{(9)}}{dt} \le -\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(9)}, (v_0)^{(9)}$ :-

For $0 < \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (v_1)^{(9)} < (\bar{v}_1)^{(9)}$

$$v^{(9)}(t) \ge \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(v_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (v_2)^{(9)}}}$$

it follows $(v_0)^{(9)} \le v^{(9)}(t) \le (v_9)^{(9)}$

In the same manner , we get

$$v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{v}_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (\bar{v}_2)^{(9)}}}$$

From which we deduce $(v_0)^{(9)} \le v^{(9)}(t) \le (\bar{v}_1)^{(9)}$

If $0 < (v_1)^{(9)} < (v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}} < (\bar{v}_1)^{(9)}$ we find like in the previous case,

$$(v_1)^{(9)} \le \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}} \le v^{(9)}(t) \le$$

$$\frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (\bar{v}_1)^{(9)}$$

If $0 < (v_1)^{(9)} \le (\bar{v}_1)^{(9)} \le \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$ , we obtain

$$(v_1)^{(9)} \le v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (v_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(9)}(t)$ :-

$$(m_2)^{(9)} \le v^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{v^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}$, $\boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ 430

$with\ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations 431

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$

| | |
|---|---|
| $with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | |
| We can prove the following<br><br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br><br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br><br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br><br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$ | 435 |

| | |
|---|---|
| $with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |

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|---|---|
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27})\big)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\big((G_{27})\big)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\big((G_{27})\big)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |

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|---|---|
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |

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| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(i) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) +$ | 489 |

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| $(a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that | 496 |

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| there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{32}=\frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}=\frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{36}=\frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}=\frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |
| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*})=0$. With this value , we obtain from the three first equations | 501 |

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| $G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G) + (b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*}) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19}) + (b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br>$[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ | 506 |

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| is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution | 511 |

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| $G_{14}^{*}$ given by $\varphi(G^{*})=0$ , $T_{14}^{*}$ given by $f(T_{14}^{*})=0$ and<br>$G_{13}^{*}=\frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*}=\frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br>$T_{13}^{*}=\frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*}=\frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br>Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*})=0$ , $T_{17}^{*}$ given by $f(T_{17}^{*})=0$ and | 512 |
| $G_{16}^{*}=\frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*}=\frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*}=\frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*}=\frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*})=0$ , $T_{21}^{*}$ given by $f(T_{21}^{*})=0$ and<br>$G_{20}^{*}=\frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*}=\frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br>$T_{20}^{*}=\frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*}=\frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^{*}$ given by $\varphi(G_{27})=0$ , $T_{25}^{*}$ given by $f(T_{25}^{*})=0$ and<br>$G_{24}^{*}=\frac{(a_{24})^{(4)}G_{25}^{*}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*}=\frac{(a_{26})^{(4)}G_{25}^{*}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*}=\frac{(b_{24})^{(4)}T_{25}^{*}}{[(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*}=\frac{(b_{26})^{(4)}T_{25}^{*}}{[(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}((G_{27})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*})=0$ , $T_{29}^{*}$ given by $f(T_{29}^{*})=0$ and<br>$G_{28}^{*}=\frac{(a_{28})^{(5)}G_{29}^{*}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*}=\frac{(a_{30})^{(5)}G_{29}^{*}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |

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| $T_{28}^{*} = \frac{(b_{28})^{(5)}T_{29}^{*}}{[(b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)}T_{29}^{*}}{[(b_{30}^{'})^{(5)}-(b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br>$G_{32}^{*} = \frac{(a_{32})^{(6)}G_{33}^{*}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)}G_{33}^{*}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)}T_{33}^{*}}{[(b_{32}^{'})^{(6)}-(b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)}T_{33}^{*}}{[(b_{34}^{'})^{(6)}-(b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br>$G_{36}^{*} = \frac{(a_{36})^{(7)}G_{37}^{*}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)}G_{37}^{*}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br>$T_{36}^{*} = \frac{(b_{36})^{(7)}T_{37}^{*}}{[(b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)}T_{37}^{*}}{[(b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br>$G_{40}^{*} = \frac{(a_{40})^{(8)}G_{41}^{*}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)}G_{41}^{*}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br>$T_{40}^{*} = \frac{(b_{40})^{(8)}T_{41}^{*}}{[(b_{40}^{'})^{(8)}-(b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)}T_{41}^{*}}{[(b_{42}^{'})^{(8)}-(b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)}G_{45}^{*}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)}G_{45}^{*}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)}T_{45}^{*}}{[(b_{44}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)}T_{45}^{*}}{[(b_{46}^{'})^{(9)}-(b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | 523 A |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^*) = (q_{14})^{(1)} \quad , \frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^*) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)}G_{13}^*\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^*\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^*\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^*\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^*\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^*\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)} \quad , \frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |

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| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ $and$ $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)}$ , $\frac{\partial(b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_{j}\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_{j}\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_{j}\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 548 |

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| $\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^{*}) = (q_{25})^{(4)}$ , $\frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^{*}) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^{*}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^{*}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^{*}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^{*}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^{*}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^{*}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^{*}) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^{*}) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^{*}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^{*}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^{*}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^{*}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^{*}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^{*}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 563 |

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| **Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^*\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^*\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^*\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^*\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}((G_{39})^{**}) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |

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| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^{*}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^{*}\mathbb{G}_{j}\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^{*}\mathbb{G}_{j}\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^{*}\mathbb{G}_{j}\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^{*}) = (q_{41})^{(8)} \quad , \frac{\partial(b_i^{''})^{(8)}}{\partial G_j}\left((G_{43})^{*}\right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^{*}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^{*}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^{*}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^{*}\mathbb{G}_{j}\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b_{41}^{'})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T_{41}^{*}\mathbb{G}_{j}\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b_{42}^{'})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T_{42}^{*}\mathbb{G}_{j}\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |

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| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \ \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b_{44}^{'})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T_{44}^*\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b_{45}^{'})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T_{45}^*\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b_{46}^{'})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T_{46}^*\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)$<br>$\left[\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(q_{13})^{(1)}G_{13}^*\right)\Big]$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T_{14}^* + (b_{14})^{(1)}s_{(13),(14)}T_{14}^*\Big)$<br>$+\left(\left((\lambda)^{(1)} + (a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^* + (a_{13})^{(1)}(q_{14})^{(1)}G_{14}^*\right)$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^* + (b_{14})^{(1)}s_{(13),(13)}T_{13}^*\Big)$<br>$\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$<br>$\left((\lambda)^{(1)}\right)^2 + \left((b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$<br>$+\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$<br>$+\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^*\right)$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^* + (b_{14})^{(1)}s_{(13),(15)}T_{13}^*\Big)\} = 0$ | |

$$+$$

$$((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)})\{((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)})$$

$$\left[\left(((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)})(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)\right]$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(17)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^{*}_{17}\right)$$

$$+\left(((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)})(q_{16})^{(2)}\mathrm{G}^{*}_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(16)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^{*}_{16}\right)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left(\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(18)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(18)}\mathrm{T}^{*}_{16}\right)\} = 0$$

$$+$$

$$((\lambda)^{(3)} + (b'_{22})^{(3)} - (r_{22})^{(3)})\{((\lambda)^{(3)} + (a'_{22})^{(3)} + (p_{22})^{(3)})$$

$$\left[\left(((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)})(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)\right]$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(21)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(21)}T^{*}_{21}\right)$$

$$+\left(((\lambda)^{(3)} + (a'_{21})^{(3)} + (p_{21})^{(3)})(q_{20})^{(3)}G^{*}_{20} + (a_{20})^{(3)}(q_{21})^{(1)}G^{*}_{21}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(20)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(20)}T^{*}_{20}\right)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((b'_{20})^{(3)} + (b'_{21})^{(3)} - (r_{20})^{(3)} + (r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left(\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(22)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(22)}T^{*}_{20}\right)\} = 0$$

$$+$$

$$\left((\lambda)^{(4)}+(b'_{26})^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a'_{26})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a'_{24})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G^{*}_{25}+(a_{25})^{(4)}(q_{24})^{(4)}G^{*}_{24}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(25)}T^{*}_{25}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a'_{25})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G^{*}_{24}+(a_{24})^{(4)}(q_{25})^{(4)}G^{*}_{25}\right)$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(24)}T^{*}_{24}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((a'_{24})^{(4)}+(a'_{25})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((b'_{24})^{(4)}+(b'_{25})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^{2}+\left((a'_{24})^{(4)}+(a'_{25})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a'_{24})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G^{*}_{25}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G^{*}_{24}\right)$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(26)}T^{*}_{24}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(5)}+(b'_{30})^{(5)}-(r_{30})^{(5)}\right)\{\left((\lambda)^{(5)}+(a'_{30})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a'_{28})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G^{*}_{29}+(a_{29})^{(5)}(q_{28})^{(5)}G^{*}_{28}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(29)}T^{*}_{29}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a'_{29})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G^{*}_{28}+(a_{28})^{(5)}(q_{29})^{(5)}G^{*}_{29}\right)$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(28)}T^{*}_{28}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((a'_{28})^{(5)}+(a'_{29})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((b'_{28})^{(5)}+(b'_{29})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^{2}+\left((a'_{28})^{(5)}+(a'_{29})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a'_{28})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G^{*}_{29}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G^{*}_{28}\right)$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(30)}T^{*}_{28}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\right)\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\right)\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(8)} + (b'_{42})^{(8)} - (r_{42})^{(8)})\{((\lambda)^{(8)} + (a'_{42})^{(8)} + (p_{42})^{(8)})$$

$$\left[\left(((\lambda)^{(8)} + (a'_{40})^{(8)} + (p_{40})^{(8)})(q_{41})^{(8)}G^*_{41} + (a_{41})^{(8)}(q_{40})^{(8)}G^*_{40}\right)\right]$$

$$\left(((\lambda)^{(8)} + (b'_{40})^{(8)} - (r_{40})^{(8)})s_{(41),(41)}T^*_{41} + (b_{41})^{(8)}s_{(40),(41)}T^*_{41}\right)$$

$$+\left(((\lambda)^{(8)} + (a'_{41})^{(8)} + (p_{41})^{(8)})(q_{40})^{(8)}G^*_{40} + (a_{40})^{(8)}(q_{41})^{(8)}G^*_{41}\right)$$

$$\left(((\lambda)^{(8)} + (b'_{40})^{(8)} - (r_{40})^{(8)})s_{(41),(40)}T^*_{41} + (b_{41})^{(8)}s_{(40),(40)}T^*_{40}\right)$$

$$\left(((\lambda)^{(8)})^2 + (\,(a'_{40})^{(8)} + (a'_{41})^{(8)} + (p_{40})^{(8)} + (p_{41})^{(8)})\,(\lambda)^{(8)}\right)$$

$$\left(((\lambda)^{(8)})^2 + (\,(b'_{40})^{(8)} + (b'_{41})^{(8)} - (r_{40})^{(8)} + (r_{41})^{(8)})\,(\lambda)^{(8)}\right)$$

$$+\left(((\lambda)^{(8)})^2 + (\,(a'_{40})^{(8)} + (a'_{41})^{(8)} + (p_{40})^{(8)} + (p_{41})^{(8)})\,(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+((\lambda)^{(8)} + (a'_{40})^{(8)} + (p_{40})^{(8)})\,((a_{42})^{(8)}(q_{41})^{(8)}G^*_{41} + (a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^*_{40})$$

$$\left(((\lambda)^{(8)} + (b'_{40})^{(8)} - (r_{40})^{(8)})s_{(41),(42)}T^*_{41} + (b_{41})^{(8)}s_{(40),(42)}T^*_{40}\right)\} = 0$$

$$+$$

$$(\lambda)^{(9)} + (b'_{46})^{(9)} - (r_{46})^{(9)})\{((\lambda)^{(9)} + (a'_{46})^{(9)} + (p_{46})^{(9)})$$

$$\left[\left(((\lambda)^{(9)} + (a'_{44})^{(9)} + (p_{44})^{(9)})(q_{45})^{(9)}G^*_{45} + (a_{45})^{(9)}(q_{44})^{(9)}G^*_{44}\right)\right]$$

$$\left(((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)})s_{(45),(45)}T^*_{45} + (b_{45})^{(9)}s_{(44),(45)}T^*_{45}\right)$$

$$+\left(((\lambda)^{(9)} + (a'_{45})^{(9)} + (p_{45})^{(9)})(q_{44})^{(9)}G^*_{44} + (a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\right)$$

$$\left(((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)})s_{(45),(44)}T^*_{45} + (b_{45})^{(9)}s_{(44),(44)}T^*_{44}\right)$$

$$\left(((\lambda)^{(9)})^2 + (\,(a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)})\,(\lambda)^{(9)}\right)$$

$$\left(((\lambda)^{(9)})^2 + (\,(b'_{44})^{(9)} + (b'_{45})^{(9)} - (r_{44})^{(9)} + (r_{45})^{(9)})\,(\lambda)^{(9)}\right)$$

$$+\left(((\lambda)^{(9)})^2 + (\,(a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)})\,(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+((\lambda)^{(9)} + (a'_{44})^{(9)} + (p_{44})^{(9)})\,((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45} + (a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44})$$

$\left(\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\right)\} = 0$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i \in R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i\ X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor

bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k \tag{9}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 9. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,...x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \in I, x \in K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow *x.t restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\,\delta\,(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\mathcal{A}(\tilde{A}), C\backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population *i* is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int_0^t (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\to\infty}{limsup}\frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \epsilon R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v=au_i(x)+w$ with $a \epsilon R$ and $w \epsilon F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |B_i(t,x)u_i(x)|(a+ \alpha+ exp(-\beta t)|w|)$. Hence, $\underset{t\to\infty}{limsup}\frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\underset{t\to\infty}{limsup}\frac{1}{t} In\ B_i(t,x)\ u_i(x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \underset{t\to\infty}{limsup}\frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a- \alpha+ exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \underset{t\to\infty}{limsup}\frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \underset{t\to\infty}{limsup}\frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In } b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) =0$ implies

$$\frac{d}{dt}b_i(t,x) = \lim_{s\to 0}\frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

=

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \le 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \le \gamma$ for all $t \ge 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \operatorname{In} |B_i(t,x)x_i| = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \operatorname{In} |\, (x.t)_i\, |$$

$$\le \operatorname{limsup}_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0. If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\ge 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\operatorname{limsup}_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x) =$ Integral $h(x.t)d\,\mu(x)$for all $t \ge 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure*. $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$ , $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x) \rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1 \le j \le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |\mathrm{h}(x.t) - h(x^n.t)|d\mu_n(x)| + |\int_c h(x^n.t)|d\mu_n(x) - \int_c \mathrm{h}(x)d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x)d\mu_n(x) - \int_c h(x)d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $\mathrm{h}_i^n(x) = \langle \mathrm{A}_i^n(x)\mathrm{u}_i^n(x), \mathrm{u}_i^n(x)\rangle$ where $\mathrm{u}_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $\mathrm{A}_i^n(x) \to \mathrm{A}_i(x)$ uniformly for $x \in N_1((\mathrm{A}))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $\mathrm{u}_i^n(x) \to \mathrm{u}_i(x)$ converges uniformly for $x \in N_1(\Lambda(\mathrm{A}))$. Given $\in > 0$, choose $n$ sufficiently large so that $|\mathrm{h}_i^n(x) - \mathrm{h}_i(x)| \leq \in$ for all $x \in N_1(\Lambda(\mathrm{A}))$ and $|\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$\lambda_i(\mu) \leq \int_c |\mathrm{h}_i(x) - \mathrm{h}_i^n(x)|d\mu(x) + |\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)|$

$+ \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)$

$\leq \quad 2\in$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\, \mathrm{ds}$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument

implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i \, (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

(6) $\min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_{i} Pi\lambda i(\mu)$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$ $\sum_{i} P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_{1,}\ldots.,\tilde{A}_{k})$ by

$$\tilde{A}_i(x) = \begin{cases} \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} & if\ i \epsilon I \\ if\ i \notin I \end{cases}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t >$ $0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in Kand\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c\ II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set $K$ if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$?then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$?and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume x* $\mapsto$ *A(x) is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists j* $\in$ { 1,….,k} *such that $M_i$ ----- { $x \in C : x_l \gg 0$ for $l \neq j$} Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_iM_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----{1,…,k} such that $\mu(---_i \in_I C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures μ with support in ∂C, $\lambda_i(\mu)>0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $$\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j \qquad i = 1,..,k \qquad j=1,\ldots,m$$

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge

to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j$ ($x^j_i$) = 0 i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product ($E_1^1$ ,…,$E_1^m$), and the matrix $A^2$ ($E_1$) is close to the diagonal matrix diag($f^i_2(E_1{}^j)$)$^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f_2^j(E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial$C is given by two equilibria {0,$E_1$}, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j\ x_2^j\ - \alpha^j\ x_3^j\ ) + \sum d_1^{jk}\ \ x_1^k\ - e_1^j\ x_1^j$

$\frac{dx_2^j}{dt} = x_2^j(1 - \alpha^j\ x_1^j\ - x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_2^{jk}\ \ x_2^k\ - e_2^j\ x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1 - \beta^j\ x_1^j\ - \alpha^j x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_3^{jk}\ \ x_3^k\ - e_3^j\ x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all *j*. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all *i, j*) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all *i*, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2\ (E_1)$and $\lambda_3\ (E_1)$are close to $\max_j\ 1 - a^j$ and $\max_j\ 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j\ 1 - a^j$ and $\max_j\ 1 - \beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1 - a^j) > -\max_j\ (1 - \beta^j)$$

Equivalently,

$$2 > \min_j\ a^j + \min_j\ \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i\left(r_i - \sum_s a_{is}x_s\right), \qquad i = 1,\dots,k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$$

It has been shown [12,ch.16.1,16.2] That for $k = 3$,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \ \ with\ \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing

holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & O \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left[ \beta(1 - y - z)\frac{P}{\epsilon + P} - r - m(1 - y) - b(N)\ y \right]$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*′ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-}\ y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + Kp\, x_i x_i - 1}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords

Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation

of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1$ *(t)* and $P_2$ *(t)* denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial

alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.25) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.26) $$D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.27) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_{1,}$ $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_{i,}$ $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic ($as$ $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 9. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, \text{R}^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1, 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) \quad P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^\infty f_i(a + s, P(t + S)) ds \right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[ - \int_o^x f_1(a + s, p(t + s)) ds \right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0) \exp\left[ - \int_o^t f_1\left(a - t + s, p_1(s)\right) ds \right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(\text{a+s}, p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da+\int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$$t> 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in[-\tau, 0], i = 1,2,$

(2.8) $\beta_i(\text{t}) = \int_0^t \beta_i(\text{a}) b_i(\text{t-a}, P_i(t)) M_i.(a, t, P_i(t))da$

$$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$$

$$i = 1, 2, \text{t} > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t)\, \phi(0) d_i^*]t \quad for\ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T],R^+), \quad B_i \in C([0,T],R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau,T] = \{f \in C([-\tau,T],R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau,T]$. Then for any

$(x, y) \in C^+[-\tau,T] \times C^+[-\tau,T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau,T], \quad C_T^2 = C^+[-\tau,T] \times C^+[-\tau,T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau,T]$ *and* $t \in [0,T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$B: C_T^2 \to C_T^2$ *and* $\Pi C_T^2 \to C_T^2$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = P_t \left\{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right)$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Pi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Pi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Pi(P_t^{(1)}) - \Pi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Pi(P^{(1)}(t)) - \Pi(P^{(2)}(t)| \le \int_0^t B_1^{(1)}(a) \left\{ M_1(a, t, P_t^{(1)}) - M_1(a, t, P_t^{(2)}) \right\} da$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left( \int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_* ds \right) da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o) \left[ \int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds \right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \|_t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$=\int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\mathrm{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \, /\!/ P_t^{(1)} - P_t^{(2)} /\!/ \text{ for } a \, \varepsilon [o, t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \leq \beta_1 \Phi_1(0) || P_t^{(1)} - P_t^{(2)} ||_t^* + \Phi_1(0) \beta_1 d_1 || P_t^{(1)} - P_t^{(2)} ||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) \, e^{\delta 1^a} \beta_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) \, e^{\delta 1^a} /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| B_1^{*d}\sigma$$

$$+ \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \left\{ B_1 + B_2 d_1 a + B_1^* \left(B_2 a + d_1 a^2/2\right) e^{\delta 1^a} \right\}$$

$$\leq \beta_1^* \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| do + \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \to \beta_1$ as $a \to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{g_1(a) + 0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a)) - b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a)) do \right|$$

$$\left| B_1^* \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi\left(P_t^{(1)} - \pi\left(P_t^{(2)}\right) /\!/ \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^*)$$

As $t^* \to o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp P_2^*(a)}{da} + f_i(a, P_1^*, P_2^*) P_i^*(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^* = \int_o^\infty P_i^*(a) da, i = 1,\ 2,$$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i=1.\ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s,P_1^*,P_2^*)ds\right], i=1,\quad 2, a\geq 0,$

Which together with (3.3) lead to

$$1=\int_o^\infty b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^\infty b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1,P_2) = \int_o^\infty b_i(a,P_1,P_2)\exp\left[-\int_o^\infty f_i(s,P_1,P_2)ds\right]da,$

$i=1,\ 2,\quad P_1\geq 0,\quad P_2\geq 0,$

**And suppose the following hold:**

(3.7) $A_{4.}\quad F_{i.}\ (0,\ 0)>1,\ i=1,2$:

$A_{5.}$ *there exists a positive constant c for which*

$$F_1(C,0)< F_{2.}(C,0),$$

(3.8) $$F_1(0.C)< F_{2.}(0,C),$$

Then there exists a unique pair $(P_1^*,P_2^*)$ of real numbers $P_1^*>o,$

$P_2^*>0$ such that

(3,9) $F_1(P_1^*,P_2^*)=1=F_2(P_1^*,P_2^*)$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a)=[P_2^*\exp[-f_i(s,P_1^*,P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s,P_1^*,P_2^*)ds]da),$

$i=1,2,\quad a\geq 0,$

Proof Consider the elements of the surfaces defined by

Z= $F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0,$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

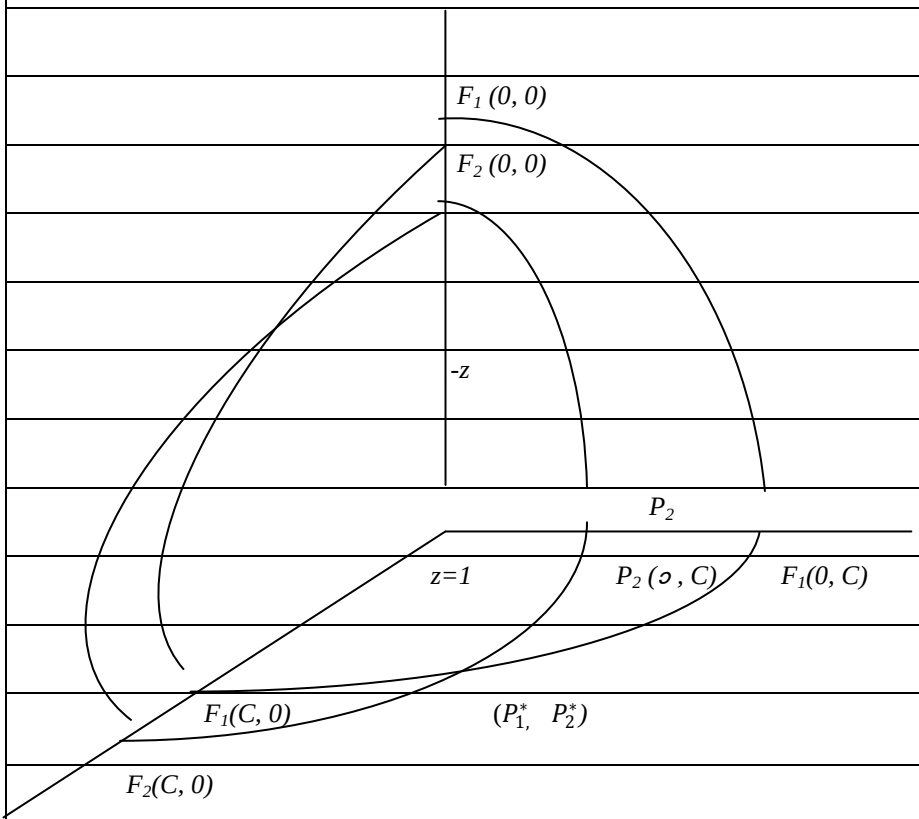
$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0,$ These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, F_1(P_1, P_2 \geq 0,$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11) $$\begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12) $$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square=1}^{2} \frac{\partial b_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}da \end{cases}$$

Where $i$ = 1,2 and (3.13) $$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14) $$\begin{cases} u_i(a,t) = u\square(a)\exp[\lambda t], \quad \square(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty \square(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d\square}{da} + \lambda\square = -f_i\left(a, p_1^*, p_2^*\right)\square(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_{i\square}^*\right\},$$

(3.15)

$$\square(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\square(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*} p_{i\square}^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^{*} = p_{1}^{*}, \quad p_{12}^{*} = p_{2}^{*} \exp\{-\lambda\tau\},$$

$$p_{21}^{*} = p_{1}^{*}, \exp\{-\lambda\tau\}, \ p_{22}^{*} = p_{2}^{*}.$$

If we integrate the first of (3.15) and use the fact that $\mu_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\mu_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\mu_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\mu_i(a) = \{\mu_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\mu_i\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\mu_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\mu_i\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\mu_i\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\mu_i\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)}\int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}da, \quad i = 1,\ 2,$$

Now supplying $\mu_1(a)$ and $\mu_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^*\exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^*\exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^* \exp(-\lambda\tau) + (\lambda - A_{22})\, p_2^* = g_{21}(\lambda)\, p_1^* \exp(-\lambda\tau) + g_{22}(\lambda)\, p_2^*$$

Where

(3.22) $$A_{i} = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_{}^*} - \frac{\partial f_i}{\partial p_{}^*}\right]da,$$

(3.23) $$g_{i}(\lambda) = \left[p_i^*(0)c_i\right]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\, da\right\}$$
$$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_{}^*} - b_i(a, p_1^*, p_2^*)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{}^*}\, e^{\lambda s}\, ds\right)\right]da\right\}$$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{}^*}\, e^{\lambda s}\, ds\right)da, \quad i = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*), \quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})\, e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$$S_2(\lambda, \tau) = \left[A_{21}\, g_{12}(\lambda) + A_{12}\, g_{21}(\lambda) + g_{12}(\lambda)\, g_{21}(\lambda)\right]\exp[-2\lambda\tau]$$

$$-A_{11}\, g_{22}(\lambda) - A_{22}\, g_{11}(\lambda) - g_{11}(\lambda)\, g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6$. $0 < \beta_i^* < d_i^*$ $(i = 1, 2.)$ $(see(A_1) and (A_2))$;

$A_7$. $\{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \text{[illegible]}(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \left|\right.$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \,| - |\, A_{11} + A_{22} + \| S_1(0) \| \,|^2 \right\}/2,$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational

ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the

Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have

obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is:

individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## SECTION A(2)

## Quantum Perceptron Models

### INTRODUCTION—VARIABLES USED

**Quantum Perceptron Models Nathan Wiebe, Ashish Kapoor, Krysta M Svore (Submitted on 15 Feb 2016)**

(1) **Nathan Wiebe, Ashish Kapoor, Krysta M Svore** demonstrate (eb) how quantum computation can provide (eb) non-trivial improvements in the computational and statistical complexity of (e) the perceptron model.
(2) Authors develop two quantum algorithms for (e) perceptron learning.
(3) The first algorithm exploits (e&eb) quantum information processing to determine (eb) a separating hyperplane using (e) a number of steps sublinear in (eb) the number of data points N, namely O (N−−√).
(4) The second algorithm illustrates (eb) how the classical mistake bound of (e) O (1γ2) can be further improved to (e) O (1γ√) through quantum means, where γ denotes the margin.
**(5)** Such improvements are achieved through (e&eb) the application of quantum amplitude

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| amplification to (e) the version space interpretation of (e) the perceptron model. Subjects: Quantum Physics (quant-ph); Learning (cs.LG); Machine Learning (stat.ML) Cite as: arXiv: 1602.04799 [quant-ph] (or arXiv: 1602.04799v1 [quant-ph] for this version) **Quantum Perceptron Models Nathan Wiebe, Ashish Kapoor, Krysta M Svore (Submitted on 15 Feb 2016)** | |
| **NOTATION** | |
| **Module One**<br>**Nathan Wiebe, Ashish Kapoor, Krysta M Svore** demonstrate (eb) how quantum computation can provide (eb) non-trivial improvements in the computational and statistical complexity of (e) the perceptron model. | |
| $G_{13}$ : Category one of quantum computation<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of non-trivial improvements in the computational and statistical complexity of the perceptron model<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>**Nathan Wiebe, Ashish Kapoor, Krysta M Svore** demonstrate how quantum computation can provide non-trivial improvements in the computational and statistical complexity of (e) the **perceptron model** | |
| $G_{16}$ : Category one of quantum computation can provide non-trivial improvements in the computational and statistical complexity; **perceptron model**<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of **perceptron model;** quantum computation can provide non-trivial improvements in the computational and statistical complexity<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>Authors develop two quantum algorithms for (e) perceptron learning | |
| $G_{20}$ : Category one of **two quantum algorithms;** perceptron learning<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of perceptron learning; **two quantum algorithms** | |

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| $T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>The first algorithm exploits (e&eb) quantum information processing to determine (eb) a separating hyperplane using (e) a number of steps sublinear in (eb) the number of data points N, namely O (N−−√). | |
| $G_{24}$ : Category one of **first algorithm;** quantum information processing to determine (eb) a separating hyperplane using (e) a number of steps sublinear in (eb) the number of data points N, namely O (N−−√).<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of quantum information processing to determine (eb) a separating hyperplane using (e) a number of steps sublinear in (eb) the number of data points N, namely O (N−−√).;**first algorithm**<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>The first algorithm exploits quantum information processing to determine (eb) a separating hyperplane using (e) a number of steps sublinear in (eb) the number of data points N, namely O (N−−√). | |
| $G_{28}$ : Category one of first algorithm exploits quantum information processing<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of separating hyperplane using (e) a number of steps sublinear in (eb) the number of data points N, namely O (N−−√).<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>The first algorithm exploits quantum information processing to determine a separating hyperplane using (e) a number of steps sublinear in (eb) the number of data points N, namely O (N−−√). | |
| $G_{32}$ : Category one of first algorithm exploits quantum information processing to determine a separating hyperplane using (e) a number of steps sublinear<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |

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| $T_{32}$ : Category one of number of data points N, namely O (N−−√).<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>The second algorithm illustrates (eb) how the classical mistake bound of (e) O (1γ2) can be further improved to (e) O (1γ√) through quantum means, where γ denotes the margin | |
| $G_{36}$ : Category one of second algorithm<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of classical mistake bound of (e) O (1γ2) can be further improved to (e) O (1γ√) through quantum means, where γ denotes the margin<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The second algorithm illustrates how the classical mistake bound of O (1γ2) can be further improved to (e) O (1γ√) through quantum means, where γ denotes the margin | |
| $G_{40}$ : Category one of **second algorithm illustrates how the classical mistake bound of O (1γ2) can be further improved;** O (1γ√) through quantum means, where γ denotes the margin<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of O (1γ√) through quantum means, where γ denotes the margin **;second algorithm illustrates how the classical mistake bound of O (1γ2) can be further improved**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Such improvements are achieved through (e&eb) the application of quantum amplitude amplification to (e) the version space interpretation of (e) the perceptron model.<br><br>Subjects: Quantum Physics (quant-ph); Learning (cs.LG); Machine Learning (stat.ML) Cite as: arXiv: 1602.04799 [quant-ph] (or arXiv: 1602.04799v1 [quant-ph] for this version) **Quantum Perceptron Models Nathan Wiebe, Ashish Kapoor, Krysta M Svore (Submitted on 15 Feb 2016)** | |

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| $G_{44}$ : Category one of **improvements are achieved;** application of quantum amplitude amplification to (e) the version space interpretation of (e) the perceptron model<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of application of quantum amplitude amplification to (e) the version space interpretation of (e) the perceptron model; **improvements are achieved**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a_{13}^{'})^{(1)}, (a_{14}^{'})^{(1)}, (a_{15}^{'})^{(1)}, (b_{13}^{'})^{(1)}, (b_{14}^{'})^{(1)}, (b_{15}^{'})^{(1)}, (a_{16}^{'})^{(2)}, (a_{17}^{'})^{(2)}, (a_{18}^{'})^{(2)}$, $(b_{16}^{'})^{(2)}, (b_{17}^{'})^{(2)}, (b_{18}^{'})^{(2)}$<br>$, (a_{20}^{'})^{(3)}, (a_{21}^{'})^{(3)}, (a_{22}^{'})^{(3)}, (b_{20}^{'})^{(3)}, (b_{21}^{'})^{(3)}, (b_{22}^{'})^{(3)}$<br>$(a_{24}^{'})^{(4)}, (a_{25}^{'})^{(4)}, (a_{26}^{'})^{(4)}, (b_{24}^{'})^{(4)}, (b_{25}^{'})^{(4)}, (b_{26}^{'})^{(4)}, (b_{28}^{'})^{(5)}, (b_{29}^{'})^{(5)}, (b_{30}^{'})^{(5)}$ $(a_{28}^{'})^{(5)}, (a_{29}^{'})^{(5)}, (a_{30}^{'})^{(5)}$ ,<br>$(a_{32}^{'})^{(6)}, (a_{33}^{'})^{(6)}, (a_{34}^{'})^{(6)}, (b_{32}^{'})^{(6)}, (b_{33}^{'})^{(6)}, (b_{34}^{'})^{(6)}$<br>$(a_{36}^{'})^{(7)}, (a_{37}^{'})^{(7)}, (a_{38}^{'})^{(7)}, (b_{36}^{'})^{(7)}, (b_{37}^{'})^{(7)}, (b_{38}^{'})^{(7)}$,<br>$(a_{40}^{'})^{(8)}, (a_{41}^{'})^{(8)}, (a_{42}^{'})^{(8)}, (b_{40}^{'})^{(8)}, (b_{41}^{'})^{(8)}, (b_{42}^{'})^{(8)}$,<br>$(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)} G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right] G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)} G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right] G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)} G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right] G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)} T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right] T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)} T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right] T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)} T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right] T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14}, t) =$ First augmentation factor | |

| | |
|---|---|
| $-(b_{13}^{''})^{(1)}(G,t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17},t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17},t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17},t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19}),t)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}((G_{19}),t)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19}),t)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17},t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}((G_{19}),t) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |

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| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29}, t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}, t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}), t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}), t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}), t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29}, t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}), t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33}, t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}, t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}), t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}), t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}), t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33}, t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}, t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}, t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}), t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}), t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}), t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37}, t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}, t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}, t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}), t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}), t)\right]T_{41}$ | *47* |

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| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | 48 |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{13}$ | 55 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{array}{ccc} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{14}$ | 56 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{15}$ | 57 |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3<br>$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3 | |

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| $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\left[-(b_{13}^{''})^{(1)}(G,t)\right] & \left[-(b_{16}^{''})^{(2,2,)}(G_{19},t)\right] & \left[-(b_{20}^{''})^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{36}^{''})^{(7,7,)}(G_{39},t)\right] & \left[-(b_{40}^{''})^{(8,8)}(G_{43},t)\right] & \left[-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\left[-(b_{14}^{''})^{(1)}(G,t)\right] & \left[-(b_{17}^{''})^{(2,2,)}(G_{19},t)\right] & \left[-(b_{21}^{''})^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{37}^{''})^{(7,7,)}(G_{39},t)\right] & \left[-(b_{41}^{''})^{(8,8)}(G_{43},t)\right] & \left[-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\left[-(b_{15}^{''})^{(1)}(G,t)\right] & \left[-(b_{18}^{''})^{(2,2,)}(G_{19},t)\right] & \left[-(b_{22}^{''})^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{38}^{''})^{(7,7,)}(G_{39},t)\right] & \left[-(b_{42}^{''})^{(8,8)}(G_{43},t)\right] & \left[-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{15}$ | *60* |
| Where $-(b_{13}^{''})^{(1)}(G,t)$, $-(b_{14}^{''})^{(1)}(G,t)$, $-(b_{15}^{''})^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{16}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{20}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{37}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}^{'})^{(2)}\left[+(a_{16}^{''})^{(2)}(T_{17},t)\right] & \left[+(a_{13}^{''})^{(1,1,)}(T_{14},t)\right] & \left[+(a_{20}^{''})^{(3,3,3)}(T_{21},t)\right] \\ \left[+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)\right] & \left[+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)\right] & \left[+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)\right] \\ \left[+(a_{36}^{''})^{(7,7,7)}(T_{37},t)\right] & \left[+(a_{40}^{''})^{(8,8,8)}(T_{41},t)\right] & \left[+(a_{44}^{''})^{(9,9)}(T_{45},t)\right] \end{bmatrix} G_{16}$ | *61* |

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| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[\begin{array}{c} (a_{17}^{'})^{(2)}\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)}\ \boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)}\ \boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)}\ \boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)}\ \boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)} \end{array}\right] G_{17}$ | 62 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[\begin{array}{c} (a_{18}^{'})^{(2)}\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)}\ \boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)}\ \boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)}\ \boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)}\ \boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)} \end{array}\right] G_{18}$ | 63 |
| Where $\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)}$ are seventh augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)}$ are eight augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)}$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[\begin{array}{c} (b_{16}^{'})^{(2)}\boxed{-(b_{16}^{''})^{(2)}(G_{19},t)}\ \boxed{-(b_{13}^{''})^{(1,1,)}(G,t)}\ \boxed{-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)}\ \boxed{-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)}\ \boxed{-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7)}(G_{39},t)}\ \boxed{-(b_{40}^{''})^{(8,8,8)}(G_{43},t)}\ \boxed{-(b_{44}^{''})^{(9,9)}(G_{47},t)} \end{array}\right] T_{16}$ | 64 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[\begin{array}{c} (b_{17}^{'})^{(2)}\boxed{-(b_{17}^{''})^{(2)}(G_{19},t)}\ \boxed{-(b_{14}^{''})^{(1,1,)}(G,t)}\ \boxed{-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)}\ \boxed{-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)}\ \boxed{-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,7)}(G_{39},t)}\ \boxed{-(b_{41}^{''})^{(8,8,8)}(G_{43},t)}\ \boxed{-(b_{45}^{''})^{(9,9)}(G_{47},t)} \end{array}\right] T_{17}$ | 65 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[\begin{array}{c} (b_{18}^{'})^{(2)}\boxed{-(b_{18}^{''})^{(2)}(G_{19},t)}\ \boxed{-(b_{15}^{''})^{(1,1,)}(G,t)}\ \boxed{-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)}\ \boxed{-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)}\ \boxed{-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,7)}(G_{39},t)}\ \boxed{-(b_{42}^{''})^{(8,8,8)}(G_{43},t)}\ \boxed{-(b_{46}^{''})^{(9,9)}(G_{47},t)} \end{array}\right] T_{18}$ | 66 |
| where $\boxed{-(b_{16}^{''})^{(2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3 | |

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| $-(b_{13}^{''})^{(1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3<br>$-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[ \begin{array}{c} (a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t) + (a_{16}^{''})^{(2,2,2)}(T_{17},t) + (a_{13}^{''})^{(1,1,1,)}(T_{14},t) \\ + (a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t) + (a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t) + (a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ + (a_{36}^{''})^{(7,7,7,7)}(T_{37},t) + (a_{40}^{''})^{(8,8,8,8)}(T_{41},t) + (a_{44}^{''})^{(9,9,9)}(T_{45},t) \end{array} \right] G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[ \begin{array}{c} (a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t) + (a_{17}^{''})^{(2,2,2)}(T_{17},t) + (a_{14}^{''})^{(1,1,1,)}(T_{14},t) \\ + (a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t) + (a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t) + (a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ + (a_{37}^{''})^{(7,7,7,7)}(T_{37},t) + (a_{41}^{''})^{(8,8,8,8)}(T_{41},t) + (a_{45}^{''})^{(9,9,9)}(T_{45},t) \end{array} \right] G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[ \begin{array}{c} (a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t) + (a_{18}^{''})^{(2,2,2)}(T_{17},t) + (a_{15}^{''})^{(1,1,1,)}(T_{14},t) \\ + (a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t) + (a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t) + (a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ + (a_{38}^{''})^{(7,7,7,7)}(T_{37},t) + (a_{42}^{''})^{(8,8,8,8)}(T_{41},t) + (a_{46}^{''})^{(9,9,9)}(T_{45},t) \end{array} \right] G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation | |

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| coefficients for category 1, 2 and 3 $+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3 $+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3 $+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt}=(b_{20})^{(3)}T_{21}-\begin{bmatrix}(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23},t) & -(b_{16}^{''})^{(2,2,2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,1,)}(G,t)\\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)\\ -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9)}(G_{47},t)\end{bmatrix}T_{20}$ | *70* |
| $\frac{dT_{21}}{dt}=(b_{21})^{(3)}T_{20}-\begin{bmatrix}(b_{21}^{'})^{(3)}-(b_{21}^{''})^{(3)}(G_{23},t) & -(b_{17}^{''})^{(2,2,2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,1,)}(G,t)\\ -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)\\ -(b_{37}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9)}(G_{47},t)\end{bmatrix}T_{21}$ | *71* |
| $\frac{dT_{22}}{dt}=(b_{22})^{(3)}T_{21}-\begin{bmatrix}(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23},t) & -(b_{18}^{''})^{(2,2,2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,1,)}(G,t)\\ -(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)\\ -(b_{38}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9)}(G_{47},t)\end{bmatrix}T_{22}$ | *72* |
| $-(b_{20}^{''})^{(3)}(G_{23},t)$, $-(b_{21}^{''})^{(3)}(G_{23},t)$, $-(b_{22}^{''})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3* $-(b_{16}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3* $-(b_{13}^{''})^{(1,1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3* $-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3* $-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3* $-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3* $-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 $-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3 $-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a'_{24})^{(4)} & +(a''_{24})^{(4)}(T_{25},t) & +(a''_{28})^{(5,5,)}(T_{29},t) & +(a''_{32})^{(6,6,)}(T_{33},t) \\ & +(a''_{13})^{(1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3)}(T_{21},t) \\ & +(a''_{36})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a'_{25})^{(4)} & +(a''_{25})^{(4)}(T_{25},t) & +(a''_{29})^{(5,5,)}(T_{29},t) & +(a''_{33})^{(6,6)}(T_{33},t) \\ & +(a''_{14})^{(1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3)}(T_{21},t) \\ & +(a''_{37})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a'_{26})^{(4)} & +(a''_{26})^{(4)}(T_{25},t) & +(a''_{30})^{(5,5,)}(T_{29},t) & +(a''_{34})^{(6,6,)}(T_{33},t) \\ & +(a''_{15})^{(1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a''_{24})^{(4)}(T_{25},t)$, $(a''_{25})^{(4)}(T_{25},t)$, $(a''_{26})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a''_{28})^{(5,5,)}(T_{29},t)$, $+(a''_{29})^{(5,5,)}(T_{29},t)$, $+(a''_{30})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3

$+(a''_{32})^{(6,6,)}(T_{33},t)$, $+(a''_{33})^{(6,6,)}(T_{33},t)$, $+(a''_{34})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1)}(T_{14},t)$ *are fourth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{16})^{(2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2)}(T_{17},t)$ *are fifth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{20})^{(3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3)}(T_{21},t)$ *are sixth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{36})^{(7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7)}(T_{37},t)$ *are seventh augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{40})^{(8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8)}(T_{41},t)$ *are eighth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{46})^{(9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9)}(T_{45},t)$, $+(a''_{44})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b'_{24})^{(4)} & -(b''_{24})^{(4)}(G_{27},t) & -(b''_{28})^{(5,5,)}(G_{31},t) & -(b''_{32})^{(6,6,)}(G_{35},t) \\ & -(b''_{13})^{(1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b'_{25})^{(4)} & -(b''_{25})^{(4)}(G_{27},t) & -(b''_{29})^{(5,5,)}(G_{31},t) & -(b''_{33})^{(6,6,)}(G_{35},t) \\ & -(b''_{14})^{(1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b'_{26})^{(4)}\boxed{-(b''_{26})^{(4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{26} \quad (78)$$

Where $\boxed{-(b''_{24})^{(4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4)}(G_{27},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,)}(G_{31},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,)}(G_{35},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3

$$\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a'_{28})^{(5)}\boxed{+(a''_{28})^{(5)}(T_{29},t)} & \boxed{+(a''_{24})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{32})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{28} \quad (79)$$

$$\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a'_{29})^{(5)}\boxed{+(a''_{29})^{(5)}(T_{29},t)} & \boxed{+(a''_{25})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{33})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{29} \quad (80)$$

$$\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a'_{30})^{(5)}\boxed{+(a''_{30})^{(5)}(T_{29},t)} & \boxed{+(a''_{26})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{34})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{30} \quad (81)$$

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|---|---|
| *Where* $+(a_{28}^{''})^{(5)}(T_{29},t)$ , $+(a_{29}^{''})^{(5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5)}(T_{29},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3<br>*And* $+(a_{24}^{''})^{(4,4,)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,)}(T_{25},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3<br>$+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$ , $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3<br>$+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3<br>$+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3<br>$+(a_{40}^{''})^{(8,8\ ,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3<br>$+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[\begin{matrix}(b_{28}^{'})^{(5)}\boxed{-(b_{28}^{''})^{(5)}(G_{31},t)}\ \boxed{-(b_{24}^{''})^{(4,4,)}(G_{27},t)}\ \boxed{-(b_{32}^{''})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)}\ \boxed{-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)}\ \boxed{-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)}\ \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)}\ \boxed{-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)}\end{matrix}\right]T_{28}$ | *82* |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[\begin{matrix}(b_{29}^{'})^{(5)}\boxed{-(b_{29}^{''})^{(5)}(G_{31},t)}\ \boxed{-(b_{25}^{''})^{(4,4,)}(G_{27},t)}\ \boxed{-(b_{33}^{''})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)}\ \boxed{-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)}\ \boxed{-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)}\ \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)}\ \boxed{-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)}\end{matrix}\right]T_{29}$ | *83* |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[\begin{matrix}(b_{30}^{'})^{(5)}\boxed{-(b_{30}^{''})^{(5)}(G_{31},t)}\ \boxed{-(b_{26}^{''})^{(4,4,)}(G_{27},t)}\ \boxed{-(b_{34}^{''})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)}\ \boxed{-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)}\ \boxed{-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)}\ \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)}\ \boxed{-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)}\end{matrix}\right]T_{30}$ | *84* |
| *where* $\boxed{-(b_{28}^{''})^{(5)}(G_{31},t)}$ , $\boxed{-(b_{29}^{''})^{(5)}(G_{31},t)}$ , $\boxed{-(b_{30}^{''})^{(5)}(G_{31},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b_{24}^{''})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,)}(G_{27},t)}$ *are second detrition coefficients for category* 1,2 *and* 3<br>$\boxed{-(b_{32}^{''})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6)}(G_{35},t)}$ *are third detrition coefficients for category* 1,2 *and* 3<br>$\boxed{-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)}$ , $\boxed{-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)}$ are fourth detrition coefficients for category 1,2, and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients | |

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| for category 1,2, and 3<br>$-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3<br>$-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3<br>$-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3<br>$-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3 | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,4,)}(T_{25},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{32}$ | *85* |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a_{33}^{'})^{(6)}+(a_{33}^{''})^{(6)}(T_{33},t) & +(a_{29}^{''})^{(5,5,5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,4,)}(T_{25},t) \\ +(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ +(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{33}$ | *86* |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \begin{bmatrix} (a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33},t) & +(a_{30}^{''})^{(5,5,5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,4,)}(T_{25},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ +(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34}$ | *87* |
| $+(a_{32}^{''})^{(6)}(T_{33},t)$, $+(a_{33}^{''})^{(6)}(T_{33},t)$, $+(a_{34}^{''})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{28}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients<br>$+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients<br>$+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients<br>$+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients<br>$+(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients<br>$+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation | |

coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b'_{32})^{(6)}\boxed{-(b''_{32})^{(6)}(G_{35},t)} & \boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b'_{33})^{(6)}\boxed{-(b''_{33})^{(6)}(G_{35},t)} & \boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)}\boxed{-(b''_{34})^{(6)}(G_{35},t)} & \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{34} \quad (90)$$

$\boxed{-(b''_{32})^{(6)}(G_{35},t)}, \boxed{-(b''_{33})^{(6)}(G_{35},t)}, \boxed{-(b''_{34})^{(6)}(G_{35},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)}, \boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)}, \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)}$ *are second detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)}$ *are third detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a'_{36})^{(7)}\boxed{+(a''_{36})^{(7)}(T_{37},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}')^{(7)}\boxed{+(a_{37}'')^{(7)}(T_{37},t)} & \boxed{+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}')^{(7)}\boxed{+(a_{38}'')^{(7)}(T_{37},t)} & \boxed{+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad (93)$$

Where $\boxed{(a_{36}'')^{(7)}(T_{37},t)}$, $\boxed{(a_{37}'')^{(7)}(T_{37},t)}$, $\boxed{(a_{38}'')^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{40}'')^{(8,8,8,8,8,8,8,8,)}(T_{41},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}')^{(7)}\boxed{-(b_{36}'')^{(7)}(G_{39},t)} & \boxed{-(b_{16}'')^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}'')^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}'')^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}'')^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}'')^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}'')^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}'')^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13} \quad (94)$$

$$\frac{dT_{37}}{dt} =$$

$$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b'_{37})^{(7)}\boxed{-(b''_{37})^{(7)}(G_{39},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} =$$

$$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b'_{38})^{(7)}\boxed{-(b''_{38})^{(7)}(G_{39},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a'_{40})^{(8)}\boxed{+(a''_{40})^{(8)}(T_{41},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a'_{41})^{(8)}\boxed{+(a''_{41})^{(8)}(T_{41},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a'_{42})^{(8)}\boxed{+(a''_{42})^{(8)}(T_{41},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{40})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b'_{40})^{(8)}\boxed{-(b''_{40})^{(8)}(G_{43},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b'_{41})^{(8)}\boxed{-(b''_{41})^{(8)}(G_{43},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b'_{42})^{(8)}\boxed{-(b''_{42})^{(8)}(G_{43},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13} \quad 96$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[\begin{array}{ccc} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[\begin{array}{ccc} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{14}$$

$$\frac{dT_{46}}{dt} =$$

$$(b_{46})^{(9)}T_{45} - \begin{bmatrix} (b'_{46})^{(9)}\boxed{-(b''_{46})^{(9)}(G_{47},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{44})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a'_i)^{(1)}, (a''_i)^{(1)}, (b_i)^{(1)}, (b'_i)^{(1)}, (b''_i)^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a''_i)^{(1)}, (b''_i)^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

$$(a''_i)^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$$

$$(b''_i)^{(1)}(G,t) \le (r_i)^{(1)} \le (b'_i)^{(1)} \le (\hat{B}_{13})^{(1)}$$

$lim_{T_2\to\infty}(a''_i)^{(1)}(T_{14},t) = (p_i)^{(1)}$ 98

$\lim_{G\to\infty}(b''_i)^{(1)}(G,t) = (r_i)^{(1)}$

**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :

Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$

| | |
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| They satisfy Lipschitz condition:<br>$\|(a_i'')^{(1)}(T_{14}', t) - (a_i'')^{(1)}(T_{14}, t)\| \leq (\hat{k}_{13})^{(1)}\|T_{14} - T_{14}'\|e^{-(\hat{M}_{13})^{(1)}t}$<br>$\|(b_i'')^{(1)}(G', t) - (b_i'')^{(1)}(G, t)\| < (\hat{k}_{13})^{(1)}\|G - G'\|e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(1)}(T_{14}', t)$ and $(a_i'')^{(1)}(T_{14}, t)$ . $(T_{14}', t)$ and $(T_{14}, t)$ are points belonging to the interval $[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}]$. It is to be noted that $(a_i'')^{(1)}(T_{14}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i'')^{(1)}(T_{14}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)},$ are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i')^{(1)}, (b_i)^{(1)}, (b_i')^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i, j = 16,17,18$ | |
| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17}, t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19}, t) \leq (r_i)^{(2)} \leq (b_i')^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty}(a_i'')^{(2)}(T_{17}, t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty}(b_i'')^{(2)}((G_{19}), t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ : | 106 |

| | |
|---|---|
| Where $\boxed{(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}}$ are positive constants and $\boxed{i = 16,17,18}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'}, t) - (a_i^{''})^{(2)}(T_{17}, t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}^{'}\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'}, t) - (b_i^{''})^{(2)}((G_{19}), t)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})^{'}\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'}, t)$ and $(a_i^{''})^{(2)}(T_{17}, t)$ . $(T_{17}^{'}, t)$ and $(T_{17}, t)$ are points belonging to the interval $\left[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}\right]$. It is to be noted that $(a_i^{''})^{(2)}(T_{17}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i^{''})^{(2)}(T_{17}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br><br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br><br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br><br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i^{'})^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i, j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(r_i)^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(3)}(T_{21}, t) = (p_i)^{(3)}$ | 113 |

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|---|---|
| $\lim_{G\to\infty}(b_i'')^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(3)}(T_{21}',t) - (a_i'')^{(3)}(T_{21},t)\rvert \le (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}'\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i'')^{(3)}(G_{23}',t) - (b_i'')^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}'\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(3)}(T_{21}',t)$ and$(a_i'')^{(3)}(T_{21},t)$ . $(T_{21}',t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i'')^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i'')^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i')^{(3)}, (b_i)^{(3)}, (b_i')^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i')^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i')^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i')^{(4)}, (a_i'')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (b_i'')^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i'')^{(4)}, (b_i'')^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i'')^{(4)}(T_{25},t) \le (p_i)^{(4)} \le (\hat{A}_{24})^{(4)}$<br><br>$(b_i'')^{(4)}\big((G_{27}),t\big) \le (r_i)^{(4)} \le (b_i')^{(4)} \le (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2\to\infty}(a_i'')^{(4)}(T_{25},t) = (p_i)^{(4)}$ | 118 |

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| $\lim_{G\to\infty}(b_i'')^{(4)}\left((G_{27}),t\right) = \ (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, \ (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24{,}25{,}26}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(4)}(T_{25}',t) - (a_i'')^{(4)}(T_{25},t)\rvert \le (\hat{k}_{24})^{(4)}\lvert T_{25} - T_{25}'\rvert e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\lvert(b_i'')^{(4)}((G_{27})',t) - (b_i'')^{(4)}\left((G_{27}),t\right)\rvert < (\hat{k}_{24})^{(4)}\lvert\lvert(G_{27}) - (G_{27})'\rvert\rvert e^{-(\hat{M}_{24})^{(4)}t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(4)}(T_{25}',t)$ and $(a_i'')^{(4)}(T_{25},t)$ . $(T_{25}',t)$ and $(T_{25},t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i'')^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i'')^{(4)}(T_{25},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (p_i)^{(4)}, \ (r_i)^{(4)}, i = 24{,}25{,}26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[\,(a_i)^{(4)} + (a_i')^{(4)} + \ (\hat{A}_{24})^{(4)} + \ (\hat{P}_{24})^{(4)}\,(\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i')^{(4)} + \ (\hat{B}_{24})^{(4)} + \ (\hat{Q}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i')^{(5)}, (a_i'')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (b_i'')^{(5)} > 0, \quad i,j = 28{,}29{,}30$<br><br>The functions $(a_i'')^{(5)}, (b_i'')^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}, \ (r_i)^{(5)}$:<br><br>$(a_i'')^{(5)}(T_{29},t) \le (p_i)^{(5)} \le (\hat{A}_{28})^{(5)}$<br><br>$(b_i'')^{(5)}\left((G_{31}),t\right) \le \ (r_i)^{(5)} \le (b_i')^{(5)} \le (\hat{B}_{28})^{(5)}$ | 122 |

$lim_{T_2\to\infty}(a_i'')^{(5)}(T_{29},t) = (p_i)^{(5)}$

$\lim_{G\to\infty}(b_i'')^{(5)}(G_{31},t) = (r_i)^{(5)}$

**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ : 123

Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$

They satisfy Lipschitz condition: 124

$$|(a_i'')^{(5)}(T_{29}',t) - (a_i'')^{(5)}(T_{29},t)| \le (\hat{k}_{28})^{(5)}|T_{29} - T_{29}'|e^{-(\hat{M}_{28})^{(5)}t}$$

$$|(b_i'')^{(5)}((G_{31})',t) - (b_i'')^{(5)}((G_{31}),t)| < (\hat{k}_{28})^{(5)}||(G_{31}) - (G_{31})'||e^{-(\hat{M}_{28})^{(5)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(5)}(T_{29}',t)$ and $(a_i'')^{(5)}(T_{29},t)$ . $(T_{29}',t)$ and $(T_{29},t)$ are points belonging to the interval $\left[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\right]$. It is to be noted that $(a_i'')^{(5)}(T_{29},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i'')^{(5)}(T_{29},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ : 125

$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)},$ are positive constants

$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$$

**Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ : 126

There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i')^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i')^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

Where we suppose

$(a_i)^{(6)}, (a_i')^{(6)}, (a_i'')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (b_i'')^{(6)} > 0, \quad i,j = 32,33,34$ 127

The functions $(a_i'')^{(6)}, (b_i'')^{(6)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(6)}, (r_i)^{(6)}$:

$$(a_i'')^{(6)}(T_{33},t) \le (p_i)^{(6)} \le (\hat{A}_{32})^{(6)}$$

| | |
|---|---|
| $(b_i^{''})^{(6)}((G_{35}),t) \leq \ (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(6)}(T_{33},t) = (p_i)^{(6)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(6)}\left((G_{35}),t\right) = \ (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'},t) - (a_i^{''})^{(6)}(T_{33},t)\rvert \leq (\hat{k}_{32})^{(6)}\lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})^{'},t) - (b_i^{''})^{(6)}\left((G_{35}),t\right)\rvert < (\hat{k}_{32})^{(6)}\lvert\lvert(G_{35}) - (G_{35})^{'}\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'},t)$ and$(a_i^{''})^{(6)}(T_{33},t)$ . $(T_{33}^{'},t)$ and $(T_{33},t)$ are points belonging to the interval $\left[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\right]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[\ (a_i)^{(6)} + (a_i^{'})^{(6)} + \ (\hat{A}_{32})^{(6)} + \ (\hat{P}_{32})^{(6)}\ (\hat{k}_{32})^{(6)}] < 1$<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[\ (b_i)^{(6)} + (b_i^{'})^{(6)} + \ (\hat{B}_{32})^{(6)} + \ (\hat{Q}_{32})^{(6)}\ \ (\hat{k}_{32})^{(6)}] < 1$ | 130 |
| Where we suppose | |
| (CCC) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i,j = 36,37,38$<br><br>(DDD) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$: | 131 |

$(a_i^{''})^{(7)}(T_{37},t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$

$(b_i^{''})^{(7)}(G_{39},t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$

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*(EEE)* $\lim_{T_2\to\infty} (a_i^{''})^{(7)}(T_{37},t) = (p_i)^{(7)}$

*(FFF)*

$\lim_{G\to\infty}(b_i^{''})^{(7)}\left((G_{39}),t\right) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

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They satisfy Lipschitz condition:

$|(a_i^{''})^{(7)}(T_{37}^{'},t) - (a_i^{''})^{(7)}(T_{37},t)| \leq (\hat{k}_{36})^{(7)}|T_{37} - T_{37}^{'}|e^{-(\hat{M}_{36})^{(7)}t}$

$|(b_i^{''})^{(7)}((G_{39})^{'},t) - (b_i^{''})^{(7)}\left((G_{39}),t\right)| < (\hat{k}_{36})^{(7)}||(G_{39}) - (G_{39})^{'}||e^{-(\hat{M}_{36})^{(7)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'},t)$ and$(a_i^{''})^{(7)}(T_{37},t)$ . $(T_{37}^{'},t)$ and $(T_{37},t)$ are points belonging to the interval $\left[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}\right]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

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**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :

(GGG) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$, are positive constants

$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$

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**Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :

(HHH) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities

$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$

$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$

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| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}\big((G_{43}), t\big) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})^{'}, t) - (b_i^{''})^{(8)}\big((G_{43}), t\big)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})^{'}\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $\big[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}\big]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i^{'})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (p_i)^{(8)}$, $(r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |

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| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i,j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45},t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47},t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2\to\infty}(a_i'')^{(9)}(T_{45},t) = (p_i)^{(9)}$<br>$\lim_{G\to\infty}(b_i'')^{(9)}(G_{47},t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}',t) - (a_i'')^{(9)}(T_{45},t)\rvert \le (\hat{k}_{44})^{(9)}\lvert T_{45} - T_{45}'\rvert e^{-(\hat{M}_{44})^{(9)}t}$<br>$\lvert(b_i'')^{(9)}((G_{47})',t) - (b_i'')^{(9)}((G_{47}),t)\rvert < (\hat{k}_{44})^{(9)}\lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}',t)$ and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i'')^{(9)}(T_{45},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :<br>$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants<br>$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$ | |
| **Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :<br>There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and (\hat{B}_{44})^{(9)}$ and the constants | |

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| $(a_i)^{(9)}, (a_i')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}, i = 44,45,46,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i')^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$<br>$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i')^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$ | |
| **Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>Definition of $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 147 |
| **Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$<br>$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | 148 |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |

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| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le \left(\hat{Q}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le \left(\hat{P}_{13}\right)^{(1)}$ , $T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le \left(\hat{P}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |
| $0 \le T_i(t) - T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 157 |
| By<br><br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}\left(s_{(13)}\right) - \left((a_{13}')^{(1)} + a_{13}'')^{(1)}\left(T_{14}\left(s_{(13)}\right), s_{(13)}\right)\right) G_{13}\left(s_{(13)}\right)\right] ds_{(13)}$ | 158 |

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| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{16})^{(2)}$, $T_i^0 \le (\hat{Q}_{16})^{(2)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} G_{17}(s_{(16)}) - \left((a_{16}')^{(2)} + a_{16}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{16}(s_{(16)})\right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[(a_{17})^{(2)} G_{16}(s_{(16)}) - \left((a_{17}')^{(2)} + (a_{17}'')^{(2)}(T_{17}(s_{(16)}), s_{(17)})\right) G_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{20})^{(3)}$ , $T_i^0 \leq (\hat{Q}_{20})^{(3)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a'_{20})^{(3)} + a''_{20})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a'_{21})^{(3)} + (a''_{21})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a'_{22})^{(3)} + (a''_{22})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b'_{20})^{(3)} - (b''_{20})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b'_{21})^{(3)} - (b''_{21})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b'_{22})^{(3)} - (b''_{22})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{24})^{(4)}$ , $T_i^0 \leq (\hat{Q}_{24})^{(4)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |

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| $\overline{\mathrm{T}}_{26}(\mathrm{t}) = \mathrm{T}_{26}^{0} + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{28})^{(5)}$ , $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a_{28}^{'})^{(5)} + a_{28}^{''})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\overline{\mathrm{T}}_{30}(\mathrm{t}) = \mathrm{T}_{30}^{0} + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{32})^{(6)}$ , $T_i^0 \le (\hat{Q}_{32})^{(6)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a_{32}^{'})^{(6)} + a_{32}^{''})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |

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| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a_{34}')^{(6)} + (a_{34}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b_{32}')^{(6)} - (b_{32}'')^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b_{33}')^{(6)} - (b_{33}'')^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b_{34}')^{(6)} - (b_{34}'')^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{36})^{(7)}$, $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a_{36}')^{(7)} + a_{36}'')^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a_{37}')^{(7)} + (a_{37}'')^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a_{38}')^{(7)} + (a_{38}'')^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}')^{(7)} - (b_{36}'')^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}')^{(7)} - (b_{37}'')^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}')^{(7)} - (b_{38}'')^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ | |

which satisfy

$G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{40})^{(8)}$ , $T_i^0 \leq (\hat{Q}_{40})^{(8)}$,

$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$

$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$

By 166

$$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{40})^{(8)} + a''_{40})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{40}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[ (a_{41})^{(8)} G_{40}(s_{(40)}) - \left( (a'_{41})^{(8)} + (a''_{41})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{41}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[ (a_{42})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{42})^{(8)} + (a''_{42})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{42}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[ (b_{40})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{40})^{(8)} - (b''_{40})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{40}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[ (b_{41})^{(8)} T_{40}(s_{(40)}) - \left( (b'_{41})^{(8)} - (b''_{41})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{41}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[ (b_{42})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{42})^{(8)} - (b''_{42})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{42}(s_{(40)}) \right] ds_{(40)}$$

Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$

**Proof:** 166 A

Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy

$G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$,

$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$

$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$

By

$$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[ (a_{44})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{44}(s_{(44)}) \right] ds_{(44)}$$

$$\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[ (a_{45})^{(9)} G_{44}(s_{(44)}) - \left( (a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{45}(s_{(44)}) \right] ds_{(44)}$$

$$\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[ (a_{46})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{46}(s_{(44)}) \right] ds_{(44)}$$

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| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that<br>$(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} \left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$<br>$\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}} \left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$ | 171 |
| From which it follows that | 172 |

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| $(G_{20}(t)-G_{20}^0)e^{-(\hat{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that $G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\hat{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$ $\left(1+(a_{24})^{(4)}t\right)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that $(G_{24}(t)-G_{24}^0)e^{-(\hat{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$ $\left(1+(a_{28})^{(5)}t\right)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left(e^{(\hat{M}_{28})^{(5)}t}-1\right)$ | |
| From which it follows that $(G_{28}(t)-G_{28}^0)e^{-(\hat{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 5 | 175 |
| The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0+(\hat{P}_{32})^{(6)}e^{(\hat{M}_{32})^{(6)}s_{(32)}}\right)\right] ds_{(32)} =$ $\left(1+(a_{32})^{(6)}t\right)G_{33}^0 + \frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)}t}-1\right)$ | 176 |
| From which it follows that $(G_{32}(t)-G_{32}^0)e^{-(\hat{M}_{32})^{(6)}t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^0)e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^0}{G_{33}^0}\right)}+(\hat{P}_{32})^{(6)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 6 | 177 |

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| Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$ | |
| (j) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} \left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$<br>$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$ | 178 |
| From which it follows that<br>$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left[\left((\hat{P}_{36})^{(7)} + G_{37}^0\right) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 7 | |
| The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} \left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$<br>$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$ | 180 |
| From which it follows that<br>$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left[\left((\hat{P}_{40})^{(8)} + G_{41}^0\right) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} \left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$<br>$\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left(e^{(\hat{M}_{44})^{(9)} t} - 1\right)$ | |
| From which it follows that<br>$(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left[\left((\hat{P}_{44})^{(9)} + G_{45}^0\right) e^{\left(-\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0}\right)} + (\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ and to choose | 182 |

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| $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{13})^{(1)}\left\|G_{14}^{(1)}-G_{14}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$<br>$\int_0^t\{(a_{13}')^{(1)}\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{-(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$G_{13}^{(2)}\left\\|(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}'')^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right\\|\ e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$<br>Where $s_{(13)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | |
| $\left\|G^{(1)}-G^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}t}\leq$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}\left((a_{13})^{(1)}+(a_{13}')^{(1)}+(\hat{A}_{13})^{(1)}+(\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)},T^{(1)};\ G^{(2)},T^{(2)}\right)\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a_{13}'')^{(1)}$ and $(b_{13}'')^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by | |

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| $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(1)} - (a_i^{''})^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a_{14}^{'})^{(1)}G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}^{'})^{(1)}$<br><br>If $G_{14}\ or\ G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}\ and\ G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t), t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t), t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right),\quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t), t) = (b_{15}^{'})^{(1)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |

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| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)}, (T_{19})^{(1)}), ((G_{19})^{(2)}, (T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)}\left\|G_{17}^{(1)} - G_{17}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\,ds_{(16)} +$<br>$\int_0^t\{(a_{16}')^{(2)}\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{-(\hat{M}_{16})^{(2)}s_{(16)}} +$<br>$(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}} +$<br>$G_{16}^{(2)}\left\|(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}'')^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}t} \leq$<br>$\frac{1}{(\hat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}\right)d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}'')^{(2)}$ and $(b_{16}'')^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | 199 |

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| condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ and $(\widehat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)}t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1, \left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)}G_{17}$ and by integrating<br><br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1/(a_{17}^{'})^{(2)}$<br><br>In the same way , one can obtain<br><br>$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}^{'})^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | 201 |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(2)}((G_{19})(t), t)) = (b_{17}^{'})^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i^{''})^{(2)}((G_{19})(t), t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}^{''})^{(2)}\left((G_{19})(t), t\right) = (b_{18}^{'})^{(2)}$ | 205 |

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| We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)}, (T_{23})^{(1)}), ((G_{23})^{(2)}, (T_{23})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{20})^{(3)}t}\}$ | 211 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :(\widetilde{(G_{23})}, \widetilde{(T_{23})}) = \mathcal{A}^{(3)}((G_{23}), (T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)}\left\|G_{21}^{(1)} - G_{21}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)} +$<br>$\int_0^t\{(a'_{20})^{(3)}\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{-(\hat{M}_{20})^{(3)}s_{(20)}} +$<br>$(a''_{20})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right)\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}} +$<br>$G_{20}^{(2)}\left\|(a''_{20})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a''_{20})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}t} \leq$<br>$\frac{1}{(\hat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a'_{20})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)}, (T_{23})^{(1)};\ (G_{23})^{(2)}, (T_{23})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |

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| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ $and$ $(\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(3)}t)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2\ and\ \left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}\ and\ G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)}G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br><br>If $G_{21}\ or\ G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ $is$ bounded, from below, the same property holds for $G_{21}\ and\ G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t), t\right)) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t), t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded. | 220 |

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| The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}'')^{(3)}\left((G_{23})(t),t\right)=(b_{22}')^{(3)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}<1$ and to choose<br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)}+\left((\widehat{P}_{24})^{(4)}+G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right]\le(\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)}+T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)}+(\widehat{Q}_{24})^{(4)}\right]\le(\widehat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{24})^{(4)}\left\|G_{25}^{(1)}-G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}ds_{(24)}+$<br>$\int_0^t\{(a_{24}')^{(4)}\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$(a_{24}'')^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$G_{24}^{(2)}\left\\|(a_{24}'')^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}'')^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\\|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)}-(G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t}\le$<br>$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}')^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$ | 226 |

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| And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i = 24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(4)} - (a_i^{''})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(4)}t\right)} > 0$ for t > 0 | 228 |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ :<br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if<br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}^{'})^{(4)}G_{25}$ and by integrating<br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a_{25}^{'})^{(4)}$<br>In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a_{26}^{'})^{(4)}$<br>If $G_{25}\ or\ G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}\ and\ G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t), t)) = (b_{25}^{'})^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i^{''})^{(4)}((G_{27})(t), t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results | 232 |

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right), \quad t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42

Analogous inequalities hold also for $G_{29}, G_{30}, T_{28}, T_{29}, T_{30}$

It is now sufficient to take $\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ and to choose

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$(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ large to have

$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[(\hat{P}_{28})^{(5)} + \left((\hat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{28})^{(5)} \qquad 234$$

$$\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{28})^{(5)}\right] \leq (\hat{Q}_{28})^{(5)} \qquad 235$$

In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric 236

$$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right) = \mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$

It results

$$\left|\tilde{G}_{28}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{28})^{(5)}\left|G_{29}^{(1)} - G_{29}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)} +$$

$$\int_0^t\{(a_{28}^{'})^{(5)}\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{-(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$G_{28}^{(2)}\left|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right) - (a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right|\ e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$$

Where $s_{(28)}$ represents integrand that is integrated over the interval $[0, t]$

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| From the hypotheses on it follows | |
| $\left\|(G_{31})^{(1)} - (G_{31})^{(2)}\right\| e^{-(\widehat{M}_{28})^{(5)}t} \leq$ $\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)} + (a'_{28})^{(5)} + (\widehat{A}_{28})^{(5)} + (\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right) d\left(((G_{31})^{(1)}, (T_{31})^{(1)};\ (G_{31})^{(2)}, (T_{31})^{(2)})\right)$ And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a''_{28})^{(5)}$ and $(b''_{28})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ and $(\widehat{Q}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(5)}$ and $(b''_i)^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t) = 0$ and $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(5)} - (a''_i)^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(5)}t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ and $\left((\widehat{M}_{28})^{(5)}\right)_3$ : **Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ and $G_{30}$ . indeed if $G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a'_{29})^{(5)} G_{29}$ and by integrating $G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a'_{29})^{(5)}$ In the same way , one can obtain $G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a'_{30})^{(5)}$ If $G_{29}$ or $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ is bounded, from below, the same property holds for $G_{29}$ and $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |

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| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t),t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1-e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}, G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}\left[(\hat{P}_{32})^{(6)} + \left((\hat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{32})^{(6)}\right] \leq (\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{32})^{(6)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{35})}, \widetilde{(T_{35})}$ : $\left(\widetilde{(G_{35})}, \widetilde{(T_{35})}\right) = \mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br><br>It results | 247 |

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| $\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{32})^{(6)} \left\|G_{33}^{(1)}-G_{33}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)} s_{(32)}} e^{(\widehat{M}_{32})^{(6)} s_{(32)}}\, ds_{(32)} +$<br>$\int_0^t \{(a_{32}')^{(6)} \left\|G_{32}^{(1)}-G_{32}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)} s_{(32)}} e^{-(\widehat{M}_{32})^{(6)} s_{(32)}} +$<br>$(a_{32}'')^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)} s_{(32)}} e^{(\widehat{M}_{32})^{(6)} s_{(32)}} +$<br>$G_{32}^{(2)} \left\|(a_{32}'')^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)-(a_{32}'')^{(6)}\left(T_{33}^{(2)}, s_{(32)}\right)\right\| \; e^{-(\widehat{M}_{32})^{(6)} s_{(32)}} e^{(\widehat{M}_{32})^{(6)} s_{(32)}}\} ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)} t} \leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a_{32}')^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right) d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a_{32}'')^{(6)}$ and $(b_{32}'')^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)} e^{(\widehat{M}_{32})^{(6)} t}$ *and* $(\widehat{Q}_{32})^{(6)} e^{(\widehat{M}_{32})^{(6)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(6)}$ and $(b_i'')^{(6)}, i = 32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 249 |
| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(6)} - (a_i'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(6)} t\right)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ :<br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if<br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}')^{(6)} G_{33}$ and by integrating<br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}')^{(6)}$<br>In the same way , one can obtain<br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}')^{(6)}$ | 251 |

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|---|---|
| If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i^{''})^{(6)}\big((G_{35})(t),t\big) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ large to have | 255 |
| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[(\hat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote | 258 |

**Definition of** $\widetilde{(G_{39})}, \widetilde{(T_{39})} : \left( \widetilde{(G_{39})}, \widetilde{(T_{39})} \right) = \mathcal{A}^{(7)}((G_{39}), (T_{39}))$
It results

$\left|\tilde{G}_{36}^{(1)} - \tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{36})^{(7)} \left|G_{37}^{(1)} - G_{37}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}}\, ds_{(36)} +$
$\int_0^t \{(a_{36}^{'})^{(7)} \left|G_{36}^{(1)} - G_{36}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} +$
$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) \left|G_{36}^{(1)} - G_{36}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}} +$
$G_{36}^{(2)} \left|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) - (a_{36}^{''})^{(7)}\left(T_{37}^{(2)}, s_{(36)}\right)\right| \; e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}} \} ds_{(36)}$

Where $s_{(36)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on it follows

$\left|(G_{39})^{(1)} - (G_{39})^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} t} \leq$
$\frac{1}{(\widehat{M}_{36})^{(7)}} \left((a_{36})^{(7)} + (a_{36}^{'})^{(7)} + (\widehat{A}_{36})^{(7)} + (\widehat{P}_{36})^{(7)} (\widehat{k}_{36})^{(7)}\right) d\left(((G_{39})^{(1)}, (T_{39})^{(1)};\ (G_{39})^{(2)}, (T_{39})^{(2)})\right)$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

**Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)} e^{(\widehat{M}_{36})^{(7)} t}$ *and* $(\widehat{Q}_{36})^{(7)} e^{(\widehat{M}_{36})^{(7)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36{,}37{,}38$ depend only on $T_{37}$ and respectively on $(G_{39})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition.

**Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$

it results

$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)} (T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$

$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0

**Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2$ *and* $\left((\widehat{M}_{36})^{(7)}\right)_3$ :

**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}$ *and* $G_{38}$ . indeed if

$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating

$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)} \left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$

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In the same way , one can obtain

$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}')^{(7)}$

If $G_{37}$ *or* $G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively.

**Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}$ *and* $G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. 263

**Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(7)}((G_{39})(t),t)) = (b_{37}')^{(7)}$ then $T_{37} \to \infty$. 264

**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :

Indeed let $t_7$ be so that for $t > t_7$

$(b_{37})^{(7)} - (b_i'')^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$

Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to 265

$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results

$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right), \quad t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.
The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}'')^{(7)}\left((G_{39})(t),t\right) = (b_{38}')^{(7)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations

It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose 266
$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have

$$\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)} + \left((\widehat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{40})^{(8)} \quad 267$$

$$\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\widehat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{40})^{(8)}\right] \leq (\widehat{Q}_{40})^{(8)} \quad 268$$

In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric

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| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$ $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\,ds_{(40)}+$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\le$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ *and* $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i=40,41,42$ depend only on $\mathrm{T}_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$<br><br>it results<br><br>$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(8)}-(a_i^{''})^{(8)}(T_{41}(s_{(40)}),s_{(40)})\}ds_{(40)}\right]}\ge 0$<br><br>$T_i\,(t)\ge T_i^0e^{\left(-(b_i^{'})^{(8)}t\right)}>0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1,\left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ : | 276 |

**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if

$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a'_{41})^{(8)}G_{41}$ and by integrating

$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1/(a'_{41})^{(8)}$

In the same way , one can obtain

$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2/(a'_{42})^{(8)}$

If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively.

**Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. 277

**Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t),t)) = (b'_{41})^{(8)}$ then $T_{41} \to \infty$. 278

**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :

Indeed let $t_8$ be so that for $t > t_8$

$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t),t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$

Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to 279

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right),\ t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded. The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t),t(t),t\right) = (b'_{42})^{(8)}$

It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have 279 A

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\left(((G_{47})^{(1)},(T_{47})^{(1)}),((G_{47})^{(2)},(T_{47})^{(2)})\right)=$$
$$sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})},\widetilde{(T_{47})}:\left(\widetilde{(G_{47})},\widetilde{(T_{47})}\right)=\mathcal{A}^{(9)}\left((G_{47}),(T_{47})\right)$
It results

$$\left|\tilde{G}_{44}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{44})^{(9)}\left|G_{45}^{(1)}-G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)}+$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)-(a_{44}^{''})^{(9)}\left(T_{45}^{(2)},s_{(44)}\right)\right|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t}\le$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)}+(a_{44}^{'})^{(9)}+(\widehat{A}_{44})^{(9)}+(\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)},(T_{47})^{(1)};\ (G_{47})^{(2)},(T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i=44{,}45{,}46$ depend only on $\mathrm{T}_{45}$ and respectively on $(G_{47})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$

From 99 to 44 it results

$$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(9)}-(a_i^{''})^{(9)}(T_{45}(s_{(44)}),s_{(44)})\}ds_{(44)}\right]}\ge 0$$

$$T_i(t)\ge T_i^0e^{(-(b_i^{'})^{(9)}t)}>0\quad\text{for t}>0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1,\left((\widehat{M}_{44})^{(9)}\right)_2$ *and* $\left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)} G_{45}$ and by integrating

$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$

In the same way , one can obtain

$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t),t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t),t\right) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right),\quad t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\left((G_{47})(t),t\right) = (b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} - (a_{13}^{''})^{(1)}(T_{14},t) + (a_{14}^{''})^{(1)}(T_{14},t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(v_1)^{(1)}, (v_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, v^{(1)}, u^{(1)}$ : 281

By $(v_1)^{(1)} > 0$ , $(v_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$ , $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

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| **Definition of** $(\bar{v}_1)^{(1)},,(\bar{v}_2)^{(1)},(\bar{u}_1)^{(1)},(\bar{u}_2)^{(1)}$ :<br><br>By $(\bar{v}_1)^{(1)} > 0$ , $(\bar{v}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$ , $(\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$ | 282 |
| **Definition of** $(m_1)^{(1)}$ , $(m_2)^{(1)}$ , $(\mu_1)^{(1)},(\mu_2)^{(1)},(\nu_0)^{(1)}$ :-<br><br>If we define $(m_1)^{(1)}$ , $(m_2)^{(1)}$ , $(\mu_1)^{(1)},(\mu_2)^{(1)}$ by<br>$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)},\ if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$<br><br>$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}\ , if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)},$<br><br>and $\boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$<br><br>$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)},\ if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$ | 283 |
| and analogously<br><br>$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)},\ if\ (u_0)^{(1)} < (u_1)^{(1)}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}\ , if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)},$<br><br>and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$<br><br>are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br><br>where $(p_i)^{(1)}$ is defined by equation<br><br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le\right.$<br><br>$\left.\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a'_{15})^{(1)})}\left[e^{(S_1)^{(1)}t} - e^{-(a'_{15})^{(1)}t}\right] + G_{15}^0 e^{-(a'_{15})^{(1)}t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}$ | 288 |

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| $\frac{(b_{15})^{(1)}T_{13}^{0}}{(\mu_1)^{(1)}((R_1)^{(1)}-(b_{15}^{'})^{(1)})}\left[e^{(R_1)^{(1)}t}-e^{-(b_{15}^{'})^{(1)}t}\right]+T_{15}^{0}e^{-(b_{15}^{'})^{(1)}t}\le T_{15}(t)\le$ $\frac{(a_{15})^{(1)}T_{13}^{0}}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t}-e^{-(R_2)^{(1)}t}\right]+T_{15}^{0}e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)},(S_2)^{(1)},(R_1)^{(1)},(R_2)^{(1)}$:- Where $(S_1)^{(1)}=(a_{13})^{(1)}(m_2)^{(1)}-(a_{13}^{'})^{(1)}$ $(S_2)^{(1)}=(a_{15})^{(1)}-(p_{15})^{(1)}$ $(R_1)^{(1)}=(b_{13})^{(1)}(\mu_2)^{(1)}-(b_{13}^{'})^{(1)}$ $(R_2)^{(1)}=(b_{15}^{'})^{(1)}-(r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ : $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)}\le-(a_{16}^{'})^{(2)}+(a_{17}^{'})^{(2)}-(a_{16}^{''})^{(2)}(\mathrm{T}_{17},t)+(a_{17}^{''})^{(2)}(\mathrm{T}_{17},t)\le-(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)}\le-(b_{16}^{'})^{(2)}+(b_{17}^{'})^{(2)}-(b_{16}^{''})^{(2)}\big((G_{19}),t\big)-(b_{17}^{''})^{(2)}\big((G_{19}),t\big)\le-(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(v_1)^{(2)},(v_2)^{(2)},(u_1)^{(2)},(u_2)^{(2)}$ : | 295 |
| By $(v_1)^{(2)}>0$ , $(v_2)^{(2)}<0$ and respectively $(u_1)^{(2)}>0$ , $(u_2)^{(2)}<0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(v^{(2)}\big)^2+(\sigma_1)^{(2)}v^{(2)}-(a_{16})^{(2)}=0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2+(\tau_1)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ and | 298 |
| **Definition of** $(\bar{v}_1)^{(2)},,(\bar{v}_2)^{(2)},(\bar{u}_1)^{(2)},(\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{v}_1)^{(2)}>0$ , $(\bar{v}_2)^{(2)}<0$ and respectively $(\bar{u}_1)^{(2)}>0$ , $(\bar{u}_2)^{(2)}<0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(v^{(2)}\big)^2+(\sigma_2)^{(2)}v^{(2)}-(a_{16})^{(2)}=0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2+(\tau_2)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ | 302 |
| **Definition of** $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)}=(v_0)^{(2)},(m_1)^{(2)}=(v_1)^{(2)}$, ***if*** $(v_0)^{(2)}<(v_1)^{(2)}$ | 305 |

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| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , $\boldsymbol{if}$ $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, $\boldsymbol{if}$ $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, $\boldsymbol{if}$ $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , $\boldsymbol{if}$ $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$,<br>and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, $\boldsymbol{if}$ $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)} t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)} t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)} t}\right] + G_{18}^0 e^{-(S_2)^{(2)} t} \le G_{18}(t) \le \right.$<br>$\left.\frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a'_{18})^{(2)})}\left[e^{(S_1)^{(2)} t} - e^{-(a'_{18})^{(2)} t}\right] + G_{18}^0 e^{-(a'_{18})^{(2)} t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b'_{18})^{(2)})}\left[e^{(R_1)^{(2)} t} - e^{-(b'_{18})^{(2)} t}\right] + T_{18}^0 e^{-(b'_{18})^{(2)} t} \le T_{18}(t) \le$<br>$\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)} + (r_{16})^{(2)} + (R_2)^{(2)})}\left[e^{((R_1)^{(2)} + (r_{16})^{(2)})t} - e^{-(R_2)^{(2)} t}\right] + T_{18}^0 e^{-(R_2)^{(2)} t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a'_{16})^{(2)}$<br>$(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b'_{16})^{(2)}$ | 318 |

$(R_2)^{(2)} = (b_{18}')^{(2)} - (r_{18})^{(2)}$

**Behavior of the solutions** 319

**Theorem 3:** If we denote and define

**Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ :

$(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying

$$-(\sigma_2)^{(3)} \le -(a_{20}')^{(3)} + (a_{21}')^{(3)} - (a_{20}'')^{(3)}(T_{21}, t) + (a_{21}'')^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$$

$$-(\tau_2)^{(3)} \le -(b_{20}')^{(3)} + (b_{21}')^{(3)} - (b_{20}'')^{(3)}(G_{23}, t) - (b_{21}'')^{(3)}((G_{23}), t) \le -(\tau_1)^{(3)}$$

**Definition of** $(v_1)^{(3)}, (v_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ : 320

By $(v_1)^{(3)} > 0$ , $(v_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$ , $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\left(u^{(3)}\right)^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and

By $(\bar{v}_1)^{(3)} > 0$ , $(\bar{v}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$ , $(\bar{u}_2)^{(3)} < 0$ the

roots of the equations $(a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\left(u^{(3)}\right)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$

**Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :- 321

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (v_0)^{(3)}, (m_1)^{(3)} = (v_1)^{(3)}$, ***if*** $(v_0)^{(3)} < (v_1)^{(3)}$

$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (\bar{v}_1)^{(3)}$ , ***if*** $(v_1)^{(3)} < (v_0)^{(3)} < (\bar{v}_1)^{(3)}$,

and $\boxed{(v_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (v_0)^{(3)}$, ***if*** $(\bar{v}_1)^{(3)} < (v_0)^{(3)}$

and analogously 322

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$ , ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

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| $G_{20}^0 e^{((S_1)^{(3)}-(p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)}t}$<br>$(p_i)^{(3)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)}-(p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)}G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)}-(p_{20})^{(3)}-(S_2)^{(3)})}\left[e^{((S_1)^{(3)}-(p_{20})^{(3)})t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \le G_{22}(t) \le \right.$<br>$\left.\frac{(a_{22})^{(3)}G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)}-(a'_{22})^{(3)})}[e^{(S_1)^{(3)}t} - e^{-(a'_{22})^{(3)}t}] + G_{22}^0 e^{-(a'_{22})^{(3)}t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)}+(r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)}+(r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)}T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)}-(b'_{22})^{(3)})}\left[e^{(R_1)^{(3)}t} - e^{-(b'_{22})^{(3)}t}\right] + T_{22}^0 e^{-(b'_{22})^{(3)}t} \le T_{22}(t) \le$<br>$\frac{(a_{22})^{(3)}T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)}+(r_{20})^{(3)}+(R_2)^{(3)})}\left[e^{((R_1)^{(3)}+(r_{20})^{(3)})t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$ | 327 |
| **Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:-<br>Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$<br>$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$<br>$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$<br>$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$ | 328 |
| **Behavior of the solutions of equation**<br>**Theorem:** If we denote and define<br>**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :<br>$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying<br>$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$<br>$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ :<br>By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}(v^{(4)})^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$<br>and $(b_{25})^{(4)}(u^{(4)})^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and | 329 |

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| **Definition of** $(\bar{v}_1)^{(4)},,(\bar{v}_2)^{(4)},(\bar{u}_1)^{(4)},(\bar{u}_2)^{(4)}$ :<br><br>By $(\bar{v}_1)^{(4)}>0$ , $(\bar{v}_2)^{(4)}<0$ and respectively $(\bar{u}_1)^{(4)}>0$ , $(\bar{u}_2)^{(4)}<0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_2)^{(4)}v^{(4)}-(a_{24})^{(4)}=0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2+(\tau_2)^{(4)}u^{(4)}-(b_{24})^{(4)}=0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)},(v_0)^{(4)}$ :-<br><br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)}$ by<br><br>$(m_2)^{(4)}=(v_0)^{(4)},(m_1)^{(4)}=(v_1)^{(4)}$, ***if*** $(v_0)^{(4)}<(v_1)^{(4)}$<br><br>$(m_2)^{(4)}=(v_1)^{(4)},(m_1)^{(4)}=(\bar{v}_1)^{(4)}$ , ***if*** $(v_4)^{(4)}<(v_0)^{(4)}<(\bar{v}_1)^{(4)}$,<br>and $\boxed{(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}$<br><br>$(m_2)^{(4)}=(v_4)^{(4)},(m_1)^{(4)}=(v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)}<(v_0)^{(4)}$ | 330 |
| and analogously<br><br>$(\mu_2)^{(4)}=(u_0)^{(4)},(\mu_1)^{(4)}=(u_1)^{(4)}$, ***if*** $(u_0)^{(4)}<(u_1)^{(4)}$<br><br>$(\mu_2)^{(4)}=(u_1)^{(4)},(\mu_1)^{(4)}=(\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)}<(u_0)^{(4)}<(\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)}=\frac{T_{24}^0}{T_{25}^0}}$<br><br>$(\mu_2)^{(4)}=(u_1)^{(4)},(\mu_1)^{(4)}=(u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)}<(u_0)^{(4)}$ where $(u_1)^{(4)},(\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{24}^0e^{((S_1)^{(4)}-(p_{24})^{(4)})t}\le G_{24}(t)\le G_{24}^0e^{(S_1)^{(4)}t}$<br><br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}}G_{24}^0e^{((S_1)^{(4)}-(p_{24})^{(4)})t}\le G_{25}(t)\le\frac{1}{(m_2)^{(4)}}G_{24}^0e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)}G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t}-e^{-(S_2)^{(4)}t}\right]+G_{26}^0e^{-(S_2)^{(4)}t}\le G_{26}(t)\le\right.$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0e^{(R_1)^{(4)}t}\le T_{24}(t)\le T_{24}^0e^{((R_1)^{(4)}+(r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}}T_{24}^0e^{(R_1)^{(4)}t}\le T_{24}(t)\le\frac{1}{(\mu_2)^{(4)}}T_{24}^0e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)}T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}^{'})^{(4)})}\left[e^{(R_1)^{(4)}t}-e^{-(b_{26}^{'})^{(4)}t}\right]+T_{26}^0e^{-(b_{26}^{'})^{(4)}t}\le T_{26}(t)\le$<br><br>$\frac{(a_{26})^{(4)}T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t}-e^{-(R_2)^{(4)}t}\right]+T_{26}^0e^{-(R_2)^{(4)}t}$ | 336 |

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|---|---|
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a'_{24})^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b'_{24})^{(4)}$<br>$(R_2)^{(4)} = (b'_{26})^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a'_{28})^{(5)} + (a'_{29})^{(5)} - (a''_{28})^{(5)}(T_{29}, t) + (a''_{29})^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b'_{28})^{(5)} + (b'_{29})^{(5)} - (b''_{28})^{(5)}((G_{31}), t) - (b''_{29})^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ :<br>By $(v_1)^{(5)} > 0$, $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | 339 |
| **Definition of** $(\bar{v}_1)^{(5)},, (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ :<br>By $(\bar{v}_1)^{(5)} > 0$, $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$, $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :-<br>If we define $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}$ by<br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$, ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}}}$<br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$ | 341 |

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| $(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , $\boldsymbol{if}$ $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$, and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$ $(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}, \boldsymbol{if}$ $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | |
| Then the solution of global equations satisfies the inequalities $G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)} t}$ where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \le G_{30}(t) \le$ *(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b_{30}')^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b_{30}')^{(5)} t}\right] + T_{30}^0 e^{-(b_{30}')^{(5)} t} \le T_{30}(t) \le$ $\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:- Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a_{28}')^{(5)}$ $(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$ $(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b_{28}')^{(5)}$ $(R_2)^{(5)} = (b_{30}')^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ : $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying $-(\sigma_2)^{(6)} \le -(a_{32}')^{(6)} + (a_{33}')^{(6)} - (a_{32}'')^{(6)}(T_{33}, t) + (a_{33}'')^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$ | 349 |

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| $-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}((G_{35}),t) - (b''_{33})^{(6)}((G_{35}),t) \le -(\tau_1)^{(6)}$ | |
| **Definition of** $(v_1)^{(6)}, (v_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, v^{(6)}, u^{(6)}$ :<br><br>By $(v_1)^{(6)} > 0$ , $(v_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_1)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\left(u^{(6)}\right)^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{v}_1)^{(6)},, (\bar{v}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br><br>By $(\bar{v}_1)^{(6)} > 0$ , $(\bar{v}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_2)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\left(u^{(6)}\right)^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (v_0)^{(6)}$ :-<br><br>If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br><br>$(m_2)^{(6)} = (v_0)^{(6)}, (m_1)^{(6)} = (v_1)^{(6)}$, ***if*** $(v_0)^{(6)} < (v_1)^{(6)}$<br><br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (\bar{v}_6)^{(6)}$ , ***if*** $(v_1)^{(6)} < (v_0)^{(6)} < (\bar{v}_1)^{(6)}$,<br>and $\boxed{(v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$<br><br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (v_0)^{(6)}$, ***if*** $(\bar{v}_1)^{(6)} < (v_0)^{(6)}$ | 351 |
| and analogously<br><br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br><br>$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$<br><br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G^0_{32}e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)}G^0_{32}}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G^0_{34}e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |

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|---|---|
| $\boxed{T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)}-(b_{34}')^{(6)})}\left[e^{(R_1)^{(6)}t} - e^{-(b_{34}')^{(6)}t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t} - e^{-(R_2)^{(6)}t}\right] + T_{34}^0 e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:- Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$ $(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$ $(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$ $(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ : $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying $-(\sigma_2)^{(7)} \le -(a_{36}')^{(7)} + (a_{37}')^{(7)} - (a_{36}'')^{(7)}(T_{37}, t) + (a_{37}'')^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$ $-(\tau_2)^{(7)} \le -(b_{36}')^{(7)} + (b_{37}')^{(7)} - (b_{36}'')^{(7)}((G_{39}), t) - (b_{37}'')^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)}, (v_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, v^{(7)}, u^{(7)}$ : By $(v_1)^{(7)} > 0, (v_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(v^{(7)})^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)}, , (\bar{v}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ : By $(\bar{v}_1)^{(7)} > 0, (\bar{v}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(v^{(7)})^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ **Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (v_0)^{(7)}$ :- | 362 |

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, $\boldsymbol{if}\ (\nu_0)^{(7)} < (\nu_1)^{(7)}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, $\boldsymbol{if}\ (\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,

and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, $\boldsymbol{if}\ (\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$

and analogously

$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, $\boldsymbol{if}\ (u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, $\boldsymbol{if}\ (u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,

and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}, \boldsymbol{if}\ (\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

(363)

Then the solution of global equations satisfies the inequalities

$$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)} t}$$

where $(p_i)^{(7)}$ is defined by equation

(364)

$$\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)} t} \quad (365)$$

$$\left(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})}\left[e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)} t}\right] + G_{38}^0 e^{-(S_2)^{(7)} t} \le G_{38}(t) \le \right.$$

$$\left. \frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a_{38}')^{(7)})}\left[e^{(S_1)^{(7)} t} - e^{-(a_{38}')^{(7)} t}\right] + G_{38}^0 e^{-(a_{38}')^{(7)} t}\right) \quad (366)$$

$$\boxed{T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}} \quad (367)$$

$$\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t} \quad (368)$$

$$\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b_{38}')^{(7)})}\left[e^{(R_1)^{(7)} t} - e^{-(b_{38}')^{(7)} t}\right] + T_{38}^0 e^{-(b_{38}')^{(7)} t} \le T_{38}(t) \le \quad (369)$$

$$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)} + (r_{36})^{(7)} + (R_2)^{(7)})}\left[e^{((R_1)^{(7)} + (r_{36})^{(7)})t} - e^{-(R_2)^{(7)} t}\right] + T_{38}^0 e^{-(R_2)^{(7)} t}$$

**Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:- 370

Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a'_{36})^{(7)}$

$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$

$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b'_{36})^{(7)}$

$(R_2)^{(7)} = (b'_{38})^{(7)} - (r_{38})^{(7)}$

**Behavior of the solutions of equation** 371

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :

$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying

$-(\sigma_2)^{(8)} \le -(a'_{40})^{(8)} + (a'_{41})^{(8)} - (a''_{40})^{(8)}(T_{41}, t) + (a''_{41})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$

$-(\tau_2)^{(8)} \le -(b'_{40})^{(8)} + (b'_{41})^{(8)} - (b''_{40})^{(8)}\big((G_{43}), t\big) - (b''_{41})^{(8)}\big((G_{43}), t\big) \le -(\tau_1)^{(8)}$

**Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : 372

By $(v_1)^{(8)} > 0, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$
and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and

**Definition of** $(\bar{v}_1)^{(8)}, , (\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :

By $(\bar{v}_1)^{(8)} > 0, (\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0, (\bar{u}_2)^{(8)} < 0$ the

roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$

and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$

**Definition of** $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :-

If we define $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by

$(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)}$, ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}$, ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)}$,

| | |
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| and $\boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br><br>$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (v_0)^{(8)},$ ***if*** $(\bar{v}_1)^{(8)} < (v_0)^{(8)}$ | |
| and analogously<br><br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},$ ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)}$,<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)},$ ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{40}(t) \leq G_{40}^0 e^{(S_1)^{(8)} t}$<br><br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{41}(t) \leq \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \leq G_{42}(t) \leq \right.$<br>$\left.\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)} t} - e^{-(a'_{42})^{(8)} t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)} t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \leq T_{40}(t) \leq T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \leq T_{40}(t) \leq \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b'_{42})^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b'_{42})^{(8)} t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)} t} \leq T_{42}(t) \leq$<br><br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br><br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br><br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$ | 381 |

$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$

$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$

**Behavior of the solutions of equation 37 to 92** 382

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$

$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}\big((G_{47}), t\big) - (b''_{45})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$

**Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :

By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(9)}, , (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{\nu}_1)^{(9)} > 0$ , $(\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}$, ***if*** $(\nu_0)^{(9)} < (\nu_1)^{(9)}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}$, ***if*** $(\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T^0_{44}}{T^0_{45}}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^{0}e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^{0}e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^{0}e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}}G_{44}^{0}e^{(S_1)^{(9)}t}$$

(

$$\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^{0}e^{-(S_2)^{(9)}t} \le G_{46}(t) \le$$

$$\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}^{'})^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}^{'})^{(9)}t}]+\ G_{46}^{0}e^{-(a_{46}^{'})^{(9)}t})$$

$$\boxed{T_{44}^{0}e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^{0}e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}}T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^{0}}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}^{'})^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}^{'})^{(9)}t}\right]+T_{46}^{0}e^{-(b_{46}^{'})^{(9)}t} \le T_{46}(t) \le$$

$$\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a_{44}^{'})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b_{44}^{'})^{(9)}$

$(R_2)^{(9)} = (b_{46}^{'})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a_{13}^{'})^{(1)} - (a_{14}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14},t)\right) - (a_{14}^{''})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}(v^{(1)})^{2} + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}(v^{(1)})^{2} + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

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For $0 < \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (\nu_1)^{(1)} < (\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t) \geq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(\nu_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\nu_1)^{(1)}$

In the same manner , we get 384

$$\nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{\nu}_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\bar{\nu}_1)^{(1)}$

If $0 < (\nu_1)^{(1)} < (\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{\nu}_1)^{(1)}$ we find like in the previous case, 385

$$(\nu_1)^{(1)} \leq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}} \leq \nu^{(1)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (C)^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

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| Particular case :<br><br>If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case<br><br>Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then<br><br>$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$ if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$. | |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$ | 387 |
| **Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ | 388 |
| It follows<br><br>$-\left((a_{17})^{(2)}(\nu^{(2)})^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \le \frac{d\nu^{(2)}}{dt} \le -\left((a_{17})^{(2)}(\nu^{(2)})^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$ | 389 |
| From which one obtains<br><br>**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-<br><br>For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$<br><br>$\nu^{(2)}(t) \ge \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}$ , $\boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$<br><br>it follows $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\nu_1)^{(2)}$ | 390 |
| In the same manner , we get<br><br>$\nu^{(2)}(t) \le \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$ | 391 |
| From which we deduce $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\bar{\nu}_1)^{(2)}$ | 392 |
| If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case,<br><br>$(\nu_1)^{(2)} \le \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \le \nu^{(2)}(t) \le$ | 393 |

| | |
|---|---|
| $\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$ | |
| If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br><br>$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :-<br><br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right) - (a_{21}^{''})^{(3)}(T_{21},t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| | 400 |

From which one obtains

For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

In the same manner , we get

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$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

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If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case,

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

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If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$ if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

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In the same manner , we get

$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

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$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}'')^{(5)} = (b_{29}'')^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then

$(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(v_1)^{(5)}$ and $(\bar{v}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain 412

$$\frac{dv^{(6)}}{dt} = (a_{32})^{(6)} - \left((a'_{32})^{(6)} - (a'_{33})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right) - (a''_{33})^{(6)}(T_{33},t)v^{(6)} - (a_{33})^{(6)}v^{(6)}$$

**Definition of** $v^{(6)}$ :- $\boxed{v^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_2)^{(6)}v^{(6)} - (a_{32})^{(6)}\right) \le \frac{dv^{(6)}}{dt} \le -\left((a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_1)^{(6)}v^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(6)}, (v_0)^{(6)}$ :-

For $0 < \boxed{(v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}} < (v_1)^{(6)} < (\bar{v}_1)^{(6)}$

$$v^{(6)}(t) \ge \frac{(v_1)^{(6)} + (C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)} - (v_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)} - (v_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(v_1)^{(6)} - (v_0)^{(6)}}{(v_0)^{(6)} - (v_2)^{(6)}}}$$

it follows $(v_0)^{(6)} \le v^{(6)}(t) \le (v_1)^{(6)}$

In the same manner , we get 413

$$v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)} + (\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{v}_1)^{(6)} - (v_0)^{(6)}}{(v_0)^{(6)} - (\bar{v}_2)^{(6)}}}$$

From which we deduce $(v_0)^{(6)} \le v^{(6)}(t) \le (\bar{v}_1)^{(6)}$

If $0 < (v_1)^{(6)} < (v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}} < (\bar{v}_1)^{(6)}$ we find like in the previous case, 414

$$(v_1)^{(6)} \le \frac{(v_1)^{(6)} + (C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)} - (v_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)} - (v_2)^{(6)}\right)t\right]}} \le v^{(6)}(t) \le$$

$$\frac{(\bar{v}_1)^{(6)} + (\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}} \le (\bar{v}_1)^{(6)}$$

If $0 < (v_1)^{(6)} \le (\bar{v}_1)^{(6)} \le \boxed{(v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$ , we obtain 415

$$(v_1)^{(6)} \le v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)} + (\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}} \le (v_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}$, $\boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

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$$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$$

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If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain

$$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(C)^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

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**Definition of** $u^{(7)}(t)$ :-

$$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}, \quad \boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}'')^{(7)} = (a_{37}'')^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}'')^{(7)} = (b_{37}'')^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right) - (a_{41}^{''})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \le \frac{dv^{(8)}}{dt} \le -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(8)}, (v_0)^{(8)}$ :-

For $0 < \boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (v_1)^{(8)} < (\bar{v}_1)^{(8)}$

$$v^{(8)}(t) \ge \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(v_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (v_2)^{(8)}}}$$

it follows $(v_0)^{(8)} \le v^{(8)}(t) \le (v_1)^{(8)}$

In the same manner , we get

$$v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{v}_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (\bar{v}_2)^{(8)}}}$$

From which we deduce $(v_0)^{(8)} \le v^{(8)}(t) \le (\bar{v}_8)^{(8)}$

If $0 < (v_1)^{(8)} < (v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{v}_1)^{(8)}$ we find like in the previous case,

$$(v_1)^{(8)} \le \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}} \le v^{(8)}(t) \le$$

$$\frac{(\bar{v}_1)^{(8)} + (C)^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (\bar{v}_1)^{(8)}$$

If $0 < (v_1)^{(8)} \le (\bar{v}_1)^{(8)} \le \boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain

$$(v_1)^{(8)} \le v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (v_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

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**Definition of** $v^{(8)}(t)$ :-

$(m_2)^{(8)} \leq v^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{v^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $v^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)}-(\nu_0)^{(9)}}{(\nu_0)^{(9)}-(\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \le \frac{(\nu_1)^{(9)}+(C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}} \le \nu^{(9)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \le (\bar{\nu}_1)^{(9)} \le \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(\nu_1)^{(9)} \le \nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \le \nu^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \le u^{(9)}(t) \le (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)}G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

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| $(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,<br>$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$<br>$with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$ | 433 |

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| $(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$ | 436 A |

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| $(a'_{44})^{(9)}(a'_{45})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a'_{45})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$ $(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$, $(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b'_{44})^{(9)}(r_{45})^{(9)} - (b'_{45})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$ $with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b'_{13})^{(1)} - (b''_{13})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b'_{14})^{(1)} - (b''_{14})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b'_{15})^{(1)} - (b''_{15})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b'_{16})^{(2)} - (b''_{16})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b'_{17})^{(2)} - (b''_{17})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b'_{18})^{(2)} - (b''_{18})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a'_{20})^{(3)} + (a''_{20})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a'_{21})^{(3)} + (a''_{21})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a'_{22})^{(3)} + (a''_{22})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b'_{20})^{(3)} - (b''_{20})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b'_{21})^{(3)} - (b''_{21})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b'_{22})^{(3)} - (b''_{22})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) +$ | 485 |

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| $(a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | |
| **Proof:**<br><br>(j) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |

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| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492<br>A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations | 498 |

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| $G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a'_{28})^{(5)}+(a''_{28})^{(5)}(T^*_{29})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a'_{30})^{(5)}+(a''_{30})^{(5)}(T^*_{29})]}$ | |
| **Definition and uniqueness of** $T^*_{33}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T^*_{33}$ for which $f(T^*_{33}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a'_{32})^{(6)}+(a''_{32})^{(6)}(T^*_{33})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a'_{34})^{(6)}+(a''_{34})^{(6)}(T^*_{33})]}$ | 499 |
| **Definition and uniqueness of** $T^*_{37}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T^*_{37}$ for which $f(T^*_{37}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a'_{36})^{(7)}+(a''_{36})^{(7)}(T^*_{37})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a'_{38})^{(7)}+(a''_{38})^{(7)}(T^*_{37})]}$ | 500 |
| **Definition and uniqueness of** $T^*_{41}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T^*_{41}$ for which $f(T^*_{41}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a'_{40})^{(8)}+(a''_{40})^{(8)}(T^*_{41})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a'_{42})^{(8)}+(a''_{42})^{(8)}(T^*_{41})]}$ | 501 |
| **Definition and uniqueness of** $T^*_{45}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T^*_{45}$ for which $f(T^*_{45}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a'_{44})^{(9)}+(a''_{44})^{(9)}(T^*_{45})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a'_{46})^{(9)}+(a''_{46})^{(9)}(T^*_{45})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br><br>$\left[(b'_{13})^{(1)}(b''_{14})^{(1)}(G) + (b'_{14})^{(1)}(b''_{13})^{(1)}(G)\right]+(b''_{13})^{(1)}(G)(b''_{14})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that $\varphi$ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that | 502 |

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| there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*})=0$ | |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br>$\varphi(G_{19})=(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)}-(b_{16})^{(2)}(b_{17})^{(2)}-$<br>$\left[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19})+(b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})\right]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19})=0$ | 503 |
| Where in $(G_{19})(G_{16},G_{17},G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*})=0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br>$\varphi(G_{23})=(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)}-(b_{20})^{(3)}(b_{21})^{(3)}-$<br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23})+(b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23})=0$<br>Where in $G_{23}(G_{20},G_{21},G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*})=0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br>$\varphi(G_{27})=(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)}-(b_{24})^{(4)}(b_{25})^{(4)}-$<br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27})+(b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27})=0$<br>Where in $(G_{27})(G_{24},G_{25},G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*})=0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31})=(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)}-(b_{28})^{(5)}(b_{29})^{(5)}-$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31})+(b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right]+(b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31})=0$<br>Where in $(G_{31})(G_{28},G_{29},G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*})=0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35})=(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)}-(b_{32})^{(6)}(b_{33})^{(6)}-$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35})+(b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right]+(b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35})=0$<br>Where in $(G_{35})(G_{32},G_{33},G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that | 508 |

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| there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0\,, T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0\,, T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |

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| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*}) = 0$ , $T_{21}^{*}$ given by $f(T_{21}^{*}) = 0$ and<br><br>$G_{20}^{*} = \frac{(a_{20})^{(3)} G_{21}^{*}}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*} = \frac{(a_{22})^{(3)} G_{21}^{*}}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br><br>$T_{20}^{*} = \frac{(b_{20})^{(3)} T_{21}^{*}}{[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*} = \frac{(b_{22})^{(3)} T_{21}^{*}}{[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br><br>$G_{25}^{*}$ given by $\varphi(G_{27}) = 0$ , $T_{25}^{*}$ given by $f(T_{25}^{*}) = 0$ and<br><br>$G_{24}^{*} = \frac{(a_{24})^{(4)} G_{25}^{*}}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*} = \frac{(a_{26})^{(4)} G_{25}^{*}}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*} = \frac{(b_{24})^{(4)} T_{25}^{*}}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*} = \frac{(b_{26})^{(4)} T_{25}^{*}}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br><br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*}) = 0$ , $T_{29}^{*}$ given by $f(T_{29}^{*}) = 0$ and<br><br>$G_{28}^{*} = \frac{(a_{28})^{(5)} G_{29}^{*}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*} = \frac{(a_{30})^{(5)} G_{29}^{*}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |
| $T_{28}^{*} = \frac{(b_{28})^{(5)} T_{29}^{*}}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)} T_{29}^{*}}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br><br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br><br>$G_{32}^{*} = \frac{(a_{32})^{(6)} G_{33}^{*}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)} G_{33}^{*}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)} T_{33}^{*}}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)} T_{33}^{*}}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br><br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and | 522 |

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| $G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)} (T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)} (T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)} ((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)} ((G_{39})^{*})]}$ | |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)} (T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)} (T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)} ((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)} ((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br><br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br><br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)} (T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)} (T_{45}^{*})]}$<br><br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)} ((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)} ((G_{47})^{*})]}$ | 523 A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:**_Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}} (T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j} (G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)} \mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*} \mathbb{T}_{14}$ | 525 |

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| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^{*}\mathbb{G}_{j}\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^{*}\mathbb{G}_{j}\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^{*}\mathbb{G}_{j}\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 532 |
| $\frac{\partial(a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial(b_i^{''})^{(2)}}{\partial G_j}((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^{*}\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | 540 |

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| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)} \ , \ \frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^*\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^*\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^*\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^*\mathbb{G}_j\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^*\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^*\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ *and* $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)} \ , \ \frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^*) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right)$ | 554 |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^*) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^*\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^*\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^*\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |

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| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{37})^{(7)}}{\partial T_{37}}(T^*_{37}) = (q_{37})^{(7)}$ , $\frac{\partial(b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a'_{36})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G^*_{36}\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a'_{37})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G^*_{37}\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. | |

| | |
|---|---|
| Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \ , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}((G_{43})^*) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^*\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^*\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b_{41}^{'})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T_{41}^*\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b_{42}^{'})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T_{42}^*\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS** **Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. **Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |

| | |
|---|---|
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$ $\left[\left(\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\right]$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\right)$ $+\left(\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\right)$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((b'_{13})^{(1)} + (b'_{14})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $+\left(\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$ $+\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G^*_{13}\right)$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T^*_{14} + (b_{14})^{(1)}s_{(13),(15)}T^*_{13}\right)\} = 0$ $+$ $\left((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)}\right)\{\left((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)}\right)$ $\left[\left(\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}^*_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^*_{16}\right)\right]$ $\left(\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^*_{17}\right)$ $+\left(\left((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}^*_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^*_{17}\right)$ $\left(\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^*_{16}\right)$ $\left(\left((\lambda)^{(2)}\right)^2 + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ $\left(\left((\lambda)^{(2)}\right)^2 + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ | |

$$+\left(\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\right)\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left(\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\right)$$

$$+\left(\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((b_{20}')^{(3)}+(b_{21}')^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}')^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}')^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}')^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}')^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}')^{(4)}+(a_{25}')^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((b_{24}')^{(4)}+(b_{25}')^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\}\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left((\lambda)^{(6)}\right)^2+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G^*_{33}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G^*_{32}\right)$$

$$\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T^*_{33}+(b_{33})^{(6)}s_{(32),(34)}T^*_{32}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b'_{38})^{(7)}-(r_{38})^{(7)}\}\{(\lambda)^{(7)}+(a'_{38})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(q_{36})^{(7)}G^*_{36}\right)\Big]$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T^*_{37}+(b_{37})^{(7)}s_{(36),(37)}T^*_{37}\Big)$$

$$+\left((\lambda)^{(7)}+(a'_{37})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G^*_{36}+(a_{36})^{(7)}(q_{37})^{(7)}G^*_{37}\Big)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T^*_{37}+(b_{37})^{(7)}s_{(36),(36)}T^*_{36}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^*_{36}\right)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^*_{37}+(b_{37})^{(7)}s_{(36),(38)}T^*_{36}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\}\{(\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^*_{40}\right)\Big]$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^*_{41}+(b_{41})^{(8)}s_{(40),(41)}T^*_{41}\Big)$$

$$+\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^*_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^*_{41}\Big)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^*_{41}+(b_{41})^{(8)}s_{(40),(40)}T^*_{40}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^*_{40}\right)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^*_{41}+(b_{41})^{(8)}s_{(40),(42)}T^*_{40}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^*_{44}\right)\right]$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^*_{45}+(b_{45})^{(9)}s_{(44),(45)}T^*_{45}\Big)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^*_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45}+(b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left(\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45}+(b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

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# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i(x)\, xi$ where $x_i \in R^n_i$, $x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i(x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i(\mu)$ of the $A_i(x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k \tag{10}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to

replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 10. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon\ C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \ i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0}$

$\overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow *x.t restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set *K* is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** *x.t* is defined for all $t \geq 0$

**A4:** *x.t* is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\mathcal{A}(\tilde{A}), C \backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \longrightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) in\ ||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population *i* is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^{t}_{0} (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\to\infty}{limsup} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v = au_i(x) + w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |B_i(t,x)u_i(x)|(a + \alpha + exp(-\beta t)|w|)$. Hence, $\limsup_{t\to\infty} \frac{1}{t} In\, B_i(t,x)\, v| \leq$

$$\limsup_{t\to\infty} \frac{1}{t} In\, B_i(t,x)\, u_i(x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\, |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\, u_i(x)| = In\, |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\, u_i(x.t)||\, B_i(t,x)\, u_i(x)|$$

$$= \text{In } b_i(s,t.x)\, b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x - bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

$$=$$

$$= (A_i(x.t)u_i(x.s), u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s), u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } | (x.t)_i |$$

$$\leq \limsup_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x) =$ Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ *- almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of

irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x)u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$ where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ *and* $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as $n\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\le \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x)u_i^n(x), u_i^n(x)\rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)|d\mu(x) + |\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\,d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\,ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i\,dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\,d\,v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak$^*$ limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \{ \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i \epsilon I}{if\ i \notin I}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ *whenever* $x.[0,t] \in C\backslash W, it\ follows\ that\ \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t$ −invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently *K* is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s =$

$s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_s$. If $r(x) = s(x), then\ x \in M_r$. Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume $x \mapsto A(x)$ is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists $j \in \{1,\ldots.,k\}$ such that $M_i$ ----- $\{x \in C : x_l \gg 0 \text{ for } l \neq j\}$ Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_i M_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----{1,…,k} such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, *$\lambda_i(\mu) > 0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, ..., x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda(0)$, which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i(0) > 0$ for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the

system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j \ (x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1{}^1, \ldots, E_1{}^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i{}_2(E_1{}^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$$

$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > 1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2$ $(E_1)$and $\lambda_3$ $(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\ldots,k;\ j = 1,\ldots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i\, a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i\, x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \qquad i = 1, \ldots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \ldots, k$$

It has been shown [12,ch.16.1,16.2] That for *k* =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is

the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1-y) - b(N)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*′ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\curlywedge_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\curlywedge_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\curlywedge_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + K p\, x_i x_i - 1}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions

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# Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a

qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and

orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t\geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a\geq 0)$ at time $t$ so that we have

(1.28) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t\geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.29) $$D\rho_i(a, t) = \limsup_{h\to 0} \frac{\rho_i(a+h,t+h)-\rho_i(a,t)}{h},\ i = 1, 2.$$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.30) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s\in[-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s\in[-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1, \varphi_2, f_1, f_2, b_1, b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for

each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as* $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 10. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'}\ - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, \text{R}^-),$$

$A_3$. $\varphi_i\ (\cdot\ , s) \in L^2\ (R^+, R^+)\ \cap C(R^+, R^+)$ for each $s \in\ [-\tau, 0]$,

$i, = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[-\int_0^\infty f_i(a + s, P(t + S))ds\right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[-\int_o^x f_1(a + s, p(t + s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0)\exp\left[-\int_o^t f_1(a - t + s, p_1(s))\, ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

*(2.3)* $p_i(a,t) = B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i and\ L_i\ (i = 1,2)$ as follows:

$$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a,t,\ p_i(t)) = \exp[-\int_o^t f_1(a+s,\ p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

$$(2.6)\ p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da + \int_o^\infty \varphi_i(a, 0)\ L_i(a,t,p_i(t)da,$$

$$t > 0, i = 1,2,$$

$$(2.7)\ p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$$

$$(2.8)\ \beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.\left(a,t,P_i(t)\right)da$$

$$+\int_0^\infty b_i\left(a+t, P_i(t)\right)\varphi_i(a,0)L_i\left(a,t,P_i(t)\right)da$$

$$i = 1, 2, t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t) \phi(0) d_i^*] t \quad for \ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T], R^+), \quad B_i \in C([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in C([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau, T]$. Then for any

$(x, y) \in C^+[-\tau, T] \times C^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau, T], \quad C_T^2 = C^+[-\tau, T] \times C^+[-\tau, T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for $i = 1,2$.

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \pi C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \quad P_t = \ (P_{1t}, P_{2t}) \in C^2_{t*} \mid \|P_t - \Phi_t\|_{t*} \le r^*$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)\, da \ for\ s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)\, da \ for\ s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Phi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^* \right] \quad t^*$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for t $\le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \le \left| \int_0^t B_1^{(1)}(a) \left\{ M_1(a, t, P_t^{(1)}) - M_1(a, t, P_t^{(2)}) \right\} da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a, t, P_t^{(2)}) da \right|$$

$$+ \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \; for \; x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$|J_1| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \| ds\right) da$$

$$\leq d_1 \beta_1^* \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$|J_3| \leq \int_o^\infty \varphi(a,o)\left[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds\right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \| t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\text{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right]\right] d\sigma$$

$$+ \int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^{\infty} \varphi(\sigma,o) b_1\left(\sigma+a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma,a,P_t^{(1)}(a)\right)-L_1\left(\sigma,a,P_t^{(2)}(a)\right)\right]d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right)-b_1\left(\sigma+a, P_t^{(2)}(a)\right)\right| \le B_1 \,/\!/\, P_t^{(1)}-P_t^{(2)} /\!/ \text{ for } a\,\varepsilon[o,t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \le \beta_1 \Phi_1(0)||P_t^{(1)}-P_t^{(2)}||_t^{*}+\Phi_1(0)\beta_1 d_1||P_t^{(1)}-P_t^{(2)}||_t^{*a},$$

$$|R_2(a)|=[\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a))-b_1(a-\sigma, P_t^{(2)}(a))\}M_1(\sigma, a, P_t^{(1)}(a))$$

$$+b_1(a,-\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a))-M_1(a-\sigma, P_t^{(2)}(a))\}]d\sigma]$$

$$+b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a))-M_1(a-\sigma, P_t^{(2)}(a))\}]d\sigma]$$

$$\le \int_0^a [\beta_1^{*}\Phi_1(0)\, e^{\delta 1^a}\beta_1 \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}+\beta_1 d_1 \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}(a-o)]do$$

$$\le \beta_1^{*}\Phi_1(0)\, e^{\delta 1^a} \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\{\beta_1 a+d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a)\right|$$

$$\le \int_o^a \left| B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma)\right| B_1^{*d}\sigma$$

$$+\Phi_1(0) /\!/ P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\left\{B_1+B_2 d_1 a+B_1^{*}\left(B_2 a+d_1 a^2/2\right)e^{\delta 1^a}\right\}$$

$$\le \beta_1^{*}\int_o^a \left| B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma)\right| do+\left(r^{*}+/\!/\Phi_t /\!/_{t*}\right)\| P_t^{(1)}-P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a)\to \beta_1$ as $a\to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a)\right| \le \left(r^{*}+/\!/\Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))\text{-}b_1(a\text{-}\sigma, P_t^{(1)}(a))\, M_1(\sigma, a, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)}\text{-}P_t^{(2)} /\!/_{t*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right) /\!/ \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^{*})$$

As $t^{*} \to o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^{*}, P_t^{*}$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp\, P_2^{*}(a)}{da} + f_i(\mathrm{a}, P_1^{*}, P_2^{*})\, P_i^{*}(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^{*} = \int_o^{\infty} P_i^{*}(a)da, i = 1,\ 2,$$

$$(3,3)\ P_i^{*} = \int_o^{\infty} b_i(a,\ P_1^{*},\ P_2^{*}) P_i^{*}(a)da,\ i = 1.\ \ 2$$

All solutions of (3.1)are of the from

$$(3.4)\ P_i^{*}(a) = P_i^{*}(o) \exp\left[ - \int_o^{\infty} f_i(s, P_1^{*}, P_2^{*})ds \right], i = 1, \quad 2, a \geq 0,$$

Which together with (3.3) lead to

$$1=\int_o^{\infty} b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^{\infty} b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $$F_{i.}\ (P_1,P_2) = \int_o^{\infty} b_i(a,P_1,P_2)\exp\left[-\int_o^{\infty} f_i\,(s,P_1,P_2)ds\right]da,$$

$$i = 1,\ 2, \quad P_1 \geq 0, \quad P_2 \geq 0,$$

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_5$. *there exists a positive constant c for which*

$$F_1(C,0) < F_{2.}(C,0),$$

(3.8) $$F_1(0.C) < F_{2.}(0,C),$$

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $$F_1(P_1^*,P_2^*) = 1 = F_2(P_1^*,P_2^*)$$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $$P_2^*(a) = [P_2^*\exp[-f_i\,(s,P_1^*,P_2^*)ds])/(\int_o^{\infty}\exp[-\int_o^a f_i\,(s,P_1^*,P_2^*)ds]da),$$

$$i = 1,\ 2, \quad a \geq 0,$$

Proof Consider the elements of the surfaces defined by

$$Z= F_1(P_1,P_2), z = F_2(P_1,P_2),\ P_1 \geq 0, P_2 \geq 0,$$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i/\partial P_j < 0\ and$ , by ($A_2$), $\partial b_i/\partial P_j \leq 0$ for all

$$P_1 \geq 0, P_2 \geq 0, \quad i = 1,\ 2,$$

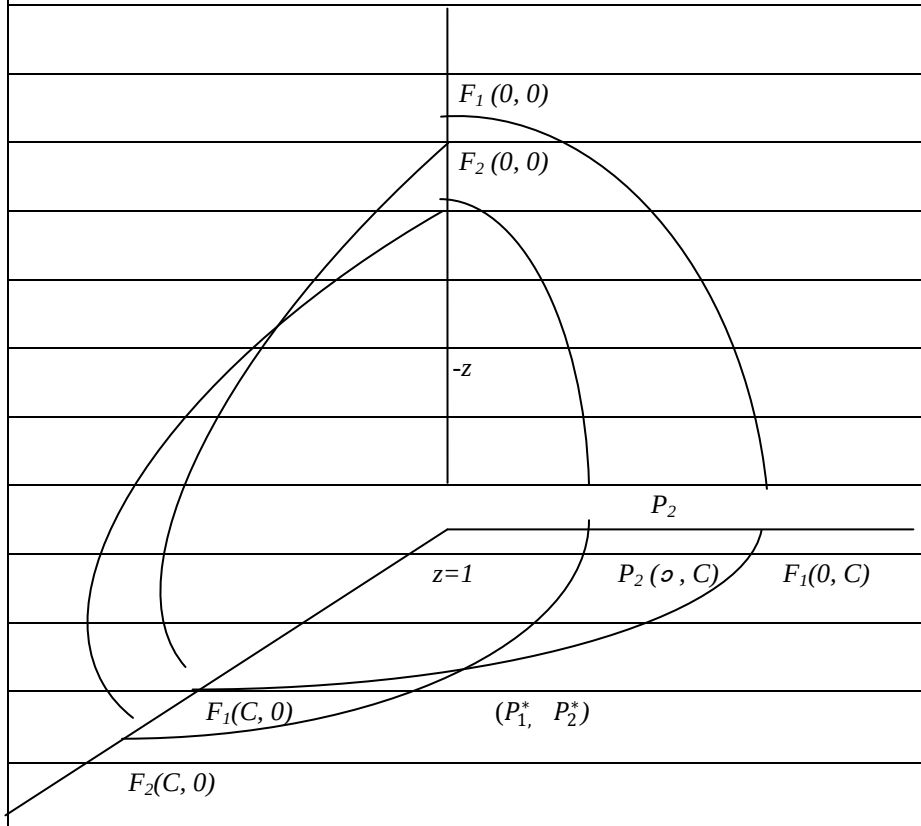
$j = 1,\ 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= 1, $P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$$F_1(P_1, P_2 = 1 \; and \; F_2(P_1, P_2) = 1, \; F_1(P_1, P_2 \geq 0,$$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

$$(3.11) \quad \begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*}\, p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square 1}^{2} \frac{\partial b_i}{\partial p_{\square}^*}\, p_{i\square}(t)\right\} da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_{\square}(a)\exp[\lambda t], \quad \square(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty \square(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d\square}{da} + \lambda\square = -f_i\left(a, p_1^*, p_2^*\right)\square(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*}\, p_{i\square}^*\right\},$$

(3.15)

$$\square(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\square(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*}\, p_{i\square}^*\right\} da,$$

Where $i = 1, 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \quad p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\Phi_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\Phi_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\Phi_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\Phi_i(a) = \{\Phi_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_j\right\} e^{\lambda s} ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\Phi_i(a)\left[p_i^*(a)/p_i^*(0)c_i \exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_j\right\} da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a}\left\{\int_0^a \left(\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_j\right) e^{\lambda s} ds\right\} da\right]$$

$$- p_i^*(a) e^{-\lambda a}\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_j\right\} e^{\lambda s} ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\Phi_1(a)$ and $\Phi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^* \exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^* \exp(-\lambda\tau) + (\lambda - A_{22})p_2^* = g_{21}(\lambda)p_1^* \exp(-\lambda\tau) + g_{22}(\lambda)p_2^*$$

Where

(3.22) $A_{ij} = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_j^*} - \frac{\partial f_i}{\partial p_j^*}\right]da,$

(3.23) $g_{ij}(\lambda) = [p_i^*(0)c_i]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\,da\right\}$
$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_j^*} - b_i(a, p_1^*, p_2^*)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)\right]da\right\}$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)da,\quad i, j = 1, 2,$$

(3.24) $\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*),\quad i, = 1, 2.$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix}\lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})\,e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda))\end{bmatrix} = 0$

Or equivalently

(3.25) $\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = [A_{21}\,g_{12}(\lambda) + A_{12}\,g_{21}(\lambda) + g_{12}(\lambda)\,g_{21}(\lambda)]\exp[-2\lambda\tau]$

$-A_{11}\,g_{22}(\lambda) - A_{22}\,g_{11}(\lambda) - g_{11}(\lambda)\,g_{22}(\lambda)$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i, = 1, 2.)\ (see(A_1)\ and\ (A_2))$;

$A_7.\ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\},$

$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|$ ,

$\| S_2(0) \| = |A_{11}\,g_{12}(0)| + |A_{12}\,g_{21}(0)| + |g_{12}(0)\,g_{21}(0)|$

$+ |A_{11}\,g_{22}(0)| + |A_{22}\,g_{11}(0)| + |g_{11}(0)\,g_{22}(0)|$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i(a, t) = \phi_i(a) \exp[\lambda t] \to 0$ as $t \to \infty$ for $i = 1, 2$ and $a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda$ $\mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$

$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \left|\right.$

$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$

$\geq [(|\lambda| - P)^2 + (q)]^2$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \{4|(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| | - |A_{11} + A_{22} + \| S_1(0) \| |^2\}/2,$

It will now follow, from (A) and (3.26), $F_1(\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder

concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power

obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy

withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation

from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## SECTION A(3)

## Nondeterministic Quantum Communication Complexity

### INTRODUCTION—VARIABLES USED

**Nondeterministic quantum communication complexity: the cyclic equality game and iterated matrix multiplication Harry Buhrman, Matthias Christandl, Jeroen Zuiddam (Submitted on 11 Mar 2016 (v1), last revised 2 Apr 2016 (this version, v2))**

(1) **Harry Buhrman, Matthias Christandl, Jeroen Zuiddam** study nondeterministic multiparty quantum communication with (e&eb) a quantum generalization of broadcasts.
(2) Authors show that, with number-in-hand classical inputs, the communication complexity of (e) a Boolean function in this communication model equals (=) the logarithm of (e) the support rank of (e) the corresponding tensor, whereas (e) the approximation complexity in (eb) this model equals (=)the logarithm of the border support rank.
(3) This characterisation allow (eb)s us to prove a log-rank conjecture posed by (e) Villagra et al. for (e) nondeterministic multiparty quantum communication with (e&eb) message-passing.
(4) The support rank characterization of (e) the communication model connects quantum communication complexity intimately to (e&eb) the theory of asymptotic entanglement transformation and (e&eb) algebraic complexity theory.
(5) In this context, authors introduce (eb) the graph wise equality problem. For a cycle graph, the complexity of this communication problem is closely related to (e&eb) the complexity of the computational problem of multiplying matrices, or more precisely, it equals (=) the logarithm of the asymptotic support rank of (e) the iterated matrix multiplication tensor.

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| (6) Authors employ Strassen's laser method to show that (eb) asymptotically there exist nontrivial protocols for (e) every odd-player cyclic equality problem.<br>**(7)** They exhibit an efficient protocol for (e) the 5-player problem for small inputs, and authors show how Young flattenings yield (eb) nontrivial complexity lower bounds. MSC classes: 68Q17, 15A69, 16Z05, 68Q12, 81P45 ACM classes: E.4Cite as: arXiv: 1603.03757 [quant-ph] (or arXiv: 1603.03757v2 [quant-ph] for this version) **Nondeterministic quantum communication complexity: the cyclic equality game and iterated matrix multiplication Harry Buhrman, Matthias Christandl, Jeroen Zuiddam (Submitted on 11 Mar 2016 (v1), last revised 2 Apr 2016 (this version, v2))** | |
| **NOTATION** | |
| **Module One**<br>**Harry Buhrman, Matthias Christandl, Jeroen Zuiddam** study nondeterministic multiparty quantum communication with (e&eb) a quantum generalization of broadcasts | |
| $G_{13}$ : Category one of **nondeterministic multiparty quantum communication;** quantum generalization of broadcasts<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of quantum generalization of broadcasts **;nondeterministic multiparty quantum communication**<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>Authors show that, with number-in-hand classical inputs, the communication complexity of (e) a Boolean function in this communication model equals (=) the logarithm of (e) the support rank of (e) the corresponding tensor, whereas (e) the approximation complexity in (eb) this model equals (=)the logarithm of the border support rank | |
| $G_{16}$ : Category one of Boolean function in this communication model equals (=) the logarithm of (e) the support rank of (e) the corresponding tensor, whereas (e) the approximation complexity in (eb) this model equals (=)the logarithm of the border support rank<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of number-in-hand classical inputs, the communication complexity<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three** | |

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| Authors show that, with number-in-hand classical inputs, the communication complexity of a Boolean function in this communication model equals (=) the logarithm of (e) the support rank of (e) the corresponding tensor, whereas (e) the approximation complexity in (eb) this model equals (=)the logarithm of the border support rank | |
| $G_{20}$ : Category one of number-in-hand classical inputs, the communication complexity of a Boolean function in this communication model<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of logarithm of (e) the support rank of (e) the corresponding tensor, whereas (e) the approximation complexity in (eb) this model equals (=)the logarithm of the border support rank<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Authors show that, with number-in-hand classical inputs, the communication complexity of a Boolean function in this communication model equals the logarithm of the support rank of (e) the corresponding tensor, whereas (e) the approximation complexity in (eb) this model equals (=)the logarithm of the border support rank | |
| $G_{24}$ : Category one of corresponding tensor, whereas (e) the approximation complexity in (eb) this model equals (=)the logarithm of the border support rank<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of number-in-hand classical inputs, the communication complexity of a Boolean function in this communication model equals the logarithm of the support rank<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Authors show that, with number-in-hand classical inputs, the communication complexity of a Boolean function in this communication model equals the logarithm of the support rank of the corresponding tensor, whereas (e) the approximation complexity in (eb) this model equals (=)the logarithm of the border support rank | |
| $G_{28}$ : Category one of approximation complexity in (eb) this model equals (=)the logarithm of the border support rank<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |

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| $T_{28}$ : Category one of number-in-hand classical inputs, the communication complexity of a Boolean function in this communication model equals the logarithm of the support rank of the corresponding tensor<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Authors show that, with number-in-hand classical inputs, the communication complexity of a Boolean function in this communication model equals the logarithm of the support rank of the corresponding tensor, whereas the approximation complexity in this model equals (=)the logarithm of the border support rank | |
| $G_{32}$ : Category one of number-in-hand classical inputs, the communication complexity of a Boolean function in this communication model equals the logarithm of the support rank of the corresponding tensor, whereas the approximation complexity in this model<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of logarithm of the border support rank<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>This characterisation allow (eb)s us to prove a log-rank conjecture posed by (e) Villagra et al. for (e) nondeterministic multiparty quantum communication with (e&eb) message-passing | |
| $G_{36}$ : Category one of characterisation<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of prove a log-rank conjecture posed by (e) Villagra et al. for (e) nondeterministic multiparty quantum communication with (e&eb) message-passing<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>This characterisation allows us to prove a log-rank conjecture posed by Villagra et al. for nondeterministic multiparty quantum communication with (e&eb) message-passing | |

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| $G_{40}$ : Category one of **characterisation allows us to prove a log-rank conjecture posed by Villagra et al. for nondeterministic multiparty quantum communication;** message-passing<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of message-passing **;characterisation allows us to prove a log-rank conjecture posed by Villagra et al. for nondeterministic multiparty quantum communication**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The support rank characterization of the communication model connects quantum communication complexity intimately to (e&eb) the theory of asymptotic entanglement transformation and (e&eb) algebraic complexity theory | |
| $G_{44}$ : Category one of **support rank characterization of the communication model connects quantum communication complexity;** theory of asymptotic entanglement transformation and (e&eb) algebraic complexity theory<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of theory of asymptotic entanglement transformation and (e&eb) algebraic complexity theory **;support rank characterization of the communication model connects quantum communication complexity**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION A(3)-2**<br><br>**Complexity Of This Communication Problem** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Nondeterministic quantum communication complexity: the cyclic equality game and iterated matrix multiplication Harry Buhrman, Matthias Christandl, Jeroen Zuiddam (Submitted on 11 Mar 2016 (v1), last revised 2 Apr 2016 (this version, v2))**<br><br>(1) In this context, authors introduce (eb) the graph wise equality problem. For a cycle graph, the | |

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| complexity of this communication problem is closely related to (e&eb) the complexity of the computational problem of multiplying matrices, or more precisely, it equals (=) the logarithm of the asymptotic support rank of (e) the iterated matrix multiplication tensor.<br>**(2)** Authors employ Strassen's laser method to show that (eb) asymptotically there exist nontrivial protocols for (e) every odd-player cyclic equality problem.<br>**(3)** They exhibit an efficient protocol for (e) the 5-player problem for small inputs, and authors show how Young flattenings yield (eb) nontrivial complexity lower bounds. MSC classes: 68Q17, 15A69, 16Z05, 68Q12, 81P45 ACM classes: E.4Cite as: arXiv: 1603.03757 [quant-ph] (or arXiv: 1603.03757v2 [quant-ph] for this version) **Nondeterministic quantum communication complexity: the cyclic equality game and iterated matrix multiplication Harry Buhrman, Matthias Christandl, Jeroen Zuiddam (Submitted on 11 Mar 2016 (v1), last revised 2 Apr 2016 (this version, v2))** | |
| **NOTATION** | |
| **Module One**<br>The support rank characterization of the communication model connects quantum communication complexity intimately to the theory of asymptotic entanglement transformation and (e&eb) algebraic complexity theory | |
| $G_{13}$ : Category one of **rank characterization of the communication model connects quantum communication complexity intimately to the theory of asymptotic entanglement transformation;** algebraic complexity theory<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of algebraic complexity theory **;rank characterization of the communication model connects quantum communication complexity intimately to the theory of asymptotic entanglement transformation**<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>In this context, authors introduce (eb) the graph wise equality problem.<br>the complexity for a cycle graph of this communication problem is closely related to (e&eb) the complexity of the computational problem of multiplying matrices, or more precisely, it equals (=) the logarithm of the asymptotic support rank of (e) the iterated matrix multiplication tensor | |
| $G_{16}$ : Category one of **complexity for a cycle graph of this communication problem;** complexity of the computational problem of multiplying matrices, or more precisely, it equals (=) the logarithm of the asymptotic support rank of (e) the iterated matrix multiplication tensor<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of complexity of the computational problem of multiplying matrices, or more precisely, it equals (=) the logarithm of the asymptotic support rank of (e) the iterated matrix multiplication tensor | |

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| **;complexity for a cycle graph of this communication problem**<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>the complexity for a cycle graph of this communication problem is closely related to the complexity of the computational problem of multiplying matrices, or more precisely, it equals (=) the logarithm of the asymptotic support rank of (e) the iterated matrix multiplication tensor | |
| $G_{20}$ : Category one of complexity for a cycle graph of this communication problem is closely related to the complexity of the computational problem of multiplying matrices, or more precisely<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of logarithm of the asymptotic support rank of (e) the iterated matrix multiplication tensor<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>complexity for a cycle graph of this communication problem is closely related to the complexity of the computational problem of multiplying matrices, or more precisely, it equals the logarithm of the asymptotic support rank of (e) the iterated matrix multiplication tensor | |
| $G_{24}$ : Category one of iterated matrix multiplication tensor<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of complexity for a cycle graph of this communication problem is closely related to the complexity of the computational problem of multiplying matrices, or more precisely, it equals the logarithm of the asymptotic support rank<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Authors employ Strassen's laser method to show that (eb) asymptotically there exist nontrivial protocols for (e) every odd-player cyclic equality problem | |
| $G_{28}$ : Category one of Strassen's laser method<br><br>$G_{29}$ : Category two of SAS | |

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| $G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of asymptotically there exist nontrivial protocols for (e) every odd-player cyclic equality problem<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Authors employ Strassen's laser method to show that asymptotically there exist nontrivial protocols for (e) every odd-player cyclic equality problem | |
| $G_{32}$ : Category one of every odd-player cyclic equality problem<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of Strassen's laser method to show that asymptotically there exist nontrivial protocols<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>nontrivial protocols for every odd-player cyclic equality problem exhibit an efficient protocol for (e) the 5-player problem for small inputs, and authors show how Young flattenings yield (eb) nontrivial complexity lower bounds.<br><br>MSC classes: 68Q17, 15A69, 16Z05, 68Q12, 81P45 ACM classes: E.4Cite as: arXiv: 1603.03757 [quant-ph] (or arXiv: 1603.03757v2 [quant-ph] for this version) **Nondeterministic quantum communication complexity: the cyclic equality game and iterated matrix multiplication Harry Buhrman, Matthias Christandl, Jeroen Zuiddam (Submitted on 11 Mar 2016 (v1), last revised 2 Apr 2016 (this version, v2))** | |
| $G_{36}$ : Category one of nontrivial protocols for every odd-player cyclic equality problem<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of efficient protocol for (e) the 5-player problem for small inputs, and authors show how Young flattenings yield (eb) nontrivial complexity lower bounds<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |

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| **Module eight**<br><br>Nontrivial protocols for every odd-player cyclic equality problem exhibit an efficient protocol for (e) the 5-player problem for small inputs, and authors show how Young flattenings yield (eb) nontrivial complexity lower bounds. | |
| $G_{40}$ : Category one of5-player problem for small inputs, and authors show how Young flattenings yield (eb) nontrivial complexity lower bounds.<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of nontrivial protocols for every odd-player cyclic equality problem exhibit an efficient protocol<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Nontrivial protocols for every odd-player cyclic equality problem exhibit an efficient protocol for the 5-player problem for small inputs, and authors show how Young flattenings yield (eb) nontrivial complexity lower bounds. | |
| $G_{44}$ : Category one of Nontrivial protocols for every odd-player cyclic equality problem exhibit an efficient protocol for the 5-player problem for small inputs, and authors show how Young flattenings<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of nontrivial complexity lower bounds.<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION A(4)**<br><br>**Quantum Algorithms For Finding Constant-Sized Sub-Hyper Graphs** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Theoretical Computer Science Volume 609, Part 3, 4 January 2016, Pages 569–582 Computing and Combinatorics Conference Quantum algorithms for finding constant-sized sub-hyper graphs** | |

**François Le Galla, , Harumichi Nishimurab, , Seiichiro Tanic**

(1) Authors develop a general framework to construct (eb) quantum algorithms that detect if a 3-uniform hypergraph given as input contains (e) a sub-hypergraph isomorphic to (e&eb) a prespecified constant-sized hypergraph.
(2) This framework is based on (e) the concept of nested quantum walks recently proposed by Jeffery, Kothari and Magniez (2013) [12], and extends the (e&eb) methodology designed by Lee, Magniez and Santha (2013) [18] for similar problems over graphs.
(3) As applications, authors obtain a quantum algorithm for (e) finding a 4-clique in a 3-uniform hypergraph on n vertices with (e&eb) query complexity O(n1.883), and a quantum algorithm for determining if (e) a ternary operator over a set of size n is associative with (e&eb) query complexity O(n2.113). Keywords Quantum algorithms; Hypergraphs; Quantum walk

**Quantum algorithms for Gibbs sampling and hitting-time estimation Anirban Narayan Chowdhury, Rolando D. Somma (Submitted on 9 Mar 2016)**

(4) **Anirban Narayan Chowdhury, Rolando D. Somma** present quantum algorithms for (e) solving two problems regarding stochastic processes.
(5) The first algorithm prepares (eb) the thermal Gibbs state of a quantum system and runs in time almost linear in (eb) Nβ/Z−−−−−√ and polynomial in log (1/ϵ), where (e) N is the Hilbert space dimension, β is (=) the inverse temperature, Z is (=) the partition function, and ϵ is (=) the desired precision of (e) the output state.
(6) Quantum algorithm exponentially improves (eb) the dependence on 1/ϵ and quadratically improves (eb) the dependence on β of (e) known quantum algorithms for this problem.
(7) The second algorithm estimates the hitting time of (e) a Markov chain. For a sparse stochastic matrix P, it runs in time almost linear in (eb)1/ (ϵΔ3/2), where (e) ϵ is the absolute precision in the estimation and Δ is a parameter determined by (e) P, and whose inverse is (=) an upper bound of the hitting time.
(8) Quantum algorithm quadratically improves (eb) the dependence on 1/ϵ and 1/Δ of the analog classical algorithm for (e) hitting-time estimation.
**(9)** Both algorithms use tools recently developed in the context of Hamiltonian simulation, spectral gap amplification, and solving linear systems of equations. Subjects: Quantum Physics (quant-ph) Report number: LA-UR-16-21218 Cite as: arXiv: 1603.02940 [quant-ph] (or arXiv: 1603.02940v1 [quant-ph] for this version)

**NOTATION**

**Module One**

Authors develop a general framework to construct (eb) quantum algorithms that detect if a 3-uniform hypergraph given as input contains (e) a sub-hypergraph isomorphic to (e&eb) a prespecified constant-sized hypergraph

$G_{13}$ : Category one of general framework to construct

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of quantum algorithms that detect if a 3-uniform hypergraph given as input contains (e) a sub-hypergraph isomorphic to (e&eb) a prespecified constant-sized hypergraph

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| $T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>Authors develop a general framework to construct quantum algorithms that detect if a 3-uniform hypergraph given as input contains (e) a sub-hypergraph isomorphic to (e&eb) a prespecified constant-sized hypergraph | |
| $G_{16}$ : Category one of sub-hypergraph isomorphic to (e&eb) a prespecified constant-sized hypergraph<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of general framework to construct quantum algorithms that detect if a 3-uniform hypergraph given as input contains<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>Authors develop a general framework to construct quantum algorithms that detect if a 3-uniform hypergraph given as input contains sub-hypergraph isomorphic to (e&eb) a prespecified constant-sized hypergraph | |
| $G_{20}$ : Category one of **general framework to construct quantum algorithms that detect if a 3-uniform hypergraph given as input contains sub-hypergraph isomorphic;** prespecified constant-sized hypergraph<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of prespecified constant-sized hypergraph **;general framework to construct quantum algorithms that detect if a 3-uniform hypergraph given as input contains sub-hypergraph isomorphic**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>This framework is based on (e) the concept of nested quantum walks recently proposed by Jeffery, Kothari and Magniez (2013) [12], and extends the (e&eb) methodology designed by Lee, Magniez and Santha (2013) [18] for similar problems over graphs | |
| $G_{24}$ : Category one of concept of nested quantum walks recently proposed by Jeffery, Kothari and Magniez (2013) [12], and extends the (e&eb) methodology designed by Lee, Magniez and Santha (2013) [18] for similar problems over graphs<br><br>$G_{25}$ : Category two of SAS | |

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| $G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of framework<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>This framework is based on the concept of nested quantum walks recently proposed by Jeffery, Kothari and Magniez (2013) [12], and extends the (e&eb) methodology designed by Lee, Magniez and Santha (2013) [18] for similar problems over graphs | |
| $G_{28}$ : Category one of **framework is based on the concept of nested quantum walks recently proposed by Jeffery, Kothari and Magniez (2013) [12], ;** methodology designed by Lee, Magniez and Santha (2013) [18] for similar problems over graphs<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of methodology designed by Lee, Magniez and Santha (2013) [18] for similar problems over graphs; **framework is based on the concept of nested quantum walks recently proposed by Jeffery, Kothari and Magniez (2013) [12]**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>As applications, authors obtain a quantum algorithm for (e) finding a 4-clique in a 3-uniform hypergraph on n vertices with (e&eb) query complexity O(n1.883), and a quantum algorithm for determining if (e) a ternary operator over a set of size n is associative with (e&eb) query complexity O(n2.113). Keywords Quantum algorithms; Hypergraphs; Quantum walk | |
| $G_{32}$ : Category one of finding a 4-clique in a 3-uniform hypergraph on n vertices with (e&eb) query complexity O(n1.883), and a quantum algorithm for determining if (e) a ternary operator over a set of size n is associative with (e&eb) query complexity O(n2.113). Keywords Quantum algorithms; Hypergraphs; Quantum walk<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of quantum algorithm<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |

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| **Module seven**<br><br>As applications, authors obtain a quantum algorithm for finding a 4-clique in a 3-uniform hypergraph on n vertices with (e&eb) query complexity O(n1.883), and a quantum algorithm for determining if (e) a ternary operator over a set of size n is associative with (e&eb) query complexity O(n2.113). Keywords Quantum algorithms; Hypergraphs; Quantum walk | |
| $G_{36}$ : Category one of quantum algorithm for finding a 4-clique in a 3-uniform hypergraph on n vertices; **query complexity O(n1.883), and a quantum algorithm for determining if (e) a ternary operator over a set of size n is associative with (e&eb) query complexity O(n2.113). Keywords Quantum algorithms; Hypergraphs; Quantum walk;**<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of **query complexity O(n1.883), and a quantum algorithm for determining if (e) a ternary operator over a set of size n is associative with (e&eb) query complexity O(n2.113). Keywords Quantum algorithms;** Hypergraphs; Quantum walk; quantum algorithm for finding a 4-clique in a 3-uniform hypergraph on n vertices<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>As applications, authors obtain a quantum algorithm for finding a 4-clique in a 3-uniform hypergraph on n vertices with query complexity O(n1.883), and a quantum algorithm for determining if (e) a ternary operator over a set of size n is associative with (e&eb) query complexity O(n2.113). Keywords Quantum algorithms; Hypergraphs; Quantum walk | |
| $G_{40}$ : Category one of ternary operator over a set of size n is associative with (e&eb) query complexity O(n2.113).<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of a quantum algorithm for finding a 4-clique in a 3-uniform hypergraph on n vertices with query complexity O(n1.883), and a quantum algorithm for determining<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>As applications, authors obtain a quantum algorithm for finding a 4-clique in a 3-uniform hypergraph on n vertices with query complexity O(n1.883), and a quantum algorithm for determining if a ternary operator | |

over a set of size n is associative with (e&eb) query complexity O(n2.113).

Keywords Quantum algorithms; Hypergraphs; Quantum walk

$G_{44}$ : Category one of **quantum algorithm for finding a 4-clique in a 3-uniform hypergraph on n vertices with query complexity O(n1.883), and a quantum algorithm for determining if a ternary operator over a set of size n is associative;** query complexity O(n2.113).

$G_{45}$ : Category two of SAS

$G_{46}$ : Category three of SAS

$T_{44}$ : Category one of query complexity O(n2.113).;**quantum algorithm for finding a 4-clique in a 3-uniform hypergraph on n vertices with query complexity O(n1.883), and a quantum algorithm for determining if a ternary operator over a set of size n is associative**

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

**The Coefficients:**

$(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:
$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$, $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$
$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,
$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$
$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$
$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$
$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$

are Accentuation coefficients
$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$
$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$
$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}\ (a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,
$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$
$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,
$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,
$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,
are Dissipation coefficients

**Module Numbered One**

**The differential system of this model is now (Module Numbered one)**

$$\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right]G_{13} \quad (1)$$

$$\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14}, t)\right]G_{14} \quad (2)$$

$$\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14}, t)\right]G_{15} \quad (3)$$

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| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G,t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G,t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14},t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G,t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17},t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17},t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17},t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17},t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}),t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{26}$ | 24 |

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| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}\big((G_{27}),t\big) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}\big((G_{31}),t\big)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}\big((G_{31}),t\big)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}\big((G_{31}),t\big)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}\big((G_{31}),t\big) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}\big((G_{35}),t\big)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}\big((G_{35}),t\big)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}\big((G_{35}),t\big)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}\big((G_{39}),t\big)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}\big((G_{39}),t\big)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}\big((G_{39}),t\big)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |

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| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}\left((G_{43}),t\right)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}\left((G_{43}),t\right)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}\left((G_{43}),t\right)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}\left((G_{47}),t\right) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{array}{ccc} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient | |

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| for category 1, 2 and 3 $\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3 $\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3 $\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\boxed{-(b_{15}^{''})^{(1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | *60* |
| Where $\boxed{-(b_{13}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |

$$\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a'_{16})^{(2)}\boxed{+(a''_{16})^{(2)}(T_{17},t)}\ \boxed{+(a''_{13})^{(1,1,)}(T_{14},t)}\ \boxed{+(a''_{20})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a''_{28})^{(5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a''_{32})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{36})^{(7,7,7)}(T_{37},t)}\ \boxed{+(a''_{40})^{(8,8,8)}(T_{41},t)}\ \boxed{+(a''_{44})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{16} \quad (61)$$

$$\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a'_{17})^{(2)}\boxed{+(a''_{17})^{(2)}(T_{17},t)}\ \boxed{+(a''_{14})^{(1,1,)}(T_{14},t)}\ \boxed{+(a''_{21})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a''_{29})^{(5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a''_{33})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{37})^{(7,7,7)}(T_{37},t)}\ \boxed{+(a''_{41})^{(8,8,8)}(T_{41},t)}\ \boxed{+(a''_{45})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{17} \quad (62)$$

$$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a'_{18})^{(2)}\boxed{+(a''_{18})^{(2)}(T_{17},t)}\ \boxed{+(a''_{15})^{(1,1,)}(T_{14},t)}\ \boxed{+(a''_{22})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a''_{30})^{(5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a''_{34})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7)}(T_{37},t)}\ \boxed{+(a''_{42})^{(8,8,8)}(T_{41},t)}\ \boxed{+(a''_{46})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{18} \quad (63)$$

Where $\boxed{+(a''_{16})^{(2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2)}(T_{17},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,)}(T_{14},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{36})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7)}(T_{37},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{40})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8)}(T_{41},t)}$ are eight augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{44})^{(9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9,9)}(T_{45},t)}$ are ninth augmentation coefficient for category 1, 2 and 3

$$\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b'_{16})^{(2)}\boxed{-(b''_{16})^{(2)}(G_{19},t)}\ \boxed{-(b''_{13})^{(1,1,)}(G,t)}\ \boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4)}(G_{27},t)}\ \boxed{-(b''_{28})^{(5,5,5,5,5)}(G_{31},t)}\ \boxed{-(b''_{32})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)}\ \boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)}\ \boxed{-(b''_{44})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{16} \quad (64)$$

$$\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b'_{17})^{(2)}\boxed{-(b''_{17})^{(2)}(G_{19},t)}\ \boxed{-(b''_{14})^{(1,1,)}(G,t)}\ \boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4)}(G_{27},t)}\ \boxed{-(b''_{29})^{(5,5,5,5,5)}(G_{31},t)}\ \boxed{-(b''_{33})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)}\ \boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)}\ \boxed{-(b''_{45})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{17} \quad (65)$$

$$\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b'_{18})^{(2)} & \boxed{-(b''_{18})^{(2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,)}(G,t)} & \boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{18} \quad (66)$$

where $\boxed{-(b''_{16})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{17})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{18})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,)}(G,t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{44})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2 and 3

$$\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a'_{20})^{(3)} & \boxed{+(a''_{20})^{(3)}(T_{21},t)} & \boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{20} \quad (67)$$

$$\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a'_{21})^{(3)} & \boxed{+(a''_{21})^{(3)}(T_{21},t)} & \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{21} \quad (68)$$

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a'_{22})^{(3)} & \boxed{+(a''_{22})^{(3)}(T_{21},t)} & \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{22} \quad (69)$$

$\boxed{+(a''_{20})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)}$ are third augmentation coefficients

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| for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)} & -(b_{20}^{''})^{(3)}(G_{23},t) & -(b_{16}^{''})^{(2,2,2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,1,)}(G,t) \\ & -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ & -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{20}$ | *70* |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)} & -(b_{21}^{''})^{(3)}(G_{23},t) & -(b_{17}^{''})^{(2,2,2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,1,)}(G,t) \\ & -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ & -(b_{37}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21}$ | *71* |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b_{22}^{'})^{(3)} & -(b_{22}^{''})^{(3)}(G_{23},t) & -(b_{18}^{''})^{(2,2,2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,1,)}(G,t) \\ & -(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ & -(b_{38}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{22}$ | *72* |
| $-(b_{20}^{''})^{(3)}(G_{23},t)$, $-(b_{21}^{''})^{(3)}(G_{23},t)$, $-(b_{22}^{''})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*<br>$-(b_{16}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*<br>$-(b_{13}^{''})^{(1,1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*<br>$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 | |

$\boxed{-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{46}^{''})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt}=(a_{24})^{(4)}G_{25}-\begin{bmatrix}(a_{24}^{'})^{(4)}\boxed{+(a_{24}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)}\\ \boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt}=(a_{25})^{(4)}G_{24}-\begin{bmatrix}(a_{25}^{'})^{(4)}\boxed{+(a_{25}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6)}(T_{33},t)}\\ \boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt}=(a_{26})^{(4)}G_{25}-\begin{bmatrix}(a_{26}^{'})^{(4)}\boxed{+(a_{26}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)}\\ \boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{26} \quad (75)$$

$\boxed{(a_{24}^{''})^{(4)}(T_{25},t)}$, $\boxed{(a_{25}^{''})^{(4)}(T_{25},t)}$, $\boxed{(a_{26}^{''})^{(4)}(T_{25},t)}$ *are first augmentation coefficients category* 1, 2 3

$\boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)}$
*are fourth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)}$
*are fifth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)}$
*are sixth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)}$
*are seventh augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)}$
*are eighth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)}$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)}\boxed{-(b_{24}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,)}(G_{35},t)}\\ \boxed{-(b_{13}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)}\\ \boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}\end{bmatrix}T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b'_{25})^{(4)} & -(b''_{25})^{(4)}(G_{27},t) & -(b''_{29})^{(5,5,)}(G_{31},t) & -(b''_{33})^{(6,6,)}(G_{35},t) \\ & -(b''_{14})^{(1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b'_{26})^{(4)} & -(b''_{26})^{(4)}(G_{27},t) & -(b''_{30})^{(5,5,)}(G_{31},t) & -(b''_{34})^{(6,6,)}(G_{35},t) \\ & -(b''_{15})^{(1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{26} \quad (78)$$

Where $-(b''_{24})^{(4)}(G_{27},t)$, $-(b''_{25})^{(4)}(G_{27},t)$, $-(b''_{26})^{(4)}(G_{27},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b''_{28})^{(5,5,)}(G_{31},t)$, $-(b''_{29})^{(5,5,)}(G_{31},t)$, $-(b''_{30})^{(5,5,)}(G_{31},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b''_{32})^{(6,6,)}(G_{35},t)$, $-(b''_{33})^{(6,6,)}(G_{35},t)$, $-(b''_{34})^{(6,6,)}(G_{35},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b''_{13})^{(1,1,1,1)}(G,t)$, $-(b''_{14})^{(1,1,1,1)}(G,t)$, $-(b''_{15})^{(1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b''_{16})^{(2,2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b''_{20})^{(3,3,3,3)}(G_{23},t)$, $-(b''_{21})^{(3,3,3,3)}(G_{23},t)$, $-(b''_{22})^{(3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b''_{36})^{(7,7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)$, $-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b''_{40})^{(8,8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b''_{46})^{(9,9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a'_{28})^{(5)} & +(a''_{28})^{(5)}(T_{29},t) & +(a''_{24})^{(4,4,)}(T_{25},t) & +(a''_{32})^{(6,6,6)}(T_{33},t) \\ & +(a''_{13})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{28} \quad (79)$$

$$\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a'_{29})^{(5)} & +(a''_{29})^{(5)}(T_{29},t) & +(a''_{25})^{(4,4,)}(T_{25},t) & +(a''_{33})^{(6,6,6)}(T_{33},t) \\ & +(a''_{14})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{29} \quad (80)$$

$$\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a'_{30})^{(5)} & +(a''_{30})^{(5)}(T_{29},t) & +(a''_{26})^{(4,4,)}(T_{25},t) & +(a''_{34})^{(6,6,6)}(T_{33},t) \\ & +(a''_{15})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{30} \quad (81)$$

Where $+(a''_{28})^{(5)}(T_{29},t)$, $+(a''_{29})^{(5)}(T_{29},t)$, $+(a''_{30})^{(5)}(T_{29},t)$ are first augmentation coefficients for category 1, 2 and 3

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| $And\ \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)}\ ,\boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)}\ ,\boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)}\ are\ second\ augmentation\ coefficient\ for\ category\ 1,2\ and\ 3$ $\boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)}\ ,\boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)}\ ,\boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)}\ are\ third\ augmentation\ coefficient\ for\ category\ 1,2\ and\ 3$ $\boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)}\ ,\boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)}\ ,\boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3 $\boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)}\ ,\boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)}\ ,\boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation coefficients for category 1,2,and 3 $\boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)}\ ,\boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)}\ ,\boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)}$ are sixth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}\ ,\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}\ ,\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1,2, 3 $\boxed{+(a_{40}^{''})^{(8,8\ ,8,8,8,8)}(T_{41},t)}\ ,\boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}\ ,\boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$ are eighth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)}\ ,\boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)}\ ,\boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1,2, 3 | |
| $\frac{dT_{28}}{dt}=(b_{28})^{(5)}T_{29}-\left[\begin{array}{ccc}(b_{28}^{'})^{(5)}\boxed{-(b_{28}^{''})^{(5)}(G_{31},t)} & \boxed{-(b_{24}^{''})^{(4,4,)}(G_{27},t)} & \boxed{-(b_{32}^{''})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)}\end{array}\right]T_{28}$ | 82 |
| $\frac{dT_{29}}{dt}=(b_{29})^{(5)}T_{28}-\left[\begin{array}{ccc}(b_{29}^{'})^{(5)}\boxed{-(b_{29}^{''})^{(5)}(G_{31},t)} & \boxed{-(b_{25}^{''})^{(4,4,)}(G_{27},t)} & \boxed{-(b_{33}^{''})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)}\end{array}\right]T_{29}$ | 83 |
| $\frac{dT_{30}}{dt}=(b_{30})^{(5)}T_{29}-\left[\begin{array}{ccc}(b_{30}^{'})^{(5)}\boxed{-(b_{30}^{''})^{(5)}(G_{31},t)} & \boxed{-(b_{26}^{''})^{(4,4,)}(G_{27},t)} & \boxed{-(b_{34}^{''})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)}\end{array}\right]T_{30}$ | 84 |
| $where\ \boxed{-(b_{28}^{''})^{(5)}(G_{31},t)}\ ,\boxed{-(b_{29}^{''})^{(5)}(G_{31},t)}\ ,\boxed{-(b_{30}^{''})^{(5)}(G_{31},t)}\ are\ first\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{24}^{''})^{(4,4,)}(G_{27},t)}\ ,\boxed{-(b_{25}^{''})^{(4,4,)}(G_{27},t)}\ ,\boxed{-(b_{26}^{''})^{(4,4,)}(G_{27},t)}\ are\ second\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{32}^{''})^{(6,6,6)}(G_{35},t)}\ ,\boxed{-(b_{33}^{''})^{(6,6,6)}(G_{35},t)}\ ,\boxed{-(b_{34}^{''})^{(6,6,6)}(G_{35},t)}\ are\ third\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)}\ ,\boxed{-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)}\ ,\boxed{-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)}$ are fourth detrition coefficients for category 1,2, and 3 $\boxed{-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)}\ ,\boxed{-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)}\ ,\boxed{-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1,2, and 3 $\boxed{-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)}\ ,\boxed{-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)}\ ,\boxed{-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients | |

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|---|---|
| for category 1,2, and 3 $[-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)]$, $[-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)]$, $[-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)]$ are seventh detrition coefficients for category 1,2, and 3 $[-(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)]$, $[-(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)]$, $[-(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)]$ are eighth detrition coefficients for category 1,2, and 3 $[-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)]$, $[-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)]$, $[-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)]$ are ninth detrition coefficients for category 1,2, and 3 | |
| $\frac{dG_{32}}{dt}=(a_{32})^{(6)}G_{33}$ $-\left[\begin{matrix}(a_{32}^{'})^{(6)}[+(a_{32}^{''})^{(6)}(T_{33},t)][+(a_{28}^{''})^{(5,5,5)}(T_{29},t)][+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)]\\ [+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)][+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)][+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)]\\ [+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)][+(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)][+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)]\end{matrix}\right]G_{32}$ | *85* |
| $\frac{dG_{33}}{dt}=(a_{33})^{(6)}G_{32}-\left[\begin{matrix}(a_{33}^{'})^{(6)}[+(a_{33}^{''})^{(6)}(T_{33},t)][+(a_{29}^{''})^{(5,5,5)}(T_{29},t)][+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)]\\ [+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)][+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)][+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)]\\ [+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)][+(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)][+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)]\end{matrix}\right]G_{33}$ | *86* |
| $\frac{dG_{34}}{dt}=(a_{34})^{(6)}G_{33}-\left[\begin{matrix}(a_{34}^{'})^{(6)}[+(a_{34}^{''})^{(6)}(T_{33},t)][+(a_{30}^{''})^{(5,5,5)}(T_{29},t)][+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)]\\ [+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)][+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)][+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)]\\ [+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)][+(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)][+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)]\end{matrix}\right]G_{34}$ | *87* |
| $[+(a_{32}^{''})^{(6)}(T_{33},t)]$, $[+(a_{33}^{''})^{(6)}(T_{33},t)]$, $[+(a_{34}^{''})^{(6)}(T_{33},t)]$ *are first augmentation coefficients for category* 1, 2 *and* 3 $[+(a_{28}^{''})^{(5,5,5)}(T_{29},t)]$, $[+(a_{29}^{''})^{(5,5,5)}(T_{29},t)]$, $[+(a_{30}^{''})^{(5,5,5)}(T_{29},t)]$ *are second augmentation coefficients for category* 1, 2 *and* 3 $[+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)]$, $[+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)]$, $[+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)]$ *are third augmentation coefficients for category* 1, 2 *and* 3 $[+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)]$, $[+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)]$, $[+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)]$ - are fourth augmentation coefficients $[+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)]$, $[+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)]$, $[+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)]$ - fifth augmentation coefficients $[+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)]$, $[+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)]$, $[+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)]$ sixth augmentation coefficients $[+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)]$, $[+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)]$, $[+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)]$ seventh augmentation coefficients $[+(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)]$, $[+(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)]$, $[+(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)]$ Eighth augmentation coefficients $[+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)]$, $[+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)]$, $[+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)]$ ninth augmentation coefficients | |

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & \boxed{-(b''_{32})^{(6)}(G_{35},t)} & \boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)} & \\ \boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t)} & \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & \boxed{-(b''_{33})^{(6)}(G_{35},t)} & \boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)} & \\ \boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)} & \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & \boxed{-(b''_{34})^{(6)}(G_{35},t)} & \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)} & \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)} & \end{bmatrix} T_{34} \quad (90)$$

$\boxed{-(b''_{32})^{(6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6)}(G_{35},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)}$ *are second detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)}$ *are third detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a'_{36})^{(7)} & \boxed{+(a''_{36})^{(7)}(T_{37},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} & \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} & \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}')^{(7)}\boxed{+(a_{37}'')^{(7)}(T_{37},t)} & \boxed{+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}')^{(7)}\boxed{+(a_{38}'')^{(7)}(T_{37},t)} & \boxed{+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad (93)$$

Where $\boxed{(a_{36}'')^{(7)}(T_{37},t)}$, $\boxed{(a_{37}'')^{(7)}(T_{37},t)}$, $\boxed{(a_{38}'')^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{40}'')^{(8,8,8,8,8,8,8,8,)}(T_{41},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}')^{(7)}\boxed{-(b_{36}'')^{(7)}(G_{39},t)} & \boxed{-(b_{16}'')^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}'')^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}'')^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}'')^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}'')^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}'')^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}'')^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13} \quad (94)$$

$$\frac{dT_{37}}{dt} =$$

$$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b'_{37})^{(7)}\boxed{-(b''_{37})^{(7)}(G_{39},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} =$$

$$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b'_{38})^{(7)}\boxed{-(b''_{38})^{(7)}(G_{39},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a'_{40})^{(8)}\boxed{+(a''_{40})^{(8)}(T_{41},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a'_{41})^{(8)}\boxed{+(a''_{41})^{(8)}(T_{41},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a'_{42})^{(8)}\boxed{+(a''_{42})^{(8)}(T_{41},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{40})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b'_{40})^{(8)}\boxed{-(b''_{40})^{(8)}(G_{43},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b'_{41})^{(8)}\boxed{-(b''_{41})^{(8)}(G_{43},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b'_{42})^{(8)}\boxed{-(b''_{42})^{(8)}(G_{43},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} =$$

$$(b_{46})^{(9)}T_{45} - \begin{bmatrix} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

$(a_i^{''})^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$

$(b_i^{''})^{(1)}(G,t) \le (r_i)^{(1)} \le (b_i^{'})^{(1)} \le (\hat{B}_{13})^{(1)}$

$lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$ 98

$\lim_{G\to\infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$

**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :

Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$

| | |
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| They satisfy Lipschitz condition:<br>$\|(a_i'')^{(1)}(T_{14}', t) - (a_i'')^{(1)}(T_{14}, t)\| \leq (\hat{k}_{13})^{(1)}\|T_{14} - T_{14}'\|e^{-(\hat{M}_{13})^{(1)}t}$<br>$\|(b_i'')^{(1)}(G', t) - (b_i'')^{(1)}(G, t)\| < (\hat{k}_{13})^{(1)}\|G - G'\|e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(1)}(T_{14}', t)$ and $(a_i'')^{(1)}(T_{14}, t)$ . $(T_{14}', t)$ and $(T_{14}, t)$ are points belonging to the interval $[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}]$. It is to be noted that $(a_i'')^{(1)}(T_{14}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i'')^{(1)}(T_{14}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)},$ are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i')^{(1)}, (b_i)^{(1)}, (b_i')^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i, j = 16,17,18$ | |
| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17}, t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19}, t) \leq (r_i)^{(2)} \leq (b_i')^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty}(a_i'')^{(2)}(T_{17}, t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty}(b_i'')^{(2)}((G_{19}), t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ : | 106 |

| | |
|---|---|
| Where $\boxed{(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}}$ are positive constants and $\boxed{i = 16,17,18}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'}, t) - (a_i^{''})^{(2)}(T_{17}, t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}^{'}\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'}, t) - (b_i^{''})^{(2)}((G_{19}), t)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})^{'}\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'}, t)$ and $(a_i^{''})^{(2)}(T_{17}, t)$ . $(T_{17}^{'}, t)$ and $(T_{17}, t)$ are points belonging to the interval $\left[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}\right]$. It is to be noted that $(a_i^{''})^{(2)}(T_{17}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i^{''})^{(2)}(T_{17}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br><br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br><br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br><br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i^{'})^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i, j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(r_i)^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(3)}(T_{21}, t) = (p_i)^{(3)}$ | 113 |

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|---|---|
| $\lim_{G\to\infty}(b_i'')^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(3)}(T_{21}',t) - (a_i'')^{(3)}(T_{21},t)\rvert \le (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}'\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i'')^{(3)}(G_{23}',t) - (b_i'')^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}'\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(3)}(T_{21}',t)$ and$(a_i'')^{(3)}(T_{21},t)$ . $(T_{21}',t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i'')^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i'')^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i')^{(3)}, (b_i)^{(3)}, (b_i')^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i')^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i')^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i')^{(4)}, (a_i'')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (b_i'')^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i'')^{(4)}, (b_i'')^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i'')^{(4)}(T_{25},t) \le (p_i)^{(4)} \le (\hat{A}_{24})^{(4)}$<br><br>$(b_i'')^{(4)}\big((G_{27}),t\big) \le (r_i)^{(4)} \le (b_i')^{(4)} \le (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2\to\infty}(a_i'')^{(4)}(T_{25},t) = (p_i)^{(4)}$ | 118 |

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|---|---|
| $\lim_{G\to\infty}(b_i'')^{(4)}\left((G_{27}),t\right) = \ (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, \ (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24{,}25{,}26}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(4)}(T_{25}',t) - (a_i'')^{(4)}(T_{25},t)\rvert \le (\hat{k}_{24})^{(4)}\lvert T_{25} - T_{25}'\rvert e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\lvert(b_i'')^{(4)}((G_{27})',t) - (b_i'')^{(4)}\left((G_{27}),t\right)\rvert < (\hat{k}_{24})^{(4)}\lvert\lvert(G_{27}) - (G_{27})'\rvert\rvert e^{-(\hat{M}_{24})^{(4)}t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(4)}(T_{25}',t)$ and $(a_i'')^{(4)}(T_{25},t)$ . $(T_{25}',t)$ and $(T_{25},t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i'')^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i'')^{(4)}(T_{25},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (p_i)^{(4)}, \ (r_i)^{(4)}, i = 24{,}25{,}26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i')^{(4)} + \ (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)}(\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i')^{(4)} + \ (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i')^{(5)}, (a_i'')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (b_i'')^{(5)} > 0, \quad i,j = 28{,}29{,}30$<br><br>The functions $(a_i'')^{(5)}, (b_i'')^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}, \ (r_i)^{(5)}$:<br><br>$(a_i'')^{(5)}(T_{29},t) \le (p_i)^{(5)} \le (\hat{A}_{28})^{(5)}$<br><br>$(b_i'')^{(5)}\left((G_{31}),t\right) \le \ (r_i)^{(5)} \le (b_i')^{(5)} \le (\hat{B}_{28})^{(5)}$ | 122 |

$lim_{T_2\to\infty}(a_i'')^{(5)}(T_{29},t) = (p_i)^{(5)}$

$\lim_{G\to\infty}(b_i'')^{(5)}(G_{31},t) = (r_i)^{(5)}$

**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ : 123

Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$

They satisfy Lipschitz condition: 124

$$|(a_i'')^{(5)}(T_{29}', t) - (a_i'')^{(5)}(T_{29}, t)| \le (\hat{k}_{28})^{(5)}|T_{29} - T_{29}'|e^{-(\hat{M}_{28})^{(5)}t}$$

$$|(b_i'')^{(5)}((G_{31})', t) - (b_i'')^{(5)}((G_{31}), t)| < (\hat{k}_{28})^{(5)}||(G_{31}) - (G_{31})'||e^{-(\hat{M}_{28})^{(5)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(5)}(T_{29}', t)$ and $(a_i'')^{(5)}(T_{29}, t)$ . $(T_{29}', t)$ and $(T_{29}, t)$ are points belonging to the interval $[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}]$. It is to be noted that $(a_i'')^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i'')^{(5)}(T_{29}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ : 125

$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants

$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$$

**Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ : 126

There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i')^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i')^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

Where we suppose

$(a_i)^{(6)}, (a_i')^{(6)}, (a_i'')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (b_i'')^{(6)} > 0, \quad i, j = 32,33,34$ 127

The functions $(a_i'')^{(6)}, (b_i'')^{(6)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(6)}, (r_i)^{(6)}$:

$$(a_i'')^{(6)}(T_{33}, t) \le (p_i)^{(6)} \le (\hat{A}_{32})^{(6)}$$

| | |
|---|---|
| $(b_i^{''})^{(6)}((G_{35}),t) \leq \ (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(6)}(T_{33},t) = (p_i)^{(6)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(6)}\left((G_{35}),t\right) = \ (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'},t) - (a_i^{''})^{(6)}(T_{33},t)\rvert \leq (\hat{k}_{32})^{(6)}\lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})^{'},t) - (b_i^{''})^{(6)}\left((G_{35}),t\right)\rvert < (\hat{k}_{32})^{(6)}\lvert\lvert(G_{35}) - (G_{35})^{'}\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'},t)$ and$(a_i^{''})^{(6)}(T_{33},t)$ . $(T_{33}^{'},t)$ and $(T_{33},t)$ are points belonging to the interval $\left[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\right]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + \ (\hat{A}_{32})^{(6)} + \ (\hat{P}_{32})^{(6)}\ (\hat{k}_{32})^{(6)}] < 1$<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[\ (b_i)^{(6)} + (b_i^{'})^{(6)} + \ (\hat{B}_{32})^{(6)} + \ (\hat{Q}_{32})^{(6)}\ \ (\hat{k}_{32})^{(6)}] < 1$ | 130 |
| Where we suppose | |
| (III) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i,j = 36,37,38$<br><br>(JJJ) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$: | 131 |

$(a_i^{''})^{(7)}(T_{37},t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$

$(b_i^{''})^{(7)}(G_{39},t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$

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*(KKK)* $\lim_{T_2\to\infty} (a_i^{''})^{(7)}(T_{37},t) = (p_i)^{(7)}$

*(LLL)*

$\lim_{G\to\infty}(b_i^{''})^{(7)}\left((G_{39}),t\right) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

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They satisfy Lipschitz condition:

$|(a_i^{''})^{(7)}(T_{37}^{'},t) - (a_i^{''})^{(7)}(T_{37},t)| \leq (\hat{k}_{36})^{(7)}|T_{37} - T_{37}^{'}|e^{-(\hat{M}_{36})^{(7)}t}$

$|(b_i^{''})^{(7)}((G_{39})^{'},t) - (b_i^{''})^{(7)}\left((G_{39}),t\right)| < (\hat{k}_{36})^{(7)}||(G_{39}) - (G_{39})^{'}||e^{-(\hat{M}_{36})^{(7)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'},t)$ and$(a_i^{''})^{(7)}(T_{37},t)$ . $(T_{37}^{'},t)$ and $(T_{37},t)$ are points belonging to the interval $\left[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}\right]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

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**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :

(MMM) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$, are positive constants

$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$

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**Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :

(NNN) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities

$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$

$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$

| | |
|---|---|
| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})^{'}, t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})^{'}\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $\left[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}\right]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i^{'})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (p_i)^{(8)}$, $(r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |

| | |
|---|---|
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i,j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45},t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47},t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2\to\infty}(a_i'')^{(9)}(T_{45},t) = (p_i)^{(9)}$<br>$\lim_{G\to\infty}(b_i'')^{(9)}(G_{47},t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}',t) - (a_i'')^{(9)}(T_{45},t)\rvert \le (\hat{k}_{44})^{(9)}\lvert T_{45} - T_{45}'\rvert e^{-(\hat{M}_{44})^{(9)}t}$<br>$\lvert(b_i'')^{(9)}((G_{47})',t) - (b_i'')^{(9)}((G_{47}),t)\rvert < (\hat{k}_{44})^{(9)}\lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}',t)$ and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i'')^{(9)}(T_{45},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :<br>$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants<br>$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$ | |
| **Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :<br>There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and (\hat{B}_{44})^{(9)}$ and the constants | |

$(a_i)^{(9)}, (a_i')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}, i = 44,45,46,$
satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i')^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i')^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0), T_i(0)$ :

$$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}, \quad \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}, \quad \boxed{T_i(0) = T_i^0 > 0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0), T_i(0)$

$$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}, \quad G_i(0) = G_i^0 > 0$$

$$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}, \quad T_i(0) = T_i^0 > 0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}, \quad G_i(0) = G_i^0 > 0$$

$$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}, \quad T_i(0) = T_i^0 > 0$$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

**Definition of** $G_i(0), T_i(0)$ :

$$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}, \quad \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}, \quad \boxed{T_i(0) = T_i^0 > 0}$$

**Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 151

**Definition of** $G_i(0), T_i(0)$ :

$$G_i(t) \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}, \quad \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}, \quad \boxed{T_i(0) = T_i^0 > 0}$$

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|---|---|
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le \left(\hat{Q}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le \left(\hat{P}_{13}\right)^{(1)}$ , $T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le \left(\hat{P}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |
| $0 \le T_i(t) - T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 157 |
| By<br><br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}\left(s_{(13)}\right) - \left((a_{13}')^{(1)} + a_{13}'')^{(1)}\left(T_{14}\left(s_{(13)}\right), s_{(13)}\right)\right) G_{13}\left(s_{(13)}\right)\right] ds_{(13)}$ | 158 |

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|---|---|
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{16})^{(2)}$, $T_i^0 \le (\hat{Q}_{16})^{(2)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} G_{17}(s_{(16)}) - \left((a_{16}')^{(2)} + a_{16}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{16}(s_{(16)})\right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[(a_{17})^{(2)} G_{16}(s_{(16)}) - \left((a_{17}')^{(2)} + (a_{17}'')^{(2)}(T_{17}(s_{(16)}), s_{(17)})\right) G_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{20})^{(3)}$ , $T_i^0 \leq (\hat{Q}_{20})^{(3)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a_{20}^{'})^{(3)} + a_{20}^{''})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{24})^{(4)}$ , $T_i^0 \leq (\hat{Q}_{24})^{(4)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a_{24}^{'})^{(4)} + a_{24}^{''})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |

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| $\overline{T}_{26}(t) = T_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b_{26}')^{(4)} - (b_{26}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{28})^{(5)}$ , $T_i^0 \leq (\hat{Q}_{28})^{(5)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a_{28}')^{(5)} + a_{28}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a_{29}')^{(5)} + (a_{29}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a_{30}')^{(5)} + (a_{30}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\overline{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b_{28}')^{(5)} - (b_{28}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\overline{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b_{29}')^{(5)} - (b_{29}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\overline{T}_{30}(t) = T_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b_{30}')^{(5)} - (b_{30}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a_{32}')^{(6)} + a_{32}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a_{33}')^{(6)} + (a_{33}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |

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| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{36})^{(7)}$, $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a_{36}^{'})^{(7)} + a_{36}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ | |

which satisfy

$G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{40})^{(8)}$ , $T_i^0 \leq (\hat{Q}_{40})^{(8)}$,

$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$

$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$

By 166

$$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{40})^{(8)} + a''_{40})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{40}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[ (a_{41})^{(8)} G_{40}(s_{(40)}) - \left( (a'_{41})^{(8)} + (a''_{41})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{41}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[ (a_{42})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{42})^{(8)} + (a''_{42})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{42}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[ (b_{40})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{40})^{(8)} - (b''_{40})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{40}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[ (b_{41})^{(8)} T_{40}(s_{(40)}) - \left( (b'_{41})^{(8)} - (b''_{41})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{41}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[ (b_{42})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{42})^{(8)} - (b''_{42})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{42}(s_{(40)}) \right] ds_{(40)}$$

Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$

**Proof:** 166 A

Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy

$G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$,

$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$

$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$

By

$$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[ (a_{44})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{44}(s_{(44)}) \right] ds_{(44)}$$

$$\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[ (a_{45})^{(9)} G_{44}(s_{(44)}) - \left( (a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{45}(s_{(44)}) \right] ds_{(44)}$$

$$\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[ (a_{46})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{46}(s_{(44)}) \right] ds_{(44)}$$

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| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that<br>$(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} \left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$<br>$\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}} \left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$ | 171 |
| From which it follows that | 172 |

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| $(G_{20}(t)-G_{20}^0)e^{-(\hat{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that $G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\hat{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$ $\left(1+(a_{24})^{(4)}t\right)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that $(G_{24}(t)-G_{24}^0)e^{-(\hat{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$ $\left(1+(a_{28})^{(5)}t\right)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left(e^{(\hat{M}_{28})^{(5)}t}-1\right)$ | |
| From which it follows that $(G_{28}(t)-G_{28}^0)e^{-(\hat{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 5 | 175 |
| The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0+(\hat{P}_{32})^{(6)}e^{(\hat{M}_{32})^{(6)}s_{(32)}}\right)\right] ds_{(32)} =$ $\left(1+(a_{32})^{(6)}t\right)G_{33}^0 + \frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)}t}-1\right)$ | 176 |
| From which it follows that $(G_{32}(t)-G_{32}^0)e^{-(\hat{M}_{32})^{(6)}t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^0)e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^0}{G_{33}^0}\right)}+(\hat{P}_{32})^{(6)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 6 | 177 |

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| Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$ | |
| (k) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} \left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$<br>$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$ | 178 |
| From which it follows that<br>$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left[\left((\hat{P}_{36})^{(7)} + G_{37}^0\right) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 7 | |
| The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} \left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$<br>$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$ | 180 |
| From which it follows that<br>$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left[\left((\hat{P}_{40})^{(8)} + G_{41}^0\right) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} \left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$<br>$\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left(e^{(\hat{M}_{44})^{(9)} t} - 1\right)$ | |
| From which it follows that<br>$(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left[\left((\hat{P}_{44})^{(9)} + G_{45}^0\right) e^{\left(-\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0}\right)} + (\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ and to choose | 182 |

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| $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{13})^{(1)}\left\|G_{14}^{(1)}-G_{14}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$<br>$\int_0^t\{(a_{13}')^{(1)}\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{-(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$G_{13}^{(2)}\left\\|(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}'')^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right\\|\ e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$<br>Where $s_{(13)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | |
| $\left\|G^{(1)}-G^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}t}\leq$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}\left((a_{13})^{(1)}+(a_{13}')^{(1)}+(\hat{A}_{13})^{(1)}+(\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)},T^{(1)};\ G^{(2)},T^{(2)}\right)\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a_{13}'')^{(1)}$ and $(b_{13}'')^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by | |

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| $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(1)} - (a_i^{''})^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a_{14}^{'})^{(1)}G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}^{'})^{(1)}$<br><br>If $G_{14}\ or\ G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}\ and\ G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t), t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t), t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right),\quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t), t) = (b_{15}^{'})^{(1)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |

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| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)}, (T_{19})^{(1)}), ((G_{19})^{(2)}, (T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)}\left\|G_{17}^{(1)} - G_{17}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\,ds_{(16)} +$<br>$\int_0^t\{(a_{16}')^{(2)}\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{-(\hat{M}_{16})^{(2)}s_{(16)}} +$<br>$(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}} +$<br>$G_{16}^{(2)}\left\|(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}'')^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}t} \leq$<br>$\frac{1}{(\hat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}\right)d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}'')^{(2)}$ and $(b_{16}'')^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | 199 |

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| condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ and $(\widehat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)}t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1, \left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)}G_{17}$ and by integrating<br><br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1/(a_{17}^{'})^{(2)}$<br><br>In the same way , one can obtain<br><br>$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}^{'})^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | 201 |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(2)}((G_{19})(t), t)) = (b_{17}^{'})^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i^{''})^{(2)}((G_{19})(t), t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}^{''})^{(2)}\left((G_{19})(t), t\right) = (b_{18}^{'})^{(2)}$ | 205 |

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| We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)}, (T_{23})^{(1)}), ((G_{23})^{(2)}, (T_{23})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{20})^{(3)}t}\}$ | 211 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :(\widetilde{(G_{23})}, \widetilde{(T_{23})}) = \mathcal{A}^{(3)}((G_{23}), (T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)}\left\|G_{21}^{(1)} - G_{21}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)} +$<br>$\int_0^t\{(a'_{20})^{(3)}\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{-(\hat{M}_{20})^{(3)}s_{(20)}} +$<br>$(a''_{20})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right)\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}} +$<br>$G_{20}^{(2)}\left\|(a''_{20})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a''_{20})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}t} \leq$<br>$\frac{1}{(\hat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a'_{20})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)}, (T_{23})^{(1)};\ (G_{23})^{(2)}, (T_{23})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |

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| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ $and$ $(\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(3)}t)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2\ and\ \left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}\ and\ G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)}G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br><br>If $G_{21}\ or\ G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ $is$ bounded, from below, the same property holds for $G_{21}\ and\ G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t), t\right)) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t), t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded. | 220 |

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| The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right)=(b_{22}^{'})^{(3)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}<1$ and to choose<br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)}+\left((\hat{P}_{24})^{(4)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\hat{Q}_{24})^{(4)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{24})^{(4)}\right]\leq(\hat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{24})^{(4)}\left\|G_{25}^{(1)}-G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}ds_{(24)}+$<br>$\int_0^t\{(a_{24}^{'})^{(4)}\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$G_{24}^{(2)}\left\|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)}-(G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t}\leq$<br>$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$ | 226 |

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| And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i = 24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(4)} - (a_i^{''})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(4)}t\right)} > 0$ for t > 0 | 228 |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ :<br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if<br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}^{'})^{(4)}G_{25}$ and by integrating<br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a_{25}^{'})^{(4)}$<br>In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a_{26}^{'})^{(4)}$<br>If $G_{25}\ or\ G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}\ and\ G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t), t)) = (b_{25}^{'})^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i^{''})^{(4)}((G_{27})(t), t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results | 232 |

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right), \quad t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42

Analogous inequalities hold also for $G_{29}, G_{30}, T_{28}, T_{29}, T_{30}$

It is now sufficient to take $\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ and to choose

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$(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ large to have

$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[(\hat{P}_{28})^{(5)} + \left((\hat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{28})^{(5)} \qquad 234$$

$$\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{28})^{(5)}\right] \leq (\hat{Q}_{28})^{(5)} \qquad 235$$

In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric 236

$$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right) = \mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$

It results

$$\left|\tilde{G}_{28}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{28})^{(5)}\left|G_{29}^{(1)} - G_{29}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)} +$$

$$\int_0^t\{(a_{28}^{'})^{(5)}\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{-(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$G_{28}^{(2)}\left|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right) - (a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right|\ e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$$

Where $s_{(28)}$ represents integrand that is integrated over the interval $[0, t]$

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| From the hypotheses on it follows | |
| $\left\|(G_{31})^{(1)} - (G_{31})^{(2)}\right\| e^{-(\widehat{M}_{28})^{(5)}t} \leq$ $\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)} + (a'_{28})^{(5)} + (\widehat{A}_{28})^{(5)} + (\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right) d\left(((G_{31})^{(1)}, (T_{31})^{(1)};\ (G_{31})^{(2)}, (T_{31})^{(2)})\right)$ And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a''_{28})^{(5)}$ and $(b''_{28})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ and $(\widehat{Q}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(5)}$ and $(b''_i)^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t) = 0$ and $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(5)} - (a''_i)^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(5)}t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ and $\left((\widehat{M}_{28})^{(5)}\right)_3$ : **Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ and $G_{30}$ . indeed if $G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a'_{29})^{(5)} G_{29}$ and by integrating $G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a'_{29})^{(5)}$ In the same way , one can obtain $G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a'_{30})^{(5)}$ If $G_{29}$ or $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ is bounded, from below, the same property holds for $G_{29}$ and $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |

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| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t),t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1-e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}, G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}\left[(\hat{P}_{32})^{(6)} + \left((\hat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{32})^{(6)}\right] \leq (\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{32})^{(6)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{35})}, \widetilde{(T_{35})}$ : $\left(\widetilde{(G_{35})}, \widetilde{(T_{35})}\right) = \mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br><br>It results | 247 |

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| $\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{32})^{(6)} \left\|G_{33}^{(1)}-G_{33}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\, ds_{(32)} +$<br>$\int_0^t \{(a_{32}^{'})^{(6)} \left\|G_{32}^{(1)}-G_{32}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$(a_{32}^{''})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$G_{32}^{(2)} \left\|(a_{32}^{''})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right) - (a_{32}^{''})^{(6)}\left(T_{33}^{(2)}, s_{(32)}\right)\right\| \; e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\} ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}t} \leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)} + (a_{32}^{'})^{(6)} + (\widehat{A}_{32})^{(6)} + (\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right) d\left(((G_{35})^{(1)}, (T_{35})^{(1)};\ (G_{35})^{(2)}, (T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a_{32}^{''})^{(6)}$ and $(b_{32}^{''})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)} e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)} e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i = 32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 249 |
| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(6)} - (a_i^{''})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(6)}t\right)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ :<br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if<br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}^{'})^{(6)} G_{33}$ and by integrating<br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}^{'})^{(6)}$<br>In the same way , one can obtain<br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}^{'})^{(6)}$ | 251 |

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| If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i^{''})^{(6)}\big((G_{35})(t),t\big) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ large to have | 255 |
| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[(\hat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote | 258 |

**Definition of** $\widetilde{(G_{39})}, \widetilde{(T_{39})} : \left( \widetilde{(G_{39})}, \widetilde{(T_{39})} \right) = \mathcal{A}^{(7)}((G_{39}), (T_{39}))$
It results

$\left|\tilde{G}_{36}^{(1)} - \tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{36})^{(7)} \left|G_{37}^{(1)} - G_{37}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}}\, ds_{(36)} +$
$\int_0^t \{(a_{36}^{'})^{(7)} \left|G_{36}^{(1)} - G_{36}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} +$
$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) \left|G_{36}^{(1)} - G_{36}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}} +$
$G_{36}^{(2)} \left|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) - (a_{36}^{''})^{(7)}\left(T_{37}^{(2)}, s_{(36)}\right)\right| \; e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}} \} ds_{(36)}$

Where $s_{(36)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on it follows

$\left|(G_{39})^{(1)} - (G_{39})^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} t} \leq$
$\frac{1}{(\widehat{M}_{36})^{(7)}} \left((a_{36})^{(7)} + (a_{36}^{'})^{(7)} + (\widehat{A}_{36})^{(7)} + (\widehat{P}_{36})^{(7)} (\widehat{k}_{36})^{(7)}\right) d\left(((G_{39})^{(1)}, (T_{39})^{(1)};\ (G_{39})^{(2)}, (T_{39})^{(2)})\right)$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

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**Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)} e^{(\widehat{M}_{36})^{(7)} t}$ *and* $(\widehat{Q}_{36})^{(7)} e^{(\widehat{M}_{36})^{(7)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36{,}37{,}38$ depend only on $T_{37}$ and respectively on $(G_{39})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition.

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**Remark 32:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$

it results

$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)} (T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$

$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0

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**Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2$ *and* $\left((\widehat{M}_{36})^{(7)}\right)_3$ :

**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}$ *and* $G_{38}$ . indeed if

$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating

$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)} \left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$

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In the same way , one can obtain

$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}')^{(7)}$

If $G_{37}$ *or* $G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively.

**Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}$ *and* $G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. 263

**Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(7)}((G_{39})(t),t)) = (b_{37}')^{(7)}$ then $T_{37} \to \infty$. 264

**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :

Indeed let $t_7$ be so that for $t > t_7$

$(b_{37})^{(7)} - (b_i'')^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$

Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to 265

$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results

$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right), \quad t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.
The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}'')^{(7)}\left((G_{39})(t),t\right) = (b_{38}')^{(7)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations

It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose 266
$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have

$$\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)} + \left((\widehat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{40})^{(8)} \quad 267$$

$$\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\widehat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{40})^{(8)}\right] \leq (\widehat{Q}_{40})^{(8)} \quad 268$$

In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric

| | |
|---|---|
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$ $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\,ds_{(40)}+$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\le$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ *and* $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i=40,41,42$ depend only on $\mathrm{T}_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$<br><br>it results<br><br>$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(8)}-(a_i^{''})^{(8)}(T_{41}(s_{(40)}),s_{(40)})\}ds_{(40)}\right]}\ge 0$<br><br>$T_i\,(t)\ge T_i^0e^{\left(-(b_i^{'})^{(8)}t\right)}>0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1,\left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ : | 276 |

**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if

$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a'_{41})^{(8)}G_{41}$ and by integrating

$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1/(a'_{41})^{(8)}$

In the same way , one can obtain

$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2/(a'_{42})^{(8)}$

If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively.

**Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. 277

**Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t),t)) = (b'_{41})^{(8)}$ then $T_{41} \to \infty$. 278

**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :

Indeed let $t_8$ be so that for $t > t_8$

$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t),t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$

Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to 279

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right),\ t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded. The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t),t(t),t\right) = (b'_{42})^{(8)}$

It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have 279 A

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\left(((G_{47})^{(1)},(T_{47})^{(1)}),((G_{47})^{(2)},(T_{47})^{(2)})\right)=$$
$$sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})},\widetilde{(T_{47})}:\left(\widetilde{(G_{47})},\widetilde{(T_{47})}\right)=\mathcal{A}^{(9)}\left((G_{47}),(T_{47})\right)$
It results

$$\left|\tilde{G}_{44}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{44})^{(9)}\left|G_{45}^{(1)}-G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)}+$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)-(a_{44}^{''})^{(9)}\left(T_{45}^{(2)},s_{(44)}\right)\right|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t}\le$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)}+(a_{44}^{'})^{(9)}+(\widehat{A}_{44})^{(9)}+(\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)},(T_{47})^{(1)};\ (G_{47})^{(2)},(T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i=44{,}45{,}46$ depend only on $\mathrm{T}_{45}$ and respectively on $(G_{47})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$

From 99 to 44 it results

$$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(9)}-(a_i^{''})^{(9)}(T_{45}(s_{(44)}),s_{(44)})\}ds_{(44)}\right]}\ge 0$$

$$T_i(t)\ge T_i^0e^{(-(b_i^{'})^{(9)}t)}>0\quad\text{for t}>0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1,\left((\widehat{M}_{44})^{(9)}\right)_2$ *and* $\left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}$ *and* $G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)} G_{45}$ and by integrating

$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$

In the same way , one can obtain

$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$

If $G_{45}$ *or* $G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}$ *and* $G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t),t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t),t\right) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\left((G_{47})(t),t\right) = (b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} - (a_{13}^{''})^{(1)}(T_{14},t) + (a_{14}^{''})^{(1)}(T_{14},t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(v_1)^{(1)}, (v_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, v^{(1)}, u^{(1)}$ : 281

By $(v_1)^{(1)} > 0$ , $(v_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$ , $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

| | |
|---|---|
| **Definition of** $(\bar{v}_1)^{(1)},,(\bar{v}_2)^{(1)},(\bar{u}_1)^{(1)},(\bar{u}_2)^{(1)}$ :<br><br>By $(\bar{v}_1)^{(1)}>0$ , $(\bar{v}_2)^{(1)}<0$ and respectively $(\bar{u}_1)^{(1)}>0$ , $(\bar{u}_2)^{(1)}<0$ the roots of the equations $(a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_2)^{(1)}v^{(1)}-(a_{13})^{(1)}=0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2+(\tau_2)^{(1)}u^{(1)}-(b_{13})^{(1)}=0$ | 282 |
| **Definition of** $(m_1)^{(1)}$ , $(m_2)^{(1)}$ , $(\mu_1)^{(1)},(\mu_2)^{(1)},(\nu_0)^{(1)}$ :-<br><br>If we define $(m_1)^{(1)}$ , $(m_2)^{(1)}$ , $(\mu_1)^{(1)},(\mu_2)^{(1)}$ by<br>$(m_2)^{(1)}=(\nu_0)^{(1)},(m_1)^{(1)}=(\nu_1)^{(1)},\ if\ (\nu_0)^{(1)}<(\nu_1)^{(1)}$<br><br>$(m_2)^{(1)}=(\nu_1)^{(1)},(m_1)^{(1)}=(\bar{\nu}_1)^{(1)}\ ,if\ (\nu_1)^{(1)}<(\nu_0)^{(1)}<(\bar{\nu}_1)^{(1)},$<br><br>and $\boxed{(\nu_0)^{(1)}=\frac{G_{13}^0}{G_{14}^0}}$<br><br>$(\ m_2)^{(1)}=(\nu_1)^{(1)},(m_1)^{(1)}=(\nu_0)^{(1)},\ if\ (\bar{\nu}_1)^{(1)}<(\nu_0)^{(1)}$ | 283 |
| and analogously<br><br>$(\mu_2)^{(1)}=(u_0)^{(1)},(\mu_1)^{(1)}=(u_1)^{(1)},\ if\ (u_0)^{(1)}<(u_1)^{(1)}$<br><br>$(\mu_2)^{(1)}=(u_1)^{(1)},(\mu_1)^{(1)}=(\bar{u}_1)^{(1)}\ ,if\ (u_1)^{(1)}<(u_0)^{(1)}<(\bar{u}_1)^{(1)},$<br><br>and $\boxed{(u_0)^{(1)}=\frac{T_{13}^0}{T_{14}^0}}$<br><br>$(\ \mu_2)^{(1)}=(u_1)^{(1)},(\mu_1)^{(1)}=(u_0)^{(1)},if\ (\bar{u}_1)^{(1)}<(u_0)^{(1)}$ where $(u_1)^{(1)},(\bar{u}_1)^{(1)}$<br><br>are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{13}^0e^{((S_1)^{(1)}-(p_{13})^{(1)})t}\le G_{13}(t)\le G_{13}^0e^{(S_1)^{(1)}t}$<br><br>where $(p_i)^{(1)}$ is defined by equation<br><br>$\frac{1}{(m_1)^{(1)}}G_{13}^0e^{((S_1)^{(1)}-(p_{13})^{(1)})t}\le G_{14}(t)\le\frac{1}{(m_2)^{(1)}}G_{13}^0e^{(S_1)^{(1)}t}$ | 285 |
| $(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t}-e^{-(S_2)^{(1)}t}\right]+G_{15}^0e^{-(S_2)^{(1)}t}\le G_{15}(t)\le$<br><br>$\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a_{15}^{'})^{(1)})}[e^{(S_1)^{(1)}t}-e^{-(a_{15}^{'})^{(1)}t}]+\ G_{15}^0e^{-(a_{15}^{'})^{(1)}t})$ | 286 |
| $\boxed{T_{13}^0e^{(R_1)^{(1)}t}\le T_{13}(t)\le T_{13}^0e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}}T_{13}^0e^{(R_1)^{(1)}t}\le T_{13}(t)\le\frac{1}{(\mu_2)^{(1)}}T_{13}^0e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |

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| $\frac{(b_{15})^{(1)}T_{13}^{0}}{(\mu_1)^{(1)}((R_1)^{(1)}-(b_{15}^{'})^{(1)})}\left[e^{(R_1)^{(1)}t}-e^{-(b_{15}^{'})^{(1)}t}\right]+T_{15}^{0}e^{-(b_{15}^{'})^{(1)}t}\leq T_{15}(t)\leq$ $\frac{(a_{15})^{(1)}T_{13}^{0}}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t}-e^{-(R_2)^{(1)}t}\right]+T_{15}^{0}e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)},(S_2)^{(1)},(R_1)^{(1)},(R_2)^{(1)}$:- Where $(S_1)^{(1)}=(a_{13})^{(1)}(m_2)^{(1)}-(a_{13}^{'})^{(1)}$ $(S_2)^{(1)}=(a_{15})^{(1)}-(p_{15})^{(1)}$ $(R_1)^{(1)}=(b_{13})^{(1)}(\mu_2)^{(1)}-(b_{13}^{'})^{(1)}$ $(R_2)^{(1)}=(b_{15}^{'})^{(1)}-(r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ : $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)}\leq-(a_{16}^{'})^{(2)}+(a_{17}^{'})^{(2)}-(a_{16}^{''})^{(2)}(\mathrm{T}_{17},t)+(a_{17}^{''})^{(2)}(\mathrm{T}_{17},t)\leq-(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)}\leq-(b_{16}^{'})^{(2)}+(b_{17}^{'})^{(2)}-(b_{16}^{''})^{(2)}\big((G_{19}),t\big)-(b_{17}^{''})^{(2)}\big((G_{19}),t\big)\leq-(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(v_1)^{(2)},(v_2)^{(2)},(u_1)^{(2)},(u_2)^{(2)}$ : | 295 |
| By $(v_1)^{(2)}>0$ , $(v_2)^{(2)}<0$ and respectively $(u_1)^{(2)}>0$ , $(u_2)^{(2)}<0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(v^{(2)}\big)^2+(\sigma_1)^{(2)}v^{(2)}-(a_{16})^{(2)}=0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2+(\tau_1)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ and | 298 |
| **Definition of** $(\bar{v}_1)^{(2)},,(\bar{v}_2)^{(2)},(\bar{u}_1)^{(2)},(\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{v}_1)^{(2)}>0$ , $(\bar{v}_2)^{(2)}<0$ and respectively $(\bar{u}_1)^{(2)}>0$ , $(\bar{u}_2)^{(2)}<0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(v^{(2)}\big)^2+(\sigma_2)^{(2)}v^{(2)}-(a_{16})^{(2)}=0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2+(\tau_2)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ | 302 |
| **Definition of** $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)}=(v_0)^{(2)},(m_1)^{(2)}=(v_1)^{(2)}$, ***if*** $(v_0)^{(2)}<(v_1)^{(2)}$ | 305 |

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| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , $\boldsymbol{if}$ $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, $\boldsymbol{if}$ $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, $\boldsymbol{if}$ $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , $\boldsymbol{if}$ $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$,<br>and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, $\boldsymbol{if}$ $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)} t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)} t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)} t}\right] + G_{18}^0 e^{-(S_2)^{(2)} t} \le G_{18}(t) \le \right.$<br>$\left. \frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a'_{18})^{(2)})}\left[e^{(S_1)^{(2)} t} - e^{-(a'_{18})^{(2)} t}\right] + G_{18}^0 e^{-(a'_{18})^{(2)} t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b'_{18})^{(2)})}\left[e^{(R_1)^{(2)} t} - e^{-(b'_{18})^{(2)} t}\right] + T_{18}^0 e^{-(b'_{18})^{(2)} t} \le T_{18}(t) \le$<br>$\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)} + (r_{16})^{(2)} + (R_2)^{(2)})}\left[e^{((R_1)^{(2)} + (r_{16})^{(2)})t} - e^{-(R_2)^{(2)} t}\right] + T_{18}^0 e^{-(R_2)^{(2)} t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a'_{16})^{(2)}$<br>$(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b'_{16})^{(2)}$ | 318 |

$(\mathrm{R}_2)^{(2)} = (b_{18}^{'})^{(2)} - (r_{18})^{(2)}$

**Behavior of the solutions** 319

**Theorem 3:** If we denote and define

**Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ :

$(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying

$$-(\sigma_2)^{(3)} \le -(a_{20}^{'})^{(3)} + (a_{21}^{'})^{(3)} - (a_{20}^{''})^{(3)}(T_{21}, t) + (a_{21}^{''})^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$$

$$-(\tau_2)^{(3)} \le -(b_{20}^{'})^{(3)} + (b_{21}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t) - (b_{21}^{''})^{(3)}\big((G_{23}), t\big) \le -(\tau_1)^{(3)}$$

**Definition of** $(\nu_1)^{(3)}, (\nu_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ : 320

By $(\nu_1)^{(3)} > 0$ , $(\nu_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$ , $(u_2)^{(3)} < 0$ the roots of the equations

$(a_{21})^{(3)}\big(\nu^{(3)}\big)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\big(u^{(3)}\big)^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and

By $(\bar{\nu}_1)^{(3)} > 0$ , $(\bar{\nu}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$ , $(\bar{u}_2)^{(3)} < 0$ the

roots of the equations $(a_{21})^{(3)}\big(\nu^{(3)}\big)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\big(u^{(3)}\big)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$

**Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :- 321

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ , ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,

and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously 322

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$ , ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

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| $G_{20}^{0}e^{((S_1)^{(3)}-(p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^{0}e^{(S_1)^{(3)}t}$<br>$(p_i)^{(3)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(3)}}G_{20}^{0}e^{((S_1)^{(3)}-(p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}}G_{20}^{0}e^{(S_1)^{(3)}t}$ | 323 |
| $(\frac{(a_{22})^{(3)}G_{20}^{0}}{(m_1)^{(3)}((S_1)^{(3)}-(p_{20})^{(3)}-(S_2)^{(3)})}\left[e^{((S_1)^{(3)}-(p_{20})^{(3)})t}-e^{-(S_2)^{(3)}t}\right]+G_{22}^{0}e^{-(S_2)^{(3)}t} \le G_{22}(t) \le$<br>$\frac{(a_{22})^{(3)}G_{20}^{0}}{(m_2)^{(3)}((S_1)^{(3)}-(a_{22}^{'})^{(3)})}[e^{(S_1)^{(3)}t}-e^{-(a_{22}^{'})^{(3)}t}]+G_{22}^{0}e^{-(a_{22}^{'})^{(3)}t})$ | 324 |
| $\boxed{T_{20}^{0}e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^{0}e^{((R_1)^{(3)}+(r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}}T_{20}^{0}e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}}T_{20}^{0}e^{((R_1)^{(3)}+(r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)}T_{20}^{0}}{(\mu_1)^{(3)}((R_1)^{(3)}-(b_{22}^{'})^{(3)})}\left[e^{(R_1)^{(3)}t}-e^{-(b_{22}^{'})^{(3)}t}\right]+T_{22}^{0}e^{-(b_{22}^{'})^{(3)}t} \le T_{22}(t) \le$<br>$\frac{(a_{22})^{(3)}T_{20}^{0}}{(\mu_2)^{(3)}((R_1)^{(3)}+(r_{20})^{(3)}+(R_2)^{(3)})}\left[e^{((R_1)^{(3)}+(r_{20})^{(3)})t}-e^{-(R_2)^{(3)}t}\right]+T_{22}^{0}e^{-(R_2)^{(3)}t}$ | 327 |
| **Definition of** $(S_1)^{(3)},(S_2)^{(3)},(R_1)^{(3)},(R_2)^{(3)}$:-<br>Where $(S_1)^{(3)}=(a_{20})^{(3)}(m_2)^{(3)}-(a_{20}^{'})^{(3)}$<br>$(S_2)^{(3)}=(a_{22})^{(3)}-(p_{22})^{(3)}$<br>$(R_1)^{(3)}=(b_{20})^{(3)}(\mu_2)^{(3)}-(b_{20}^{'})^{(3)}$<br>$(R_2)^{(3)}=(b_{22}^{'})^{(3)}-(r_{22})^{(3)}$ | 328 |
| **Behavior of the solutions of equation**<br>**Theorem:** If we denote and define<br>**Definition of** $(\sigma_1)^{(4)},(\sigma_2)^{(4)},(\tau_1)^{(4)},(\tau_2)^{(4)}$ :<br>$(\sigma_1)^{(4)},(\sigma_2)^{(4)},(\tau_1)^{(4)},(\tau_2)^{(4)}$ four constants satisfying<br>$-(\sigma_2)^{(4)} \le -(a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}-(a_{24}^{''})^{(4)}(T_{25},t)+(a_{25}^{''})^{(4)}(T_{25},t) \le -(\sigma_1)^{(4)}$<br>$-(\tau_2)^{(4)} \le -(b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27}),t)-(b_{25}^{''})^{(4)}((G_{27}),t) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)},(v_2)^{(4)},(u_1)^{(4)},(u_2)^{(4)},v^{(4)},u^{(4)}$ :<br>By $(v_1)^{(4)}>0$ , $(v_2)^{(4)}<0$ and respectively $(u_1)^{(4)}>0$ , $(u_2)^{(4)}<0$ the roots of the equations $(a_{25})^{(4)}(v^{(4)})^2+(\sigma_1)^{(4)}v^{(4)}-(a_{24})^{(4)}=0$<br>and $(b_{25})^{(4)}(u^{(4)})^2+(\tau_1)^{(4)}u^{(4)}-(b_{24})^{(4)}=0$ and | 329 |

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| **Definition of** $(\bar{v}_1)^{(4)},,(\bar{v}_2)^{(4)},(\bar{u}_1)^{(4)},(\bar{u}_2)^{(4)}$ :<br><br>By $(\bar{v}_1)^{(4)}>0$ , $(\bar{v}_2)^{(4)}<0$ and respectively $(\bar{u}_1)^{(4)}>0$ , $(\bar{u}_2)^{(4)}<0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_2)^{(4)}v^{(4)}-(a_{24})^{(4)}=0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2+(\tau_2)^{(4)}u^{(4)}-(b_{24})^{(4)}=0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)},(v_0)^{(4)}$ :-<br><br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)}$ by<br><br>$(m_2)^{(4)}=(v_0)^{(4)},(m_1)^{(4)}=(v_1)^{(4)}$, ***if*** $(v_0)^{(4)}<(v_1)^{(4)}$<br><br>$(m_2)^{(4)}=(v_1)^{(4)},(m_1)^{(4)}=(\bar{v}_1)^{(4)}$ , ***if*** $(v_4)^{(4)}<(v_0)^{(4)}<(\bar{v}_1)^{(4)}$,<br>and $\boxed{(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}$<br><br>$(m_2)^{(4)}=(v_4)^{(4)},(m_1)^{(4)}=(v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)}<(v_0)^{(4)}$ | 330 |
| and analogously<br><br>$(\mu_2)^{(4)}=(u_0)^{(4)},(\mu_1)^{(4)}=(u_1)^{(4)}$, ***if*** $(u_0)^{(4)}<(u_1)^{(4)}$<br><br>$(\mu_2)^{(4)}=(u_1)^{(4)},(\mu_1)^{(4)}=(\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)}<(u_0)^{(4)}<(\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)}=\frac{T_{24}^0}{T_{25}^0}}$<br><br>$(\mu_2)^{(4)}=(u_1)^{(4)},(\mu_1)^{(4)}=(u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)}<(u_0)^{(4)}$ where $(u_1)^{(4)},(\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{24}^0e^{((S_1)^{(4)}-(p_{24})^{(4)})t}\le G_{24}(t)\le G_{24}^0e^{(S_1)^{(4)}t}$<br><br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}}G_{24}^0e^{((S_1)^{(4)}-(p_{24})^{(4)})t}\le G_{25}(t)\le\frac{1}{(m_2)^{(4)}}G_{24}^0e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)}G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t}-e^{-(S_2)^{(4)}t}\right]+G_{26}^0e^{-(S_2)^{(4)}t}\le G_{26}(t)\le\right.$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0e^{(R_1)^{(4)}t}\le T_{24}(t)\le T_{24}^0e^{((R_1)^{(4)}+(r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}}T_{24}^0e^{(R_1)^{(4)}t}\le T_{24}(t)\le\frac{1}{(\mu_2)^{(4)}}T_{24}^0e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)}T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}^{'})^{(4)})}\left[e^{(R_1)^{(4)}t}-e^{-(b_{26}^{'})^{(4)}t}\right]+T_{26}^0e^{-(b_{26}^{'})^{(4)}t}\le T_{26}(t)\le$<br><br>$\frac{(a_{26})^{(4)}T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t}-e^{-(R_2)^{(4)}t}\right]+T_{26}^0e^{-(R_2)^{(4)}t}$ | 336 |

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| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a'_{24})^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b'_{24})^{(4)}$<br>$(R_2)^{(4)} = (b'_{26})^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a'_{28})^{(5)} + (a'_{29})^{(5)} - (a''_{28})^{(5)}(T_{29}, t) + (a''_{29})^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b'_{28})^{(5)} + (b'_{29})^{(5)} - (b''_{28})^{(5)}((G_{31}), t) - (b''_{29})^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ :<br>By $(v_1)^{(5)} > 0$, $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | 339 |
| **Definition of** $(\bar{v}_1)^{(5)},, (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ :<br>By $(\bar{v}_1)^{(5)} > 0$, $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$, $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :-<br>If we define $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}$ by<br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$, ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}}}$<br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$ | 341 |

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| $(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , $\boldsymbol{if}$ $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$, and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$ $(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}, \boldsymbol{if}$ $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | |
| Then the solution of global equations satisfies the inequalities $G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)} t}$ where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \le G_{30}(t) \le$ *(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b_{30}')^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b_{30}')^{(5)} t}\right] + T_{30}^0 e^{-(b_{30}')^{(5)} t} \le T_{30}(t) \le$ $\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:- Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a_{28}')^{(5)}$ $(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$ $(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b_{28}')^{(5)}$ $(R_2)^{(5)} = (b_{30}')^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ : $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying $-(\sigma_2)^{(6)} \le -(a_{32}')^{(6)} + (a_{33}')^{(6)} - (a_{32}'')^{(6)}(T_{33}, t) + (a_{33}'')^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$ | 349 |

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| $-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}\big((G_{35}),t\big) - (b''_{33})^{(6)}\big((G_{35}),t\big) \le -(\tau_1)^{(6)}$ | |
| **Definition of** $(\nu_1)^{(6)}, (\nu_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, \nu^{(6)}, u^{(6)}$ :<br><br>By $(\nu_1)^{(6)} > 0$ , $(\nu_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{\nu}_1)^{(6)},, (\bar{\nu}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br><br>By $(\bar{\nu}_1)^{(6)} > 0$ , $(\bar{\nu}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (\nu_0)^{(6)}$ :-<br><br>If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br><br>$(m_2)^{(6)} = (\nu_0)^{(6)}, (m_1)^{(6)} = (\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)} < (\nu_1)^{(6)}$<br><br>$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)} < (\nu_0)^{(6)} < (\bar{\nu}_1)^{(6)}$,<br>and $\boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$<br><br>$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)} < (\nu_0)^{(6)}$ | 351 |
| and analogously<br><br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br><br>$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$<br><br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G^0_{32}e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)}G^0_{32}}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G^0_{34}e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |

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| $\boxed{T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)}T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)}-(b_{34}')^{(6)})}\left[e^{(R_1)^{(6)}t} - e^{-(b_{34}')^{(6)}t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)}T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t} - e^{-(R_2)^{(6)}t}\right] + T_{34}^0 e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:- Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$ $(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$ $(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$ $(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ :<br><br>$(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying<br><br>$-(\sigma_2)^{(7)} \le -(a_{36}')^{(7)} + (a_{37}')^{(7)} - (a_{36}'')^{(7)}(T_{37}, t) + (a_{37}'')^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$<br><br>$-(\tau_2)^{(7)} \le -(b_{36}')^{(7)} + (b_{37}')^{(7)} - (b_{36}'')^{(7)}\big((G_{39}), t\big) - (b_{37}'')^{(7)}\big((G_{39}), t\big) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)}, (v_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, v^{(7)}, u^{(7)}$ :<br><br>By $(v_1)^{(7)} > 0, (v_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(v^{(7)}\big)^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)},, (\bar{v}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ :<br><br>By $(\bar{v}_1)^{(7)} > 0, (\bar{v}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(v^{(7)}\big)^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$<br>**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (v_0)^{(7)}$ :- | 362 |

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,

and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$

and analogously

$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,

and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

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Then the solution of global equations satisfies the inequalities

$$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)} t}$$

where $(p_i)^{(7)}$ is defined by equation

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$$\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)} t}$$

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$$\left(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})}\left[e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)} t}\right] + G_{38}^0 e^{-(S_2)^{(7)} t} \le G_{38}(t) \le \right.$$

$$\left. \frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a_{38}')^{(7)})}\left[e^{(S_1)^{(7)} t} - e^{-(a_{38}')^{(7)} t}\right] + G_{38}^0 e^{-(a_{38}')^{(7)} t}\right)$$

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$$\boxed{T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}}$$

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$$\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}$$

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$$\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b_{38}')^{(7)})}\left[e^{(R_1)^{(7)} t} - e^{-(b_{38}')^{(7)} t}\right] + T_{38}^0 e^{-(b_{38}')^{(7)} t} \le T_{38}(t) \le$$

$$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)} + (r_{36})^{(7)} + (R_2)^{(7)})}\left[e^{((R_1)^{(7)} + (r_{36})^{(7)})t} - e^{-(R_2)^{(7)} t}\right] + T_{38}^0 e^{-(R_2)^{(7)} t}$$

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**Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:- 370

Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a'_{36})^{(7)}$

$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$

$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b'_{36})^{(7)}$

$(R_2)^{(7)} = (b'_{38})^{(7)} - (r_{38})^{(7)}$

**Behavior of the solutions of equation** 371

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :

$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying

$-(\sigma_2)^{(8)} \le -(a'_{40})^{(8)} + (a'_{41})^{(8)} - (a''_{40})^{(8)}(T_{41}, t) + (a''_{41})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$

$-(\tau_2)^{(8)} \le -(b'_{40})^{(8)} + (b'_{41})^{(8)} - (b''_{40})^{(8)}\big((G_{43}), t\big) - (b''_{41})^{(8)}\big((G_{43}), t\big) \le -(\tau_1)^{(8)}$

**Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : 372

By $(v_1)^{(8)} > 0, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$
and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and

**Definition of** $(\bar{v}_1)^{(8)}, , (\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :

By $(\bar{v}_1)^{(8)} > 0, (\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0, (\bar{u}_2)^{(8)} < 0$ the

roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$

and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$

**Definition of** $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :-

If we define $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by

$(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)}$, ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}$, ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)}$,

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| and $\boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br><br>$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (v_0)^{(8)},\ \boldsymbol{if}\ (\bar{v}_1)^{(8)} < (v_0)^{(8)}$ | |
| and analogously<br><br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},\ \boldsymbol{if}\ (u_0)^{(8)} < (u_1)^{(8)}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}, \boldsymbol{if}\ (u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}, \boldsymbol{if}\ (\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)} t}$<br><br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \le G_{42}(t) \le \right.$<br>$\left.\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)} t} - e^{-(a'_{42})^{(8)} t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)} t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b'_{42})^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b'_{42})^{(8)} t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)} t} \le T_{42}(t) \le$<br><br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br><br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br><br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$ | 381 |

$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$

$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$

**Behavior of the solutions of equation 37 to 92** 382

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$

$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}\big((G_{47}), t\big) - (b''_{45})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$

**Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :

By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(9)}, , (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{\nu}_1)^{(9)} > 0$ , $(\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}$, ***if*** $(\nu_0)^{(9)} < (\nu_1)^{(9)}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}$, ***if*** $(\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T^0_{44}}{T^0_{45}}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^{0}e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^{0}e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^{0}e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}}G_{44}^{0}e^{(S_1)^{(9)}t}$$

(

$$\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^{0}e^{-(S_2)^{(9)}t} \le G_{46}(t) \le$$

$$\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}^{'})^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}^{'})^{(9)}t}]+\ G_{46}^{0}e^{-(a_{46}^{'})^{(9)}t})$$

$$\boxed{T_{44}^{0}e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^{0}e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}}T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^{0}}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}^{'})^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}^{'})^{(9)}t}\right]+T_{46}^{0}e^{-(b_{46}^{'})^{(9)}t} \le T_{46}(t) \le$$

$$\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a_{44}^{'})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b_{44}^{'})^{(9)}$

$(R_2)^{(9)} = (b_{46}^{'})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain 383

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a_{13}^{'})^{(1)} - (a_{14}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14},t)\right) - (a_{14}^{''})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}(v^{(1)})^{2} + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}(v^{(1)})^{2} + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (\nu_1)^{(1)} < (\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t) \geq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(\nu_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\nu_1)^{(1)}$

In the same manner , we get 384

$$\nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{\nu}_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\bar{\nu}_1)^{(1)}$

If $0 < (\nu_1)^{(1)} < (\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{\nu}_1)^{(1)}$ we find like in the previous case, 385

$$(\nu_1)^{(1)} \leq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}} \leq \nu^{(1)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$ if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \le \frac{d\nu^{(2)}}{dt} \le -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \ge \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \le \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \le \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \le \nu^{(2)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$$

If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain

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$$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(2)}(t)$ :-

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$$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}, \quad \boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$$

In a completely analogous way, we obtain

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**Definition of** $u^{(2)}(t)$ :-

$$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}, \quad \boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

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If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$

Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then

$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right) - (a_{21}^{''})^{(3)}(T_{21},t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$$

**Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$

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It follows

$$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$$

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From which one obtains

For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

In the same manner , we get

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$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

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If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case,

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

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If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$ if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

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$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}'')^{(5)} = (b_{29}'')^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then

$(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a'_{32})^{(6)} - (a'_{33})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right) - (a''_{33})^{(6)}(T_{33},t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \le \frac{d\nu^{(6)}}{dt} \le -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \ge \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \le \nu^{(6)}(t) \le (\nu_1)^{(6)}$

In the same manner , we get

$$\nu^{(6)}(t) \le \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \le \nu^{(6)}(t) \le (\bar{\nu}_1)^{(6)}$

If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

$$(\nu_1)^{(6)} \le \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}} \le \nu^{(6)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \le (\bar{\nu}_1)^{(6)}$$

If $0 < (\nu_1)^{(6)} \le (\bar{\nu}_1)^{(6)} \le \boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$ , we obtain

$$(\nu_1)^{(6)} \le \nu^{(6)}(t) \le \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \le (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

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$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}$, $\boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

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$$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$$

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If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain

$$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(C)^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

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**Definition of** $u^{(7)}(t)$ :-

$$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}, \quad \boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}'')^{(7)} = (a_{37}'')^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}'')^{(7)} = (b_{37}'')^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a'_{40})^{(8)} - (a'_{41})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right) - (a''_{41})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \le \frac{dv^{(8)}}{dt} \le -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(8)}, (v_0)^{(8)}$ :-

For $0 < \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}} < (v_1)^{(8)} < (\bar{v}_1)^{(8)}$

$$v^{(8)}(t) \ge \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(v_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (v_2)^{(8)}}}$$

it follows $(v_0)^{(8)} \le v^{(8)}(t) \le (v_1)^{(8)}$

In the same manner , we get

$$v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{v}_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (\bar{v}_2)^{(8)}}}$$

From which we deduce $(v_0)^{(8)} \le v^{(8)}(t) \le (\bar{v}_8)^{(8)}$

If $0 < (v_1)^{(8)} < (v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}} < (\bar{v}_1)^{(8)}$ we find like in the previous case,

$$(v_1)^{(8)} \le \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}} \le v^{(8)}(t) \le$$

$$\frac{(\bar{v}_1)^{(8)} + (C)^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (\bar{v}_1)^{(8)}$$

If $0 < (v_1)^{(8)} \le (\bar{v}_1)^{(8)} \le \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$ , we obtain

$$(v_1)^{(8)} \le v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (v_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

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**Definition of** $v^{(8)}(t)$ :-

$(m_2)^{(8)} \leq v^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{v^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $v^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)v^{(9)} - (a_{45})^{(9)}v^{(9)}$$

**Definition of** $v^{(9)}$ :- $\boxed{v^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)}\right) \leq \frac{dv^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$v^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq v^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)}-(\nu_0)^{(9)}}{(\nu_0)^{(9)}-(\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \le \frac{(\nu_1)^{(9)}+(C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}} \le \nu^{(9)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \le (\bar{\nu}_1)^{(9)} \le \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(\nu_1)^{(9)} \le \nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \le \nu^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \le u^{(9)}(t) \le (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)}G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

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| $(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,<br>$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$<br>$with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$ | 433 |

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| $(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$ | 436 A |

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| $(a'_{44})^{(9)}(a'_{45})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a'_{45})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b'_{44})^{(9)}(r_{45})^{(9)} - (b'_{45})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b'_{13})^{(1)} - (b''_{13})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b'_{14})^{(1)} - (b''_{14})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b'_{15})^{(1)} - (b''_{15})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b'_{16})^{(2)} - (b''_{16})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b'_{17})^{(2)} - (b''_{17})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b'_{18})^{(2)} - (b''_{18})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a'_{20})^{(3)} + (a''_{20})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a'_{21})^{(3)} + (a''_{21})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a'_{22})^{(3)} + (a''_{22})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b'_{20})^{(3)} - (b''_{20})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b'_{21})^{(3)} - (b''_{21})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b'_{22})^{(3)} - (b''_{22})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) +$ | 485 |

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| $(a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | |
| **Proof:**<br><br>(k) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |

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| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492<br>A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations | 498 |

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| $G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a'_{28})^{(5)}+(a''_{28})^{(5)}(T^*_{29})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a'_{30})^{(5)}+(a''_{30})^{(5)}(T^*_{29})]}$ | |
| **Definition and uniqueness of** $T^*_{33}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T^*_{33}$ for which $f(T^*_{33}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a'_{32})^{(6)}+(a''_{32})^{(6)}(T^*_{33})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a'_{34})^{(6)}+(a''_{34})^{(6)}(T^*_{33})]}$ | 499 |
| **Definition and uniqueness of** $T^*_{37}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T^*_{37}$ for which $f(T^*_{37}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a'_{36})^{(7)}+(a''_{36})^{(7)}(T^*_{37})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a'_{38})^{(7)}+(a''_{38})^{(7)}(T^*_{37})]}$ | 500 |
| **Definition and uniqueness of** $T^*_{41}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T^*_{41}$ for which $f(T^*_{41}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a'_{40})^{(8)}+(a''_{40})^{(8)}(T^*_{41})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a'_{42})^{(8)}+(a''_{42})^{(8)}(T^*_{41})]}$ | 501 |
| **Definition and uniqueness of** $T^*_{45}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T^*_{45}$ for which $f(T^*_{45}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a'_{44})^{(9)}+(a''_{44})^{(9)}(T^*_{45})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a'_{46})^{(9)}+(a''_{46})^{(9)}(T^*_{45})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br><br>$\left[(b'_{13})^{(1)}(b''_{14})^{(1)}(G) + (b'_{14})^{(1)}(b''_{13})^{(1)}(G)\right]+(b''_{13})^{(1)}(G)(b''_{14})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that $\varphi$ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that | 502 |

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| there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*})=0$ | |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br>$\varphi(G_{19})=(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)}-(b_{16})^{(2)}(b_{17})^{(2)}-$<br>$\left[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19})+(b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})\right]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19})=0$ | 503 |
| Where in $(G_{19})(G_{16},G_{17},G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*})=0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br>$\varphi(G_{23})=(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)}-(b_{20})^{(3)}(b_{21})^{(3)}-$<br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23})+(b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23})=0$<br>Where in $G_{23}(G_{20},G_{21},G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*})=0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br>$\varphi(G_{27})=(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)}-(b_{24})^{(4)}(b_{25})^{(4)}-$<br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27})+(b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27})=0$<br>Where in $(G_{27})(G_{24},G_{25},G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*})=0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31})=(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)}-(b_{28})^{(5)}(b_{29})^{(5)}-$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31})+(b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right]+(b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31})=0$<br>Where in $(G_{31})(G_{28},G_{29},G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*})=0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35})=(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)}-(b_{32})^{(6)}(b_{33})^{(6)}-$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35})+(b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right]+(b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35})=0$<br>Where in $(G_{35})(G_{32},G_{33},G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that | 508 |

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| there exists a unique $G_{33}^{*}$ such that $\varphi(G_{35}{}^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi(G_{39}{}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi(G_{43}{}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0\,, T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0\,, T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |

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| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^*$ given by $\varphi((G_{23})^*) = 0$ , $T_{21}^*$ given by $f(T_{21}^*) = 0$ and<br>$G_{20}^* = \frac{(a_{20})^{(3)} G_{21}^*}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^*)]}$ , $G_{22}^* = \frac{(a_{22})^{(3)} G_{21}^*}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^*)]}$<br>$T_{20}^* = \frac{(b_{20})^{(3)} T_{21}^*}{[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}{}^*)]}$ , $T_{22}^* = \frac{(b_{22})^{(3)} T_{21}^*}{[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}{}^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^*$ given by $\varphi(G_{27}) = 0$ , $T_{25}^*$ given by $f(T_{25}^*) = 0$ and<br>$G_{24}^* = \frac{(a_{24})^{(4)} G_{25}^*}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}^*)]}$ , $G_{26}^* = \frac{(a_{26})^{(4)} G_{25}^*}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}^*)]}$ | 516 |
| $T_{24}^* = \frac{(b_{24})^{(4)} T_{25}^*}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27})^*)]}$ , $T_{26}^* = \frac{(b_{26})^{(4)} T_{25}^*}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^*$ given by $\varphi((G_{31})^*) = 0$ , $T_{29}^*$ given by $f(T_{29}^*) = 0$ and<br>$G_{28}^* = \frac{(a_{28})^{(5)} G_{29}^*}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^*)]}$ , $G_{30}^* = \frac{(a_{30})^{(5)} G_{29}^*}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^*)]}$ | 518 |
| $T_{28}^* = \frac{(b_{28})^{(5)} T_{29}^*}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^*)]}$ , $T_{30}^* = \frac{(b_{30})^{(5)} T_{29}^*}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^*$ given by $\varphi((G_{35})^*) = 0$ , $T_{33}^*$ given by $f(T_{33}^*) = 0$ and<br>$G_{32}^* = \frac{(a_{32})^{(6)} G_{33}^*}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^*)]}$ , $G_{34}^* = \frac{(a_{34})^{(6)} G_{33}^*}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^*)]}$ | 520 |
| $T_{32}^* = \frac{(b_{32})^{(6)} T_{33}^*}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^*)]}$ , $T_{34}^* = \frac{(b_{34})^{(6)} T_{33}^*}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^*$ given by $\varphi((G_{39})^*) = 0$ , $T_{37}^*$ given by $f(T_{37}^*) = 0$ and | 522 |

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| $G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)} (T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)} (T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)} ((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)} ((G_{39})^{*})]}$ | |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)} (T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)} (T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)} ((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)} ((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br><br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br><br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)} (T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)} (T_{45}^{*})]}$<br><br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)} ((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)} ((G_{47})^{*})]}$ | 523 A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:**_Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}} (T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j} (G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)} \mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*} \mathbb{T}_{14}$ | 525 |

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| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^{*}\mathbb{G}_{j}\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^{*}\mathbb{G}_{j}\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^{*}\mathbb{G}_{j}\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 532 |
| $\frac{\partial(a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial(b_i^{''})^{(2)}}{\partial G_j}((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^{*}\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | 540 |

$$G_i = G_i^* + \mathbb{G}_i \qquad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)} \quad , \frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^*\mathbb{T}_{21} \quad (541)$$

$$\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^*\mathbb{T}_{21} \quad (542)$$

$$\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^*\mathbb{T}_{21} \quad (543)$$

$$\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^*\mathbb{G}_j\right) \quad (544)$$

$$\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^*\mathbb{G}_j\right) \quad (545)$$

$$\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^*\mathbb{G}_j\right) \quad (546)$$

**ASYMPTOTIC STABILITY ANALYSIS** (547)

**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ *and* $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- (548)

$$G_i = G_i^* + \mathbb{G}_i \qquad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)} \quad , \frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^*) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25} \quad (549)$$

$$\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25} \quad (550)$$

$$\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25} \quad (551)$$

$$\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right) \quad (552)$$

$$\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right) \quad (553)$$

$$\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right) \quad (554)$$

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^*) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^*\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^*\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^*\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |

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| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{37})^{(7)}}{\partial T_{37}}(T^*_{37}) = (q_{37})^{(7)}$ , $\frac{\partial(b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a'_{36})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G^*_{36}\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a'_{37})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G^*_{37}\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. | |

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| Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \ , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}((G_{43})^*) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^*\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^*\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b_{41}^{'})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T_{41}^*\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b_{42}^{'})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T_{42}^*\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS** **Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. **Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |

| | |
|---|---|
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$ $\left[\left(\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\right]$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\right)$ $+\left(\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\right)$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((b'_{13})^{(1)} + (b'_{14})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $+\left(\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$ $+\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G^*_{13}\right)$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T^*_{14} + (b_{14})^{(1)}s_{(13),(15)}T^*_{13}\right)\} = 0$ $+$ $\left((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)}\right)\{\left((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)}\right)$ $\left[\left(\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}^*_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^*_{16}\right)\right]$ $\left(\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^*_{17}\right)$ $+\left(\left((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}^*_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^*_{17}\right)$ $\left(\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^*_{16}\right)$ $\left(\left((\lambda)^{(2)}\right)^2 + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ $\left(\left((\lambda)^{(2)}\right)^2 + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ | |

$$+\left(\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\right)\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left(\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\right)$$

$$+\left(\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((b_{20}')^{(3)}+(b_{21}')^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}')^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}')^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}')^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}')^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}')^{(4)}+(a_{25}')^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((b_{24}')^{(4)}+(b_{25}')^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\}\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^2+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G^*_{33}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G^*_{32}\right)$$

$$\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T^*_{33}+(b_{33})^{(6)}s_{(32),(34)}T^*_{32}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b'_{38})^{(7)}-(r_{38})^{(7)}\}\{\left((\lambda)^{(7)}+(a'_{38})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(q_{36})^{(7)}G^*_{36}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T^*_{37}+(b_{37})^{(7)}s_{(36),(37)}T^*_{37}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a'_{37})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G^*_{36}+(a_{36})^{(7)}(q_{37})^{(7)}G^*_{37}\right)$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T^*_{37}+(b_{37})^{(7)}s_{(36),(36)}T^*_{36}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^2+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^*_{36}\right)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^*_{37}+(b_{37})^{(7)}s_{(36),(38)}T^*_{36}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^*_{40}\right)\right]$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^*_{41}+(b_{41})^{(8)}s_{(40),(41)}T^*_{41}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^*_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^*_{41}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^*_{41}+(b_{41})^{(8)}s_{(40),(40)}T^*_{40}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^*_{40}\right)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^*_{41}+(b_{41})^{(8)}s_{(40),(42)}T^*_{40}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^*_{44}\right)\right]$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^*_{45}+(b_{45})^{(9)}s_{(44),(45)}T^*_{45}\Big)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^*_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45}+(b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left(\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45}+(b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

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# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i(x)\, xi$ where $x_i \in R^n_i$, $x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i(x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i(\mu)$ of the $A_i(x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k \tag{11}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to

replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 11. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon\ C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \ i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0}$

$\overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow *x.t restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set *K* is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** *x.t* is defined for all $t \geq 0$

**A4:** *x.t* is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\mathcal{A}(\tilde{A}), C \backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \longrightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) In$ $||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population $i$ persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population $i$ is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) In \ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^{t}_{0} (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\to\infty}{limsup} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v = au_i(x) + w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |B_i(t,x)u_i(x)|(a + \alpha + exp(-\beta t)|w|)$. Hence, $\limsup_{t\to\infty} \frac{1}{t} In\, B_i(t,x)\, v| \leq$

$$\limsup_{t\to\infty} \frac{1}{t} In\, B_i(t,x)\, u_i(x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\, |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\, u_i(x)| = In\, |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\, u_i(x.t)||\, B_i(t,x)\, u_i(x)|$$

$$= \text{In } b_i(s,t.x)\, b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x - bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

$$=$$

$$= (A_i(x.t)u_i(x.s), u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s), u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } | (x.t)_i |$$

$$\leq \limsup_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x) =$ Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of

irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x)u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$ where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ and $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as $n\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$. Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\le \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x)u_i^n(x), u_i^n(x)\rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)|d\mu(x) + |\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\,d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\,ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i\,dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\,d\,v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak$^*$ limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^\infty$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \{ \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i \epsilon I}{if\ i \notin I}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ *whenever* $x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t$ −invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently *K* is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s =$

$s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_s$. If $r(x) = s(x), then\ x \in M_r$. Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume $x \mapsto A(x)$ is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists $j \in \{1,\ldots,k\}$ such that $M_i$ ----- $\{x \in C : x_l \gg 0 \text{ for } l \neq j\}$ Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_i M_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, *$\lambda_i(\mu) > 0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, ..., x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda(0)$, which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i(0) > 0$ for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the

system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j \ (x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1{}^1, \ldots, E_1{}^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i{}_2(E_1{}^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$$

$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > 1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria $E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$ and $\lambda_3(E_1)$ are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j, \qquad i = 1,\ldots,k;\ j = 1,\ldots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i\, a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i\, x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \qquad i = 1, \ldots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \ldots, k$$

It has been shown [12,ch.16.1,16.2] That for $k = 3$,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is

the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1-y) - b(N)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1 = N$, $x_2 = p$, $x_3 = (y,z)'$ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\curlywedge_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\curlywedge_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\curlywedge_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + K p\, x_i x_i - 1}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions

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# Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a

qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and

orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t\geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a\geq 0)$ at time $t$ so that we have

(1.31) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t\geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.32) $$D\rho_i(a, t) = \limsup_{h\to 0} \frac{\rho_i(a+h,t+h)-\rho_i(a,t)}{h},\ i = 1, 2.$$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.33) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s\in[-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s\in[-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1, \varphi_2, f_1, f_2, b_1, b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for

each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as* $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 11. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'}\ - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, \text{R}^-),$$

$A_3$. $\varphi_i\ (\cdot\ , s) \in L^2\ (R^+, R^+)\ \cap C(R^+, R^+)$ for each $s \in\ [-\tau, 0]$,

$i, = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[-\int_0^\infty f_i(a + s, P(t + S))ds\right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[-\int_o^x f_1(a + s, p(t + s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0)\exp\left[-\int_o^t f_1(a - t + s, p_1(s))\, ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

*(2.3)* $p_i(a,t) = B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i and\ L_i\ (i = 1,2)$ as follows:

$$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a,t,\ p_i(t)) = \exp[-\int_o^t f_1(a+s,\ p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

$$(2.6)\ p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da + \int_o^\infty \varphi_i(a, 0)\ L_i(a,t,p_i(t)da,$$

$$t > 0, i = 1,2,$$

$$(2.7)\ p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$$

$$(2.8)\ \beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.\left(a,t,P_i(t)\right)da$$

$$+\int_0^\infty b_i\left(a+t, P_i(t)\right)\varphi_i(a,0)L_i\left(a,t,P_i(t)\right)da$$

$$i = 1, 2, t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t) \phi(0) d_i^*] t \quad for \ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T], R^+), \quad B_i \in C([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in C([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau, T]$. Then for any

$(x, y) \in C^+[-\tau, T] \times C^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau, T], \quad C_T^2 = C^+[-\tau, T] \times C^+[-\tau, T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for $i = 1,2$.

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \pi C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \quad P_t = \ (P_{1t}, P_{2t}) \in C^2_{t*} \mid \|P_t - \Phi_t\|_{t*} \le r^*$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)\, da \ for\ s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)\, da \ for\ s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Phi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^* \right] \ t^*$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for t $\le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \le \left| \int_0^t B_1^{(1)}(a) \left\{ M_1(a, t, P_t^{(1)}) - M_1(a, t, P_t^{(2)}) \right\} da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a, t, P_t^{(2)}) da \right|$$

$$+ \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \; for \; x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \| ds\right) da$$

$$\leq d_1 \beta_1^* \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o)\left[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds\right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \| t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\text{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right]\right] d\sigma$$

$$+ \int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^{\infty} \varphi(\sigma,o) b_1\left(\sigma+a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma,a,P_t^{(1)}(a)\right)-L_1\left(\sigma,a,P_t^{(2)}(a)\right)\right]d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right)-b_1\left(\sigma+a, P_t^{(2)}(a)\right)\right| \le B_1 \,/\!/\, P_t^{(1)}-P_t^{(2)} /\!/ \text{ for } a\,\varepsilon[o,t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \le \beta_1 \Phi_1(0)||P_t^{(1)}-P_t^{(2)}||_t^{*}+\Phi_1(0)\beta_1 d_1||P_t^{(1)}-P_t^{(2)}||_t^{*a},$$

$$|R_2(a)|=[\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a))-b_1(a-\sigma, P_t^{(2)}(a))\}M_1(\sigma, a, P_t^{(1)}(a))$$

$$+b_1(a,-\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a))-M_1(a-\sigma, P_t^{(2)}(a))\}]d\sigma]$$

$$+b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a))-M_1(a-\sigma, P_t^{(2)}(a))\}]d\sigma]$$

$$\le \int_0^a [\beta_1^{*}\Phi_1(0)\, e^{\delta 1^a}\beta_1 \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}+\beta_1 d_1 \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}(a-o)]do$$

$$\le \beta_1^{*}\Phi_1(0)\, e^{\delta 1^a} \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\{\beta_1 a+d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a)\right|$$

$$\le \int_o^a \left| B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma)\right| B_1^{*d}\sigma$$

$$+\Phi_1(0) /\!/ P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\left\{B_1+B_2 d_1 a+B_1^{*}\left(B_2 a+d_1 a^2/2\right)e^{\delta 1^a}\right\}$$

$$\le \beta_1^{*}\int_o^a \left| B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma)\right| do+\left(r^{*}+/\!/\Phi_t /\!/_{t*}\right)\| P_t^{(1)}-P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a)\to \beta_1$ as $a\to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a)\right| \le \left(r^{*}+/\!/\Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))\text{-}b_1(a\text{-}\sigma, P_t^{(1)}(a))\, M_1(\sigma, a, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + /\!/ \Phi_t /\!/_{t*}\right) /\!/ \; P_t^{(1)}\text{-}P_t^{(2)} /\!/_{t*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right) /\!/ \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^{*})$$

As $t^{*} \to o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^{*}, P_t^{*}$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp\, P_2^{*}(a)}{da} + f_i(\mathrm{a}, P_1^{*}, P_2^{*})\, P_i^{*}(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^{*} = \int_o^{\infty} P_i^{*}(a)da, i = 1,\ 2,$$

$$(3,3)\ P_i^{*} = \int_o^{\infty} b_i(a,\ P_1^{*},\ P_2^{*}) P_i^{*}(a)da,\ i = 1.\ \ 2$$

All solutions of (3.1)are of the from

$$(3.4)\ P_i^{*}(a) = P_i^{*}(o) \exp\left[ - \int_o^{\infty} f_i\,(s, P_1^{*}, P_2^{*})ds \right], i = 1, \quad 2, a \geq 0,$$

Which together with (3.3) lead to

$$1=\int_o^{\infty} b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^{\infty} b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $$F_{i.}\ (P_1,P_2) = \int_o^{\infty} b_i(a,P_1,P_2)\exp\left[-\int_o^{\infty} f_i\,(s,P_1,P_2)ds\right]da,$$

$$i = 1,\ 2, \quad P_1 \geq 0, \quad P_2 \geq 0,$$

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_5$. *there exists a positive constant c for which*

$$F_1(\mathrm{C},0) < F_{2.}(\mathrm{C},0),$$

(3.8) $$F_1(0.\mathrm{C}) < F_{2.}(0,\mathrm{C}),$$

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $$F_1(P_1^*,P_2^*) = 1 = F_2(P_1^*,P_2^*)$$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $$P_2^*(a) = [P_2^*\exp[-f_i\,(s,P_1^*,P_2^*)ds])/(\int_o^{\infty}\exp[-\int_o^a f_i\,(s,P_1^*,P_2^*)ds]da),$$

$$i = 1,\ 2, \quad a \geq 0,$$

Proof Consider the elements of the surfaces defined by

$$\mathrm{Z}= F_1(P_1,P_2), z = F_2(P_1,P_2),\ P_1 \geq 0, P_2 \geq 0,$$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i/\partial P_j < 0\ and$ , by ($A_2$), $\partial b_i/\partial P_j \leq 0$ for all

$$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$$

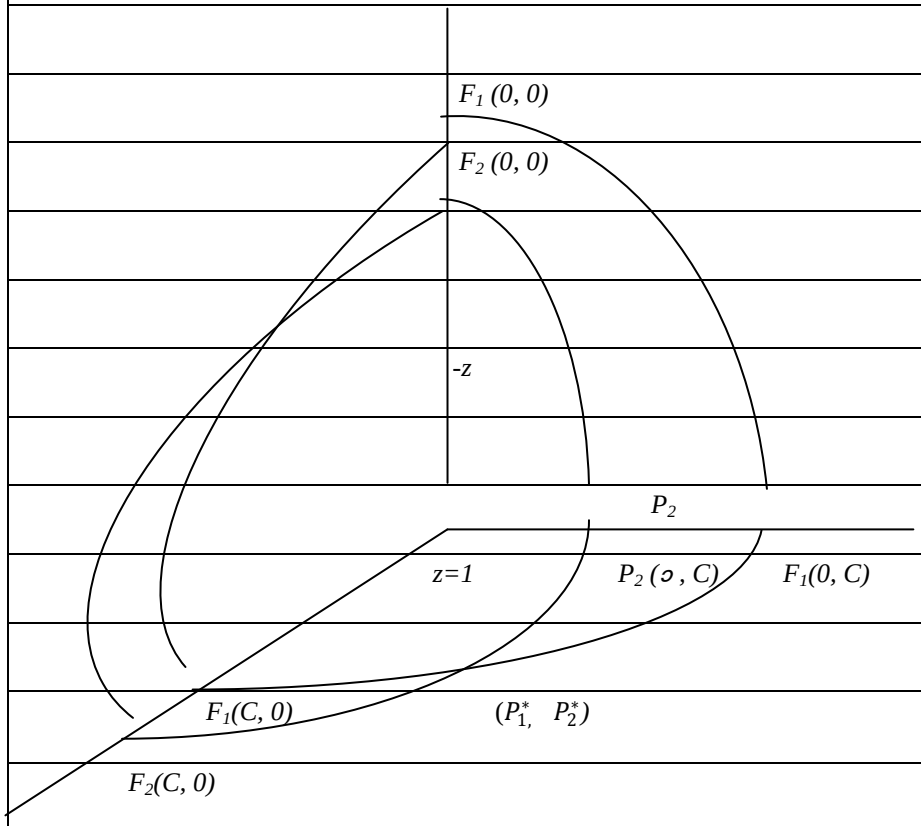
$j = 1,\ 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= 1, $P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$$F_1(P_1, P_2 = 1 \; and \; F_2(P_1, P_2) = 1, \; F_1(P_1, P_2 \geq 0,$$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p_1^*, p_2^*)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p_1^*, p_2^*$, we let

$$(3.11) \quad \begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*}\, p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square=1}^{2} \frac{\partial b_i}{\partial p_{\square}^*}\, p_{i\square}(t)\right\} da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_{\square}(a)\exp[\lambda t], \quad \square(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty \square(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d\square}{da} + \lambda\square = -f_i\left(a, p_1^*, p_2^*\right)\square(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*}\, p_{i\square}^*\right\},$$

(3.15)

$$\square(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\square(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*}\, p_{i\square}^*\right\} da,$$

Where $i = 1, 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \quad p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\Phi_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\Phi_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\Phi_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\Phi_i(a) = \{\Phi_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_j\right\} e^{\lambda s} ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\Phi_i(a)\left[p_i^*(a)/p_i^*(0)c_i \exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_j\right\} da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a}\left\{\int_0^a \left(\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_j\right) e^{\lambda s} ds\right\} da\right]$$

$$- p_i^*(a) e^{-\lambda a}\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_j\right\} e^{\lambda s} ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\Phi_1(a)$ and $\Phi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^* \exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^* \exp(-\lambda\tau) + (\lambda - A_{22})p_2^* = g_{21}(\lambda)p_1^* \exp(-\lambda\tau) + g_{22}(\lambda)p_2^*$$

Where

(3.22) $$A_{ij} = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_j^*} - \frac{\partial f_i}{\partial p_j^*}\right]da,$$

(3.23) $$g_{ij}(\lambda) = \left[p_i^*(0)c_i\right]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\,da\right\}$$
$$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_j^*} - b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)\right]da\right\}$$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)da,\quad i, j = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i\left(a, p_1^*, p_2^*\right) - f_i\left(a, p_1^*, p_2^*\right),\quad i, = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $\left(p_1^*, p_2^*\right)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix}\lambda - \left(A_{11} + g_{11}(\lambda)\right) & -\left(A_{12} + g_{12}\right)e^{-\lambda\tau} \\ -\left(A_{21} + g_{21}\right)e^{-\lambda\tau} & \lambda - \left(A_{22} + g_{22}(\lambda)\right)\end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda,\tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda,\tau) = \left[A_{21}\,g_{12}(\lambda) + A_{12}\,g_{21}(\lambda) + g_{12}(\lambda)\,g_{21}(\lambda)\right]\exp[-2\lambda\tau]$

$-A_{11}\,g_{22}(\lambda) - A_{22}\,g_{11}(\lambda) - g_{11}(\lambda)\,g_{22}(\lambda)$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6$. $0 < \beta_i^* < d_i^*$ $(i, = 1, 2.)$ $\left(see(A_1)\ and\ (A_2)\right)$;

$A_7$. $\{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|$ ,

$\| S_2(0) \| = |A_{11}\,g_{12}(0)| + |A_{12}\,g_{21}(0)| + |g_{12}(0)\,g_{21}(0)|$

$+ |A_{11}\,g_{22}(0)| + |A_{22}\,g_{11}(0)| + |g_{11}(0)\,g_{22}(0)|$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i(a, t) = \phi_i(a) \exp[\lambda t] \to 0$ *as* $t \to \infty$ for $i = 1, 2$ *and* $a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$$F_1(\lambda, \tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau];$$

$$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau),$$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$, $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda$ $\mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - \left| F_2(\lambda, \tau) \right|$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2,$$

$$q = \{4 | (A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| | - | A_{11} + A_{22} + \| S_1(0) \| |^2\}/2,$$

It will now follow, from (A) and (3.26), $F_1(\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder

concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power

obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy

withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation

from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## SECTION A(4)-2

## Quantum Algorithms For Gibbs Sampling And Hitting-Time Estimation

### INTRODUCTION—VARIABLES USED

**Quantum algorithms for Gibbs sampling and hitting-time estimation Anirban Narayan Chowdhury, Rolando D. Somma (Submitted on 9 Mar 2016)**

(1) **Anirban Narayan Chowdhury, Rolando D. Somma** present quantum algorithms for (e) solving two problems regarding stochastic processes.

(2) The first algorithm prepares (eb) the thermal Gibbs state of a quantum system and runs in time almost linear in (eb) Nβ/Z−−−−−√ and polynomial in log (1/ϵ), where (e) N is the Hilbert space dimension, β is (=) the inverse temperature, Z is (=) the partition function, and ϵ is (=) the desired precision of (e) the output state.

(3) Quantum algorithm exponentially improves (eb) the dependence on 1/ϵ and quadratically improves (eb) the dependence on β of (e) known quantum algorithms for this problem.

(4) The second algorithm estimates the hitting time of (e) a Markov chain. For a sparse stochastic matrix P, it runs in time almost linear in (eb)1/ (ϵΔ3/2), where (e) ϵ is the absolute precision in the estimation and Δ is a parameter determined by (e) P, and whose inverse is (=) an upper bound of the hitting time.

**(5)** Quantum algorithm quadratically improves (eb) the dependence on 1/ϵ and 1/Δ of the analog classical algorithm for (e) hitting-time estimation.

**(6)** Both algorithms use tools recently developed in the context of Hamiltonian simulation, spectral gap amplification, and solving linear systems of equations. Subjects: Quantum Physics (quant-ph) Report number: LA-UR-16-21218 Cite as: arXiv: 1603.02940 [quant-ph] (or arXiv:

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| 1603.02940v1 [quant-ph] for this version) | |
| **NOTATION** | |
| **Module One**<br>**Anirban Narayan Chowdhury, Rolando D. Somma** present quantum algorithms for (e) solving two problems regarding stochastic processes | |
| $G_{13}$ : Category one of solving two problems regarding stochastic processes<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of quantum algorithms<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>The first algorithm prepares (eb) the thermal Gibbs state of a quantum system and runs in time almost linear in (eb) Nβ/Z−−−−−√ and polynomial in log (1/ϵ), where (e) N is the Hilbert space dimension, β is (=) the inverse temperature, Z is (=) the partition function, and ϵ is (=) the desired precision of (e) the output state. | |
| $G_{16}$ : Category one of first algorithm<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of thermal Gibbs state of a quantum system and runs in time almost linear in (eb) Nβ/Z−−−−−√ and polynomial in log (1/ϵ), where (e) N is the Hilbert space dimension, β is (=) the inverse temperature, Z is (=) the partition function, and ϵ is (=) the desired precision of (e) the output state<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>The first algorithm prepares the thermal Gibbs state of a quantum system and runs in time almost linear in (eb) Nβ/Z−−−−−√ and polynomial in log (1/ϵ), where (e) N is the Hilbert space dimension, β is (=) the inverse temperature, Z is (=) the partition function, and ϵ is (=) the desired precision of (e) the output state | |
| $G_{20}$ : Category one of first algorithm prepares the thermal Gibbs state of a quantum system and runs in time almost linear<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of Nβ/Z−−−−−√ and polynomial in log (1/ϵ), where (e) N is the Hilbert space dimension, β is (=) the inverse temperature, Z is (=) the partition function, and ϵ is (=) the desired precision of (e) the | |

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| output state<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>The first algorithm prepares the thermal Gibbs state of a quantum system and runs in time almost linear in Nβ/Z−−−−−√ and polynomial in log (1/ϵ), where (e) N is the Hilbert space dimension, β is (=) the inverse temperature, Z is (=) the partition function, and ϵ is (=) the desired precision of (e) the output state | |
| $G_{24}$ : Category one of N is the Hilbert space dimension, β is (=) the inverse temperature, Z is (=) the partition function, and ϵ is (=) the desired precision of (e) the output state<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of first algorithm prepares the thermal Gibbs state of a quantum system and runs in time almost linear in Nβ/Z−−−−−√ and polynomial in log (1/ϵ)<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>The first algorithm prepares the thermal Gibbs state of a quantum system and runs in time almost linear in Nβ/Z−−−−−√ and polynomial in log (1/ϵ), where N is the Hilbert space dimension, β is the inverse temperature, Z is the partition function, and ϵ is (=) the desired precision of the output state | |
| $G_{28}$ : Category one of first algorithm prepares the thermal Gibbs state of a quantum system and runs in time almost linear in Nβ/Z−−−−−√ and polynomial in log (1/ϵ), where N is the Hilbert space dimension, β is the inverse temperature, Z is the partition function, and ϵ<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of desired precision of the output state<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Quantum algorithm exponentially improves (eb) the dependence on 1/ϵ and quadratically improves (eb) the dependence on β of (e) known quantum algorithms for this problem | |
| $G_{32}$ : Category one of Quantum algorithm exponentially | |

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| $G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of dependence on 1/ϵ and quadratically improves (eb) the dependence on β of (e) known quantum algorithms for this problem<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Quantum algorithm exponentially improves the dependence on 1/ϵ and quadratically improves (eb) the dependence on β of (e) known quantum algorithms for this problem | |
| $G_{36}$ : Category one of Quantum algorithm exponentially improves the dependence on 1/ϵ and quadratically improves<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of dependence on β of known quantum algorithms for this problem<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The second algorithm estimates the hitting time of (e) a Markov chain.<br><br>For a sparse stochastic matrix P, it runs in time almost linear in (eb)1/ (ϵΔ3/2), where ϵ is the absolute precision in the estimation and Δ is a parameter determined by P, and whose inverse is (=) an upper bound of the hitting time | |
| $G_{40}$ : Category one of sparse stochastic matrix P, it runs in time almost linear<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of1/ (ϵΔ3/2), where ϵ is the absolute precision in the estimation and Δ is a parameter determined by P, and whose inverse is (=) an upper bound of the hitting time<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine** | |

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| For a sparse stochastic matrix P, it runs in time almost linear in (eb)1/ (ϵΔ3/2), where ϵ is the absolute precision in the estimation and Δ is a parameter determined by P, and whose inverse is (=) an upper bound of the hitting time | |
| $G_{44}$ : Category one of sparse stochastic matrix P, it runs in time almost linear in (eb)1/ (ϵΔ3/2), where ϵ is the absolute precision in the estimation and Δ is a parameter determined by P, and whose inverse<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of upper bound of the hitting time<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION A(5)**<br><br>**Jarzynski's Theorem For Lattice Gauge Theory** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Jarzynski's theorem for lattice gauge theory Michele Caselle, Gianluca Costagliola, Alessandro Nada, Marco Panero, Arianna Toniato (Submitted on 19 Apr 2016)**<br><br>(1) Jarzynski's theorem is a well-known equality in statistical mechanics, which relates fluctuations in the work performed during (e&eb) a non-equilibrium transformation of a system, to (e) the free-energy difference between (e&eb) two equilibrium states.<br>(2) In this article, authors extend Jarzynski's theorem to (e&eb) lattice gauge theory, and present examples of applications for two challenging computational problems, namely the calculation of (e) interface free energies and the determination of (e) the equation of state.<br>**(3)** Authors conclude with a discussion of further applications of interest in (e&eb) QCD and in other strongly coupled gauge theories, in particular for the Schrödinger functional and for simulations at finite density using (e) reweighting techniques. Subjects: High Energy Physics - Lattice (hep-lat); Statistical Mechanics (cond-mat.stat-mech); High Energy Physics - Theory (hep-th) Report number: CP3-Origins-2016-020 DNRF90, DIAS-2016-20 Cite as: arXiv: 1604.05544 [hep-lat] (or arXiv: 1604.05544v1 [hep-lat] for this version) | |
| **NOTATION** | |
| **Module One**<br>Jarzynski's theorem is a well-known equality in statistical mechanics, which relates fluctuations in the work performed during a non-equilibrium transformation of a system, to (e) the free-energy difference between two equilibrium states | |
| $G_{13}$ : Category one of **Jarzynski's theorem is a well-known equality in statistical mechanics, which relates fluctuations in the work performed during a non-equilibrium transformation of a system** ; | |

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| free-energy difference between two equilibrium states<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of free-energy difference between two equilibrium states; **Jarzynski's theorem is a well-known equality in statistical mechanics, which relates fluctuations in the work performed during a non-equilibrium transformation of a system**<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>In this article, authors extend Jarzynski's theorem to (e&eb) lattice gauge theory, and present examples of applications for two challenging computational problems, namely the calculation of (e) interface free energies and the determination of (e) the equation of state | |
| $G_{16}$ : Category one of **Jarzynski's theorem;** lattice gauge theory, and present examples of applications for two challenging computational problems, namely the calculation of (e) interface free energies and the determination of (e) the equation of state<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of lattice gauge theory, and present examples of applications for two challenging computational problems, namely the calculation of (e) interface free energies and the determination of (e) the equation of state; **Jarzynski's theorem**<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>In this article, authors extend Jarzynski's theorem to lattice gauge theory, and present examples of applications for two challenging computational problems, namely the calculation of (e) interface free energies and the determination of (e) the equation of state | |
| $G_{20}$ : Category one of interface free energies and the determination of (e) the equation of state<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of Jarzynski's theorem to lattice gauge theory, and present examples of applications for two challenging computational problems, namely the calculation<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |

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| **Module four**<br><br>In this article, authors extend Jarzynski's theorem to lattice gauge theory, and present examples of applications for two challenging computational problems, namely the **calculation of interface free energies** and the determination of the equation of state | |
| $G_{24}$ : Category one of equation of state<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of Jarzynski's theorem to lattice gauge theory, and present examples of applications for two challenging computational problems, namely the calculation of (e) interface free energies and the determination<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five** | |
| $G_{28}$ : Category one of **calculation of interface free energies;** determination of the equation of state<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of determination of the equation of state; **calculation of interface free energies**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six** | |
| $G_{32}$ : Category one of quantum walk; **determination of the equation of state**<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of **determination of the equation of state;** quantum walk<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Authors conclude with a discussion of further applications of interest in (e&eb) QCD and in other strongly coupled gauge theories, in particular for the Schrödinger functional and for simulations at finite density | |

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| using (e) reweighting techniques.<br><br>Subjects: High Energy Physics - Lattice (hep-lat); Statistical Mechanics (cond-mat.stat-mech); High Energy Physics - Theory (hep-th) Report number: CP3-Origins-2016-020 DNRF90, DIAS-2016-20 Cite as: arXiv: 1604.05544 [hep-lat] (or arXiv: 1604.05544v1 [hep-lat] for this version) | |
| $G_{36}$ : Category one of **further applications of interest;** QCD and in other strongly coupled gauge theories, in particular for the Schrödinger functional and for simulations at finite density using (e) reweighting techniques.<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of QCD and in other strongly coupled gauge theories, in particular for the Schrödinger functional and for simulations at finite density using (e) reweighting techniques; **further applications of interest**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Authors conclude with a discussion of further applications of interest in QCD and in other strongly coupled gauge theories, in particular for the Schrödinger functional and for simulations at finite density using reweighting techniques | |
| $G_{40}$ : Category one of **further applications of interest in QCD and in other strongly coupled gauge theories;** Schrödinger functional<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of Schrödinger functional; **further applications of interest in QCD and in other strongly coupled gauge theories**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine** | |
| $G_{44}$ : Category one of **further applications of interest in QCD and in other strongly coupled gauge theories;** simulations at finite density using reweighting techniques<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |

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| $T_{44}$ : Category one of simulations at finite density using reweighting techniques **;further applications of interest in QCD and in other strongly coupled gauge theories;**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION A(6)

## Dark-Soliton-Like Excitations And Yang-Gaudin Gas Of Attractively Interacting Fermions

### INTRODUCTION—VARIABLES USED

**Dark-soliton-like excitations in the Yang-Gaudin gas of attractively interacting fermions Sophie S. Shamailov, Joachim Brand (Submitted on 15 Mar 2016 (v1), last revised 3 May 2016 (this version, v2))**

(1) Yrast states are (=) the lowest energy states at (eb) given non-zero momentum and provide (eb) a natural extension of the concept of dark solitons to (e) strongly-interacting one-dimensional quantum gases.
(2) Here authors study the yrast states of (e) the balanced spin-12 Fermi gas with (e&eb) attractive delta-function interactions in one dimension with (e&eb) the exactly solvable Yang-Gaudin model. The corresponding Bethe-ansatz equations are solved for (e) finite particle number and in the thermodynamic limit.
(3) Properties corresponding to (e&eb) the soliton-like nature of the yrast excitations are calculated including (e) the missing particle number, phase step, and inertial and physical masses.
(4) The inertial to physical mass ratio, which is related to (e&eb) the frequency of oscillations in a trapped gas, is found to be unity in the limits of strong and weak attraction and fall to ≈0.78 in the crossover regime.
(5) This result is contrasted by (e) one-dimensional mean field theory, which predicts (eb) a divergent mass ratio in the weakly attractive limit.
(6) By means of exact mapping results also predict (eb) the existence and properties of (e) dark-soliton-like excitations in (eb) the super Tonks-Girardeau gas.
**(7)** The prospects for experimental observations are briefly discussed. Subjects: Quantum Gases (cond-mat.quant-gas) Cite as: arXiv: 1603.04864 [cond-mat.quant-gas] (or arXiv: 1603.04864v2 [cond-mat.quant-gas] for this version)

### NOTATION

**Module One**

Yrast states are (=) the lowest energy states at (eb) given non-zero momentum and provide (eb) a natural extension of the concept of dark solitons to (e) strongly-interacting one-dimensional quantum gases

$G_{13}$ : Category one of Yrast states

$G_{14}$ : Category two of SAS

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| $G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of lowest energy states at (eb) given non-zero momentum and provide (eb) a natural extension of the concept of dark solitons to (e) strongly-interacting one-dimensional quantum gases<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>Yrast states are the lowest energy states at (eb) given non-zero momentum and provide (eb) a natural extension of the concept of dark solitons to (e) strongly-interacting one-dimensional quantum gases | |
| $G_{16}$ : Category one of Yrast states are the lowest energy states<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of given non-zero momentum and provide (eb) a natural extension of the concept of dark solitons to (e) strongly-interacting one-dimensional quantum gases<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>Yrast states are the lowest energy states at given non-zero momentum and provide (eb) a natural extension of the concept of dark solitons to (e) strongly-interacting one-dimensional quantum gases | |
| $G_{20}$ : Category one of Yrast states are the lowest energy states at given non-zero momentum<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of natural extension of the concept of dark solitons to (e) strongly-interacting one-dimensional quantum gases<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Yrast states are the lowest energy states at given non-zero momentum and provide a natural extension of the concept of dark solitons to (e) strongly-interacting one-dimensional quantum gases | |
| $G_{24}$ : Category one of **Yrast states are the lowest energy states at given non-zero momentum and provide a natural extension of the concept of dark solitons;** strongly-interacting one-dimensional quantum gases<br><br>$G_{25}$ : Category two of SAS | |

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| $G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of strongly-interacting one-dimensional quantum gases **;Yrast states are the lowest energy states at given non-zero momentum and provide a natural extension of the concept of dark solitons**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Here authors study the **yrast states of the balanced spin-12 Fermi gas** with (e&eb) attractive delta-function interactions in one dimension with (e&eb) the exactly solvable Yang-Gaudin model.<br><br>The corresponding Bethe-ansatz equations are solved for (e) finite particle number and in the thermodynamic limit | |
| $G_{28}$ : Category one of **yrast states of the balanced spin-12 Fermi gas;** attractive delta-function interactions in one dimension with (e&eb) the exactly solvable Yang-Gaudin model.<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of attractive delta-function interactions in one dimension with (e&eb) the exactly solvable Yang-Gaudin model. ; **yrast states of the balanced spin-12 Fermi gas**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Here authors study the **yrast states of the balanced spin-12 Fermi gas with attractive delta-function interactions in one dimension** with (e&eb) the exactly solvable Yang-Gaudin model. | |
| $G_{32}$ : Category one of **yrast states of the balanced spin-12 Fermi gas with attractive delta-function interactions in one dimension;** exactly solvable Yang-Gaudin model.<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of exactly solvable Yang-Gaudin model.;**yrast states of the balanced spin-12 Fermi gas with attractive delta-function interactions in one dimension**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |

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| **Module seven**<br><br>Properties corresponding to (e&eb) the soliton-like nature of the yrast excitations are calculated including (e) the missing particle number, phase step, and inertial and physical masses | |
| $G_{36}$ : Category one of **Properties;** soliton-like nature of the yrast excitations are calculated including (e) the missing particle number, phase step, and inertial and physical masses<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of soliton-like nature of the yrast excitations are calculated including (e) the missing particle number, phase step, and inertial and physical masses **;Properties**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Properties corresponding to the soliton-like nature of the yrast excitations are calculated including (e) the missing particle number, phase step, and inertial and physical masses | |
| $G_{40}$ : Category one of missing particle number, phase step, and inertial and physical masses<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of Properties corresponding to the soliton-like nature of the yrast excitations are calculated<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The inertial to physical mass ratio, which is related to (e&eb) the frequency of oscillations in a trapped gas, is found to be unity in the limits of strong and weak attraction and fall to ≈0.78 in the crossover regime | |
| $G_{44}$ : Category one of **inertial to physical mass ratio;** frequency of oscillations in a trapped gas, is found to be unity in the limits of strong and weak attraction and fall to ≈0.78 in the crossover regime<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |

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| $T_{44}$ : Category one of frequency of oscillations in a trapped gas, is found to be unity in the limits of strong and weak attraction and fall to ≈0.78 in the crossover regime; **inertial to physical mass ratio**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$<br>$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$<br>$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}\ (a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,<br>$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$<br>$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,<br>$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,<br>$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b'_{13})^{(1)} - (b''_{13})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b'_{14})^{(1)} - (b''_{14})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b'_{15})^{(1)} - (b''_{15})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a''_{13})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b''_{13})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |

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| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19}), t)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}((G_{19}), t)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19}), t)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}((G_{19}), t) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}, t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}, t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21}, t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23}, t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25}, t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}, t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}), t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}), t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}), t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25}, t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}), t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29}, t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}, t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}), t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}), t)\right]T_{29}$ | 29 |

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| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b'_{30})^{(5)} - (b''_{30})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a''_{28})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b''_{28})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a'_{32})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b'_{32})^{(6)} - (b''_{32})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b'_{33})^{(6)} - (b''_{33})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b'_{34})^{(6)} - (b''_{34})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a''_{32})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b'_{36})^{(7)} - (b''_{36})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b'_{37})^{(7)} - (b''_{37})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b'_{38})^{(7)} - (b''_{38})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a''_{36})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a'_{41})^{(8)} + (a''_{41})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b'_{40})^{(8)} - (b''_{40})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b'_{41})^{(8)} - (b''_{41})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b'_{42})^{(8)} - (b''_{42})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |

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| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}\left((G_{47}),t\right) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}^{'})^{(1)} & +(a_{13}^{''})^{(1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,)}(T_{17},t) & +(a_{20}^{''})^{(3,3,)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t) & & +(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)} & +(a_{14}^{''})^{(1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,)}(T_{17},t) & +(a_{21}^{''})^{(3,3,)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t) & & +(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)} & +(a_{15}^{''})^{(1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,)}(T_{17},t) & +(a_{22}^{''})^{(3,3,)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t) & & +(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{15}$ | *57* |
| Where $(a_{13}^{''})^{(1)}(T_{14},t)$, $(a_{14}^{''})^{(1)}(T_{14},t)$, $(a_{15}^{''})^{(1)}(T_{14},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{38}^{''})^{(7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3<br>$+(a_{40}^{''})^{(8,8)}(T_{41},t)$ $+(a_{41}^{''})^{(8,8)}(T_{41},t)$ $+(a_{42}^{''})^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3<br>$+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)} & -(b_{13}^{''})^{(1)}(G,t) & -(b_{16}^{''})^{(2,2,)}(G_{19},t) & -(b_{20}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t) & & -(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,)}(G_{39},t) & -(b_{40}^{''})^{(8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{13}$ | 58 |

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| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\boxed{-(b_{15}^{''})^{(1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | *60* |
| Where $\boxed{-(b_{13}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}^{'})^{(2)}\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}^{'})^{(2)}\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{17}$ | *62* |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}^{'})^{(2)}\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{18}$ | *63* |
| Where $\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)}$ are first augmentation coefficients for | |

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| category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$ , $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$ , $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9)}(T_{45},t)$ , $+(a_{45}^{''})^{(9,9)}(T_{45},t)$ , $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)} & -(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t) & \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{16}$ | *64* |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)} & -(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t) & \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{17}$ | *65* |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)} & -(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t) & \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{18}$ | *66* |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$ , $-(b_{17}^{''})^{(2)}(G_{19},t)$ , $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,)}(G,t)$ , $-(b_{14}^{''})^{(1,1,)}(G,t)$ , $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3<br>$-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$ , $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$ , $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3 | |

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| $-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[ \begin{array}{c} (a_{20}^{'})^{(3)} +(a_{20}^{''})^{(3)}(T_{21},t) \quad +(a_{16}^{''})^{(2,2,2)}(T_{17},t) \quad +(a_{13}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t) \quad +(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t) \quad +(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t) \quad +(a_{40}^{''})^{(8,8,8,8)}(T_{41},t) \quad +(a_{44}^{''})^{(9,9,9)}(T_{45},t) \end{array} \right] G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[ \begin{array}{c} (a_{21}^{'})^{(3)} +(a_{21}^{''})^{(3)}(T_{21},t) \quad +(a_{17}^{''})^{(2,2,2)}(T_{17},t) \quad +(a_{14}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t) \quad +(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t) \quad +(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7,7)}(T_{37},t) \quad +(a_{41}^{''})^{(8,8,8,8)}(T_{41},t) \quad +(a_{45}^{''})^{(9,9,9)}(T_{45},t) \end{array} \right] G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[ \begin{array}{c} (a_{22}^{'})^{(3)} +(a_{22}^{''})^{(3)}(T_{21},t) \quad +(a_{18}^{''})^{(2,2,2)}(T_{17},t) \quad +(a_{15}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t) \quad +(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t) \quad +(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7,7)}(T_{37},t) \quad +(a_{42}^{''})^{(8,8,8,8)}(T_{41},t) \quad +(a_{46}^{''})^{(9,9,9)}(T_{45},t) \end{array} \right] G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |

$$\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b'_{20})^{(3)}\boxed{-(b''_{20})^{(3)}(G_{23},t)}\boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)}\boxed{-(b''_{13})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)}\boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)}\boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7)}(G_{39},t)}\boxed{-(b''_{40})^{(8,8,8,8)}(G_{43},t)}\boxed{-(b''_{44})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{20} \quad (70)$$

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b'_{21})^{(3)}\boxed{-(b''_{21})^{(3)}(G_{23},t)}\boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)}\boxed{-(b''_{14})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)}\boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)}\boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7)}(G_{39},t)}\boxed{-(b''_{41})^{(8,8,8,8)}(G_{43},t)}\boxed{-(b''_{45})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)}\boxed{-(b''_{22})^{(3)}(G_{23},t)}\boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)}\boxed{-(b''_{15})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)}\boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)}\boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7)}(G_{39},t)}\boxed{-(b''_{42})^{(8,8,8,8)}(G_{43},t)}\boxed{-(b''_{46})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{22} \quad (72)$$

$\boxed{-(b''_{20})^{(3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3)}(G_{23},t)}$ *are first detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)}$ *are second detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{13})^{(1,1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,)}(G,t)}$ *are third detrition coefficients for category 1,2 and 3*

$\boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)}$ *are fourth detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)}$ *are fifth detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)}$ *are sixth detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{36})^{(7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7)}(G_{39},t)}$ $\boxed{-(b''_{38})^{(7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a'_{24})^{(4)}\boxed{+(a''_{24})^{(4)}(T_{25},t)}\boxed{+(a''_{28})^{(5,5,)}(T_{29},t)}\boxed{+(a''_{32})^{(6,6,)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1)}(T_{14},t)}\boxed{+(a''_{16})^{(2,2,2,2)}(T_{17},t)}\boxed{+(a''_{20})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7)}(T_{37},t)}\boxed{+(a''_{40})^{(8,8,8,8,8)}(T_{41},t)}\boxed{+(a''_{44})^{(9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a'_{25})^{(4)}\boxed{+(a''_{25})^{(4)}(T_{25},t)}\boxed{+(a''_{29})^{(5,5,)}(T_{29},t)}\boxed{+(a''_{33})^{(6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1)}(T_{14},t)}\boxed{+(a''_{17})^{(2,2,2,2)}(T_{17},t)}\boxed{+(a''_{21})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7)}(T_{37},t)}\boxed{+(a''_{41})^{(8,8,8,8,8)}(T_{41},t)}\boxed{+(a''_{45})^{(9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a'_{26})^{(4)} & +(a''_{26})^{(4)}(T_{25},t) & +(a''_{30})^{(5,5,)}(T_{29},t) & +(a''_{34})^{(6,6,)}(T_{33},t) \\ & +(a''_{15})^{(1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a''_{24})^{(4)}(T_{25},t)$, $(a''_{25})^{(4)}(T_{25},t)$, $(a''_{26})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a''_{28})^{(5,5,)}(T_{29},t)$, $+(a''_{29})^{(5,5,)}(T_{29},t)$, $+(a''_{30})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3

$+(a''_{32})^{(6,6,)}(T_{33},t)$, $+(a''_{33})^{(6,6,)}(T_{33},t)$, $+(a''_{34})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1)}(T_{14},t)$ *are fourth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{16})^{(2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2)}(T_{17},t)$ *are fifth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{20})^{(3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3)}(T_{21},t)$ *are sixth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{36})^{(7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7)}(T_{37},t)$ *are seventh augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{40})^{(8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8)}(T_{41},t)$ *are eighth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{46})^{(9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9)}(T_{45},t)$, $+(a''_{44})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b'_{24})^{(4)} & -(b''_{24})^{(4)}(G_{27},t) & -(b''_{28})^{(5,5,)}(G_{31},t) & -(b''_{32})^{(6,6,)}(G_{35},t) \\ & -(b''_{13})^{(1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b'_{25})^{(4)} & -(b''_{25})^{(4)}(G_{27},t) & -(b''_{29})^{(5,5,)}(G_{31},t) & -(b''_{33})^{(6,6,)}(G_{35},t) \\ & -(b''_{14})^{(1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b'_{26})^{(4)} & -(b''_{26})^{(4)}(G_{27},t) & -(b''_{30})^{(5,5,)}(G_{31},t) & -(b''_{34})^{(6,6,)}(G_{35},t) \\ & -(b''_{15})^{(1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{26} \quad (78)$$

*Where* $-(b''_{24})^{(4)}(G_{27},t)$, $-(b''_{25})^{(4)}(G_{27},t)$, $-(b''_{26})^{(4)}(G_{27},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{28})^{(5,5,)}(G_{31},t)$, $-(b''_{29})^{(5,5,)}(G_{31},t)$, $-(b''_{30})^{(5,5,)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b''_{32})^{(6,6,)}(G_{35},t)$, $-(b''_{33})^{(6,6,)}(G_{35},t)$, $-(b''_{34})^{(6,6,)}(G_{35},t)$ *are third detrition coefficients for category* 1, 2 *and* 3

| | |
|---|---|
| $-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2 and 3 $-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 $-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3 $-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[\begin{matrix} (a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,)}(T_{25},t) & +(a_{32}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{matrix}\right] G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[\begin{matrix} (a_{29}^{'})^{(5)}+(a_{29}^{''})^{(5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,)}(T_{25},t) & +(a_{33}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{matrix}\right] G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[\begin{matrix} (a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,)}(T_{25},t) & +(a_{34}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{matrix}\right] G_{30}$ | *81* |
| *Where* $+(a_{28}^{''})^{(5)}(T_{29},t)$, $+(a_{29}^{''})^{(5)}(T_{29},t)$, $+(a_{30}^{''})^{(5)}(T_{29},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3 *And* $+(a_{24}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,)}(T_{25},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3 $+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3 $+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3 $+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3 $+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3 $+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3 $+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation | |

coefficients for category 1,2, 3
$+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$ , $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$ , $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3

$$\frac{dT_{28}}{dt}=(b_{28})^{(5)}T_{29}-\left[\begin{array}{ccc}(b_{28}^{'})^{(5)} & -(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) \\ -(b_{32}^{''})^{(6,6,6)}(G_{35},t) & -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) \\ -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t) & -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) \\ -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t) & & \end{array}\right]T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt}=(b_{29})^{(5)}T_{28}-\left[\begin{array}{ccc}(b_{29}^{'})^{(5)} & -(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) \\ -(b_{33}^{''})^{(6,6,6)}(G_{35},t) & -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) \\ -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t) & -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) \\ -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t) & & \end{array}\right]T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt}=(b_{30})^{(5)}T_{29}-\left[\begin{array}{ccc}(b_{30}^{'})^{(5)} & -(b_{30}^{''})^{(5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,)}(G_{27},t) \\ -(b_{34}^{''})^{(6,6,6)}(G_{35},t) & -(b_{15}^{''})^{(1,1,1,1,1,)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t) \\ -(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t) & -(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t) \\ -(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t) & & \end{array}\right]T_{30} \quad (84)$$

*where* $-(b_{28}^{''})^{(5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3

$-(b_{32}^{''})^{(6,6,6)}(G_{35},t)$ , $-(b_{33}^{''})^{(6,6,6)}(G_{35},t)$ , $-(b_{34}^{''})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3

$-(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt}=(a_{32})^{(6)}G_{33}-\left[\begin{array}{ccc}(a_{32}^{'})^{(6)} & +(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) \\ +(a_{24}^{''})^{(4,4,4,)}(T_{25},t) & +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) \\ +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t) & +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) \\ +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t) & & \end{array}\right]G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a'_{33})^{(6)} & +(a''_{33})^{(6)}(T_{33},t) & +(a''_{29})^{(5,5,5)}(T_{29},t) & +(a''_{25})^{(4,4,4,)}(T_{25},t) \\ +(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients*

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|---|---|
| $for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{28}^{''})^{(5,5,5)}(G_{31},t)}, \boxed{-(b_{29}^{''})^{(5,5,5)}(G_{31},t)}, \boxed{-(b_{30}^{''})^{(5,5,5)}(G_{31},t)}\ are\ second\ detrition\ coefficients$ $for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)}\ are\ third\ detrition\ coefficients$ $for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2, and 3 | |
| $\frac{dG_{36}}{dt}$ $= (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)}\boxed{+(a_{36}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *91* |
| $\frac{dG_{37}}{dt}$ $= (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)}\boxed{+(a_{37}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *92* |
| $\frac{dG_{38}}{dt}$ $= (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)}\boxed{+(a_{38}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *93* |
| Where $\boxed{(a_{36}^{''})^{(7)}(T_{37},t)}, \boxed{(a_{37}^{''})^{(7)}(T_{37},t)}, \boxed{(a_{38}^{''})^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3 $\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3 | |

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|---|---|
| $+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ , $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ , $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$ , $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$ , $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3<br>$+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ , $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ , $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{36}}{dt} =$<br>$(b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 94 |
| $\frac{dT_{37}}{dt} =$<br>$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | |
| $\frac{dT_{38}}{dt} =$<br>$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | |
| Where $-(b_{36}^{''})^{(7)}(G_{39},t)$ , $-(b_{37}^{''})^{(7)}(G_{39},t)$ , $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ , $-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ , $-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ , $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ , $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 | |

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|---|---|
| $-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ , $-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ , $-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ , $-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ , $-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)$<br>are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ , $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ , $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3<br>$-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ , $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ , $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13}$ | *95* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}^{'})^{(8)}+(a_{41}^{''})^{(8)}(T_{41},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$ | |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$ | |
| Where $+(a_{40}^{''})^{(8)}(T_{41},t)$ , $+(a_{41}^{''})^{(8)}(T_{41},t)$ , $+(a_{42}^{''})^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ , $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ , $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ , $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ , $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ , $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ , $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 | |

$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & -(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & -(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & -(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a_{44}^{'})^{(9)} & +(a_{44}^{''})^{(9)}(T_{45},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t) \end{bmatrix} G_{13} \quad (96)$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a_{45}^{'})^{(9)} & +(a_{45}^{''})^{(9)}(T_{45},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t) \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a_{46}^{'})^{(9)} & +(a_{46}^{''})^{(9)}(T_{37},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t) \end{bmatrix} G_{15}$$

Where $+(a_{44}^{''})^{(9)}(T_{45},t)$, $+(a_{45}^{''})^{(9)}(T_{45},t)$, $+(a_{46}^{''})^{(9)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ are Seventh augmentation coefficient for category 1, 2 and 3

$+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$ are ninth

augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \begin{bmatrix} (b'_{46})^{(9)}\boxed{-(b''_{46})^{(9)}(G_{47},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{44})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$$(a_i)^{(1)}, (a'_i)^{(1)}, (a''_i)^{(1)}, (b_i)^{(1)}, (b'_i)^{(1)}, (b''_i)^{(1)} > 0, \quad i,j = 13,14,15 \qquad (97)$$

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| The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br><br>$(a_i^{''})^{(1)}(T_{14}, t) \leq (p_i)^{(1)} \leq (\hat{A}_{13})^{(1)}$<br><br>$(b_i^{''})^{(1)}(G, t) \leq (r_i)^{(1)} \leq (b_i^{'})^{(1)} \leq (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2 \to \infty}(a_i^{''})^{(1)}(T_{14}, t) = (p_i)^{(1)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(1)}(G, t) = (r_i)^{(1)}$<br><br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br><br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'}, t) - (a_i^{''})^{(1)}(T_{14}, t)\rvert \leq (\hat{k}_{13})^{(1)} \lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)} t}$<br><br>$\lvert(b_i^{''})^{(1)}(G^{'}, t) - (b_i^{''})^{(1)}(G, t)\rvert < (\hat{k}_{13})^{(1)} \lVert G - G^{'}\rVert e^{-(\hat{M}_{13})^{(1)} t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'}, t)$ and $(a_i^{''})^{(1)}(T_{14}, t)$ . $(T_{14}^{'}, t)$ and $(T_{14}, t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i^{'})^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)} (\hat{k}_{13})^{(1)}] < 1$<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i^{'})^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)} (\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |

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| $(a_i)^{(2)}, (a_i^{'})^{(2)}, (a_i^{''})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (b_i^{''})^{(2)} > 0, \quad i,j = 16,17,18$ | |
| The functions $(a_i^{''})^{(2)}, (b_i^{''})^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(\mathrm{p_i})^{(2)}$, $(\mathrm{r_i})^{(2)}$: | |
| $(a_i^{''})^{(2)}(T_{17}, t) \leq (p_i)^{(2)} \leq \left(\hat{A}_{16}\right)^{(2)}$ | 102 |
| $(b_i^{''})^{(2)}(G_{19}, t) \leq \ (r_i)^{(2)} \leq (b_i^{'})^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(2)}(T_{17}, t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty} (b_i^{''})^{(2)}\left((G_{19}), t\right) = \ (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $\boxed{(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, \ (r_i)^{(2)}}$ are positive constants and $\boxed{i = 16,17,18}$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'}, t) - (a_i^{''})^{(2)}(T_{17}, t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}^{'}\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'}, t) - (b_i^{''})^{(2)}\left((G_{19}), t\right)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})^{'}\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'}, t)$ and $(a_i^{''})^{(2)}(T_{17}, t)$ . $(T_{17}^{'}, t)$ and $(T_{17}, t)$ are points belonging to the interval $\left[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}\right]$. It is to be noted that $(a_i^{''})^{(2)}(T_{17}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i^{''})^{(2)}(T_{17}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, \ (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + \ (\hat{A}_{16})^{(2)} + \ (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |

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| $\frac{1}{(\hat{M}_{16})^{(2)}}[\ (b_i)^{(2)} + (b_i^{'})^{(2)} +\ (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)}\ (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23}, t) \leq\ (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(3)}\ (T_{21}, t) = (p_i)^{(3)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(3)}\ (G_{23}, t) =\ (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)},\ (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'}, t) - (a_i^{''})^{(3)}(T_{21}, t)\rvert \leq (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}{}^{'}, t) - (b_i^{''})^{(3)}(G_{23}, t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}{}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'}, t)$ and$(a_i^{''})^{(3)}(T_{21}, t)$ . $(T_{21}^{'}, t)$ And $(T_{21}, t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |

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| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}((G_{27}), t) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(4)}((G_{27}), t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\|(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\| \leq (\hat{k}_{24})^{(4)} \|T_{25} - T_{25}^{'}\| e^{-(\hat{M}_{24})^{(4)} t}$<br><br>$\|(b_i^{''})^{(4)}((G_{27})', t) - (b_i^{''})^{(4)}((G_{27}), t)\| < (\hat{k}_{24})^{(4)} \|(G_{27}) - (G_{27})'\| e^{-(\hat{M}_{24})^{(4)} t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants | 120 |

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| $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)} + (b_i^{'})^{(4)} + (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}, (r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}((G_{31}), t) \leq (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(5)} (T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(5)} (G_{31}, t) = (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)} \lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)} t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})', t) - (b_i^{''})^{(5)}((G_{31}), t)\rvert < (\hat{k}_{28})^{(5)} \lvert\lvert(G_{31}) - (G_{31})'\rvert\rvert e^{-(\hat{M}_{28})^{(5)} t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ : | 125 |

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| $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i')^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i')^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$ | 126 |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i')^{(6)}, (a_i'')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (b_i'')^{(6)} > 0, \quad i, j = 32,33,34$<br><br>The functions $(a_i'')^{(6)}, (b_i'')^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i'')^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i'')^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i')^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty} (a_i'')^{(6)} (T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty} (b_i'')^{(6)} ((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(6)}(T_{33}', t) - (a_i'')^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)} \lvert T_{33} - T_{33}'\rvert e^{-(\hat{M}_{32})^{(6)} t}$<br><br>$\lvert(b_i'')^{(6)}((G_{35})', t) - (b_i'')^{(6)}((G_{35}), t)\rvert < (\hat{k}_{32})^{(6)} \lvert\lvert(G_{35}) - (G_{35})'\rvert\rvert e^{-(\hat{M}_{32})^{(6)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(6)}(T_{33}', t)$ and $(a_i'')^{(6)}(T_{33}, t)$ . $(T_{33}', t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$. It is to be noted that $(a_i'')^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i'')^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |

**Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ : 129

$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)},$ are positive constants

$$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$$

**Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ : 130

There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i')^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i')^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

Where we suppose

(OOO) $(a_i)^{(7)}, (a_i')^{(7)}, (a_i'')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (b_i'')^{(7)} > 0, \quad i, j = 36,37,38$ 131

(PPP) The functions $(a_i'')^{(7)}, (b_i'')^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}$, $(r_i)^{(7)}$:

$$(a_i'')^{(7)}(T_{37}, t) \le (p_i)^{(7)} \le (\hat{A}_{36})^{(7)}$$

$$(b_i'')^{(7)}(G_{39}, t) \le (r_i)^{(7)} \le (b_i')^{(7)} \le (\hat{B}_{36})^{(7)}$$

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*(QQQ)* $\lim_{T_2 \to \infty} (a_i'')^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(RRR)*

$\lim_{G \to \infty} (b_i'')^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i'')^{(7)}(T_{37}', t) - (a_i'')^{(7)}(T_{37}, t)| \le (\hat{k}_{36})^{(7)} |T_{37} - T_{37}'| e^{-(\hat{M}_{36})^{(7)} t}$$

$$|(b_i'')^{(7)}((G_{39})', t) - (b_i'')^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)} ||(G_{39}) - (G_{39})'|| e^{-(\hat{M}_{36})^{(7)} t}$$

| | |
|---|---|
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'},t)$ and$(a_i^{''})^{(7)}(T_{37},t)$ . $(T_{37}^{'},t)$ and $(T_{37},t)$ are points belonging to the interval $\left[(\hat{k}_{36})^{(7)},(\widehat{M}_{36})^{(7)}\right]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{36})^{(7)}=1$ then the function $(a_i^{''})^{(7)}(T_{37},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)}$ :<br><br>(SSS) $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)}$, are positive constants<br><br>$\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}<1$ | 134 |
| **Definition of** $(\hat{P}_{36})^{(7)},(\hat{Q}_{36})^{(7)}$ :<br><br>(TTT) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)},(\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)},(a_i^{'})^{(7)},(b_i)^{(7)},(b_i^{'})^{(7)},(p_i)^{(7)},\ (r_i)^{(7)}, i=36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\widehat{M}_{36})^{(7)}}[(a_i)^{(7)}+(a_i^{'})^{(7)}+(\hat{A}_{36})^{(7)}+(\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}]<1$<br><br>$\frac{1}{(\widehat{M}_{36})^{(7)}}[(b_i)^{(7)}+(b_i^{'})^{(7)}+(\hat{B}_{36})^{(7)}+(\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}]<1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)},(a_i^{'})^{(8)},(a_i^{''})^{(8)},(b_i)^{(8)},(b_i^{'})^{(8)},(b_i^{''})^{(8)}>0, \quad i,j=40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)},(b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)},\ (r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41},t)\leq(p_i)^{(8)}\leq(\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}),t)\leq\ (r_i)^{(8)}\leq(b_i^{'})^{(8)}\leq(\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(8)}(T_{41},t)=(p_i)^{(8)}$ | 140 |
| $\lim_{G\to\infty}(b_i^{''})^{(8)}\left((G_{43}),t\right)=\ (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)},(\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)},(\hat{B}_{40})^{(8)},(p_i)^{(8)},\ (r_i)^{(8)}}$ are positive constants and $\boxed{i=40,41,42}$ | |

They satisfy Lipschitz condition:

$$|(a_i^{''})^{(8)}(T_{41}^{'},t)-(a_i^{''})^{(8)}(T_{41},t)| \leq (\hat{k}_{40})^{(8)}|T_{41}-T_{41}^{'}|e^{-(\hat{M}_{40})^{(8)}t} \quad (142)$$

$$|(b_i^{''})^{(8)}((G_{43})^{'},t)-(b_i^{''})^{(8)}((G_{43}),t)| < (\hat{k}_{40})^{(8)}||(G_{43})-(G_{43})^{'}||e^{-(\hat{M}_{40})^{(8)}t} \quad (143)$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'},t)$ and $(a_i^{''})^{(8)}(T_{41},t)$ . $(T_{41}^{'},t)$ and $(T_{41},t)$ are points belonging to the interval $\left[(\hat{k}_{40})^{(8)},(\hat{M}_{40})^{(8)}\right]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)}=1$ then the function $(a_i^{''})^{(8)}(T_{41},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{40})^{(8)},(\hat{k}_{40})^{(8)}$ :

$(\hat{M}_{40})^{(8)},(\hat{k}_{40})^{(8)}$, are positive constants

$$\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\ ,\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}}<1 \quad (144)$$

**Definition of** $(\hat{P}_{40})^{(8)},(\hat{Q}_{40})^{(8)}$ :

There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)},(\hat{k}_{40})^{(8)},(\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)},(a_i^{'})^{(8)},(b_i)^{(8)},(b_i^{'})^{(8)},(p_i)^{(8)},\ (r_i)^{(8)},i=40,41,42,$
Satisfy the inequalities

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)}+(a_i^{'})^{(8)}+(\hat{A}_{40})^{(8)}+(\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}]<1 \quad (145)$$

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)}+(b_i^{'})^{(8)}+(\hat{B}_{40})^{(8)}+(\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}]<1 \quad (146)$$

Where we suppose

$$(a_i)^{(9)},(a_i^{'})^{(9)},(a_i^{''})^{(9)},(b_i)^{(9)},(b_i^{'})^{(9)},(b_i^{''})^{(9)}>0,\quad i,j=44,45,46 \quad (146A)$$

The functions $(a_i^{''})^{(9)},(b_i^{''})^{(9)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(9)},\ (r_i)^{(9)}$:

$$(a_i^{''})^{(9)}(T_{45},t)\leq(p_i)^{(9)}\leq(\hat{A}_{44})^{(9)}$$

$$(b_i^{''})^{(9)}(G_{47},t)\leq\ (r_i)^{(9)}\leq(b_i^{'})^{(9)}\leq(\hat{B}_{44})^{(9)}$$

$$\lim_{T_2\to\infty}(a_i^{''})^{(9)}(T_{45},t)=(p_i)^{(9)}$$

$$\lim_{G\to\infty}(b_i^{''})^{(9)}(G_{47},t)=\ (r_i)^{(9)}$$

**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :

Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$

They satisfy Lipschitz condition:

$$|(a_i^{''})^{(9)}(T_{45}^{'}, t) - (a_i^{''})^{(9)}(T_{45}, t)| \leq (\hat{k}_{44})^{(9)}|T_{45} - T_{45}^{'}|e^{-(\hat{M}_{44})^{(9)}t}$$

$$|(b_i^{''})^{(9)}((G_{47})^{'}, t) - (b_i^{''})^{(9)}\big((G_{47}), t\big)| < (\hat{k}_{44})^{(9)}||(G_{47}) - (G_{47})^{'}||e^{-(\hat{M}_{44})^{(9)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(9)}(T_{45}^{'}, t)$ and$(a_i^{''})^{(9)}(T_{45}, t)$ . $(T_{45}^{'}, t)$ and $(T_{45}, t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i^{''})^{(9)}(T_{45}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i^{''})^{(9)}(T_{45}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)},$ are positive constants

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}} \ , \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$$

**Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and \ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}, i = 44,45,46,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0), T_i(0)$ :

$$G_i(t) \leq (\hat{P}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t} \ , \ \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \leq (\hat{Q}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t} \ , \ \boxed{T_i(0) = T_i^0 > 0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0), T_i(0)$

$$G_i(t) \leq (\hat{P}_{16})^{(2)}e^{(\hat{M}_{16})^{(2)}t} \ , \ G_i(0) = G_i^0 > 0$$

| | |
|---|---|
| $T_i(t) \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>$G_i(t) \leq (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \leq (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$ | 153 A |

| | |
|---|---|
| $T_i(t) \leq (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br><br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br><br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |

| | |
|---|---|
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[ (a_{18})^{(2)} G_{17}(s_{(16)}) - \left( (a_{18}')^{(2)} + (a_{18}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{18}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[ (b_{16})^{(2)} T_{17}(s_{(16)}) - \left( (b_{16}')^{(2)} - (b_{16}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{16}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[ (b_{17})^{(2)} T_{16}(s_{(16)}) - \left( (b_{17}')^{(2)} - (b_{17}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[ (b_{18})^{(2)} T_{17}(s_{(16)}) - \left( (b_{18}')^{(2)} - (b_{18}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{18}(s_{(16)}) \right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a_{20}')^{(3)} + a_{20}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a_{21}')^{(3)} + (a_{21}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a_{22}')^{(3)} + (a_{22}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b_{20}')^{(3)} - (b_{20}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b_{21}')^{(3)} - (b_{21}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b_{22}')^{(3)} - (b_{22}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |

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| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{24})^{(4)}$ , $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{28})^{(5)}$ , $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |

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| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b'_{29})^{(5)} - (b''_{29})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{\mathrm{T}}_{30}(\mathrm{t}) = \mathrm{T}_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b'_{30})^{(5)} - (b''_{30})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{32})^{(6)}$ , $T_i^0 \le (\hat{Q}_{32})^{(6)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a'_{32})^{(6)} + a''_{32})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b'_{32})^{(6)} - (b''_{32})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b'_{33})^{(6)} - (b''_{33})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{\mathrm{T}}_{34}(\mathrm{t}) = \mathrm{T}_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b'_{34})^{(6)} - (b''_{34})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{36})^{(7)}$ , $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |

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| By<br>$\bar{G}_{36}(t) = G_{36}^{0} + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a_{36}^{'})^{(7)} + a_{36}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^{0} + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^{0} + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^{0} + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}^{'})^{(8)} + a_{40}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^{0} + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^{0} + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^{0} + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^{0} + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |

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| $\overline{T}_{42}(t) = T_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br><br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166<br>A |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{44})^{(9)}$, $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br><br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\overline{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br><br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\bar{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br><br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left(e^{(\bar{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br><br>$(G_{13}(t) - G_{13}^0) e^{-(\bar{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br><br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |

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| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \leq G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$ $\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that $(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \leq \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{20}(t) \leq G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} \left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$ $\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}} \left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$ | 171 |
| From which it follows that $(G_{20}(t) - G_{20}^0) e^{-(\hat{M}_{20})^{(3)} t} \leq \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}} \left[\left((\hat{P}_{20})^{(3)} + G_{21}^0\right) e^{\left(-\frac{(\hat{P}_{20})^{(3)} + G_{21}^0}{G_{21}^0}\right)} + (\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that $G_{24}(t) \leq G_{24}^0 + \int_0^t \left[(a_{24})^{(4)} \left(G_{25}^0 + (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} s_{(24)}}\right)\right] ds_{(24)} =$ $\left(1 + (a_{24})^{(4)} t\right) G_{25}^0 + \frac{(a_{24})^{(4)} (\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}} \left(e^{(\hat{M}_{24})^{(4)} t} - 1\right)$ | 173 |
| From which it follows that $(G_{24}(t) - G_{24}^0) e^{-(\hat{M}_{24})^{(4)} t} \leq \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}} \left[\left((\hat{P}_{24})^{(4)} + G_{25}^0\right) e^{\left(-\frac{(\hat{P}_{24})^{(4)} + G_{25}^0}{G_{25}^0}\right)} + (\hat{P}_{24})^{(4)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{28}(t) \leq G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} \left(G_{29}^0 + (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} s_{(28)}}\right)\right] ds_{(28)} =$ $\left(1 + (a_{28})^{(5)} t\right) G_{29}^0 + \frac{(a_{28})^{(5)} (\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}} \left(e^{(\hat{M}_{28})^{(5)} t} - 1\right)$ | |

From which it follows that

$$(G_{28}(t)-G_{28}^{0})e^{-(\bar{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^{0})e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^{0}}{G_{29}^{0}}\right)}+(\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5

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The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

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$$G_{32}(t) \le G_{32}^{0}+\int_0^t\left[(a_{32})^{(6)}\left(G_{33}^{0}+(\hat{P}_{32})^{(6)}e^{(\bar{M}_{32})^{(6)}s_{(32)}}\right)\right]ds_{(32)}=$$

$$\left(1+(a_{32})^{(6)}t\right)G_{33}^{0}+\frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left(e^{(\bar{M}_{32})^{(6)}t}-1\right)$$

From which it follows that

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$$(G_{32}(t)-G_{32}^{0})e^{-(\bar{M}_{32})^{(6)}t} \le \frac{(a_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^{0})e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^{0}}{G_{33}^{0}}\right)}+(\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

(l) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

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$$G_{36}(t) \le G_{36}^{0}+\int_0^t\left[(a_{36})^{(7)}\left(G_{37}^{0}+(\hat{P}_{36})^{(7)}e^{(\bar{M}_{36})^{(7)}s_{(36)}}\right)\right]ds_{(36)}=$$

$$\left(1+(a_{36})^{(7)}t\right)G_{37}^{0}+\frac{(a_{36})^{(7)}(\hat{P}_{36})^{(7)}}{(\bar{M}_{36})^{(7)}}\left(e^{(\bar{M}_{36})^{(7)}t}-1\right)$$

From which it follows that

$$(G_{36}(t)-G_{36}^{0})e^{-(\bar{M}_{36})^{(7)}t} \le \frac{(a_{36})^{(7)}}{(\bar{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)}+G_{37}^{0})e^{\left(-\frac{(\hat{P}_{36})^{(7)}+G_{37}^{0}}{G_{37}^{0}}\right)}+(\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \le G_{40}^{0}+\int_0^t\left[(a_{40})^{(8)}\left(G_{41}^{0}+(\hat{P}_{40})^{(8)}e^{(\bar{M}_{40})^{(8)}s_{(40)}}\right)\right]ds_{(40)}=$$

$$\left(1+(a_{40})^{(8)}t\right)G_{41}^{0}+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\bar{M}_{40})^{(8)}}\left(e^{(\bar{M}_{40})^{(8)}t}-1\right)$$

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From which it follows that

$$(G_{40}(t)-G_{40}^{0})e^{-(\bar{M}_{40})^{(8)}t} \le \frac{(a_{40})^{(8)}}{(\bar{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^{0})e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^{0}}{G_{41}^{0}}\right)}+(\hat{P}_{40})^{(8)}\right]$$

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$(G_i^0)$ is as defined in the statement of theorem 8
Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$

The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)}\left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$$

$$\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)} t} - 1\right)$$

From which it follows that

$$(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{P}_{44})^{(9)} + G_{45}^0\right) e^{\left(-\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0}\right)} + (\hat{P}_{44})^{(9)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 9

Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$

It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ and to choose

$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have

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$$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)} + \left((\hat{P}_{13})^{(1)} + G_j^0\right) e^{-\left(\frac{(\hat{P}_{13})^{(1)} + G_j^0}{G_j^0}\right)}\right] \le (\hat{P}_{13})^{(1)} \quad (183)$$

$$\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)} + T_j^0\right) e^{-\left(\frac{(\hat{Q}_{13})^{(1)} + T_j^0}{T_j^0}\right)} + (\hat{Q}_{13})^{(1)}\right] \le (\hat{Q}_{13})^{(1)} \quad (184)$$

In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric (185)

$$d\left(\left(G^{(1)}, T^{(1)}\right), \left(G^{(2)}, T^{(2)}\right)\right) =$$

$$\sup_i \{\max_{t \in \mathbb{R}_+} \left|G_i^{(1)}(t) - G_i^{(2)}(t)\right| e^{-(\hat{M}_{13})^{(1)} t}, \max_{t \in \mathbb{R}_+} \left|T_i^{(1)}(t) - T_i^{(2)}(t)\right| e^{-(\hat{M}_{13})^{(1)} t}\}$$

Indeed if we denote

**Definition of** $\tilde{G}, \tilde{T}$ : $\left(\tilde{G}, \tilde{T}\right) = \mathcal{A}^{(1)}(G, T)$

It results

$$\left|\tilde{G}_{13}^{(1)}-\tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{13})^{(1)} \left|G_{14}^{(1)}-G_{14}^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\, ds_{(13)} +$$

$$\int_0^t \{(a'_{13})^{(1)} \left|G_{13}^{(1)}-G_{13}^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} +$$

$$(a''_{13})^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right)\left|G_{13}^{(1)}-G_{13}^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} e^{(\widehat{M}_{13})^{(1)}s_{(13)}} +$$

$$G_{13}^{(2)}\left|(a''_{13})^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right)-(a''_{13})^{(1)}\left(T_{14}^{(2)}, s_{(13)}\right)\right|\ e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\} ds_{(13)}$$

Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses it follows

$$\left|G^{(1)}-G^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)}t} \leq$$
$$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right) d\left(\left(G^{(1)}, T^{(1)};\ G^{(2)}, T^{(2)}\right)\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

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**Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(1)}$ and $(b''_i)^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 2:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$

From 19 to 24 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(1)} - (a''_i)^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(1)}t\right)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :

**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if

$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating

$$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1 / (a'_{14})^{(1)}$$

In the same way , one can obtain

$$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2 / (a'_{15})^{(1)}$$

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| If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t),t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1-e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right)$, $t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br><br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote | 195 |

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| **Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right) d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$ | 201 |

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| In the same way , one can obtain<br>$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | |
| **Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br>Indeed let $t_2$ be so that for $t > t_2$<br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$ | 211 |

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| $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}},\widetilde{T_{23}}:(\widetilde{(G_{23})},\widetilde{(T_{23})})=\mathcal{A}^{(3)}((G_{23}),(T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{20})^{(3)}\left\|G_{21}^{(1)}-G_{21}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)}+$<br>$\int_0^t\{(a_{20}^{'})^{(3)}\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$G_{20}^{(2)}\left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)-(a_{20}^{''})^{(3)}\left(T_{21}^{(2)},s_{(20)}\right)\right\|\ e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)}-G_{23}{}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}t}\le$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)}+(a_{20}^{'})^{(3)}+(\widehat{A}_{20})^{(3)}+(\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)},(T_{23})^{(1)};\ (G_{23})^{(2)},(T_{23})^{(2)})\right)$<br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ and\ (\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i=20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(3)}-(a_i^{''})^{(3)}(T_{21}(s_{(20)}),s_{(20)})\}ds_{(20)}\right]}\ge 0$<br>$T_i(t)\ge T_i^0e^{(-(b_i^{'})^{(3)}t)}>0$ for t > 0 | 216 |

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| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |

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| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{24})^{(4)}\left\|G_{25}^{(1)}-G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$<br>$\int_0^t\{(a_{24}^{'})^{(4)}\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$G_{24}^{(2)}\left\|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)}-(G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t}\le$<br>$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 226 |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24{,}25{,}26$ depend only on $T_{25}$ and respectively on $(G_{27})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results | 228 |

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| $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(4)} - (a_i'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(4)} t)} > 0$ for t > 0 | |
| **Definition of** $((\widehat{M}_{24})^{(4)})_1, ((\widehat{M}_{24})^{(4)})_2$ *and* $((\widehat{M}_{24})^{(4)})_3$ :<br><br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if<br><br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq ((\widehat{M}_{24})^{(4)})_1 - (a_{25}')^{(4)} G_{25}$ and by integrating<br><br>$G_{25} \leq ((\widehat{M}_{24})^{(4)})_2 = G_{25}^0 + 2(a_{25})^{(4)} ((\widehat{M}_{24})^{(4)})_1 / (a_{25}')^{(4)}$<br><br>In the same way , one can obtain<br><br>$G_{26} \leq ((\widehat{M}_{24})^{(4)})_3 = G_{26}^0 + 2(a_{26})^{(4)} ((\widehat{M}_{24})^{(4)})_2 / (a_{26}')^{(4)}$<br><br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t), t)) = (b_{25}')^{(4)}$ then $T_{25} \to \infty$.<br><br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br><br>Indeed let $t_4$ be so that for $t > t_4$<br><br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t), t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}((G_{27})(t), t) = (b_{26}')^{(4)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br><br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br><br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |

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| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)}+\left((\hat{P}_{28})^{(5)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{28})^{(5)}\right]\leq(\hat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $\widetilde{(G_{31})},\widetilde{(T_{31})}$ : $\left(\widetilde{(G_{31})},\widetilde{(T_{31})}\right)=\mathcal{A}^{(5)}((G_{31}),(T_{31}))$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}ds_{(28)}+$<br>$\int_0^t\{(a_{28}')^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}'')^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\leq$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}')^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}'')^{(5)}$ and $(b_{28}'')^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}\ and\ (\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it | 238 |

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| suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 22:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(5)} - (a_i^{''})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(5)} t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2\ and\ \left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}\ and\ G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}^{'})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a_{29}^{'})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a_{30}^{'})^{(5)}$<br><br>If $G_{29}\ or\ G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}\ and\ G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t), t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t), t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t), t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |

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| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\widehat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{32})^{(6)}\right] \leq (\widehat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)}, (T_{35})^{(1)}), ((G_{35})^{(2)}, (T_{35})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $\widetilde{(G_{35})}, \widetilde{(T_{35})}$ : $\left(\widetilde{(G_{35})}, \widetilde{(T_{35})}\right) = \mathcal{A}^{(6)}((G_{35}), (T_{35}))$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{32})^{(6)}\left\|G_{33}^{(1)} - G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\,ds_{(32)} +$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)} - G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)\left\|G_{32}^{(1)} - G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right) - (a''_{32})^{(6)}\left(T_{33}^{(2)}, s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)} - (G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t} \leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)} + (a'_{32})^{(6)} + (\widehat{A}_{32})^{(6)} + (\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)}, (T_{35})^{(1)};\ (G_{35})^{(2)}, (T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | 249 |

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| condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i = 32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(6)} - (a_i^{''})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(6)}t)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2\ and\ \left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}\ and\ G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}^{'})^{(6)}G_{33}$ and by integrating<br><br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1/(a_{33}^{'})^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2/(a_{34}^{'})^{(6)}$<br><br>If $G_{33}\ or\ G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}\ and\ G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t), t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i^{''})^{(6)}((G_{35})(t), t) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}^{'})^{(6)}$ | 254 |

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| We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ large to have | 255 |
| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[(\hat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)}, (T_{39})^{(1)}), ((G_{39})^{(2)}, (T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $(\widetilde{G_{39}}), (\widetilde{T_{39}}) : \left((\widetilde{G_{39}}), (\widetilde{T_{39}})\right) = \mathcal{A}^{(7)}((G_{39}), (T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{36})^{(7)}\left\|G_{37}^{(1)} - G_{37}^{(2)}\right\|e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{(\hat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)} +$<br>$\int_0^t\{(a'_{36})^{(7)}\left\|G_{36}^{(1)} - G_{36}^{(2)}\right\|e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{-(\hat{M}_{36})^{(7)}s_{(36)}} +$<br>$(a''_{36})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right)\left\|G_{36}^{(1)} - G_{36}^{(2)}\right\|e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{(\hat{M}_{36})^{(7)}s_{(36)}} +$<br>$G_{36}^{(2)}\left|(a''_{36})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) - (a''_{36})^{(7)}\left(T_{37}^{(2)}, s_{(36)}\right)\right|\ e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{(\hat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)} - (G_{39})^{(2)}\right\|e^{-(\hat{M}_{36})^{(7)}t} \leq$<br>$\frac{1}{(\hat{M}_{36})^{(7)}}\left((a_{36})^{(7)} + (a'_{36})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)}, (T_{39})^{(1)};\ (G_{39})^{(2)}, (T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |

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| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ $and$ $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\}ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)}t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)}G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1/(a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ $is$ bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}\,((G_{39})(t),t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |

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| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right), \quad t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t), t\right) = (b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)} + \left((\widehat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\widehat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{40})^{(8)}\right] \leq (\widehat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)}, (T_{43})^{(1)}), ((G_{43})^{(2)}, (T_{43})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})}, \widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})}, \widetilde{(T_{43})}\right) = \mathcal{A}^{(8)}((G_{43}), (T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{40})^{(8)}\left\|G_{41}^{(1)} - G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\,ds_{(40)} +$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}} +$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}} +$<br>$G_{40}^{(2)}\left\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right) - (a_{40}^{''})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)\right\| \; e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval [0, t] | 272 |

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| From the hypotheses it follows | |
| $\left\|(G_{43})^{(1)} - (G_{43})^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)}t} \leq$ $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)} + (a'_{40})^{(8)} + (\widehat{A}_{40})^{(8)} + (\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right) d\left(((G_{43})^{(1)}, (T_{43})^{(1)};\ (G_{43})^{(2)}, (T_{43})^{(2)})\right)$ And analogous inequalities for $G_i$ $and$ $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a''_{40})^{(8)}$ and $(b''_{40})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(8)}$ and $(b''_i)^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t) = 0$ $and$ $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(8)} - (a''_i)^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ $and$ $\left((\widehat{M}_{40})^{(8)}\right)_3$ : **Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ $and$ $G_{42}$ . indeed if $G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a'_{41})^{(8)} G_{41}$ and by integrating $G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a'_{41})^{(8)}$ In the same way , one can obtain $G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a'_{42})^{(8)}$ If $G_{41}$ $or$ $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}$ $and$ $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(8)}((G_{43})(t), t)) = (b'_{41})^{(8)}$ then $T_{41} \to \infty$. | 278 |

**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :

Indeed let $t_8$ be so that for $t > t_8$

$(b_{41})^{(8)} - (b_i^{''})^{(8)}\big((G_{43})(t), t\big) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$

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Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.
The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\big((G_{43})(t), t(t), t\big) = (b_{42}^{'})^{(8)}$

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It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \big((\hat{P}_{44})^{(9)} + G_j^0\big)e^{-\left(\frac{(\hat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\big((\hat{Q}_{44})^{(9)} + T_j^0\big)e^{-\left(\frac{(\hat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{44})^{(9)}\right] \leq (\hat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\Big(\big((G_{47})^{(1)}, (T_{47})^{(1)}\big), \big((G_{47})^{(2)}, (T_{47})^{(2)}\big)\Big) =$$
$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \big(\widetilde{(G_{47})}, \widetilde{(T_{47})}\big) = \mathcal{A}^{(9)}\big((G_{47}), (T_{47})\big)$
It results

$$\left|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$(a_{44}^{''})^{(9)}\big(T_{45}^{(1)}, s_{(44)}\big)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\big(T_{45}^{(1)}, s_{(44)}\big) - (a_{44}^{''})^{(9)}\big(T_{45}^{(2)}, s_{(44)}\big)\right| \; e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)} - G^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)}t} \leq \frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a'_{44})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right) d\left(((G_{47})^{(1)}, (T_{47})^{(1)};\ (G_{47})^{(2)}, (T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a''_{44})^{(9)}$ and $(b''_{44})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)} e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)} e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(9)}$ and $(b_i'')^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$

From 99 to 44 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(9)} - (a''_i)^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b'_i)^{(9)}t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2$ *and* $\left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}$ *and* $G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a'_{45})^{(9)} G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1 / (a'_{45})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2 / (a'_{46})^{(9)}$$

If $G_{45}$ *or* $G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}$ *and* $G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(9)}\left((G_{47})(t), t\right)) = (b'_{45})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\big((G_{47})(t),t\big) < \varepsilon_9, T_{44}\,(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\big((G_{47})(t),t\big) = (b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} - (a_{13}^{''})^{(1)}(T_{14}, t) + (a_{14}^{''})^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\big(\nu^{(1)}\big)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\big(u^{(1)}\big)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\big(\nu^{(1)}\big)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\big(u^{(1)}\big)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

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| $(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)},$<br>and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$<br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$<br>are defined | |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)} t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)} t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)} t}\right] + G_{15}^0 e^{-(S_2)^{(1)} t} \le G_{15}(t) \le \right.$<br>$\left.\frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a'_{15})^{(1)})}[e^{(S_1)^{(1)} t} - e^{-(a'_{15})^{(1)} t}] + G_{15}^0 e^{-(a'_{15})^{(1)} t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)} T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)} - (b'_{15})^{(1)})}\left[e^{(R_1)^{(1)} t} - e^{-(b'_{15})^{(1)} t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)} t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)} T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)} + (r_{13})^{(1)} + (R_2)^{(1)})}\left[e^{((R_1)^{(1)} + (r_{13})^{(1)})t} - e^{-(R_2)^{(1)} t}\right] + T_{15}^0 e^{-(R_2)^{(1)} t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$<br>$(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ :<br>$(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |

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| $-(\sigma_2)^{(2)} \le -(a_{16}^{'})^{(2)} + (a_{17}^{'})^{(2)} - (a_{16}^{''})^{(2)}(\mathrm{T}_{17}, t) + (a_{17}^{''})^{(2)}(\mathrm{T}_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b_{16}^{'})^{(2)} + (b_{17}^{'})^{(2)} - (b_{16}^{''})^{(2)}\big((G_{19}), t\big) - (b_{17}^{''})^{(2)}\big((G_{19}), t\big) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}$,, $(\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{\mathrm{G}_{16}^0}{\mathrm{G}_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{\mathrm{T}_{16}^0}{\mathrm{T}_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$\mathrm{G}_{16}^0 \mathrm{e}^{((\mathrm{S}_1)^{(2)} - (p_{16})^{(2)})\mathrm{t}} \le G_{16}(t) \le \mathrm{G}_{16}^0 \mathrm{e}^{(\mathrm{S}_1)^{(2)}\mathrm{t}}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |

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| $\frac{1}{(m_1)^{(2)}} \mathrm{G}_{16}^{0} \mathrm{e}^{((\mathrm{S}_1)^{(2)}-(p_{16})^{(2)})\mathrm{t}} \leq G_{17}(t) \leq \frac{1}{(m_2)^{(2)}} \mathrm{G}_{16}^{0} \mathrm{e}^{(\mathrm{S}_1)^{(2)}\mathrm{t}}$ | 311 |
| $\left(\frac{(a_{18})^{(2)}\mathrm{G}_{16}^{0}}{(m_1)^{(2)}((\mathrm{S}_1)^{(2)}-(p_{16})^{(2)}-(\mathrm{S}_2)^{(2)})}\left[\mathrm{e}^{((\mathrm{S}_1)^{(2)}-(p_{16})^{(2)})\mathrm{t}} - \mathrm{e}^{-(\mathrm{S}_2)^{(2)}\mathrm{t}}\right] + \mathrm{G}_{18}^{0}\mathrm{e}^{-(\mathrm{S}_2)^{(2)}\mathrm{t}} \leq \mathrm{G}_{18}(t) \leq \right.$ $\left.\frac{(a_{18})^{(2)}\mathrm{G}_{16}^{0}}{(m_2)^{(2)}((\mathrm{S}_1)^{(2)}-(a_{18}^{'})^{(2)})}\left[\mathrm{e}^{(\mathrm{S}_1)^{(2)}\mathrm{t}} - \mathrm{e}^{-(a_{18}^{'})^{(2)}\mathrm{t}}\right] + \mathrm{G}_{18}^{0}\mathrm{e}^{-(a_{18}^{'})^{(2)}\mathrm{t}}\right)$ | 312 |
| $\boxed{\mathrm{T}_{16}^{0}\mathrm{e}^{(\mathrm{R}_1)^{(2)}t} \leq T_{16}(t) \leq \mathrm{T}_{16}^{0}\mathrm{e}^{((\mathrm{R}_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} \mathrm{T}_{16}^{0}\mathrm{e}^{(\mathrm{R}_1)^{(2)}t} \leq T_{16}(t) \leq \frac{1}{(\mu_2)^{(2)}} \mathrm{T}_{16}^{0}\mathrm{e}^{((\mathrm{R}_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)}\mathrm{T}_{16}^{0}}{(\mu_1)^{(2)}((\mathrm{R}_1)^{(2)}-(b_{18}^{'})^{(2)})}\left[\mathrm{e}^{(\mathrm{R}_1)^{(2)}t} - \mathrm{e}^{-(b_{18}^{'})^{(2)}t}\right] + \mathrm{T}_{18}^{0}\mathrm{e}^{-(b_{18}^{'})^{(2)}t} \leq T_{18}(t) \leq$ $\frac{(a_{18})^{(2)}\mathrm{T}_{16}^{0}}{(\mu_2)^{(2)}((\mathrm{R}_1)^{(2)}+(r_{16})^{(2)}+(\mathrm{R}_2)^{(2)})}\left[\mathrm{e}^{((\mathrm{R}_1)^{(2)}+(r_{16})^{(2)})t} - \mathrm{e}^{-(\mathrm{R}_2)^{(2)}t}\right] + \mathrm{T}_{18}^{0}\mathrm{e}^{-(\mathrm{R}_2)^{(2)}t}$ | 315 |
| **Definition of** $(\mathrm{S}_1)^{(2)}, (\mathrm{S}_2)^{(2)}, (\mathrm{R}_1)^{(2)}, (\mathrm{R}_2)^{(2)}$:- | 316 |
| Where $(\mathrm{S}_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a_{16}^{'})^{(2)}$ $(\mathrm{S}_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b_{16}^{'})^{(2)}$ $(\mathrm{R}_2)^{(2)} = (b_{18}^{'})^{(2)} - (r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ : $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)} \leq -(a_{20}^{'})^{(3)} + (a_{21}^{'})^{(3)} - (a_{20}^{''})^{(3)}(T_{21}, t) + (a_{21}^{''})^{(3)}(T_{21}, t) \leq -(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)} \leq -(b_{20}^{'})^{(3)} + (b_{21}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t) - (b_{21}^{''})^{(3)}((G_{23}), t) \leq -(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(\nu_1)^{(3)}, (\nu_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ : By $(\nu_1)^{(3)} > 0$, $(\nu_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$, $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}(\nu^{(3)})^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$ and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and By $(\bar{\nu}_1)^{(3)} > 0$, $(\bar{\nu}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$, $(\bar{u}_2)^{(3)} < 0$ the | 320 |

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| roots of the equations $(a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$<br><br>and $(b_{21})^{(3)}\left(u^{(3)}\right)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ | |
| **Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :-<br><br>If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by<br>$(m_2)^{(3)} = (v_0)^{(3)}, (m_1)^{(3)} = (v_1)^{(3)}$, ***if*** $(v_0)^{(3)} < (v_1)^{(3)}$<br><br>$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (\bar{v}_1)^{(3)}$, ***if*** $(v_1)^{(3)} < (v_0)^{(3)} < (\bar{v}_1)^{(3)}$,<br><br>and $\boxed{(v_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$<br><br>$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (v_0)^{(3)}$, ***if*** $(\bar{v}_1)^{(3)} < (v_0)^{(3)}$ | 321 |
| and analogously<br><br>$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$<br><br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$<br><br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$<br><br>Then the solution of global equations satisfies the inequalities<br><br>$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)}t}$<br><br>$(p_i)^{(3)}$ is defined by equation | 322 |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \le G_{22}(t) \le\right.$<br>$\left.\frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a'_{22})^{(3)})}\left[e^{(S_1)^{(3)}t} - e^{-(a'_{22})^{(3)}t}\right] + G_{22}^0 e^{-(a'_{22})^{(3)}t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b'_{22})^{(3)})}\left[e^{(R_1)^{(3)}t} - e^{-(b'_{22})^{(3)}t}\right] + T_{22}^0 e^{-(b'_{22})^{(3)}t} \le T_{22}(t) \le$<br><br>$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$ | 327 |

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- 328

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$

$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$

$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$

**Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : 329

By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$
and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{v}_1)^{(4)} > 0$, $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$
and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (v_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}$, ***if*** $(v_0)^{(4)} < (v_1)^{(4)}$

$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}$, ***if*** $(v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)}$,
and $\boxed{(v_0)^{(4)} = \frac{G^0_{24}}{G^0_{25}}}$

$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)} < (v_0)^{(4)}$

and analogously 331

$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$

$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$, ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,

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| and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \boldsymbol{if}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)} t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)} t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\right)\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)} t}\right] + G_{26}^0 e^{-(S_2)^{(4)} t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b_{26}')^{(4)})}\left[e^{(R_1)^{(4)} t} - e^{-(b_{26}')^{(4)} t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)} t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)} + (r_{24})^{(4)} + (R_2)^{(4)})}\left[e^{((R_1)^{(4)} + (r_{24})^{(4)})t} - e^{-(R_2)^{(4)} t}\right] + T_{26}^0 e^{-(R_2)^{(4)} t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |

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| **Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ :<br><br>By $(v_1)^{(5)} > 0$ , $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$ , $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | 339 |
| **Definition of** $(\bar{v}_1)^{(5)},, (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ :<br><br>By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :-<br><br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}, (\mu_2)^{(5)}$ by<br><br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br><br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br><br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |

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| $\frac{1}{(\mu_1)^{(5)}}T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}}T_{28}^0 e^{((R_1)^{(5)}+(r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)}T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)}-(b_{30}^{'})^{(5)})}\left[e^{(R_1)^{(5)}t} - e^{-(b_{30}^{'})^{(5)}t}\right] + T_{30}^0 e^{-(b_{30}^{'})^{(5)}t} \le T_{30}(t) \le$ $\frac{(a_{30})^{(5)}T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t} - e^{-(R_2)^{(5)}t}\right] + T_{30}^0 e^{-(R_2)^{(5)}t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:- Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a_{28}^{'})^{(5)}$ $(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$ $(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b_{28}^{'})^{(5)}$ $(R_2)^{(5)} = (b_{30}^{'})^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ : $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying $-(\sigma_2)^{(6)} \le -(a_{32}^{'})^{(6)} + (a_{33}^{'})^{(6)} - (a_{32}^{''})^{(6)}(T_{33}, t) + (a_{33}^{''})^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$ $-(\tau_2)^{(6)} \le -(b_{32}^{'})^{(6)} + (b_{33}^{'})^{(6)} - (b_{32}^{''})^{(6)}\left((G_{35}), t\right) - (b_{33}^{''})^{(6)}\left((G_{35}), t\right) \le -(\tau_1)^{(6)}$ | 349 |
| **Definition of** $(v_1)^{(6)}, (v_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, v^{(6)}, u^{(6)}$ : By $(v_1)^{(6)} > 0$, $(v_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$, $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_1)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{v}_1)^{(6)},, (\bar{v}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ : By $(\bar{v}_1)^{(6)} > 0$, $(\bar{v}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$, $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_2)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ **Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (v_0)^{(6)}$ :- If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by $(m_2)^{(6)} = (v_0)^{(6)}, (m_1)^{(6)} = (v_1)^{(6)}$, ***if*** $(v_0)^{(6)} < (v_1)^{(6)}$ $(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (\bar{v}_6)^{(6)}$, ***if*** $(v_1)^{(6)} < (v_0)^{(6)} < (\bar{v}_1)^{(6)}$, | 351 |

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| and $\boxed{(v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$<br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (v_0)^{(6)},$ ***if*** $(\bar{v}_1)^{(6)} < (v_0)^{(6)}$ | |
| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},$ ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)},$ ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\right)\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34')6e(S1)6t−e−(a34')6t+ G340e−(a34')6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b_{34}')^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b_{34}')^{(6)} t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$<br>$(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** | |

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ :

$(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying

$$-(\sigma_2)^{(7)} \le -(a_{36}^{'})^{(7)} + (a_{37}^{'})^{(7)} - (a_{36}^{''})^{(7)}(T_{37}, t) + (a_{37}^{''})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$$

$$-(\tau_2)^{(7)} \le -(b_{36}^{'})^{(7)} + (b_{37}^{'})^{(7)} - (b_{36}^{''})^{(7)}\big((G_{39}), t\big) - (b_{37}^{''})^{(7)}\big((G_{39}), t\big) \le -(\tau_1)^{(7)}$$

**Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ : 361

By $(\nu_1)^{(7)} > 0, (\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ : 362

By $(\bar{\nu}_1)^{(7)} > 0, (\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$

**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,
and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$

and analogously 363

$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,
and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

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| Then the solution of global equations satisfies the inequalities<br>$G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^{0}e^{(S_1)^{(7)}t}$<br>where $(p_i)^{(7)}$ is defined by equation | 364 |
| $\frac{1}{(m_7)^{(7)}}G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}}G_{36}^{0}e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t}-e^{-(S_2)^{(7)}t}\right]+G_{38}^{0}e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}^{'})^{(7)})}[e^{(S_1)^{(7)}t}-e^{-(a_{38}^{'})^{(7)}t}]+G_{38}^{0}e^{-(a_{38}^{'})^{(7)}t})$ | 366 |
| $\boxed{T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}}T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}}T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)}T_{36}^{0}}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}^{'})^{(7)})}\left[e^{(R_1)^{(7)}t}-e^{-(b_{38}^{'})^{(7)}t}\right]+T_{38}^{0}e^{-(b_{38}^{'})^{(7)}t} \le T_{38}(t) \le$<br>$\frac{(a_{38})^{(7)}T_{36}^{0}}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t}-e^{-(R_2)^{(7)}t}\right]+T_{38}^{0}e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}^{'})^{(7)}$<br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}^{'})^{(7)}$<br>$(R_2)^{(7)} = (b_{38}^{'})^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br>$-(\sigma_2)^{(8)} \le -(a_{40}^{'})^{(8)} + (a_{41}^{'})^{(8)} - (a_{40}^{''})^{(8)}(T_{41}, t) + (a_{41}^{''})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br>$-(\tau_2)^{(8)} \le -(b_{40}^{'})^{(8)} + (b_{41}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}), t) - (b_{41}^{''})^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |

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| **Definition of** $(\nu_1)^{(8)}, (\nu_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, \nu^{(8)}, u^{(8)}$ : <br><br> By $(\nu_1)^{(8)} > 0$ , $(\nu_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0$ , $(u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$ <br> and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | 372 |
| **Definition of** $(\bar{\nu}_1)^{(8)},, (\bar{\nu}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ : <br><br> By $(\bar{\nu}_1)^{(8)} > 0$ , $(\bar{\nu}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0$ , $(\bar{u}_2)^{(8)} < 0$ the <br><br> roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$ <br><br> and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ <br><br> **Definition of** $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)}, (\mu_2)^{(8)}, (\nu_0)^{(8)}$ :- <br><br> If we define $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)}, (\mu_2)^{(8)}$ by <br><br> $(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)}$, ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$ <br><br> $(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)}$, <br> and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ <br><br> $(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)}$, ***if*** $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously <br><br> $(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)}$, ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$ <br><br> $(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)}$, <br> and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$ <br><br> $(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}$, ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities <br><br> $G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{40}(t) \leq G_{40}^0 e^{(S_1)^{(8)}t}$ <br><br> where $(p_i)^{(8)}$ is defined by equation | 375 |

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| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)}-(p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)}G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t} - e^{-(S_2)^{(8)}t}\right] + G_{42}^0 e^{-(S_2)^{(8)}t} \le G_{42}(t) \le \right.$ $\left.\frac{(a_{42})^{(8)}G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)}-(a'_{42})^{(8)})}[e^{(S_1)^{(8)}t} - e^{-(a'_{42})^{(8)}t}] + G_{42}^0 e^{-(a'_{42})^{(8)}t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)}-(b'_{42})^{(8)})}\left[e^{(R_1)^{(8)}t} - e^{-(b'_{42})^{(8)}t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)}t} \le T_{42}(t) \le$ $\frac{(a_{42})^{(8)}T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t} - e^{-(R_2)^{(8)}t}\right] + T_{42}^0 e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$<br>$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}((G_{47}), t) - (b''_{45})^{(9)}((G_{47}), t) \le -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :<br>By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$<br>and $(b_{45})^{(9)}(u^{(9)})^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(9)},, (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ : | |

By $(\bar{v}_1)^{(9)} > 0$ , $(\bar{v}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\left(u^{(9)}\right)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}$ , $(m_2)^{(9)}$ , $(\mu_1)^{(9)}$, $(\mu_2)^{(9)}$, $(v_0)^{(9)}$ :-

If we define $(m_1)^{(9)}$ , $(m_2)^{(9)}$ , $(\mu_1)^{(9)}$, $(\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (v_0)^{(9)}, (m_1)^{(9)} = (v_1)^{(9)}$, ***if*** $(v_0)^{(9)} < (v_1)^{(9)}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (\bar{v}_1)^{(9)}$ , ***if*** $(v_1)^{(9)} < (v_0)^{(9)} < (\bar{v}_1)^{(9)}$,

and $\boxed{(v_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (v_0)^{(9)}$, ***if*** $(\bar{v}_1)^{(9)} < (v_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^0 e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}} G_{44}^0 e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)} G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)} - (p_{44})^{(9)} - (S_2)^{(9)})}\left[e^{((S_1)^{(9)} - (p_{44})^{(9)})t} - e^{-(S_2)^{(9)}t}\right] + G_{46}^0 e^{-(S_2)^{(9)}t} \le G_{46}(t) \le \right.$$

$$\left. \frac{(a_{46})^{(9)} G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)} - (a_{46}')^{(9)})}\left[e^{(S_1)^{(9)}t} - e^{-(a_{46}')^{(9)}t}\right] + G_{46}^0 e^{-(a_{46}')^{(9)}t}\right)$$

$$\boxed{T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}} T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}} T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)} T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)} - (b_{46}')^{(9)})}\left[e^{(R_1)^{(9)}t} - e^{-(b_{46}')^{(9)}t}\right] + T_{46}^0 e^{-(b_{46}')^{(9)}t} \le T_{46}(t) \le$$

$\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$

**Definition of** $(S_1)^{(9)},(S_2)^{(9)},(R_1)^{(9)},(R_2)^{(9)}$:-

Where $(S_1)^{(9)}=(a_{44})^{(9)}(m_2)^{(9)}-(a_{44}^{'})^{(9)}$

$(S_2)^{(9)}=(a_{46})^{(9)}-(p_{46})^{(9)}$

$(R_1)^{(9)}=(b_{44})^{(9)}(\mu_2)^{(9)}-(b_{44}^{'})^{(9)}$

$(R_2)^{(9)}=(b_{46}^{'})^{(9)}-(r_{46})^{(9)}$

**Proof :** From global equations we obtain 383

$$\frac{dv^{(1)}}{dt}=(a_{13})^{(1)}-\left((a_{13}^{'})^{(1)}-(a_{14}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14},t)\right)-(a_{14}^{''})^{(1)}(T_{14},t)v^{(1)}-(a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)}=\frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_2)^{(1)}v^{(1)}-(a_{13})^{(1)}\right)\leq\frac{dv^{(1)}}{dt}\leq-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_1)^{(1)}v^{(1)}-(a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(1)},(\nu_0)^{(1)}$ :-

For $0<\boxed{(\nu_0)^{(1)}=\frac{G_{13}^{0}}{G_{14}^{0}}}<(\nu_1)^{(1)}<(\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t)\geq\frac{(\nu_1)^{(1)}+(C)^{(1)}(\nu_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_0)^{(1)}\right)t\right]}}{1+(C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_0)^{(1)}\right)t\right]}}\quad,\quad\boxed{(C)^{(1)}=\frac{(\nu_1)^{(1)}-(\nu_0)^{(1)}}{(\nu_0)^{(1)}-(\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)}\leq\nu^{(1)}(t)\leq(\nu_1)^{(1)}$

In the same manner , we get 384

$$\nu^{(1)}(t)\leq\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(1)}=\frac{(\bar{\nu}_1)^{(1)}-(\nu_0)^{(1)}}{(\nu_0)^{(1)}-(\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)}\leq\nu^{(1)}(t)\leq(\bar{\nu}_1)^{(1)}$

If $0<(\nu_1)^{(1)}<(\nu_0)^{(1)}=\frac{G_{13}^{0}}{G_{14}^{0}}<(\bar{\nu}_1)^{(1)}$ we find like in the previous case, 385

$$(\nu_1)^{(1)} \le \frac{(\nu_1)^{(1)}+(C)^{(1)}(\nu_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_2)^{(1)}\right)t\right]}}{1+(C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_2)^{(1)}\right)t\right]}} \le \nu^{(1)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \le (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \le (\bar{\nu}_1)^{(1)} \le \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain

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$$(\nu_1)^{(1)} \le \nu^{(1)}(t) \le \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \le (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \le \nu^{(1)}(t) \le (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \le u^{(1)}(t) \le (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)}G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)}T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}')^{(2)} - (a_{17}')^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$

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It follows

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|---|---|
| $-\left((a_{17})^{(2)}\left(v^{(2)}\right)^2+(\sigma_2)^{(2)}v^{(2)}-(a_{16})^{(2)}\right)\le\frac{dv^{(2)}}{dt}\le-\left((a_{17})^{(2)}\left(v^{(2)}\right)^2+(\sigma_1)^{(2)}v^{(2)}-(a_{16})^{(2)}\right)$ | |
| From which one obtains<br><br>**Definition of** $(\bar{v}_1)^{(2)},(v_0)^{(2)}$ :-<br><br>For $0<(v_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}<(v_1)^{(2)}<(\bar{v}_1)^{(2)}$<br><br>$v^{(2)}(t)\ge\frac{(v_1)^{(2)}+(C)^{(2)}(v_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_0)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_0)^{(2)}\right)t\right]}}$ , $\boxed{(C)^{(2)}=\frac{(v_1)^{(2)}-(v_0)^{(2)}}{(v_0)^{(2)}-(v_2)^{(2)}}}$<br><br>it follows $(v_0)^{(2)}\le v^{(2)}(t)\le(v_1)^{(2)}$ | 390 |
| In the same manner , we get<br><br>$v^{(2)}(t)\le\frac{(\bar{v}_1)^{(2)}+(\bar{C})^{(2)}(\bar{v}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(2)}=\frac{(\bar{v}_1)^{(2)}-(v_0)^{(2)}}{(v_0)^{(2)}-(\bar{v}_2)^{(2)}}}$ | 391 |
| From which we deduce $(v_0)^{(2)}\le v^{(2)}(t)\le(\bar{v}_1)^{(2)}$ | 392 |
| If $0<(v_1)^{(2)}<(v_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}<(\bar{v}_1)^{(2)}$ we find like in the previous case,<br><br>$(v_1)^{(2)}\le\frac{(v_1)^{(2)}+(C)^{(2)}(v_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_2)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_2)^{(2)}\right)t\right]}}\le v^{(2)}(t)\le$<br><br>$\frac{(\bar{v}_1)^{(2)}+(\bar{C})^{(2)}(\bar{v}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}\le(\bar{v}_1)^{(2)}$ | 393 |
| If $0<(v_1)^{(2)}\le(\bar{v}_1)^{(2)}\le(v_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(v_1)^{(2)}\le v^{(2)}(t)\le\frac{(\bar{v}_1)^{(2)}+(\bar{C})^{(2)}(\bar{v}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}\le(v_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $v^{(2)}(t)$ :-<br><br>$(m_2)^{(2)}\le v^{(2)}(t)\le(m_1)^{(2)}$, $\boxed{v^{(2)}(t)=\frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :- | 396 |

| | |
|---|---|
| $(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| From which one obtains<br><br>For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$<br><br>$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)} + (C)^{(3)}(\nu_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}$ , $\boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\nu_2)^{(3)}}}$<br><br>it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$ | 400 |
| In the same manner , we get<br><br>$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)} + (C)^{(3)}(\bar{\nu}_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\bar{\nu}_2)^{(3)}}}$<br><br>**Definition of** $(\bar{\nu}_1)^{(3)}$ :-<br><br>From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$ | 401 |

If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, 402

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain 403

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain 404

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right) - (a_{25}^{''})^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_2)^{(4)}v^{(4)}-(a_{24})^{(4)}\right)\le\frac{dv^{(4)}}{dt}\le-\left((a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_4)^{(4)}v^{(4)}-(a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(4)},(v_0)^{(4)}$ :-

For $0<\boxed{(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}<(v_1)^{(4)}<(\bar{v}_1)^{(4)}$

$$v^{(4)}(t)\ge\frac{(v_1)^{(4)}+(C)^{(4)}(v_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_0)^{(4)}\right)t\right]}}{4+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_0)^{(4)}\right)t\right]}}\quad,\quad\boxed{(C)^{(4)}=\frac{(v_1)^{(4)}-(v_0)^{(4)}}{(v_0)^{(4)}-(v_2)^{(4)}}}$$

it follows $(v_0)^{(4)}\le v^{(4)}(t)\le(v_1)^{(4)}$

In the same manner , we get

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$$v^{(4)}(t)\le\frac{(\bar{v}_1)^{(4)}+(\bar{C})^{(4)}(\bar{v}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}{4+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(4)}=\frac{(\bar{v}_1)^{(4)}-(v_0)^{(4)}}{(v_0)^{(4)}-(\bar{v}_2)^{(4)}}}$$

From which we deduce $(v_0)^{(4)}\le v^{(4)}(t)\le(\bar{v}_1)^{(4)}$

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If $0<(v_1)^{(4)}<(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}<(\bar{v}_1)^{(4)}$ we find like in the previous case,

$$(v_1)^{(4)}\le\frac{(v_1)^{(4)}+(C)^{(4)}(v_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_2)^{(4)}\right)t\right]}}{1+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_2)^{(4)}\right)t\right]}}\le v^{(4)}(t)\le$$

$$\frac{(\bar{v}_1)^{(4)}+(\bar{C})^{(4)}(\bar{v}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}\le(\bar{v}_1)^{(4)}$$

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If $0<(v_1)^{(4)}\le(\bar{v}_1)^{(4)}\le\boxed{(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(v_1)^{(4)}\le v^{(4)}(t)\le\frac{(\bar{v}_1)^{(4)}+(\bar{C})^{(4)}(\bar{v}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}\le(v_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(4)}(t)$ :-

$$(m_2)^{(4)}\le v^{(4)}(t)\le(m_1)^{(4)},\quad\boxed{v^{(4)}(t)=\frac{G_{24}(t)}{G_{25}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$$(\mu_2)^{(4)}\le u^{(4)}(t)\le(\mu_1)^{(4)},\quad\boxed{u^{(4)}(t)=\frac{T_{24}(t)}{T_{25}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}'')^{(5)} = (b_{29}'')^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}')^{(6)} - (a_{33}')^{(6)} + (a_{32}'')^{(6)}(T_{33}, t)\right) - (a_{33}'')^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

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In the same manner , we get

$$\nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\bar{\nu}_1)^{(6)}$

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If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

$$(\nu_1)^{(6)} \leq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_2)^{(6)}\right)t\right]}} \leq \nu^{(6)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\bar{\nu}_1)^{(6)}$$

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If $0 < (\nu_1)^{(6)} \leq (\bar{\nu}_1)^{(6)} \leq \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(\nu_1)^{(6)} \leq \nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

$$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}, \quad \boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}, \quad \boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)}G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a'_{36})^{(7)} - (a'_{37})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right) - (a''_{37})^{(7)}(T_{37},t)v^{(7)} - (a_{37})^{(7)}v^{(7)}$$

**Definition of** $v^{(7)}$ :- $\boxed{v^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)}\right) \le \frac{dv^{(7)}}{dt} \le -\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(7)}, (v_0)^{(7)}$ :-

For $0 < \boxed{(v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}} < (v_1)^{(7)} < (\bar{v}_1)^{(7)}$

$$v^{(7)}(t) \ge \frac{(v_1)^{(7)} + (C)^{(7)}(v_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(v_1)^{(7)} - (v_0)^{(7)}}{(v_0)^{(7)} - (v_2)^{(7)}}}$$

it follows $(v_0)^{(7)} \le v^{(7)}(t) \le (v_1)^{(7)}$

In the same manner , we get

$$v^{(7)}(t) \le \frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{v}_1)^{(7)} - (v_0)^{(7)}}{(v_0)^{(7)} - (\bar{v}_2)^{(7)}}}$$

From which we deduce $(v_0)^{(7)} \le v^{(7)}(t) \le (\bar{v}_1)^{(7)}$

If $0 < (v_1)^{(7)} < (v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}} < (\bar{v}_1)^{(7)}$ we find like in the previous case,

$$(v_1)^{(7)} \le \frac{(v_1)^{(7)} + (C)^{(7)}(v_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_2)^{(7)}\right)t\right]}} \le v^{(7)}(t) \le$$

$$\frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \le (\bar{v}_1)^{(7)}$$

If $0 < (v_1)^{(7)} \le (\bar{v}_1)^{(7)} \le \boxed{(v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$ , we obtain

$$(v_1)^{(7)} \le v^{(7)}(t) \le \frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \le (v_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(7)}(t)$ :-

$(m_2)^{(7)} \leq v^{(7)}(t) \leq (m_1)^{(7)}$, $\boxed{v^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :-

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$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $v^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)} G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

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$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right) - (a_{41}^{''})^{(8)}(T_{41}, t) v^{(8)} - (a_{41})^{(8)} v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)} v^{(8)} - (a_{40})^{(8)}\right) \leq \frac{dv^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)} v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$v^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)}-(\nu_0)^{(8)}}{(\nu_0)^{(8)}-(\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\bar{\nu}_8)^{(8)}$

If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case, 423

$$(\nu_1)^{(8)} \leq \frac{(\nu_1)^{(8)}+(C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}} \leq \nu^{(8)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\bar{\nu}_1)^{(8)}$$

If $0 < (\nu_1)^{(8)} \leq (\bar{\nu}_1)^{(8)} \leq \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain 424

$$(\nu_1)^{(8)} \leq \nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)}+(C)^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \leq \nu^{(8)}(t) \leq (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)}G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)}T_{41}(t)$ This is an important

consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{d\nu^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right) - (a''_{45})^{(9)}(T_{45},t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \leq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}} \leq \nu^{(9)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \leq (\bar{\nu}_1)^{(9)} \leq \boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$ , we obtain

$$(\nu_1)^{(9)} \leq \nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$(m_2)^{(9)} \leq \nu^{(9)}(t) \leq (m_1)^{(9)}$, $\boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}$, $\boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ 430

$with\ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations 431

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$

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| $(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$ | 435 |

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| $(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |

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| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |

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| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |

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| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(l) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) +$ | 488 |

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| $(a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first | 494 |

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| equations | |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |

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| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{40}=\frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}=\frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 501 |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{44}=\frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}=\frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G)=(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)}-(b_{13})^{(1)}(b_{14})^{(1)}-$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G)+(b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G)=0$<br><br>Where in $G(G_{13},G_{14},G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*})=0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19})=(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)}-(b_{16})^{(2)}(b_{17})^{(2)}-$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19})+(b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19})=0$ | 503 |
| Where in $(G_{19})(G_{16},G_{17},G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*})=0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23})=(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)}-(b_{20})^{(3)}(b_{21})^{(3)}-$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23})+(b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23})=0$<br><br>Where in $G_{23}(G_{20},G_{21},G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*})=0$ | 505 |

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| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right] + (b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br><br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br><br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br><br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br><br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$ | |

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| $\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0$ , $T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0$ , $T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*}) = 0$ , $T_{21}^{*}$ given by $f(T_{21}^{*}) = 0$ and<br><br>$G_{20}^{*} = \frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*} = \frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br><br>$T_{20}^{*} = \frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*} = \frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br><br>$G_{25}^{*}$ given by $\varphi(G_{27}) = 0$ , $T_{25}^{*}$ given by $f(T_{25}^{*}) = 0$ and<br><br>$G_{24}^{*} = \frac{(a_{24})^{(4)}G_{25}^{*}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*} = \frac{(a_{26})^{(4)}G_{25}^{*}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*} = \frac{(b_{24})^{(4)}T_{25}^{*}}{[(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*} = \frac{(b_{26})^{(4)}T_{25}^{*}}{[(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}((G_{27})^{*})]}$ | 517 |

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| Obviously, these values represent an equilibrium solution of global equations | |
| Finally we obtain the unique solution<br><br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*}) = 0$ , $T_{29}^{*}$ given by $f(T_{29}^{*}) = 0$ and<br><br>$G_{28}^{*} = \frac{(a_{28})^{(5)} G_{29}^{*}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*} = \frac{(a_{30})^{(5)} G_{29}^{*}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |
| $T_{28}^{*} = \frac{(b_{28})^{(5)} T_{29}^{*}}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)} T_{29}^{*}}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br><br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br><br>$G_{32}^{*} = \frac{(a_{32})^{(6)} G_{33}^{*}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)} G_{33}^{*}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)} T_{33}^{*}}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)} T_{33}^{*}}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br><br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br><br>$G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99 | 523 A |

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| $G_{45}^*$ given by $\varphi((G_{47})^*) = 0$ , $T_{45}^*$ given by $f(T_{45}^*) = 0$ and<br>$G_{44}^* = \frac{(a_{44})^{(9)} G_{45}^*}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^*)]}$ , $G_{46}^* = \frac{(a_{46})^{(9)} G_{45}^*}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^*)]}$<br>$T_{44}^* = \frac{(b_{44})^{(9)} T_{45}^*}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47})^*)]}$ , $T_{46}^* = \frac{(b_{46})^{(9)} T_{45}^*}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47})^*)]}$ | |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof:** Denote<br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^* + \mathbb{G}_i$ , $T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^*) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^*) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^* \mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^* \mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^* \mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^* \mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^* \mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^* \mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i$ , $T_i = T_i^* + \mathbb{T}_i$ | 532 |

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| $\frac{\partial(a_{17}^{''})^{(2)}}{\partial \mathrm{T}_{17}}(\mathrm{T}_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial(b_i^{''})^{(2)}}{\partial \mathrm{G}_j}((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{\mathrm{d}\mathbb{G}_{16}}{\mathrm{dt}} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}\mathrm{G}_{16}^{*}\mathbb{T}_{17}$ | 534 |
| $\frac{\mathrm{d}\mathbb{G}_{17}}{\mathrm{dt}} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}\mathrm{G}_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{\mathrm{d}\mathbb{G}_{18}}{\mathrm{dt}} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}\mathrm{G}_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{\mathrm{d}\mathbb{T}_{16}}{\mathrm{dt}} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}\mathrm{T}_{16}^{*}\mathbb{G}_j\right)$ | 537 |
| $\frac{\mathrm{d}\mathbb{T}_{17}}{\mathrm{dt}} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}\mathrm{T}_{17}^{*}\mathbb{G}_j\right)$ | 538 |
| $\frac{\mathrm{d}\mathbb{T}_{18}}{\mathrm{dt}} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}\mathrm{T}_{18}^{*}\mathbb{G}_j\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)}$ , $\frac{\partial(b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_j\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 547 |

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| **Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)}$ , $\frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^*) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ $and$ $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^*) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |

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| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^{*}\mathbb{G}_{j}\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^{*}\mathbb{G}_{j}\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^{*}\mathbb{G}_{j}\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ $and$ $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^{*}) = (q_{33})^{(6)}$ , $\frac{\partial(b_i^{''})^{(6)}}{\partial G_j}((G_{35})^{*}) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^{*}\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^{*}\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^{*}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^{*}\mathbb{G}_{j}\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^{*}\mathbb{G}_{j}\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^{*}\mathbb{G}_{j}\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ $and$ $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 572 |

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| $\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^{*}) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^{*}\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^{*}\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^{*}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^{*}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^{*}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^{*}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^{*}) = (q_{41})^{(8)}$ , $\frac{\partial (b_i^{''})^{(8)}}{\partial G_j}\left((G_{43})^{*}\right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^{*}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^{*}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^{*}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^{*}\mathbb{G}_j\right)$ | 584 |

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| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a''_i)^{(9)}$ *and* $(b''_i)^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{45})^{(9)}}{\partial T_{45}}(T^*_{45}) = (q_{45})^{(9)} \ , \ \frac{\partial(b''_i)^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a'_{44})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G^*_{44}\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a'_{45})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G^*_{45}\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a'_{46})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G^*_{46}\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$<br><br>$\left[\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\Big]$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\Big)$<br><br>$+\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\Big)$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\Big)$<br><br>$\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$ | |

$$\left((\lambda)^{(1)}\right)^2+\left((b'_{13})^{(1)}+(b'_{14})^{(1)}-(r_{13})^{(1)}+(r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$$

$$+\left(\left((\lambda)^{(1)}\right)^2+\left((a'_{13})^{(1)}+(a'_{14})^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$$

$$+\left((\lambda)^{(1)}+(a'_{13})^{(1)}+(p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G^*_{14}+(a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G^*_{13}\right)$$

$$\left((\lambda)^{(1)}+(b'_{13})^{(1)}-(r_{13})^{(1)}\right)s_{(14),(15)}T^*_{14}+(b_{14})^{(1)}s_{(13),(15)}T^*_{13}\Big)\}=0$$

$$+$$

$$(\lambda)^{(2)}+(b'_{18})^{(2)}-(r_{18})^{(2)}\}\{\left((\lambda)^{(2)}+(a'_{18})^{(2)}+(p_{18})^{(2)}\right)$$

$$\left[\left((\lambda)^{(2)}+(a'_{16})^{(2)}+(p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}^*_{17}+(a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^*_{16}\right)\Big]$$

$$\left((\lambda)^{(2)}+(b'_{16})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}^*_{17}+(b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^*_{17}\Big)$$

$$+\left(\left((\lambda)^{(2)}+(a'_{17})^{(2)}+(p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}^*_{16}+(a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^*_{17}\right)$$

$$\left((\lambda)^{(2)}+(b'_{16})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}^*_{17}+(b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^*_{16}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((a'_{16})^{(2)}+(a'_{17})^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((b'_{16})^{(2)}+(b'_{17})^{(2)}-(r_{16})^{(2)}+(r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left(\left((\lambda)^{(2)}\right)^2+\left((a'_{16})^{(2)}+(a'_{17})^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a'_{16})^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}^*_{17}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}^*_{16}\right)$$

$$\left((\lambda)^{(2)}+(b'_{16})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}^*_{17}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}^*_{16}\Big)\}=0$$

$$+$$

$$(\lambda)^{(3)}+(b'_{22})^{(3)}-(r_{22})^{(3)}\}\{\left((\lambda)^{(3)}+(a'_{22})^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left((\lambda)^{(3)}+(a'_{20})^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G^*_{21}+(a_{21})^{(3)}(q_{20})^{(3)}G^*_{20}\right)\Big]$$

$$\left((\lambda)^{(3)}+(b'_{20})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T^*_{21}+(b_{21})^{(3)}s_{(20),(21)}T^*_{21}\Big)$$

$$+\left(\left((\lambda)^{(3)}+(a'_{21})^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G^*_{20}+(a_{20})^{(3)}(q_{21})^{(1)}G^*_{21}\right)$$

$$\left((\lambda)^{(3)}+(b'_{20})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T^*_{21}+(b_{21})^{(3)}s_{(20),(20)}T^*_{20}\Big)$$

$$\left((\lambda)^{(3)}\right)^2+\left((a'_{20})^{(3)}+(a'_{21})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((b'_{20})^{(3)}+(b'_{21})^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a'_{20})^{(3)}+(a'_{21})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a'_{20})^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G^*_{21}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^*_{20}\right)$$

$$\left((\lambda)^{(3)}+(b'_{20})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T^*_{21}+(b_{21})^{(3)}s_{(20),(22)}T^*_{20}\Big)\}=0$$

$$+$$

$$(\lambda)^{(4)}+(b'_{26})^{(4)}-(r_{26})^{(4)}\{\left((\lambda)^{(4)}+(a'_{26})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a'_{24})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G^*_{25}+(a_{25})^{(4)}(q_{24})^{(4)}G^*_{24}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T^*_{25}+(b_{25})^{(4)}s_{(24),(25)}T^*_{25}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a'_{25})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G^*_{24}+(a_{24})^{(4)}(q_{25})^{(4)}G^*_{25}\right)$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T^*_{25}+(b_{25})^{(4)}s_{(24),(24)}T^*_{24}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((a'_{24})^{(4)}+(a'_{25})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((b'_{24})^{(4)}+(b'_{25})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a'_{24})^{(4)}+(a'_{25})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a'_{24})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G^*_{25}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G^*_{24}\right)$$

$$\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T^*_{25}+(b_{25})^{(4)}s_{(24),(26)}T^*_{24}\Big)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b'_{30})^{(5)}-(r_{30})^{(5)}\{\left((\lambda)^{(5)}+(a'_{30})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a'_{28})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G^*_{29}+(a_{29})^{(5)}(q_{28})^{(5)}G^*_{28}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T^*_{29}+(b_{29})^{(5)}s_{(28),(29)}T^*_{29}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a'_{29})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G^*_{28}+(a_{28})^{(5)}(q_{29})^{(5)}G^*_{29}\right)$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T^*_{29}+(b_{29})^{(5)}s_{(28),(28)}T^*_{28}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((a'_{28})^{(5)}+(a'_{29})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left((\lambda)^{(5)}\right)^2+\left((b_{28}')^{(5)}+(b_{29}')^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$+\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}')^{(5)}+(a_{29}')^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}')^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}')^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}')^{(6)}-(r_{34})^{(6)}\{\left((\lambda)^{(6)}+(a_{34}')^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left((\lambda)^{(6)}+(a_{32}')^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\Big]$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\Big)$$

$$+\left((\lambda)^{(6)}+(a_{33}')^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\Big)$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((a_{32}')^{(6)}+(a_{33}')^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((b_{32}')^{(6)}+(b_{33}')^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$+\left(\left((\lambda)^{(6)}\right)^2+\left((a_{32}')^{(6)}+(a_{33}')^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}')^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b_{38}')^{(7)}-(r_{38})^{(7)}\{\left((\lambda)^{(7)}+(a_{38}')^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left((\lambda)^{(7)}+(a_{36}')^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\Big]$$

$$\left((\lambda)^{(7)}+(b_{36}')^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\Big)$$

$$+\left((\lambda)^{(7)}+(a_{37}')^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\Big)$$

$$\left((\lambda)^{(7)}+(b_{36}')^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a_{36}')^{(7)}+(a_{37}')^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left(\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^*_{36}\right)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^*_{37}+(b_{37})^{(7)}s_{(36),(38)}T^*_{36}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^*_{40}\right)\right]$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^*_{41}+(b_{41})^{(8)}s_{(40),(41)}T^*_{41}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^*_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^*_{41}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^*_{41}+(b_{41})^{(8)}s_{(40),(40)}T^*_{40}\right)$$

$$\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^*_{40}\right)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^*_{41}+(b_{41})^{(8)}s_{(40),(42)}T^*_{40}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^*_{44}\right)\right]$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^*_{45}+(b_{45})^{(9)}s_{(44),(45)}T^*_{45}\right)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^*_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\right)$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45}+(b_{45})^{(9)}s_{(44),(44)}T^*_{44}\right)$$

$$\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45}+(b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt}= A_i\ (x)\ xi$ where $x_i \in R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i \ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i \ X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions

are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$(12) \qquad \frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17].

In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 12. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon$ $C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\,(x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega\,(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \epsilon C : \; \omega\,(x) \neq \emptyset \text{ and } \omega\,(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s\,(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i\,(x)$ is continuous

**A2:** $A_i\,(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma\,(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population $i$. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility

assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C\backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^n_i$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\in \Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two

propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population *i* is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) In \ |B_i \ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^t_0(A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = limsup_{t\to\infty}\frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni}= E_{i(x)} \ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \ \leq \ a|\ B_i(t,x)u_i(x)|+ |B_i(t,x)w| \ \leq \ |\ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)|\ w|)$. Hence, $limsup_{t\to\infty}\frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$limsup_{t\to\infty}\frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty}\frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \ \geq \ a|B_i(t,x)u_i(x)|-|B_i(t,x)w|$$

$$\geq |\ B_i(t,x)u_i(x)|(a- \alpha+ exp\ (-\beta t)|\ w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty}\frac{1}{t} \text{In}\ |B_i(t,x)v| \ \geq \ \lambda i(x) \geq \limsup_{t\to\infty}\frac{1}{t} \text{In}\ |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) \ = \ In|B_i(t+s,x)\ u_i(x)|= In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

| |
|---|
| $= \text{In } \lvert B_i(s,x.t)\, u_i(x.t) \rvert \lvert B_i(t,x)\, u_i(x) \rvert$ |
| $= \text{In } b_i(s,t.x)\, b_i(t,x)$ |
| |
| Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies |
| $\frac{d}{dt} b_i(t,x) = \lim_{s \to 0} \frac{bi(t+s,x-bi(t,x)}{S-}$ |
| $= \lim_{s \to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}\lvert_{s=0}\, b_i(s,x.t)$ |
| = |
| $=(A_i(x.t)u_i(x.s),u_i(x.s),$ |
| The Fundamental Theorem of Calculus implies |
| $B_i(t,x) = \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$ |
| and the second assertion follows. |
| **Proposition 2:** *For the solutions of (2) we have:* if $x_i > 0$, then $\lambda i(x) \leq 0$. If $\lambda i(x) < 0$, then $\lim_{t \to \infty} (x.t)_i = 0$ If $\limsup_{t \to \infty}$ $(x.t)_i \rvert > 0$, then $\lambda i(x) = 0$. |
| **Proof:** First, assume that x satisfies $x_i > 0$. Since the semi flow is dissipative, there exists $\gamma > 0$ such that $\lvert x.t \rvert \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply |
| $\lambda i(x) = \limsup_{t \to \infty} \frac{1}{t} \text{In } \lvert B_i(t,x)x_i \rvert = \limsup_{t \to \infty} \frac{1}{t} \text{In } \lvert (x.t)_i \rvert$ |
| $\leq \limsup_{t \to \infty} \frac{In\, \gamma}{t} = 0$ |
| To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i = 0$, then the invariance of the faces of C imply that $(x.t)_i = 0$ for all $t \geq 0$. Alternatively, if $x_i > 0$, then Proposition I and the skew product flow imply |
| $\limsup_{t \to \infty} \frac{1}{t} In \lvert (x.t)_i \rvert = \lambda i(x) < 0.$ |
| The final assertion follows from the first two assertions. 5 |
| |

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on $\boldsymbol{C}$, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:*$\lambda_i(\mathrm{x}) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle \mathrm{A}_i(x.s)\mathrm{u}_i(x.s), \mathrm{u}_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(\mathrm{x}) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $\mathrm{h}_i(x) = \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I}\boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I}\boldsymbol{C}_i^+$such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following

equivalent conditions hold For all invariant measures $\mu$ supported by $K$ (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by $K$.

then $K$ is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and $K$ is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by $K$.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$. Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large

so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \leq \epsilon$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \leq \epsilon$, and $|h(x.t) - h(x^n.t)| \leq \epsilon$ for all $x \in N_1(\Gamma(A))$. Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$$\leq \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$$

$$\leq \quad \epsilon + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$$

$$\leq \quad 2\epsilon + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$$

$$\leq \quad 3\epsilon$$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\epsilon > 0$ and $t > 0$ are arbitrary, we have $\int_c h(x.t) d\mu(x) = \int_c h(x) d\mu(x)$ for all $t > 0$ and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x) u_i^n(x), u_i^n(x) \rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop. 3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop. 3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\epsilon > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \epsilon$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)| \leq \epsilon$.

Then

$$\lambda_i(\mu) \quad \leq \quad \int_c |h_i(x) - h_i^n(x)| d\mu(x) + |\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\, d\mu_n(x)$$

$$\leq \quad 2\epsilon$$

Since $\epsilon > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm. 2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}$

$\int_0^t \delta_{y.s}$ ds where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$ . Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}_{n=1}^{\infty}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

| |
|---|
| Min max $\lambda i$ ($\mu = \min_{\mu} \max_{p \epsilon \Delta} \sum_{i} Pi\lambda i(\mu)$ |
| |
| Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that |
| (6) min max $\lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_{i} Pi\lambda i(\mu)$ |
| |
| Where the minimum is taken over invariant probability measures $\mu$ with support in *K.* Since min$\mu$ |
| $\sum_{i} P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established. |
| |
| To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\} \backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta -$perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ [0,1]$be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C \backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by |
| |
| $\tilde{A}_i(x) = \{ \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i \epsilon I}{if\ i \notin I}$ |
| |
| Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t\ \ whenever\ x.[0,t] \in C \backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t >$ 0. $Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in Kand\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of |

$\dot{x} = \tilde{A}(x)x$ *and* $\lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\} \backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). *Since* $E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently *K* is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent*. Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume x* $\mapsto$ *A(x) is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely*

*ergodic, or There exists* $j \in \{1,\ldots,k\}$ *such that* $M_i$ ----- $\{ x \in C : x_l \gg 0 \text{ for } l \neq j \}$ *Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by* $U_i M_i$.

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure $\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures* $\mu$ *with support in* $\partial C$, $\lambda_i(\mu) > 0$ *for all* $I \in \{1,\ldots,k\} \setminus supp(\mu)$ *(2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $$\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j \qquad i = 1,..,k \qquad j=1,\ldots,m$$

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,\ldots,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda$ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0 < e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j \ (x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1{}^1, \ldots, E_1{}^m)$, and the matrix $A^2$ ($E_1$) is close to the diagonal matrix $\text{diag}(f^i_2(E_1{}^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f^j_2(E^j_1) > 0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f^j_2(E^j_1) > 0$ and $\max_j f^j_2(E^j_2) > 0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0, E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is

sufficiently slow (i.e. $0< e^{j}{}_{i}\ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j (E^j{}_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1-x_1^j - \beta^j\ x_2^j\ - \alpha^j\ x_3^j\ ) + \sum d_1^{jk}\ \ x_1^k\ - e_1^j\ x_1^j$$

$$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j\ x_1^j\ - x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_2^{jk}\ \ x_2^k\ - e_2^j\ x_2^j$$

$$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j\ x_1^j\ - \alpha^j x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_3^{jk}\ \ x_3^k\ - e_3^j\ x_3^j$$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all *j*. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j{}_i = \sum_l d^{l,j}_i$ for all *i, j*) , the maximal invariant set in $\partial$C consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0< e^j{}_i\ll 1$ for all *i*, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2$ $(E_1)$and $\lambda_3$ $(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1 - \beta^j$. Consider the Morse decomposition of $\partial$C given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1> 0$ for *i* = 1, 2, 3, 0 is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j (1 - \beta^j)$$

Equivalently,

$$2>\min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j (r_i - \sum_s a_{is} x_s^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j , \qquad i = 1, \dots, k;\ j = 1, \dots, m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$ . We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set H= $\{x \in C: x_i^j = x_i^l \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \qquad i = 1, \dots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \dots, k$$

It has been shown [12,ch.16.1,16.2] That for $k$ =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there

is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{\mathbf{x}}$ (i.e $A\dot{x} = r \;\; with\, \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \rightarrow \mathbf{E_2} \rightarrow \mathbf{E_3} \rightarrow \mathbf{E_1}$ then the following inequality holds

12) $\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within $\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as

well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that

there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has

abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left( \beta(1 - y - z)\frac{P}{\epsilon + P} - r - m(1 - y) - b(N)\ y \right)$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1 = N$, $x_2 = p$, $x_3 = (y,z)'$ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} - y_i)$$

With

$$(16)\quad F(x)_i = x_i g\,(x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\quad \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag\,(x_i, x_i) - 1 & \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon

plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational

consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical

propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$ to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.34) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.35) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.36) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],$ $i = 1, 2,$

(1.6) $P_i(s) = \int \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the

model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1$, $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as* $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 12. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^{+}\text{x}\ k_{\ 2}\text{x}\ k_{\ 1}, R^{-})$,

$A_3$. $\varphi_i\ (\cdot\ , s) \in L^{\ 2}\ (R^{+}, R^{+})\ \cap C(R^{+}, R^{+})$ for each $s \in\ [-\tau, 0]$,

$i,\ = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i$: $R^{+}$ ▯$[-\tau, T] \rightarrow R^{+}$ $(i,\ = 1,\ 2,)$ With the following properties:

$S_1$. $P_i(\cdot\ , t) \in L^{\ 1}\ (R^{+}, R^{+})$ for each $t\ \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot\ ) \in C\ ([-\tau, T], R^{+})$ for each $a\ \in R^{+}$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^{+}$ ▯$\{R^{+} \cap\ [0, T]\}$ And is continuous $(i,\ = 1,\ 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,)\ \in R^{+}$ ▯$[-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$P_i(a + x, t + x) =\ \ P_i(x), (i,\ = 1,\ 2,)$ ,

Then (1.3) considered at $(1.3)$ considered at $(a + x,\ t + x)$ becomes

$\frac{dp_{\ i}(x)}{dx} + f_i(a + x,\ P(t + x))\ \ P_i(x) = 0$

Which has a solution of the form?

$$P_i(x) = P_i(0)\ \exp\left[-\int_0^{\infty} f_{\ i}\big(a + s, P(t + S)\big)ds\right]$$

And hence

(2.1) $p_{\ 1}\ (a + x, t + x) = p_{\ 1}\ (a, t)\ exp\big[-\int_o^x\ f_{\ 1}\ \big(a + s, p(t + s)\big)ds\big]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_{\ 1}\ (a, t) =\ \varphi_{\ 1}\ (a - t, 0)$exp▯$\left[-\int_o^t\ f_{\ 1}\ \Big(a - t + s, p_{\ 1}\ (s)\Big)\ ds\right]$

$O \leq t \leq a, i = 1,2,$

With

$P_1(s)=(p_1(s),p_2(s-t)),$

$P_2(s)=(p_1(s-t),p_2(s)),$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$t > a \geq, i = 1,2,$

Where

(2.4) $B_i(t) = p_i(0, t), i = 1,2, t > 0.$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$

(2.5) $L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(a+s, p_i(s))ds]$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da + \int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$t > 0, i = 1,2,$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.(a, t, P_i(t))da$

$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$

$i = 1, 2, t > 0$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t) \phi(0) d_i^*] t \quad for \ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T],R^+), \quad B_i \in C([0,T],R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau,T] = \{f \in C([-\tau,T],R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau,T]$. Then for any

$(x, y) \in C^+[-\tau,T] \times C^+[-\tau,T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau,T], \quad C_T^2 = C^+[-\tau,T] \times C^+[-\tau,T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau,T]$ *and* $t \in [0,T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Phi C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = P_t \left\{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on [$-\tau$, $t^*$]

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Phi(P_t) - \Phi_t\|_{t*} \le \left\{ \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right\}$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ and $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for t $\le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \leq \int_0^t B_1^{(1)}(a)\left\{M_1\left(a,t,P_t^{(1)}\right) - M_1\left(a,t,P_t^{(2)}\right)\right\}da$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_t ds\right)da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o)\left[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds\right]da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \|_t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$=\int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\mathrm{X}[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \left\| P_t^{(1)} - P_t^{(2)} \right\| \text{ for } a \varepsilon [o, t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \leq \beta_1 \Phi_1(0) ||P_t^{(1)} - P_t^{(2)}||_t^* + \Phi_1(0) \beta_1 d_1 ||P_t^{(1)} - P_t^{(2)}||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) \mathrm{e}^{\delta 1^a} \beta_1 \| P_t^{(1)} - P_t^{(2)} \|_{t^*} + \beta_1 \mathrm{d}_1 \| P_t^{(1)} - P_t^{(2)} \|_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) \mathrm{e}^{\delta 1^a} \| P_t^{(1)} - P_t^{(2)} \|_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| \mathrm{B}_1^{(1)}(\sigma) - \mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{B}_1^{*\mathrm{d}} \sigma$$

$$+ \Phi_1(0)\| P_t^{(1)}\text{-}P_t^{(2)}\|_{t^*}\left\{B_1 + B_2\, d_1\, a + B_1^{*}\ \left(B_2 a + d_1 a^{2}/2\right)e^{\delta 1^{a}}\right\}$$

$$\leq \beta_1^{*}\int_o^a \left|\ B_1^{(1)}(\sigma)\text{-}\ B_1^{(2)}(\sigma)\right| \mathrm{do}+\left(\mathrm{r}^{*}+\|\Phi_t\|_{t*}\right)\|\ P_t^{(1)}\text{-}P_t^{(2)}\ \|_{t^*} g_1^{(a)}$$

Where $g_1(a)\rightarrow \beta_1$ as $a\rightarrow 0$. By Gronwall's inequality it will follow that

$$\left|B_1^{(1)}(a)\text{-}B_1^{(2)}(a)\right| \leq \left(r^{*}+\|\Phi_t\|_{t*}\right)\|\ P_t^{(1)}\text{-}P_t^{(2)}\ \|_{t^*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)|\leq \left|\ \int_0^a [B_1^{(2)}(\sigma)\ B_1^{(2)}\ (\sigma),]\ b_1(a))\text{-}b_1(a\text{-}\sigma,\ P_t^{(1)}(a))\ M_1(\sigma,\ a,\ P_t^{(1)}(a))do\right|$$

$$\left|B_1^{*}\left(r^{*}+\|\Phi_t\|_{t*}\right)\|\ P_t^{(1)}\text{-}P_t^{(2)}\ \|_{t^*} 0(a) as\ a\rightarrow 0\right.$$

*Thus*

$$\left|B_1^{(1)}(a)-B_1^{(2)}(a)\right| \leq (r^{*}+\|\Phi_t\|_{t*})\|\ P_t^{(1)}-P_t^{(2)}\|_t o(1) as \quad a\rightarrow o$$

And hence

$$\left|\pi_1(P_t^{(1)}-\pi_1\left(P_t^{(2)}\right)\right| \leq ((r^{*}+\|\Phi_t\|_{t*})\|\ P_t^{(1)}-P_t^{(2)}\|_t o(t) as \quad t\rightarrow o$$

Which together with a similar bound for $\left|\pi_2(P_t^{(1)}-\pi_2\left(P_t^{(2)}\right)\right|$ leads to

$$\|\ \pi\ (P_t^{(1)}-\pi\ \left(P_t^{(2)}\right)\|\ \leq (r^{*}+\|\Phi_t\|_{t*})\|\ P_t^{(1)}-P_t^{(2)}\|_{t*} o(t^{*})$$

As $t^{*}\rightarrow o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o\leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^{*}, P_t^{*}$ Of (1.1)-(1.6): such solutions satisfy

(3.1) $\frac{dp\,P_2^*(a)}{da}+f_i(\mathrm{a}, P_1^*, P_2^*)\,P_i^*(a) = 0,\ \ i = 1,2, a > 0,$

(3.2) $P_i^* = \int_o^\infty P_i^*(a)da, i = 1,\ 2,$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i = 1.\ \ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s, P_1^*, P_2^*)ds\right], i = 1,\quad 2, a \geq 0,$

Which together with (3.3) lead to

$1=\int_o^\infty b_1(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_1(s, P_1^*, P_2^*)ds\right]da,$

(3.5) $1=\int_o^\infty b_2(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_2(s, P_1^*, P_2^*)ds\right]da$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1, P_2) = \int_o^\infty b_i(a, P_1, P_2)\exp\left[-\int_o^\infty f_i(s, P_1, P_2)ds\right]da,$

$i$ =1, 2, $P_1 \geq 0$, $P_2 \geq 0$,

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1,2$:

$A_5$. *there exists a positive constant c for which*

$F_1$(C,0)< $F_{2.}$(C,0),

(3.8) $F_1$(0.C)< $F_{2.}$(0,C),

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $F_1(P_1^*, P_2^*) = 1 = F_2(P_1^*, P_2^*)$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a) = [P_2^* \exp[-f_i(s, P_1^*, P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s, P_1^*, P_2^*)ds]da)$,

$i$ =1, 2, $a \geq 0$,

Proof Consider the elements of the surfaces defined by

Z= $F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0$,

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

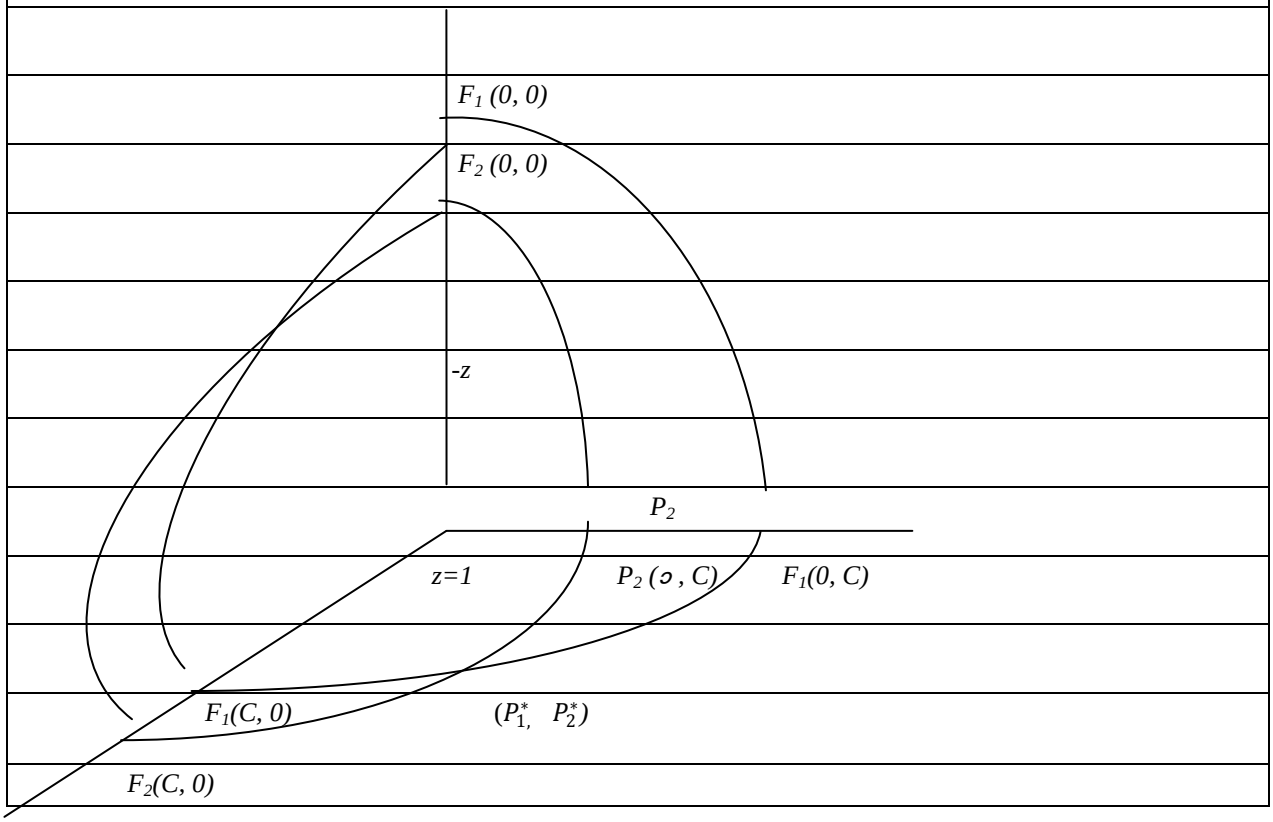
$P_1 \geq 0, P_2 \geq 0$, $i$ =1, 2,

$j$ =1, 2, Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, F_1(P_1, P_2 \geq 0$,

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.



(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11)
$$\begin{cases} p_i(a,t) = p^*_i(a) + u_i(a,t) \\ p_i(t) = p^*_i + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_i} p_i(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial b_i}{\partial p^*_i} p_i(t)\right\}da \end{cases}$$

Where $i = 1,2$ and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_i(a)\exp[\lambda t], \quad u_i(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty u_i(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{du_i}{da} + \lambda u_i = -f_i\left(a, p^*_1, p^*_2\right) u_i(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j} p^*_i\right\},$$

(3.15)

$$\mu_i(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\mu_i(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial b_i}{\partial p_j^*}\ p_i^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*,\ \exp\{-\lambda\tau\},\ p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\mu_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1\left(a, p_1^*, p_2^*\right) - f_1\left(a, p_1^*, p_2^*\right)\right]\mu_i(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2\left(a, p_1^*, p_2^*\right) - f_2\left(a, p_1^*, p_2^*\right)\right]\mu_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\mu_i(a) = \left\{\mu_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\right\} - p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\mu_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\,e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)\,e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\pi_1(a)$ and $\pi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$\left(\lambda - A_{11}\right) p_1^* - A_{12} p_2^* \exp\left(-\lambda\tau\right) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21} p_1^* \exp\left(-\lambda\tau\right) + \left(\lambda - A_{22}\right) p_2^* = g_{21}(\lambda) p_1^* \exp\left(-\lambda\tau\right) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $$A_{i\pi} = \int_0^\infty p_i^*(a) \left[\frac{\partial b_i}{\partial p_\pi^*} - \frac{\partial f_i}{\partial p_\pi^*}\right] da,$$

(3.23) $$g_{i\pi}(\lambda) = \left[p_i^*(0) c_i\right]^{-1} \left\{\int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} da\right\} \pi$$
$$\left\{\int_0^\infty \left[p_i^*(a) \frac{\partial b_i}{\partial p_\pi^*} - b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_\pi^*} e^{\lambda s} ds\right)\right] da\right\}$$

$$- \int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_\pi^*} e^{\lambda s} ds\right) da, \quad i, \pi = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i\left(a, p_1^*, p_2^*\right) - f_i\left(a, p_1^*, p_2^*\right), \quad i, = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $\left(p_1^*, p_2^*\right)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et \begin{bmatrix} \lambda - \left(A_{11} + g_{11}(\lambda)\right) & -\left(A_{12} + g_{12}\right) e^{-\lambda\tau} \\ -\left(A_{21} + g_{21}\right) e^{-\lambda\tau} & \lambda - \left(A_{22} + g_{22}(\lambda)\right) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda\left(A_{11} + A_{22}\right) + A_{11}A_{22} - A_{12}A_{21} \exp\left[-2\lambda\tau\right] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = \left[A_{21} g_{12}(\lambda) + A_{12} g_{21}(\lambda) + g_{12}(\lambda) g_{21}(\lambda)\right] \exp\left[-2\lambda\tau\right]$

$$-A_{11} g_{22}(\lambda) - A_{22} g_{11}(\lambda) - g_{11}(\lambda) g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ =1,\ 2.)\ \ (see(A_1)\ and\ (A_2))$;

$A_7.\ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = |A_{11}\, g_{12}(0)| + |A_{12}\, g_{21}(0)| + |g_{12}(0)\, g_{21}(0)|$$

$$+ |A_{11}\, g_{22}(0)| + |A_{22}\, g_{11}(0)| + |g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$$F_1(\lambda, \tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau];$$

$$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau),$$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$, $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) - F_2(\lambda, \tau) \right|$$

$$\geq |\lambda|^2 - |\lambda|\{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[(|\lambda| - P)^2 + (q)\right]^2$$

Where

$$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2,$$

$$q = \left\{4|(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \|| - |A_{11} + A_{22} + \| S_1(0) \||^2\right\}/2,$$

It will now follow, from (A) and (3.26), $F_1(\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions

of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends,

assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the “Grand design “is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner

wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the

controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## SECTION A(6)-2

## Physical Mass Ratio And The Frequency Of Oscillations

### INTRODUCTION—VARIABLES USED

**Dark-soliton-like excitations in the Yang-Gaudin gas of attractively interacting fermions Sophie S. Shamailov, Joachim Brand (Submitted on 15 Mar 2016 (v1), last revised 3 May 2016 (this version, v2))**

(1) The inertial to physical mass ratio, which is related to the frequency of oscillations in a trapped gas, is found to be (=) unity in the limits of strong and weak attraction and fall to ≈0.78 in the crossover regime.
(2) This result is contrasted by (e) one-dimensional mean field theory, which predicts (eb) a divergent

mass ratio in the weakly attractive limit.

**(3)** By means of exact mapping results also predict (eb) the existence and properties of (e) dark-soliton-like excitations in (eb) the super Tonks-Girardeau gas.

**(4)** The prospects for experimental observations are briefly discussed. Subjects: Quantum Gases (cond-mat.quant-gas) Cite as: arXiv: 1603.04864 [cond-mat.quant-gas] (or arXiv: 1603.04864v2 [cond-mat.quant-gas] for this version)

**Magnetic Solitons in a Binary Bose-Einstein Condensate Chunlei Qu, Lev P. Pitaevskii, and Sandro Stringari Phys. Rev. Lett 116, 160402 – Published 20 April 2016**

**(5)** **Chunlei Qu, Lev P. Pitaevskii, and Sandro Stringari** study solitary waves of (e) polarization (magnetic solitons) in (eb) a two-component Bose gas with (e&eb) slightly unequal repulsive intra- and (e&eb) interspin interactions.

**(6)** In experimentally relevant conditions authors obtain an analytical solution which reveals (eb) that the width and the velocity of (e) magnetic solitons are explicitly related to (e&eb)the spin healing length and (e&eb) the spin sound velocity of the Bose mixture, respectively.

**(7)** Authors calculate the profiles, the energy, and the effective mass of the solitons in the absence of (e) external fields and investigate their oscillation in (eb) a harmonic trap where (e)the oscillation period is calculated as (=) a function of the oscillation amplitude.

**(8)** The stability of magnetic solitons in two dimensions and the conditions for (e) their experimental observation are also briefly discussed. Received 19 February 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.160402 © 2016 American Physical Society

**NOTATION**

**Module One**

The inertial to physical mass ratio, which is related to the frequency of oscillations in a trapped gas, is found to be (=) unity in the limits of strong and weak attraction and fall to ≈0.78 in the crossover regime

$G_{13}$ : Category one of inertial to physical mass ratio, which is related to the frequency of oscillations in a trapped gas

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of unity in the limits of strong and weak attraction and fall to ≈0.78 in the crossover regime

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

This result inertial to physical mass ratio, which is related to the frequency of oscillations in a trapped gas, is found to be unity in the limits of strong and weak attraction and fall to ≈0.78 in the crossover regime is contrasted by (e) one-dimensional mean field theory, which predicts (eb) a divergent mass ratio in the weakly attractive limit

$G_{16}$ : Category one of **result inertial to physical mass ratio, which is related to the frequency of oscillations in a trapped gas, is found to be unity in the limits of strong and weak attraction and fall to**

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| **≈0.78 in the crossover regime is contrasted;** one-dimensional mean field theory, which predicts (eb) a divergent mass ratio in the weakly attractive limit<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of one-dimensional mean field theory, which predicts (eb) a divergent mass ratio in the weakly attractive limit **;result inertial to physical mass ratio, which is related to the frequency of oscillations in a trapped gas, is found to be unity in the limits of strong and weak attraction and fall to ≈0.78 in the crossover regime is contrasted**<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>This result inertial to physical mass ratio, which is related to the frequency of oscillations in a trapped gas, is found to be unity in the limits of strong and weak attraction and fall to ≈0.78 in the crossover regime is contrasted by one-dimensional mean field theory, which predicts (eb) a divergent mass ratio in the weakly attractive limit | |
| $G_{20}$ : Category one of inertial to physical mass ratio, which is related to the frequency of oscillations in a trapped gas, is found to be unity in the limits of strong and weak attraction and fall to ≈0.78 in the crossover regime is contrasted by one-dimensional mean field theory<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of divergent mass ratio in the weakly attractive limit<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>By means of exact mapping results also predict (eb) the existence and properties of (e) dark-soliton-like excitations in (eb) the super Tonks-Girardeau gas | |
| $G_{24}$ : Category one of exact mapping results<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of existence and properties of (e) dark-soliton-like excitations in (eb) the super Tonks-Girardeau gas<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |

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| **Module five**<br><br>By means of exact mapping results also predict the existence and properties of (e) dark-soliton-like excitations in (eb) the super Tonks-Girardeau gas | |
| $G_{28}$ : Category one of dark-soliton-like excitations in (eb) the super Tonks-Girardeau gas<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of exact mapping results also predict the existence and properties<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>By means of exact mapping results also predict the existence and properties of dark-soliton-like excitations in (eb) the super Tonks-Girardeau gas | |
| $G_{32}$ : Category one of exact mapping results also predict the existence and properties of dark-soliton-like excitations<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of super Tonks-Girardeau gas<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>**Chunlei Qu, Lev P. Pitaevskii, and Sandro Stringari** study solitary waves of (e) polarization (magnetic solitons) in (eb) a two-component Bose gas with (e&eb) slightly unequal repulsive intra- and (e&eb) interspin interactions | |
| $G_{36}$ : Category one of polarization (magnetic solitons) in (eb) a two-component Bose gas with (e&eb) slightly unequal repulsive intra- and (e&eb) interspin interactions<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of solitary waves<br><br>$T_{37}$ : Category two of SAS | |

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| $T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>**Chunlei Qu, Lev P. Pitaevskii, and Sandro Stringari** study solitary waves of polarization (magnetic solitons) in (eb) a two-component Bose gas with (e&eb) slightly unequal repulsive intra- and (e&eb) interspin interactions | |
| $G_{40}$ : Category one of solitary waves of polarization (magnetic solitons)<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of two-component Bose gas with (e&eb) slightly unequal repulsive intra- and (e&eb) interspin interactions<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>**Chunlei Qu, Lev P. Pitaevskii, and Sandro Stringari** study solitary waves of polarization (magnetic solitons) in a two-component Bose gas with (e&eb) slightly unequal repulsive intra- and (e&eb) interspin interactions | |
| $G_{44}$ : Category one of **solitary waves of polarization (magnetic solitons) in a two-component Bose gas;** slightly unequal repulsive intra- and (e&eb) interspin interactions<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of slightly unequal repulsive intra- and (e&eb) interspin interactions; **solitary waves of polarization (magnetic solitons) in a two-component Bose gas**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION A(6)-3**<br><br>**Magnetic Solitons In A Binary Bose-Einstein Condensate** | |
| **INTRODUCTION—VARIABLES USED** | |

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| **Magnetic Solitons in a Binary Bose-Einstein Condensate Chunlei Qu, Lev P. Pitaevskii, and Sandro Stringari Phys. Rev. Lett 116, 160402 – Published 20 April 2016**<br><br>**(1) Chunlei Qu, Lev P. Pitaevskii, and Sandro Stringari** study solitary waves of (e) polarization (magnetic solitons) in (eb) a two-component Bose gas with (e&eb) slightly unequal repulsive intra- and (e&eb) interspin interactions.<br>**(2)** In experimentally relevant conditions authors obtain an analytical solution which reveals (eb) that the width and the velocity of (e) magnetic solitons are explicitly related to (e&eb)the spin healing length and (e&eb) the spin sound velocity of the Bose mixture, respectively.<br>**(3)** Authors calculate the profiles, the energy, and the effective mass of the solitons in the absence of (e) external fields and investigate their oscillation in (eb) a harmonic trap where (e)the oscillation period is calculated as (=) a function of the oscillation amplitude.<br>**(4)** The stability of magnetic solitons in two dimensions and the conditions for (e) their experimental observation are also briefly discussed. Received 19 February 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.160402 © 2016 American Physical Society | |
| **NOTATION** | |
| **Module One**<br>**Chunlei Qu, Lev P. Pitaevskii, and Sandro Stringari** study solitary waves of polarization (magnetic solitons) in a two-component Bose gas with (e&eb) slightly unequal repulsive intra- and (e&eb) interspin interactions | |
| $G_{13}$ : Category one of **solitary waves of polarization (magnetic solitons) in a two-component Bose gas** ; slightly unequal repulsive intra- and (e&eb) interspin interactions<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of slightly unequal repulsive intra- and (e&eb) interspin interactions **;solitary waves of polarization (magnetic solitons) in a two-component Bose gas**<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>**Chunlei Qu, Lev P. Pitaevskii, and Sandro Stringari** study solitary waves of polarization (magnetic solitons) in a two-component Bose gas with slightly unequal repulsive intra- and (e&eb) interspin interactions | |
| $G_{16}$ : Category one of **solitary waves of polarization (magnetic solitons) in a two-component Bose gas with slightly unequal repulsive intra;** interspin interactions<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of interspin interactions **;solitary waves of polarization (magnetic solitons) in a two-component Bose gas with slightly unequal repulsive intra** | |

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| $T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>In experimentally relevant conditions authors obtain an analytical solution which reveals (eb) that the width and the velocity of (e) magnetic solitons are explicitly related to (e&eb)the spin healing length and (e&eb) the spin sound velocity of the Bose mixture, respectively | |
| $G_{20}$ : Category one of analytical solution<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of width and the velocity of (e) magnetic solitons are explicitly related to (e&eb)the spin healing length and (e&eb) the spin sound velocity of the Bose mixture, respectively<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>In experimentally relevant conditions authors obtain an analytical solution which reveals that the width and the velocity of (e) magnetic solitons are explicitly related to (e&eb)the spin healing length and (e&eb) the spin sound velocity of the Bose mixture, respectively | |
| $G_{24}$ : Category one of magnetic solitons are explicitly related to (e&eb)the spin healing length and (e&eb) the spin sound velocity of the Bose mixture, respectively<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of analytical solution which reveals that the width and the velocity<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>In experimentally relevant conditions authors obtain an analytical solution which reveals that the width and the velocity of magnetic solitons are explicitly related to (e&eb)the spin healing length and (e&eb) the spin sound velocity of the Bose mixture, respectively | |
| $G_{28}$ : Category one of **analytical solution which reveals that the width and the velocity of magnetic solitons;** spin healing length and (e&eb) the spin sound velocity of the Bose mixture, respectively<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |

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| $T_{28}$ : Category one of spin healing length and (e&eb) the spin sound velocity of the Bose mixture, respectively **;analytical solution which reveals that the width and the velocity of magnetic solitons**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>In experimentally relevant conditions authors obtain an analytical solution which reveals that the width and the velocity of magnetic solitons are explicitly related to the spin healing length and (e&eb) the spin sound velocity of the Bose mixture, respectively | |
| $G_{32}$ : Category one of **analytical solution which reveals that the width and the velocity of magnetic solitons are explicitly related to the spin healing length;** spin sound velocity of the Bose mixture, respectively<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of spin sound velocity of the Bose mixture, respectively **;analytical solution which reveals that the width and the velocity of magnetic solitons are explicitly related to the spin healing length**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Authors calculate the profiles, the energy, and the effective mass of the solitons in the absence of (e) external fields and investigate their oscillation in (eb) a harmonic trap where (e)the oscillation period is calculated as (=) a function of the oscillation amplitude | |
| $G_{36}$ : Category one of external fields and investigate their oscillation in (eb) a harmonic trap where (e)the oscillation period is calculated as (=) a function of the oscillation amplitude<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of profiles, the energy, and the effective mass of the solitons in the absence<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Authors calculate the profiles, the energy, and the effective mass of the solitons in the absence of external | |

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| fields and investigate their oscillation in (eb) a harmonic trap where (e)the oscillation period is calculated as (=) a function of the oscillation amplitude | |
| $G_{40}$ : Category one of profiles, the energy, and the effective mass of the solitons in the absence of external fields and investigate their oscillation<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of harmonic trap where (e)the oscillation period is calculated as (=) a function of the oscillation amplitude<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Authors calculate the profiles, the energy, and the effective mass of the solitons in the absence of external fields and investigate their oscillation in a harmonic trap where the oscillation period is calculated as (=) a function of the oscillation amplitude | |
| $G_{44}$ : Category one of profiles, the energy, and the effective mass of the solitons in the absence of external fields and investigate their oscillation in a harmonic trap where the oscillation period<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of function of the oscillation amplitude<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION A(7)**<br><br>**Cellular Epigenetic States** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Critical Reviews™ in Biomedical Engineering DOI: 10.1615/CritRevBiomedEng.2016016559 Multiscale modeling of cellular epigenetic states: stochasticity in molecular networks, chromatin folding in cell nuclei, and tissue pattern formation of cells Jie Liang ET AL**<br><br>(1) Genome sequences provide (eb) the overall genetic blueprint of cells, cells possessing (e) the same | |

genome can exhibit (eb) diverse phenotypes.

(2) There is a multitude of mechanisms controlling cellular (e&eb) epigenetic states and that dictate (e&eb) the behavior of cells. Among these, networks of interacting molecules, often under (e&eb) stochastic control, depending on (eb) the specific wirings of molecular components and (e&eb) the physiological conditions can have (e) a different landscape of cellular states.

(3) In addition, chromosome folding in three-dimensional space provides (eb) another important control mechanism for (e) selective activation and repression of (e) gene expression.

(4) Fully differentiated cells with (e&eb) different properties grow, divide, and interact through mechanical forces and communicate through (e&eb) signal transduction, resulting in (eb) the formation of complex tissue patterns.

(5) Developing quantitative models to study these multi-scale phenomena and to identify (e&eb) opportunities for improving human health requires (e) development of theoretical models, algorithms, and computational tools.

**(6)** Here we review recent progress made in these important directions.

**NOTATION**

**Module One**

Genome sequences provide (eb) the overall genetic blueprint of cells, cells possessing (e) the same genome can exhibit (eb) diverse phenotypes

$G_{13}$ : Category one of Genome sequences

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of overall genetic blueprint of cells, cells possessing (e) the same genome can exhibit (eb) diverse phenotypes

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

Genome sequences provide the overall genetic blueprint of cells, cells possessing (e) the same genome can exhibit (eb) diverse phenotypes

$G_{16}$ : Category one of same genome can exhibit (eb) diverse phenotypes

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

$T_{16}$ : Category one of Genome sequences provide the overall genetic blueprint of cells, cells possessing

$T_{17}$ : Category two of SAS

$T_{18}$ : Category three of SAS

**Module three**

Genome sequences provide the overall genetic blueprint of cells, cells possessing the same genome can

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| exhibit (eb) diverse phenotypes | |
| $G_{20}$ : Category one of Genome sequences provide the overall genetic blueprint of cells, cells possessing the same genome<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of diverse phenotypes<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>There is a multitude of mechanisms controlling cellular epigenetic states and that dictate (e&eb) the behavior of cells.<br><br>Among these, networks of interacting molecules, often under (e&eb) stochastic control, depending on (eb) the specific wirings of molecular components and (e&eb) the physiological conditions can have (e) a different landscape of cellular states | |
| $G_{24}$ : Category one of **multitude of mechanisms controlling cellular epigenetic states and that dictate;** stochastic control, depending on (eb) the specific wirings of molecular components and (e&eb) the physiological conditions can have (e) a different landscape of cellular states<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of stochastic control, depending on (eb) the specific wirings of molecular components and (e&eb) the physiological conditions can have (e) a different landscape of cellular states **;multitude of mechanisms controlling cellular epigenetic states and that dictate**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Among these, networks of interacting molecules, often under stochastic control, depending on (eb) the specific wirings of molecular components and (e&eb) the physiological conditions can have (e) a different landscape of cellular states | |
| $G_{28}$ : Category one of networks of interacting molecules, often under stochastic control, depending<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of specific wirings of molecular components and (e&eb) the physiological conditions | |

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| can have (e) a different landscape of cellular states<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Among these, networks of interacting molecules, often under stochastic control, depending on the specific wirings of molecular components and (e&eb) the physiological conditions can have (e) a different landscape of cellular states | |
| $G_{32}$ : Category one of **networks of interacting molecules, often under stochastic control, depending on the specific wirings of molecular components;** physiological conditions can have (e) a different landscape of cellular states<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of physiological conditions can have (e) a different landscape of cellular states **;networks of interacting molecules, often under stochastic control, depending on the specific wirings of molecular components**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Among these, networks of interacting molecules, often under stochastic control, depending on the specific wirings of molecular components and the physiological conditions can have (e) a different landscape of cellular states | |
| $G_{36}$ : Category one of different landscape of cellular states<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of networks of interacting molecules, often under stochastic control, depending on the specific wirings of molecular components and the physiological conditions<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>In addition, chromosome folding in three-dimensional space provides (eb) another important control mechanism for (e) selective activation and repression of (e) gene expression. | |

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| $G_{40}$ : Category one of chromosome folding in three-dimensional space<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of another important control mechanism for (e) selective activation and repression of (e) gene expression<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>In addition, chromosome folding in three-dimensional space provides another important control mechanism for (e) selective activation and repression of (e) gene expression | |
| $G_{44}$ : Category one of selective activation and repression of (e) gene expression<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of chromosome folding in three-dimensional space provides another important control mechanism<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION A(8)**<br><br>**High-Energy Nuclear Collisions** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**<br><br>(1) The interplay of quantum anomalies with (e&eb) magnetic field and vorticity results (eb) in a variety of novel non-dissipative transport phenomena in systems with (e&eb) chiral fermions, including the quark–gluon plasma.<br>(2) Among them is (=) the Chiral Magnetic Effect (CME)—the generation of electric current along (e&eb) an external magnetic field induced by (e) chirality imbalance.<br>(3) Because the chirality imbalance is related to (e&eb) the global topology of gauge fields, the CME | |

current is topologically protected and hence non-dissipative even in the presence of (e&eb) strong interactions.

(4) As a result, the CME and related (e&eb) quantum phenomena affect (e&eb) the hydrodynamical and (e&eb) transport behavior of strongly coupled (e&eb) quark–gluon plasma, and can be studied in relativistic heavy ion collisions where (e) strong magnetic fields are created by (e) the colliding ions.

(5) Evidence for the CME and related phenomena has been reported by (e) the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN.

(6) The goal of the present review is to provide (eb) an elementary introduction into (e&eb) the physics of anomalous chiral effects, to describe (eb) the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate (e) the existing uncertainties in (eb) the interpretation of the data.

**(7)** Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

**NOTATION**

**Module One**

The interplay of quantum anomalies with (e&eb) magnetic field and vorticity results (eb) in a variety of novel non-dissipative transport phenomena in systems with (e&eb) chiral fermions, including the quark–gluon plasma

$G_{13}$ : Category one of **quantum anomalies** ; magnetic field and vorticity results (eb) in a variety of novel non-dissipative transport phenomena in systems with (e&eb) chiral fermions, including the quark–gluon plasma

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of magnetic field and vorticity results (eb) in a variety of novel non-dissipative transport phenomena in systems with (e&eb) chiral fermions, including the quark–gluon plasma; **quantum anomalies**

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

The interplay of quantum anomalies with magnetic field and vorticity results (eb) in a variety of novel non-dissipative transport phenomena in systems with (e&eb) chiral fermions, including the quark–gluon plasma

$G_{16}$ : Category one of interplay of quantum anomalies with magnetic field and vorticity

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

$T_{16}$ : Category one of variety of novel non-dissipative transport phenomena in systems with (e&eb) chiral

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| fermions, including the quark–gluon plasma<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with (e&eb) chiral fermions, including the quark–gluon plasma | |
| $G_{20}$ : Category one of **interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems;** chiral fermions, including the quark–gluon plasma<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of chiral fermions, including the quark–gluon plasma; **interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Among them is (=) the Chiral Magnetic Effect (CME)—the generation of electric current along (e&eb) an external magnetic field induced by (e) chirality imbalance | |
| $G_{24}$ : Category one of **Chiral Magnetic Effect (CME)—the generation of electric current;** external magnetic field induced by (e) chirality imbalance<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of external magnetic field induced by (e) chirality imbalance; **Chiral Magnetic Effect (CME)—the generation of electric current**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by (e) chirality imbalance | |
| $G_{28}$ : Category one of chirality imbalance<br><br>$G_{29}$ : Category two of SAS | |

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| $G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Because the chirality imbalance is related to (e&eb) the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of (e&eb) strong interactions | |
| $G_{32}$ : Category one of **chirality imbalance;** the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of (e&eb) strong interactions<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of (e&eb) strong interactions **;chirality imbalance**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of (e&eb) strong interactions | |
| $G_{36}$ : Category one of strong interactions; chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative; strong interactions<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>As a result, the CME and related quantum phenomena affect (e&eb) the hydrodynamical and (e&eb) transport behavior of strongly coupled (e&eb) quark–gluon plasma, and can be studied in relativistic heavy | |

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| ion collisions where (e) strong magnetic fields are created by (e) the colliding ions |
| $G_{40}$ : Category one of **CME and related quantum phenomena;** hydrodynamical and (e&eb) transport behavior of strongly coupled (e&eb) quark–gluon plasma, and can be studied in relativistic heavy ion collisions where (e) strong magnetic fields are created by (e) the colliding ions<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS |
| $T_{40}$ : Category one of hydrodynamical and (e&eb) transport behavior of strongly coupled (e&eb) quark–gluon plasma, and can be studied in relativistic heavy ion collisions where (e) strong magnetic fields are created by (e) the colliding ions; **CME and related quantum phenomena**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS |
| **Module Nine**<br><br>As a result, the CME and related quantum phenomena affect the hydrodynamical and (e&eb) transport behavior of strongly coupled (e&eb) quark–gluon plasma, and can be studied in relativistic heavy ion collisions where (e) strong magnetic fields are created by (e) the colliding ions |
| $G_{44}$ : Category one of **CME and related quantum phenomena affect the hydrodynamical;** transport behavior of strongly coupled (e&eb) quark–gluon plasma, and can be studied in relativistic heavy ion collisions where (e) strong magnetic fields are created by (e) the colliding ions<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS |
| $T_{44}$ : Category one of transport behavior of strongly coupled (e&eb) quark–gluon plasma, and can be studied in relativistic heavy ion collisions where (e) strong magnetic fields are created by (e) the colliding ions; **CME and related quantum phenomena affect the hydrodynamical**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS |
| **The Coefficients:** |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)},$<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$ |

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| are Accentuation coefficients<br>$(a_{13}^{'})^{(1)}, (a_{14}^{'})^{(1)}, (a_{15}^{'})^{(1)}, (b_{13}^{'})^{(1)}, (b_{14}^{'})^{(1)}, (b_{15}^{'})^{(1)}, (a_{16}^{'})^{(2)}, (a_{17}^{'})^{(2)}, (a_{18}^{'})^{(2)}, \ (b_{16}^{'})^{(2)}, (b_{17}^{'})^{(2)}, (b_{18}^{'})^{(2)}$<br>$, (a_{20}^{'})^{(3)}, (a_{21}^{'})^{(3)}, (a_{22}^{'})^{(3)}, (b_{20}^{'})^{(3)}, (b_{21}^{'})^{(3)}, (b_{22}^{'})^{(3)}$<br>$(a_{24}^{'})^{(4)}, (a_{25}^{'})^{(4)}, (a_{26}^{'})^{(4)}, (b_{24}^{'})^{(4)}, (b_{25}^{'})^{(4)}, (b_{26}^{'})^{(4)}, (b_{28}^{'})^{(5)}, (b_{29}^{'})^{(5)}, (b_{30}^{'})^{(5)} \ (a_{28}^{'})^{(5)}, (a_{29}^{'})^{(5)}, (a_{30}^{'})^{(5)}$ ,<br>$(a_{32}^{'})^{(6)}, (a_{33}^{'})^{(6)}, (a_{34}^{'})^{(6)}, (b_{32}^{'})^{(6)}, (b_{33}^{'})^{(6)}, (b_{34}^{'})^{(6)}$<br>$(a_{36}^{'})^{(7)}, (a_{37}^{'})^{(7)}, (a_{38}^{'})^{(7)}, (b_{36}^{'})^{(7)}, (b_{37}^{'})^{(7)}, (b_{38}^{'})^{(7)}$,<br>$(a_{40}^{'})^{(8)}, (a_{41}^{'})^{(8)}, (a_{42}^{'})^{(8)}, (b_{40}^{'})^{(8)}, (b_{41}^{'})^{(8)}, (b_{42}^{'})^{(8)}$,<br>$(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19}), t)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}((G_{19}), t)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19}), t)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}((G_{19}), t) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |

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| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b'_{20})^{(3)} - (b''_{20})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b'_{21})^{(3)} - (b''_{21})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b'_{22})^{(3)} - (b''_{22})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a''_{20})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b''_{20})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a'_{24})^{(4)} + (a''_{24})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a'_{25})^{(4)} + (a''_{25})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a'_{26})^{(4)} + (a''_{26})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b'_{24})^{(4)} - (b''_{24})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b'_{25})^{(4)} - (b''_{25})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b'_{26})^{(4)} - (b''_{26})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a''_{24})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b''_{24})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a'_{28})^{(5)} + (a''_{28})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a'_{29})^{(5)} + (a''_{29})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a'_{30})^{(5)} + (a''_{30})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b'_{28})^{(5)} - (b''_{28})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b'_{29})^{(5)} - (b''_{29})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b'_{30})^{(5)} - (b''_{30})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a''_{28})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b''_{28})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a'_{32})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b'_{32})^{(6)} - (b''_{32})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b'_{33})^{(6)} - (b''_{33})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b'_{34})^{(6)} - (b''_{34})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |

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| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{13}$ | *55* |

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| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)}\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)}\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3<br>$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3<br>$\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)}\boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)}\boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)}\boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)}\boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}\boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)}\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)}\boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)}\boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)}\boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)}\boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}\boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)}\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\boxed{-(b_{15}^{''})^{(1)}(G,t)}\boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)}\boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)}\boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)}\boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}\boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)}\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | *60* |
| Where $\boxed{-(b_{13}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3 | |

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| $-(b_{16}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{20}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{37}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[\begin{array}{c} (a_{16}^{'})^{(2)}\,+(a_{16}^{''})^{(2)}(T_{17},t)\;+(a_{13}^{''})^{(1,1,)}(T_{14},t)\;+(a_{20}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)\;+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)\;+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t)\;+(a_{40}^{''})^{(8,8,8)}(T_{41},t)\;+(a_{44}^{''})^{(9,9)}(T_{45},t) \end{array}\right] G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[\begin{array}{c} (a_{17}^{'})^{(2)}\,+(a_{17}^{''})^{(2)}(T_{17},t)\;+(a_{14}^{''})^{(1,1,)}(T_{14},t)\;+(a_{21}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)\;+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)\;+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7)}(T_{37},t)\;+(a_{41}^{''})^{(8,8,8)}(T_{41},t)\;+(a_{45}^{''})^{(9,9)}(T_{45},t) \end{array}\right] G_{17}$ | *62* |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[\begin{array}{c} (a_{18}^{'})^{(2)}\,+(a_{18}^{''})^{(2)}(T_{17},t)\;+(a_{15}^{''})^{(1,1,)}(T_{14},t)\;+(a_{22}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)\;+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)\;+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7)}(T_{37},t)\;+(a_{42}^{''})^{(8,8,8)}(T_{41},t)\;+(a_{46}^{''})^{(9,9)}(T_{45},t) \end{array}\right] G_{18}$ | *63* |
| Where $+(a_{16}^{''})^{(2)}(T_{17},t)$, $+(a_{17}^{''})^{(2)}(T_{17},t)$, $+(a_{18}^{''})^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3 | |

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| $+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{16}$ | *64* |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)}-(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{17}$ | *65* |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{18}$ | *66* |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$, $-(b_{17}^{''})^{(2)}(G_{19},t)$, $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3<br>$-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |

$$\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[\begin{array}{c} (a_{20}^{'})^{(3)}\boxed{+(a_{20}^{''})^{(3)}(T_{21},t)}\ \boxed{+(a_{16}^{''})^{(2,2,2)}(T_{17},t)}\ \boxed{+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)}\ \boxed{+(a_{44}^{''})^{(9,9,9)}(T_{45},t)} \end{array}\right] G_{20} \quad (67)$$

$$\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[\begin{array}{c} (a_{21}^{'})^{(3)}\boxed{+(a_{21}^{''})^{(3)}(T_{21},t)}\ \boxed{+(a_{17}^{''})^{(2,2,2)}(T_{17},t)}\ \boxed{+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)}\ \boxed{+(a_{45}^{''})^{(9,9,9)}(T_{45},t)} \end{array}\right] G_{21} \quad (68)$$

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[\begin{array}{c} (a_{22}^{'})^{(3)}\boxed{+(a_{22}^{''})^{(3)}(T_{21},t)}\ \boxed{+(a_{18}^{''})^{(2,2,2)}(T_{17},t)}\ \boxed{+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)}\ \boxed{+(a_{46}^{''})^{(9,9,9)}(T_{45},t)} \end{array}\right] G_{22} \quad (69)$$

$\boxed{+(a_{20}^{''})^{(3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}^{''})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2)}(T_{17},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)}$ are third augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)}$ are eight augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{44}^{''})^{(9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1, 2 and 3

$$\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[\begin{array}{c} (b_{20}^{'})^{(3)}\boxed{-(b_{20}^{''})^{(3)}(G_{23},t)}\ \boxed{-(b_{16}^{''})^{(2,2,2)}(G_{19},t)}\ \boxed{-(b_{13}^{''})^{(1,1,1,)}(G,t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)}\ \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)}\ \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)}\ \boxed{-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)}\ \boxed{-(b_{44}^{''})^{(9,9,9)}(G_{47},t)} \end{array}\right] T_{20} \quad (70)$$

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[\begin{array}{c} (b_{21}^{'})^{(3)}\boxed{-(b_{21}^{''})^{(3)}(G_{23},t)}\ \boxed{-(b_{17}^{''})^{(2,2,2)}(G_{19},t)}\ \boxed{-(b_{14}^{''})^{(1,1,1,)}(G,t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)}\ \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)}\ \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)}\ \boxed{-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)}\ \boxed{-(b_{45}^{''})^{(9,9,9)}(G_{47},t)} \end{array}\right] T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)}\boxed{-(b''_{22})^{(3)}(G_{23},t)} & \boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{22} \quad (72)$$

$\boxed{-(b''_{20})^{(3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3)}(G_{23},t)}$ *are first detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)}$ *are second detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{13})^{(1,1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,)}(G,t)}$ *are third detrition coefficients for category 1,2 and 3*

$\boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)}$ *are fourth detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)}$ *are fifth detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)}$ *are sixth detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{36})^{(7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7)}(G_{39},t)}$ $\boxed{-(b''_{38})^{(7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a'_{24})^{(4)}\boxed{+(a''_{24})^{(4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{16})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a'_{25})^{(4)}\boxed{+(a''_{25})^{(4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{17})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a'_{26})^{(4)}\boxed{+(a''_{26})^{(4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{18})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{26} \quad (75)$$

$\boxed{(a''_{24})^{(4)}(T_{25},t)}$, $\boxed{(a''_{25})^{(4)}(T_{25},t)}$, $\boxed{(a''_{26})^{(4)}(T_{25},t)}$ *are first augmentation coefficients category* 1, 2 3

$\boxed{+(a''_{28})^{(5,5,)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,)}(T_{29},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a''_{32})^{(6,6,)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3

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| $+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1, 2 and 3 $+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1, 2 and 3 $+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1, 2 and 3 $+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3 $+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1, 2 and 3 $+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b_{24}^{'})^{(4)}\left[-(b_{24}^{''})^{(4)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,)}(G_{35},t)\right] \\ \left[-(b_{13}^{''})^{(1,1,1,1)}(G,t)\right] & \left[-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)\right] & \left[-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)\right] & \left[-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{24}$ | 76 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b_{25}^{'})^{(4)}\left[-(b_{25}^{''})^{(4)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,)}(G_{35},t)\right] \\ \left[-(b_{14}^{''})^{(1,1,1,1)}(G,t)\right] & \left[-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)\right] & \left[-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)\right] & \left[-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{25}$ | 77 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b_{26}^{'})^{(4)}\left[-(b_{26}^{''})^{(4)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,)}(G_{35},t)\right] \\ \left[-(b_{15}^{''})^{(1,1,1,1)}(G,t)\right] & \left[-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)\right] & \left[-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)\right] & \left[-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{26}$ | 78 |

| | |
|---|---|
| $Where\ \boxed{-(b_{24}^{''})^{(4)}(G_{27},t)}, \boxed{-(b_{25}^{''})^{(4)}(G_{27},t)}, \boxed{-(b_{26}^{''})^{(4)}(G_{27},t)}$ $are\ first\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{28}^{''})^{(5,5,)}(G_{31},t)}, \boxed{-(b_{29}^{''})^{(5,5,)}(G_{31},t)}, \boxed{-(b_{30}^{''})^{(5,5,)}(G_{31},t)}$ $are\ second\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{32}^{''})^{(6,6,)}(G_{35},t)}, \boxed{-(b_{33}^{''})^{(6,6,)}(G_{35},t)}, \boxed{-(b_{34}^{''})^{(6,6,)}(G_{35},t)}$ $are\ third\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{13}^{''})^{(1,1,1,1)}(G,t)}, \boxed{-(b_{14}^{''})^{(1,1,1,1)}(G,t)}, \boxed{-(b_{15}^{''})^{(1,1,1,1)}(G,t)}$ $are\ fourth\ \ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)}, \boxed{-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)}, \boxed{-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)}$ $are\ fifth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)}, \boxed{-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)}, \boxed{-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)}$ $are\ sixth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)}$ $are\ seventh\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)}$ $are\ eighth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)}, \boxed{-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)}, \boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[\begin{matrix} (a_{28}^{'})^{(5)}\boxed{+(a_{28}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[\begin{matrix} (a_{29}^{'})^{(5)}\boxed{+(a_{29}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[\begin{matrix} (a_{30}^{'})^{(5)}\boxed{+(a_{30}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{30}$ | *81* |
| $Where\ \boxed{+(a_{28}^{''})^{(5)}(T_{29},t)}, \boxed{+(a_{29}^{''})^{(5)}(T_{29},t)}, \boxed{+(a_{30}^{''})^{(5)}(T_{29},t)}$ $are\ first\ augmentation\ coefficients\ for\ category\ 1,2\ and\ 3$ $And\ \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)}, \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)}, \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)}$ $are\ second\ augmentation\ coefficient\ for\ category\ 1,2\ and\ 3$ $\boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)}, \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)}, \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)}$ $are\ third\ \ augmentation\ coefficient\ for\ category\ 1,2\ and\ 3$ $\boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)}, \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)}, \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3 $\boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation | |

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|---|---|
| coefficients for category 1,2,and 3 $+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3 $+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3 $+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3 $+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[\begin{matrix}(b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) & -(b_{32}^{''})^{(6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)\end{matrix}\right]T_{28}$ | *82* |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[\begin{matrix}(b_{29}^{'})^{(5)}-(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) & -(b_{33}^{''})^{(6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)\end{matrix}\right]T_{29}$ | *83* |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[\begin{matrix}(b_{30}^{'})^{(5)}-(b_{30}^{''})^{(5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,)}(G_{27},t) & -(b_{34}^{''})^{(6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ -(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)\end{matrix}\right]T_{30}$ | *84* |
| *where* $-(b_{28}^{''})^{(5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3 $-(b_{24}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3 $-(b_{32}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3 $-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3 $-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3 $-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3 $-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ , $-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3 $-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3 $-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3 | |

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)} G_{33} - \begin{bmatrix} (a'_{32})^{(6)} & +(a''_{32})^{(6)}(T_{33},t) & +(a''_{28})^{(5,5,5)}(T_{29},t) & +(a''_{24})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)} G_{32} - \begin{bmatrix} (a'_{33})^{(6)} & +(a''_{33})^{(6)}(T_{33},t) & +(a''_{29})^{(5,5,5)}(T_{29},t) & +(a''_{25})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)} G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)} T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)} T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)}\boxed{-(b''_{34})^{(6)}(G_{35},t)} & \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{34} \quad (90)$$

$\boxed{-(b''_{32})^{(6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6)}(G_{35},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)}$ *are second detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)}$ *are third detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a'_{36})^{(7)}\boxed{+(a''_{36})^{(7)}(T_{37},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a'_{37})^{(7)}\boxed{+(a''_{37})^{(7)}(T_{37},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a'_{38})^{(7)}\boxed{+(a''_{38})^{(7)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad (93)$$

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|---|---|
| Where $(a_{36}^{''})^{(7)}(T_{37},t)$, $(a_{37}^{''})^{(7)}(T_{37},t)$, $(a_{38}^{''})^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3<br>$+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{36}}{dt} =$<br>$(b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 94 |
| $\frac{dT_{37}}{dt} =$<br>$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | |
| $\frac{dT_{38}}{dt} =$<br>$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | |
| Where $\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3 | |

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| $-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3<br>$-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13}$ | *95* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}^{'})^{(8)}+(a_{41}^{''})^{(8)}(T_{41},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$ | |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$ | |
| Where $+(a_{40}^{''})^{(8)}(T_{41},t)$, $+(a_{41}^{''})^{(8)}(T_{41},t)$, $+(a_{42}^{''})^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3 | |

$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ ,$+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ ,$+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ ,$+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ ,$+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} =$$

$$(b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & \left[-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} =$$

$$(b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & \left[-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & \left[-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$ , $-(b_{37}^{''})^{(7)}(G_{39},t)$ , $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$ , $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$ , $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$ , $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$ , $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

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$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a_{44}^{'})^{(9)}\boxed{+(a_{44}^{''})^{(9)}(T_{45},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13}$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a_{45}^{'})^{(9)}\boxed{+(a_{45}^{''})^{(9)}(T_{45},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a_{46}^{'})^{(9)}\boxed{+(a_{46}^{''})^{(9)}(T_{37},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$$

Where $+(a_{44}^{''})^{(9)}(T_{45},t)$, $+(a_{45}^{''})^{(9)}(T_{45},t)$, $+(a_{46}^{''})^{(9)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth

augmentation coefficient for category 1, 2 and 3
$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ are Seventh augmentation coefficient for category 1, 2 and 3
$+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3
$+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} =$$
$$(b_{44})^{(9)}T_{45} -$$
$$\begin{bmatrix} (b_{44}^{'})^{(9)}\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} =$$
$$(b_{45})^{(9)}T_{44} - \begin{bmatrix} (b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} =$$
$$(b_{46})^{(9)}T_{45} - \begin{bmatrix} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $-(b_{44}^{''})^{(9)}(G_{47},t)$, $-(b_{45}^{''})^{(9)}(G_{47},t)$, $-(b_{46}^{''})^{(9)}(G_{47},t)$ are first detrition coefficients for category 1, 2 and 3
$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3
$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3
$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3
$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3
$-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3
$-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$ are seventh detrition coefficients for category 1, 2 and 3

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| $\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| Where we suppose | |
| $(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$<br><br>The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br><br>$(a_i^{''})^{(1)}(T_{14},t) \leq (p_i)^{(1)} \leq (\hat{A}_{13})^{(1)}$<br><br>$(b_i^{''})^{(1)}(G,t) \leq (r_i)^{(1)} \leq (b_i^{'})^{(1)} \leq (\hat{B}_{13})^{(1)}$ | *97* |
| $lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$<br><br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br><br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'},t) - (a_i^{''})^{(1)}(T_{14},t)\rvert \leq (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\lvert(b_i^{''})^{(1)}(G^{'},t) - (b_i^{''})^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G^{'}\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ | 101 |

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| satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i,j = 16,17,18$ | |
| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17}, t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19}, t) \leq (r_i)^{(2)} \leq (b_i')^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty} (a_i'')^{(2)}(T_{17}, t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty} (b_i'')^{(2)}((G_{19}), t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}$ are positive constants and $i = 16,17,18$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(2)}(T_{17}', t) - (a_i'')^{(2)}(T_{17}, t)\rvert \leq (\hat{k}_{16})^{(2)}\lvert T_{17} - T_{17}'\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i'')^{(2)}((G_{19})', t) - (b_i'')^{(2)}((G_{19}), t)\rvert < (\hat{k}_{16})^{(2)}\lvert\lvert(G_{19}) - (G_{19})'\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}', t)$ and $(a_i'')^{(2)}(T_{17}, t)$ . $(T_{17}', t)$ and $(T_{17}, t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}]$ . It is to be noted that $(a_i'')^{(2)}(T_{17}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i'')^{(2)}(T_{17}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together | |

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| with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$ satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i^{'})^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i, j = 20,21,22$<br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(3)}$, $(r_i)^{(3)}$:<br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(3)}(T_{21}, t) = (p_i)^{(3)}$<br>$\lim_{G \to \infty}(b_i^{''})^{(3)}(G_{23}, t) = (r_i)^{(3)}$<br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'}, t) - (a_i^{''})^{(3)}(T_{21}, t)\rvert \leq (\hat{k}_{20})^{(3)} \lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'}, t) - (b_i^{''})^{(3)}(G_{23}, t)\rvert < (\hat{k}_{20})^{(3)} \lvert\lvert G_{23} - G_{23}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'}, t)$ and $(a_i^{''})^{(3)}(T_{21}, t)$ . $(T_{21}^{'}, t)$ And $(T_{21}, t)$ are points belonging to the interval $[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |

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| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}((G_{27}), t) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(4)}((G_{27}), t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\|(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\| \leq (\hat{k}_{24})^{(4)} \|T_{25} - T_{25}^{'}\| e^{-(\hat{M}_{24})^{(4)} t}$<br><br>$\|(b_i^{''})^{(4)}((G_{27})', t) - (b_i^{''})^{(4)}((G_{27}), t)\| < (\hat{k}_{24})^{(4)} \|(G_{27}) - (G_{27})'\| e^{-(\hat{M}_{24})^{(4)} t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants | 120 |

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| $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)} + (b_i^{'})^{(4)} + (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}, (r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}((G_{31}), t) \leq (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(5)} (T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(5)} (G_{31}, t) = (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)} \lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)} t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})', t) - (b_i^{''})^{(5)}((G_{31}), t)\rvert < (\hat{k}_{28})^{(5)} \lvert\lvert(G_{31}) - (G_{31})'\rvert\rvert e^{-(\hat{M}_{28})^{(5)} t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ : | 125 |

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| $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$ | 126 |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(6)} (T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(6)} ((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)} \lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)} t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})', t) - (b_i^{''})^{(6)}((G_{35}), t)\rvert < (\hat{k}_{32})^{(6)} \lvert\lvert(G_{35}) - (G_{35})'\rvert\rvert e^{-(\hat{M}_{32})^{(6)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and $(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |

**Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ : 129

$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)},$ are positive constants

$$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$$

**Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ : 130

There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i')^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$$

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i')^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$$

Where we suppose

(UUU) $(a_i)^{(7)}, (a_i')^{(7)}, (a_i'')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (b_i'')^{(7)} > 0, \quad i, j = 36,37,38$ 131

(VVV) The functions $(a_i'')^{(7)}, (b_i'')^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}$, $(r_i)^{(7)}$:

$$(a_i'')^{(7)}(T_{37}, t) \le (p_i)^{(7)} \le (\hat{A}_{36})^{(7)}$$

$$(b_i'')^{(7)}(G_{39}, t) \le (r_i)^{(7)} \le (b_i')^{(7)} \le (\hat{B}_{36})^{(7)}$$

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*(WWW)* $\lim_{T_2 \to \infty} (a_i'')^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(XXX)*

$\lim_{G \to \infty} (b_i'')^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i'')^{(7)}(T_{37}', t) - (a_i'')^{(7)}(T_{37}, t)| \le (\hat{k}_{36})^{(7)}|T_{37} - T_{37}'|e^{-(\hat{M}_{36})^{(7)}t}$$

$$|(b_i'')^{(7)}((G_{39})', t) - (b_i'')^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)}||(G_{39}) - (G_{39})'||e^{-(\hat{M}_{36})^{(7)}t}$$

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| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'},t)$ and$(a_i^{''})^{(7)}(T_{37},t)$ . $(T_{37}^{'},t)$ and $(T_{37},t)$ are points belonging to the interval $\left[(\hat{k}_{36})^{(7)},(\widehat{M}_{36})^{(7)}\right]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{36})^{(7)}=1$ then the function $(a_i^{''})^{(7)}(T_{37},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)}$ : <br><br>(YYY) $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)}$, are positive constants <br><br>$\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}<1$ | 134 |
| **Definition of** $(\hat{P}_{36})^{(7)},(\hat{Q}_{36})^{(7)}$ : <br><br>(ZZZ) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)},(\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)},(a_i^{'})^{(7)},(b_i)^{(7)},(b_i^{'})^{(7)},(p_i)^{(7)},\ (r_i)^{(7)}, i=36,37,38$, satisfy the inequalities <br><br>$\frac{1}{(\widehat{M}_{36})^{(7)}}[(a_i)^{(7)}+(a_i^{'})^{(7)}+(\hat{A}_{36})^{(7)}+(\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}]<1$ <br><br>$\frac{1}{(\widehat{M}_{36})^{(7)}}[(b_i)^{(7)}+(b_i^{'})^{(7)}+(\hat{B}_{36})^{(7)}+(\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}]<1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)},(a_i^{'})^{(8)},(a_i^{''})^{(8)},(b_i)^{(8)},(b_i^{'})^{(8)},(b_i^{''})^{(8)}>0, \quad i,j=40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)},(b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)},\ (r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41},t)\leq(p_i)^{(8)}\leq(\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}),t)\leq\ (r_i)^{(8)}\leq(b_i^{'})^{(8)}\leq(\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(8)}(T_{41},t)=(p_i)^{(8)}$ | 140 |
| $\lim_{G\to\infty}(b_i^{''})^{(8)}\left((G_{43}),t\right)=\ (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)},(\hat{B}_{40})^{(8)}$ : <br><br>Where $\boxed{(\hat{A}_{40})^{(8)},(\hat{B}_{40})^{(8)},(p_i)^{(8)},\ (r_i)^{(8)}}$ are positive constants and $\boxed{i=40,41,42}$ | |

They satisfy Lipschitz condition:

$$|(a_i'')^{(8)}(T_{41}', t) - (a_i'')^{(8)}(T_{41}, t)| \leq (\hat{k}_{40})^{(8)}|T_{41} - T_{41}'|e^{-(\hat{M}_{40})^{(8)}t} \quad (142)$$

$$|(b_i'')^{(8)}((G_{43})', t) - (b_i'')^{(8)}((G_{43}), t)| < (\hat{k}_{40})^{(8)}||(G_{43}) - (G_{43})'||e^{-(\hat{M}_{40})^{(8)}t} \quad (143)$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(8)}(T_{41}', t)$ and $(a_i'')^{(8)}(T_{41}, t)$ . $(T_{41}', t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i'')^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i'')^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ :

$(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants

$$\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}, \frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1 \quad (144)$$

**Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :

There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$
Satisfy the inequalities

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1 \quad (145)$$

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1 \quad (146)$$

Where we suppose

$$(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i, j = 44,45,46 \quad (146A)$$

The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:

$$(a_i'')^{(9)}(T_{45}, t) \leq (p_i)^{(9)} \leq (\hat{A}_{44})^{(9)}$$

$$(b_i'')^{(9)}(G_{47}, t) \leq (r_i)^{(9)} \leq (b_i')^{(9)} \leq (\hat{B}_{44})^{(9)}$$

$$\lim_{T_2 \to \infty} (a_i'')^{(9)}(T_{45}, t) = (p_i)^{(9)}$$

$$\lim_{G \to \infty} (b_i'')^{(9)}(G_{47}, t) = (r_i)^{(9)}$$

**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :

Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$

They satisfy Lipschitz condition:

$$|(a_i^{''})^{(9)}(T_{45}^{'}, t) - (a_i^{''})^{(9)}(T_{45}, t)| \leq (\hat{k}_{44})^{(9)}|T_{45} - T_{45}^{'}|e^{-(\hat{M}_{44})^{(9)}t}$$

$$|(b_i^{''})^{(9)}((G_{47})^{'}, t) - (b_i^{''})^{(9)}\big((G_{47}), t\big)| < (\hat{k}_{44})^{(9)}||(G_{47}) - (G_{47})^{'}||e^{-(\hat{M}_{44})^{(9)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(9)}(T_{45}^{'}, t)$ and$(a_i^{''})^{(9)}(T_{45}, t)$ . $(T_{45}^{'}, t)$ and $(T_{45}, t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i^{''})^{(9)}(T_{45}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i^{''})^{(9)}(T_{45}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)},$ are positive constants

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}} \ , \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$$

**Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and \ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}, i = 44,45,46,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0), T_i(0)$ :

$$G_i(t) \leq (\hat{P}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t} \ , \ \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \leq (\hat{Q}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t} \ , \ \boxed{T_i(0) = T_i^0 > 0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0), T_i(0)$

$$G_i(t) \leq (\hat{P}_{16})^{(2)}e^{(\hat{M}_{16})^{(2)}t} \ , \ G_i(0) = G_i^0 > 0$$

| | |
|---|---|
| $T_i(t) \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>$G_i(t) \leq (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \leq (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$ | 153 A |

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| $T_i(t) \leq (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br><br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br><br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[ (a_{18})^{(2)} G_{17}(s_{(16)}) - \left( (a_{18}')^{(2)} + (a_{18}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{18}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[ (b_{16})^{(2)} T_{17}(s_{(16)}) - \left( (b_{16}')^{(2)} - (b_{16}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{16}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[ (b_{17})^{(2)} T_{16}(s_{(16)}) - \left( (b_{17}')^{(2)} - (b_{17}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[ (b_{18})^{(2)} T_{17}(s_{(16)}) - \left( (b_{18}')^{(2)} - (b_{18}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{18}(s_{(16)}) \right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a_{20}')^{(3)} + a_{20}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a_{21}')^{(3)} + (a_{21}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a_{22}')^{(3)} + (a_{22}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b_{20}')^{(3)} - (b_{20}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b_{21}')^{(3)} - (b_{21}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b_{22}')^{(3)} - (b_{22}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |

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| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{24})^{(4)}$ , $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{28})^{(5)}$ , $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |

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| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b'_{29})^{(5)} - (b''_{29})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{\mathrm{T}}_{30}(\mathrm{t}) = \mathrm{T}_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b'_{30})^{(5)} - (b''_{30})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{32})^{(6)}$ , $T_i^0 \le (\hat{Q}_{32})^{(6)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a'_{32})^{(6)} + a''_{32})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b'_{32})^{(6)} - (b''_{32})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b'_{33})^{(6)} - (b''_{33})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{\mathrm{T}}_{34}(\mathrm{t}) = \mathrm{T}_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b'_{34})^{(6)} - (b''_{34})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{36})^{(7)}$ , $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |

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| By<br>$\bar{G}_{36}(t) = G_{36}^{0} + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a_{36}^{'})^{(7)} + a_{36}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^{0} + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^{0} + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^{0} + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}^{'})^{(8)} + a_{40}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^{0} + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^{0} + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^{0} + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^{0} + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |

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| $\overline{T}_{42}(t) = T_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br><br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166<br>A |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{44})^{(9)}$, $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br><br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\overline{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br><br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\bar{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br><br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left(e^{(\bar{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br><br>$(G_{13}(t) - G_{13}^0) e^{-(\bar{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br><br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |

| | |
|---|---|
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \leq G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$ $\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that $(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \leq \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{20}(t) \leq G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} \left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$ $\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}} \left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$ | 171 |
| From which it follows that $(G_{20}(t) - G_{20}^0) e^{-(\hat{M}_{20})^{(3)} t} \leq \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}} \left[\left((\hat{P}_{20})^{(3)} + G_{21}^0\right) e^{\left(-\frac{(\hat{P}_{20})^{(3)} + G_{21}^0}{G_{21}^0}\right)} + (\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that $G_{24}(t) \leq G_{24}^0 + \int_0^t \left[(a_{24})^{(4)} \left(G_{25}^0 + (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} s_{(24)}}\right)\right] ds_{(24)} =$ $\left(1 + (a_{24})^{(4)} t\right) G_{25}^0 + \frac{(a_{24})^{(4)} (\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}} \left(e^{(\hat{M}_{24})^{(4)} t} - 1\right)$ | 173 |
| From which it follows that $(G_{24}(t) - G_{24}^0) e^{-(\hat{M}_{24})^{(4)} t} \leq \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}} \left[\left((\hat{P}_{24})^{(4)} + G_{25}^0\right) e^{\left(-\frac{(\hat{P}_{24})^{(4)} + G_{25}^0}{G_{25}^0}\right)} + (\hat{P}_{24})^{(4)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{28}(t) \leq G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} \left(G_{29}^0 + (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} s_{(28)}}\right)\right] ds_{(28)} =$ $\left(1 + (a_{28})^{(5)} t\right) G_{29}^0 + \frac{(a_{28})^{(5)} (\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}} \left(e^{(\hat{M}_{28})^{(5)} t} - 1\right)$ | |

From which it follows that

$$(G_{28}(t)-G_{28}^{0})e^{-(\bar{M}_{28})^{(5)}t} \leq \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^{0})e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^{0}}{G_{29}^{0}}\right)}+(\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5 175

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 176

$$G_{32}(t) \leq G_{32}^{0}+\int_0^t\left[(a_{32})^{(6)}\left(G_{33}^{0}+(\hat{P}_{32})^{(6)}e^{(\bar{M}_{32})^{(6)}s_{(32)}}\right)\right]ds_{(32)} =$$

$$\left(1+(a_{32})^{(6)}t\right)G_{33}^{0}+\frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left(e^{(\bar{M}_{32})^{(6)}t}-1\right)$$

From which it follows that 177

$$(G_{32}(t)-G_{32}^{0})e^{-(\bar{M}_{32})^{(6)}t} \leq \frac{(a_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^{0})e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^{0}}{G_{33}^{0}}\right)}+(\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

(m) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 178

$$G_{36}(t) \leq G_{36}^{0}+\int_0^t\left[(a_{36})^{(7)}\left(G_{37}^{0}+(\hat{P}_{36})^{(7)}e^{(\bar{M}_{36})^{(7)}s_{(36)}}\right)\right]ds_{(36)} =$$

$$\left(1+(a_{36})^{(7)}t\right)G_{37}^{0}+\frac{(a_{36})^{(7)}(\hat{P}_{36})^{(7)}}{(\bar{M}_{36})^{(7)}}\left(e^{(\bar{M}_{36})^{(7)}t}-1\right)$$

From which it follows that

$$(G_{36}(t)-G_{36}^{0})e^{-(\bar{M}_{36})^{(7)}t} \leq \frac{(a_{36})^{(7)}}{(\bar{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)}+G_{37}^{0})e^{\left(-\frac{(\hat{P}_{36})^{(7)}+G_{37}^{0}}{G_{37}^{0}}\right)}+(\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \leq G_{40}^{0}+\int_0^t\left[(a_{40})^{(8)}\left(G_{41}^{0}+(\hat{P}_{40})^{(8)}e^{(\bar{M}_{40})^{(8)}s_{(40)}}\right)\right]ds_{(40)} =$$

$$\left(1+(a_{40})^{(8)}t\right)G_{41}^{0}+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\bar{M}_{40})^{(8)}}\left(e^{(\bar{M}_{40})^{(8)}t}-1\right)$$ 180

From which it follows that

$$(G_{40}(t)-G_{40}^{0})e^{-(\bar{M}_{40})^{(8)}t} \leq \frac{(a_{40})^{(8)}}{(\bar{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^{0})e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^{0}}{G_{41}^{0}}\right)}+(\hat{P}_{40})^{(8)}\right]$$ 181

$(G_i^0)$ is as defined in the statement of theorem 8
Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$

The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)}\left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$$

$$\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)} t} - 1\right)$$

From which it follows that

$$(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{P}_{44})^{(9)} + G_{45}^0\right) e^{\left(-\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0}\right)} + (\hat{P}_{44})^{(9)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 9

Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$

It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ and to choose

$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have

182

$$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)} + \left((\hat{P}_{13})^{(1)} + G_j^0\right) e^{-\left(\frac{(\hat{P}_{13})^{(1)} + G_j^0}{G_j^0}\right)}\right] \le (\hat{P}_{13})^{(1)} \quad (183)$$

$$\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)} + T_j^0\right) e^{-\left(\frac{(\hat{Q}_{13})^{(1)} + T_j^0}{T_j^0}\right)} + (\hat{Q}_{13})^{(1)}\right] \le (\hat{Q}_{13})^{(1)} \quad (184)$$

In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric 185

$$d\left(\left(G^{(1)}, T^{(1)}\right), \left(G^{(2)}, T^{(2)}\right)\right) =$$

$$\sup_i \{\max_{t \in \mathbb{R}_+} \left|G_i^{(1)}(t) - G_i^{(2)}(t)\right| e^{-(\hat{M}_{13})^{(1)} t}, \max_{t \in \mathbb{R}_+} \left|T_i^{(1)}(t) - T_i^{(2)}(t)\right| e^{-(\hat{M}_{13})^{(1)} t}\}$$

Indeed if we denote

**Definition of** $\tilde{G}, \tilde{T}$ : $\left(\tilde{G}, \tilde{T}\right) = \mathcal{A}^{(1)}(G, T)$

It results

$$\left|\tilde{G}_{13}^{(1)}-\tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{13})^{(1)} \left|G_{14}^{(1)}-G_{14}^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)} s_{(13)}} e^{(\widehat{M}_{13})^{(1)} s_{(13)}}\, ds_{(13)} +$$

$$\int_0^t \{(a_{13}')^{(1)} \left|G_{13}^{(1)}-G_{13}^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)} s_{(13)}} e^{-(\widehat{M}_{13})^{(1)} s_{(13)}} +$$

$$(a_{13}'')^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right)\left|G_{13}^{(1)}-G_{13}^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)} s_{(13)}} e^{(\widehat{M}_{13})^{(1)} s_{(13)}} +$$

$$G_{13}^{(2)} \left|(a_{13}'')^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right) - (a_{13}'')^{(1)}\left(T_{14}^{(2)}, s_{(13)}\right)\right|\ e^{-(\widehat{M}_{13})^{(1)} s_{(13)}} e^{(\widehat{M}_{13})^{(1)} s_{(13)}}\} ds_{(13)}$$

Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses it follows

$$\left|G^{(1)}-G^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)} t} \leq$$
$$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a_{13}')^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)} (\widehat{k}_{13})^{(1)}\right) d\left(\left(G^{(1)}, T^{(1)};\ G^{(2)}, T^{(2)}\right)\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

(186)

**Remark 1:** The fact that we supposed $(a_{13}'')^{(1)}$ and $(b_{13}'')^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)} t}$ *and* $(\widehat{Q}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(1)}$ and $(b_i'')^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 2:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$

From 19 to 24 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(1)} - (a_i'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(1)} t\right)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :

(187)

**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if

$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a_{14}')^{(1)} G_{14}$ and by integrating

$$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)} \left((\widehat{M}_{13})^{(1)}\right)_1 / (a_{14}')^{(1)}$$

In the same way , one can obtain

$$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)} \left((\widehat{M}_{13})^{(1)}\right)_2 / (a_{15}')^{(1)}$$

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|---|---|
| If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t),t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br><br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote | 195 |

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| **Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right) d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$ | 201 |

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| In the same way , one can obtain<br>$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | |
| **Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br>Indeed let $t_2$ be so that for $t > t_2$<br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\widehat{P}_{20})^{(3)}$ and $(\widehat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[(\widehat{P}_{20})^{(3)} + \left((\widehat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[\left((\widehat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{20})^{(3)}\right] \leq (\widehat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$ | 211 |

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| $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | |
| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}},\widetilde{T_{23}}:(\widetilde{(G_{23})},\widetilde{(T_{23})})=\mathcal{A}^{(3)}((G_{23}),(T_{23}))$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{20})^{(3)}\left\|G_{21}^{(1)}-G_{21}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)}+$<br><br>$\int_0^t\{(a_{20}^{'})^{(3)}\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br><br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br><br>$G_{20}^{(2)}\left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)-(a_{20}^{''})^{(3)}\left(T_{21}^{(2)},s_{(20)}\right)\right\|\ e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)}-G_{23}{}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}t}\le$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)}+(a_{20}^{'})^{(3)}+(\widehat{A}_{20})^{(3)}+(\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)},(T_{23})^{(1)};\ (G_{23})^{(2)},(T_{23})^{(2)})\right)$<br><br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ and\ (\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i=20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$<br><br>it results<br><br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(3)}-(a_i^{''})^{(3)}(T_{21}(s_{(20)}),s_{(20)})\}ds_{(20)}\right]}\ge 0$<br><br>$T_i(t)\ge T_i^0e^{(-(b_i^{'})^{(3)}t)}>0$ for t > 0 | 216 |

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| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |

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| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{24})^{(4)}\left\|G_{25}^{(1)}-G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$<br>$\int_0^t\{(a_{24}^{'})^{(4)}\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$G_{24}^{(2)}\left\|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)}-(G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t}\le$<br>$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 226 |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24{,}25{,}26$ depend only on $\mathrm{T}_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results | 228 |

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| $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(4)} - (a_i'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(4)} t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ :<br><br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if<br><br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}')^{(4)} G_{25}$ and by integrating<br><br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)} \left((\widehat{M}_{24})^{(4)}\right)_1 / (a_{25}')^{(4)}$<br><br>In the same way , one can obtain<br><br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)} \left((\widehat{M}_{24})^{(4)}\right)_2 / (a_{26}')^{(4)}$<br><br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t), t)) = (b_{25}')^{(4)}$ then $T_{25} \to \infty$.<br><br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br><br>Indeed let $t_4$ be so that for $t > t_4$<br><br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t), t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}\left((G_{27})(t), t\right) = (b_{26}')^{(4)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br><br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br><br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |

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| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)}+\left((\hat{P}_{28})^{(5)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{28})^{(5)}\right]\leq(\hat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $\widetilde{(G_{31})},\widetilde{(T_{31})}$ : $\left(\widetilde{(G_{31})},\widetilde{(T_{31})}\right)=\mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}ds_{(28)}+$<br>$\int_0^t\{(a_{28}')^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}'')^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\leq$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}')^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}'')^{(5)}$ and $(b_{28}'')^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it | 238 |

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| suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 22:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(5)} - (a_i^{''})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(5)} t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2\ and\ \left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}\ and\ G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}^{'})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a_{29}^{'})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a_{30}^{'})^{(5)}$<br><br>If $G_{29}\ or\ G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}\ and\ G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t), t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t), t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t), t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |

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| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\widehat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{32})^{(6)}\right] \leq (\widehat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)}, (T_{35})^{(1)}), ((G_{35})^{(2)}, (T_{35})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $\widetilde{(G_{35})}, \widetilde{(T_{35})}$ : $\left(\widetilde{(G_{35})}, \widetilde{(T_{35})}\right) = \mathcal{A}^{(6)}((G_{35}), (T_{35}))$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{32})^{(6)}\left\|G_{33}^{(1)} - G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\,ds_{(32)} +$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)} - G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)\left\|G_{32}^{(1)} - G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right) - (a''_{32})^{(6)}\left(T_{33}^{(2)}, s_{(32)}\right)\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)} - (G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t} \leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)} + (a'_{32})^{(6)} + (\widehat{A}_{32})^{(6)} + (\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)}, (T_{35})^{(1)}; (G_{35})^{(2)}, (T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | 249 |

condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i = 32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 27:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$ 250

it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(6)} - (a_i^{''})^{(6)}(T_{33}(s_{(32)}),s_{(32)})\}ds_{(32)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(6)}t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2\ and\ \left((\widehat{M}_{32})^{(6)}\right)_3$ : 251

**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}\ and\ G_{34}$ . indeed if

$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}^{'})^{(6)}G_{33}$ and by integrating

$$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1/(a_{33}^{'})^{(6)}$$

In the same way , one can obtain

$$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2/(a_{34}^{'})^{(6)}$$

If $G_{33}\ or\ G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively.

**Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}\ and\ G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. 252

**Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}\ ((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$. 253

**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :

Indeed let $t_6$ be so that for $t > t_6$

$$(b_{33})^{(6)} - (b_i^{''})^{(6)}\left((G_{35})(t),t\right) < \varepsilon_6, T_{32}\,(t) > (m)^{(6)}$$

Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to 254

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right), \quad t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.
The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$

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| We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| Analogous inequalities hold also for $G_{37}, G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ large to have | 255 |
| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[(\hat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)}, (T_{39})^{(1)}), ((G_{39})^{(2)}, (T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})}, \widetilde{(T_{39})} : \left(\widetilde{(G_{39})}, \widetilde{(T_{39})}\right) = \mathcal{A}^{(7)}((G_{39}), (T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{36})^{(7)}\left|G_{37}^{(1)} - G_{37}^{(2)}\right|e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{(\hat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)} +$<br>$\int_0^t\{(a'_{36})^{(7)}\left|G_{36}^{(1)} - G_{36}^{(2)}\right|e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{-(\hat{M}_{36})^{(7)}s_{(36)}} +$<br>$(a''_{36})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right)\left|G_{36}^{(1)} - G_{36}^{(2)}\right|e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{(\hat{M}_{36})^{(7)}s_{(36)}} +$<br>$G_{36}^{(2)}\left|(a''_{36})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) - (a''_{36})^{(7)}\left(T_{37}^{(2)}, s_{(36)}\right)\right|\ e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{(\hat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on it follows | 258 |
| $\left|(G_{39})^{(1)} - (G_{39})^{(2)}\right|e^{-(\hat{M}_{36})^{(7)}t} \leq$<br>$\frac{1}{(\hat{M}_{36})^{(7)}}\left((a_{36})^{(7)} + (a'_{36})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)}, (T_{39})^{(1)};\ (G_{39})^{(2)}, (T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |

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| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ $and$ $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\}ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)}t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)}G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1/(a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ $is$ bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}\,((G_{39})(t),t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |

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| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right), \quad t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t), t\right) = (b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ and to choose<br>$(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\left[(\hat{P}_{40})^{(8)} + \left((\hat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{40})^{(8)}\right] \leq (\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)}, (T_{43})^{(1)}), ((G_{43})^{(2)}, (T_{43})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{40})^{(8)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $(\widetilde{G_{43}}), (\widetilde{T_{43}})$ : $\left((\widetilde{G_{43}}), (\widetilde{T_{43}})\right) = \mathcal{A}^{(8)}((G_{43}), (T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{40})^{(8)}\left\|G_{41}^{(1)} - G_{41}^{(2)}\right\|e^{-(\hat{M}_{40})^{(8)}s_{(40)}}e^{(\hat{M}_{40})^{(8)}s_{(40)}}\,ds_{(40)} +$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\hat{M}_{40})^{(8)}s_{(40)}}e^{-(\hat{M}_{40})^{(8)}s_{(40)}} +$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\hat{M}_{40})^{(8)}s_{(40)}}e^{(\hat{M}_{40})^{(8)}s_{(40)}} +$<br>$G_{40}^{(2)}\left\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right) - (a_{40}^{''})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)\right\| e^{-(\hat{M}_{40})^{(8)}s_{(40)}}e^{(\hat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval [0, t] | 272 |

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| From the hypotheses it follows | |
| $\left\|(G_{43})^{(1)} - (G_{43})^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)}t} \leq$ $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)} + (a'_{40})^{(8)} + (\widehat{A}_{40})^{(8)} + (\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right) d\left(((G_{43})^{(1)}, (T_{43})^{(1)};\ (G_{43})^{(2)}, (T_{43})^{(2)})\right)$ And analogous inequalities for $G_i$ $and$ $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a''_{40})^{(8)}$ and $(b''_{40})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(8)}$ and $(b''_i)^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t) = 0$ $and$ $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(8)} - (a''_i)^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ $and$ $\left((\widehat{M}_{40})^{(8)}\right)_3$ : **Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ $and$ $G_{42}$ . indeed if $G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a'_{41})^{(8)}G_{41}$ and by integrating $G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1/(a'_{41})^{(8)}$ In the same way , one can obtain $G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2/(a'_{42})^{(8)}$ If $G_{41}$ $or$ $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}$ $and$ $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(8)}((G_{43})(t), t)) = (b'_{41})^{(8)}$ then $T_{41} \to \infty$. | 278 |

**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :

Indeed let $t_8$ be so that for $t > t_8$

$(b_{41})^{(8)} - (b_i^{''})^{(8)}\big((G_{43})(t), t\big) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$

Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.
The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\big((G_{43})(t), t(t), t\big) = (b_{42}^{'})^{(8)}$

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It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have

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$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \big((\hat{P}_{44})^{(9)} + G_j^0\big)e^{-\left(\frac{(\hat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\big((\hat{Q}_{44})^{(9)} + T_j^0\big)e^{-\left(\frac{(\hat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{44})^{(9)}\right] \leq (\hat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\Big(\big((G_{47})^{(1)}, (T_{47})^{(1)}\big), \big((G_{47})^{(2)}, (T_{47})^{(2)}\big)\Big) =$$
$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \big(\widetilde{(G_{47})}, \widetilde{(T_{47})}\big) = \mathcal{A}^{(9)}\big((G_{47}), (T_{47})\big)$
It results

$$\left|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$(a_{44}^{''})^{(9)}\big(T_{45}^{(1)}, s_{(44)}\big)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\big(T_{45}^{(1)}, s_{(44)}\big) - (a_{44}^{''})^{(9)}\big(T_{45}^{(2)}, s_{(44)}\big)\right| \; e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)} - G^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)}t} \leq \frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a'_{44})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right) d\left(((G_{47})^{(1)}, (T_{47})^{(1)};\ (G_{47})^{(2)}, (T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a''_{44})^{(9)}$ and $(b''_{44})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)} e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)} e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(9)}$ and $(b''_i)^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$

From 99 to 44 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(9)} - (a''_i)^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b'_i)^{(9)}t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2$ *and* $\left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a'_{45})^{(9)} G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1 / (a'_{45})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2 / (a'_{46})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(9)}\left((G_{47})(t), t\right)) = (b'_{45})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\big((G_{47})(t),t\big) < \varepsilon_9, T_{44}\,(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\left((G_{47})(t),t\right) = (b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} - (a_{13}^{''})^{(1)}(T_{14}, t) + (a_{14}^{''})^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)}, , (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

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| $(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)},$<br>and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$<br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$<br>are defined | |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)} t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)} t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)} t}\right] + G_{15}^0 e^{-(S_2)^{(1)} t} \le G_{15}(t) \le \right.$<br>$\left.\frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a'_{15})^{(1)})}[e^{(S_1)^{(1)} t} - e^{-(a'_{15})^{(1)} t}] + G_{15}^0 e^{-(a'_{15})^{(1)} t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)} T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)} - (b'_{15})^{(1)})}\left[e^{(R_1)^{(1)} t} - e^{-(b'_{15})^{(1)} t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)} t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)} T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)} + (r_{13})^{(1)} + (R_2)^{(1)})}\left[e^{((R_1)^{(1)} + (r_{13})^{(1)})t} - e^{-(R_2)^{(1)} t}\right] + T_{15}^0 e^{-(R_2)^{(1)} t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$<br>$(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ :<br>$(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |

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| $-(\sigma_2)^{(2)} \leq -(a_{16}^{'})^{(2)} + (a_{17}^{'})^{(2)} - (a_{16}^{''})^{(2)}(\mathrm{T}_{17}, t) + (a_{17}^{''})^{(2)}(\mathrm{T}_{17}, t) \leq -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \leq -(b_{16}^{'})^{(2)} + (b_{17}^{'})^{(2)} - (b_{16}^{''})^{(2)}\big((G_{19}), t\big) - (b_{17}^{''})^{(2)}\big((G_{19}), t\big) \leq -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}$,, $(\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{\mathrm{G}_{16}^0}{\mathrm{G}_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{\mathrm{T}_{16}^0}{\mathrm{T}_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$\mathrm{G}_{16}^0 \mathrm{e}^{((\mathrm{S}_1)^{(2)} - (p_{16})^{(2)})\mathrm{t}} \leq G_{16}(t) \leq \mathrm{G}_{16}^0 \mathrm{e}^{(\mathrm{S}_1)^{(2)}\mathrm{t}}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |

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| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)}-(p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)}-(p_{16})^{(2)}-(S_2)^{(2)})}\left[e^{((S_1)^{(2)}-(p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le \right.$ $\left.\frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)}-(a_{18}')^{(2)})}\left[e^{(S_1)^{(2)}t} - e^{-(a_{18}')^{(2)}t}\right] + G_{18}^0 e^{-(a_{18}')^{(2)}t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)}t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)}t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)}-(b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t} - e^{-(b_{18}')^{(2)}t}\right] + T_{18}^0 e^{-(b_{18}')^{(2)}t} \le T_{18}(t) \le$ $\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t} - e^{-(R_2)^{(2)}t}\right] + T_{18}^0 e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a_{16}')^{(2)}$ $(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b_{16}')^{(2)}$ $(R_2)^{(2)} = (b_{18}')^{(2)} - (r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$: $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)} \le -(a_{20}')^{(3)} + (a_{21}')^{(3)} - (a_{20}'')^{(3)}(T_{21}, t) + (a_{21}'')^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)} \le -(b_{20}')^{(3)} + (b_{21}')^{(3)} - (b_{20}'')^{(3)}(G_{23}, t) - (b_{21}'')^{(3)}((G_{23}), t) \le -(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)}, (v_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$: By $(v_1)^{(3)} > 0$, $(v_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$, $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$ and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and By $(\bar{v}_1)^{(3)} > 0$, $(\bar{v}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$, $(\bar{u}_2)^{(3)} < 0$ the | 320 |

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| roots of the equations $(a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$<br><br>and $(b_{21})^{(3)}\left(u^{(3)}\right)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ | |
| **Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :-<br><br>If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by<br>$(m_2)^{(3)} = (v_0)^{(3)}, (m_1)^{(3)} = (v_1)^{(3)}$, ***if*** $(v_0)^{(3)} < (v_1)^{(3)}$<br><br>$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (\bar{v}_1)^{(3)}$, ***if*** $(v_1)^{(3)} < (v_0)^{(3)} < (\bar{v}_1)^{(3)}$,<br><br>and $\boxed{(v_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$<br><br>$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (v_0)^{(3)}$, ***if*** $(\bar{v}_1)^{(3)} < (v_0)^{(3)}$ | 321 |
| and analogously<br><br>$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$<br><br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$<br><br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$<br><br>Then the solution of global equations satisfies the inequalities<br><br>$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)}t}$<br><br>$(p_i)^{(3)}$ is defined by equation | 322 |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \le G_{22}(t) \le \right.$<br>$\left.\frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a'_{22})^{(3)})}\left[e^{(S_1)^{(3)}t} - e^{-(a'_{22})^{(3)}t}\right] + G_{22}^0 e^{-(a'_{22})^{(3)}t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b'_{22})^{(3)})}\left[e^{(R_1)^{(3)}t} - e^{-(b'_{22})^{(3)}t}\right] + T_{22}^0 e^{-(b'_{22})^{(3)}t} \le T_{22}(t) \le$<br><br>$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$ | 327 |

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- 328

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$

$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$

$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$

**Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : 329

By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$
and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{v}_1)^{(4)} > 0$, $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$
and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (v_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}$, ***if*** $(v_0)^{(4)} < (v_1)^{(4)}$

$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}$, ***if*** $(v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)}$,
and $\boxed{(v_0)^{(4)} = \frac{G^0_{24}}{G^0_{25}}}$

$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)} < (v_0)^{(4)}$

and analogously 331

$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$

$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$, ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,

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| and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \boldsymbol{if}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)} t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)} t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\right)\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)} t}\right] + G_{26}^0 e^{-(S_2)^{(4)} t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b_{26}')^{(4)})}\left[e^{(R_1)^{(4)} t} - e^{-(b_{26}')^{(4)} t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)} t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)} + (r_{24})^{(4)} + (R_2)^{(4)})}\left[e^{((R_1)^{(4)} + (r_{24})^{(4)})t} - e^{-(R_2)^{(4)} t}\right] + T_{26}^0 e^{-(R_2)^{(4)} t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |

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| **Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ :<br><br>By $(v_1)^{(5)} > 0$ , $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$ , $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | 339 |
| **Definition of** $(\bar{v}_1)^{(5)},, (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ :<br><br>By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :-<br><br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}, (\mu_2)^{(5)}$ by<br><br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br><br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br><br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |

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| $\frac{1}{(\mu_1)^{(5)}}T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}}T_{28}^0 e^{((R_1)^{(5)}+(r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)}T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)}-(b_{30}')^{(5)})}\left[e^{(R_1)^{(5)}t} - e^{-(b_{30}')^{(5)}t}\right] + T_{30}^0 e^{-(b_{30}')^{(5)}t} \le T_{30}(t) \le$<br>$\frac{(a_{30})^{(5)}T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t} - e^{-(R_2)^{(5)}t}\right] + T_{30}^0 e^{-(R_2)^{(5)}t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a_{28}')^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b_{28}')^{(5)}$<br>$(R_2)^{(5)} = (b_{30}')^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :<br>$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying<br>$-(\sigma_2)^{(6)} \le -(a_{32}')^{(6)} + (a_{33}')^{(6)} - (a_{32}'')^{(6)}(T_{33}, t) + (a_{33}'')^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$<br>$-(\tau_2)^{(6)} \le -(b_{32}')^{(6)} + (b_{33}')^{(6)} - (b_{32}'')^{(6)}((G_{35}), t) - (b_{33}'')^{(6)}((G_{35}), t) \le -(\tau_1)^{(6)}$ | 349 |
| **Definition of** $(v_1)^{(6)}, (v_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, v^{(6)}, u^{(6)}$ :<br>By $(v_1)^{(6)} > 0$, $(v_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$, $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(v^{(6)})^2 + (\sigma_1)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{v}_1)^{(6)},, (\bar{v}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br>By $(\bar{v}_1)^{(6)} > 0$, $(\bar{v}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$, $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(v^{(6)})^2 + (\sigma_2)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (v_0)^{(6)}$ :-<br>If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br>$(m_2)^{(6)} = (v_0)^{(6)}, (m_1)^{(6)} = (v_1)^{(6)}$, ***if*** $(v_0)^{(6)} < (v_1)^{(6)}$<br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (\bar{v}_6)^{(6)}$, ***if*** $(v_1)^{(6)} < (v_0)^{(6)} < (\bar{v}_1)^{(6)}$, | 351 |

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| and $\boxed{(v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$<br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (v_0)^{(6)},$ ***if*** $(\bar{v}_1)^{(6)} < (v_0)^{(6)}$ | |
| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},$ ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)},$ ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\right)\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34')6e(S1)6t−e−(a34')6t+ G340e−(a34')6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b_{34}')^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b_{34}')^{(6)} t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$<br>$(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** | |

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ :

$(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying

$$-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$$

$$-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}\big((G_{39}), t\big) - (b''_{37})^{(7)}\big((G_{39}), t\big) \le -(\tau_1)^{(7)}$$

**Definition of** $(v_1)^{(7)}, (v_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, v^{(7)}, u^{(7)}$ : 361

By $(v_1)^{(7)} > 0, (v_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(v^{(7)}\big)^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and

**Definition of** $(\bar{v}_1)^{(7)}, , (\bar{v}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ : 362

By $(\bar{v}_1)^{(7)} > 0, (\bar{v}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(v^{(7)}\big)^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$

**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (v_0)^{(7)}$ :-

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (v_0)^{(7)}, (m_1)^{(7)} = (v_1)^{(7)}$, ***if*** $(v_0)^{(7)} < (v_1)^{(7)}$

$(m_2)^{(7)} = (v_1)^{(7)}, (m_1)^{(7)} = (\bar{v}_1)^{(7)}$, ***if*** $(v_1)^{(7)} < (v_0)^{(7)} < (\bar{v}_1)^{(7)}$,
and $\boxed{(v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$

$(m_2)^{(7)} = (v_1)^{(7)}, (m_1)^{(7)} = (v_0)^{(7)}$, ***if*** $(\bar{v}_1)^{(7)} < (v_0)^{(7)}$

and analogously 363

$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,
and $\boxed{(u_0)^{(7)} = \frac{T^0_{36}}{T^0_{37}}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

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| Then the solution of global equations satisfies the inequalities<br>$G_{36}^0 e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)}t}$<br>where $(p_i)^{(7)}$ is defined by equation | 364 |
| $\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^0 e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}')^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}')^{(7)}t}] + G_{38}^0 e^{-(a_{38}')^{(7)}t})$ | 366 |
| $\boxed{T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}')^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}')^{(7)}t}\right] + T_{38}^0 e^{-(b_{38}')^{(7)}t} \le T_{38}(t) \le$<br>$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^0 e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}')^{(7)}$<br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}')^{(7)}$<br>$(R_2)^{(7)} = (b_{38}')^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br>$-(\sigma_2)^{(8)} \le -(a_{40}')^{(8)} + (a_{41}')^{(8)} - (a_{40}'')^{(8)}(T_{41}, t) + (a_{41}'')^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br>$-(\tau_2)^{(8)} \le -(b_{40}')^{(8)} + (b_{41}')^{(8)} - (b_{40}'')^{(8)}((G_{43}), t) - (b_{41}'')^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |

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| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : <br><br> By $(v_1)^{(8)} > 0$ , $(v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0$ , $(u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$ <br> and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | 372 |
| **Definition of** $(\bar{v}_1)^{(8)}$,, $(\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ : <br><br> By $(\bar{v}_1)^{(8)} > 0$ , $(\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0$ , $(\bar{u}_2)^{(8)} < 0$ the <br><br> roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$ <br><br> and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ <br><br> **Definition of** $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :- <br><br> If we define $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)}, (\mu_2)^{(8)}$ by <br><br> $(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)}$, ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$ <br><br> $(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}$ , ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)}$, <br> and $\boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ <br><br> $(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (v_0)^{(8)}$, ***if*** $(\bar{v}_1)^{(8)} < (v_0)^{(8)}$ | |
| and analogously <br><br> $(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)}$, ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$ <br><br> $(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)}$, <br> and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$ <br><br> $(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}$, ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities <br><br> $G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{40}(t) \leq G_{40}^0 e^{(S_1)^{(8)}t}$ <br><br> where $(p_i)^{(8)}$ is defined by equation | 375 |

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| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)}-(p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)}G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t} - e^{-(S_2)^{(8)}t}\right] + G_{42}^0 e^{-(S_2)^{(8)}t} \le G_{42}(t) \le \right.$ $\left.\frac{(a_{42})^{(8)}G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)}-(a'_{42})^{(8)})}[e^{(S_1)^{(8)}t} - e^{-(a'_{42})^{(8)}t}] + G_{42}^0 e^{-(a'_{42})^{(8)}t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)}-(b'_{42})^{(8)})}\left[e^{(R_1)^{(8)}t} - e^{-(b'_{42})^{(8)}t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)}t} \le T_{42}(t) \le$ $\frac{(a_{42})^{(8)}T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t} - e^{-(R_2)^{(8)}t}\right] + T_{42}^0 e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$<br>$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}((G_{47}), t) - (b''_{45})^{(9)}((G_{47}), t) \le -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :<br>By $(\nu_1)^{(9)} > 0$, $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$, $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$<br>and $(b_{45})^{(9)}(u^{(9)})^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(9)},, (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ : | |

By $(\bar{v}_1)^{(9)} > 0$ , $(\bar{v}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\left(u^{(9)}\right)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}$ , $(m_2)^{(9)}$ , $(\mu_1)^{(9)}$, $(\mu_2)^{(9)}$, $(v_0)^{(9)}$ :-

If we define $(m_1)^{(9)}$ , $(m_2)^{(9)}$ , $(\mu_1)^{(9)}$, $(\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (v_0)^{(9)}, (m_1)^{(9)} = (v_1)^{(9)}$, ***if*** $(v_0)^{(9)} < (v_1)^{(9)}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (\bar{v}_1)^{(9)}$ , ***if*** $(v_1)^{(9)} < (v_0)^{(9)} < (\bar{v}_1)^{(9)}$,

and $\boxed{(v_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (v_0)^{(9)}$, ***if*** $(\bar{v}_1)^{(9)} < (v_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^0 e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}} G_{44}^0 e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)} G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)} - (p_{44})^{(9)} - (S_2)^{(9)})}\left[e^{((S_1)^{(9)} - (p_{44})^{(9)})t} - e^{-(S_2)^{(9)}t}\right] + G_{46}^0 e^{-(S_2)^{(9)}t} \le G_{46}(t) \le \right.$$

$$\left. \frac{(a_{46})^{(9)} G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)} - (a_{46}')^{(9)})}\left[e^{(S_1)^{(9)}t} - e^{-(a_{46}')^{(9)}t}\right] + G_{46}^0 e^{-(a_{46}')^{(9)}t}\right)$$

$$\boxed{T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}} T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}} T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)} T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)} - (b_{46}')^{(9)})}\left[e^{(R_1)^{(9)}t} - e^{-(b_{46}')^{(9)}t}\right] + T_{46}^0 e^{-(b_{46}')^{(9)}t} \le T_{46}(t) \le$$

$\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$

**Definition of** $(S_1)^{(9)},(S_2)^{(9)},(R_1)^{(9)},(R_2)^{(9)}$:-

Where $(S_1)^{(9)}=(a_{44})^{(9)}(m_2)^{(9)}-(a_{44}^{'})^{(9)}$

$(S_2)^{(9)}=(a_{46})^{(9)}-(p_{46})^{(9)}$

$(R_1)^{(9)}=(b_{44})^{(9)}(\mu_2)^{(9)}-(b_{44}^{'})^{(9)}$

$(R_2)^{(9)}=(b_{46}^{'})^{(9)}-(r_{46})^{(9)}$

**Proof :** From global equations we obtain

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$$\frac{dv^{(1)}}{dt}=(a_{13})^{(1)}-\left((a_{13}^{'})^{(1)}-(a_{14}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14},t)\right)-(a_{14}^{''})^{(1)}(T_{14},t)v^{(1)}-(a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)}=\frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_2)^{(1)}v^{(1)}-(a_{13})^{(1)}\right)\leq\frac{dv^{(1)}}{dt}\leq-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_1)^{(1)}v^{(1)}-(a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(1)},(\nu_0)^{(1)}$ :-

For $0<\boxed{(\nu_0)^{(1)}=\frac{G_{13}^{0}}{G_{14}^{0}}}<(\nu_1)^{(1)}<(\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t)\geq\frac{(\nu_1)^{(1)}+(C)^{(1)}(\nu_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_0)^{(1)}\right)t\right]}}{1+(C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_0)^{(1)}\right)t\right]}}\quad,\quad\boxed{(C)^{(1)}=\frac{(\nu_1)^{(1)}-(\nu_0)^{(1)}}{(\nu_0)^{(1)}-(\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)}\leq\nu^{(1)}(t)\leq(\nu_1)^{(1)}$

In the same manner , we get

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$$\nu^{(1)}(t)\leq\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(1)}=\frac{(\bar{\nu}_1)^{(1)}-(\nu_0)^{(1)}}{(\nu_0)^{(1)}-(\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)}\leq\nu^{(1)}(t)\leq(\bar{\nu}_1)^{(1)}$

If $0<(\nu_1)^{(1)}<(\nu_0)^{(1)}=\frac{G_{13}^{0}}{G_{14}^{0}}<(\bar{\nu}_1)^{(1)}$ we find like in the previous case,

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$$(\nu_1)^{(1)} \le \frac{(\nu_1)^{(1)}+(C)^{(1)}(\nu_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_2)^{(1)}\right)t\right]}}{1+(C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_2)^{(1)}\right)t\right]}} \le \nu^{(1)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \le (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \le (\bar{\nu}_1)^{(1)} \le \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \le \nu^{(1)}(t) \le \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \le (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \le \nu^{(1)}(t) \le (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \le u^{(1)}(t) \le (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)}G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)}T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}')^{(2)} - (a_{17}')^{(2)} + (a_{16}'')^{(2)}(T_{17}, t)\right) - (a_{17}'')^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

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| $-\left((a_{17})^{(2)}\left(v^{(2)}\right)^2+(\sigma_2)^{(2)}v^{(2)}-(a_{16})^{(2)}\right)\le\frac{dv^{(2)}}{dt}\le-\left((a_{17})^{(2)}\left(v^{(2)}\right)^2+(\sigma_1)^{(2)}v^{(2)}-(a_{16})^{(2)}\right)$ | |
| From which one obtains<br><br>**Definition of** $(\bar{v}_1)^{(2)},(v_0)^{(2)}$ :-<br><br>For $0<(v_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}<(v_1)^{(2)}<(\bar{v}_1)^{(2)}$<br><br>$v^{(2)}(t)\ge\frac{(v_1)^{(2)}+(C)^{(2)}(v_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_0)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_0)^{(2)}\right)t\right]}}$ , $\boxed{(C)^{(2)}=\frac{(v_1)^{(2)}-(v_0)^{(2)}}{(v_0)^{(2)}-(v_2)^{(2)}}}$<br><br>it follows $(v_0)^{(2)}\le v^{(2)}(t)\le(v_1)^{(2)}$ | 390 |
| In the same manner , we get<br><br>$v^{(2)}(t)\le\frac{(\bar{v}_1)^{(2)}+(\bar{C})^{(2)}(\bar{v}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(2)}=\frac{(\bar{v}_1)^{(2)}-(v_0)^{(2)}}{(v_0)^{(2)}-(\bar{v}_2)^{(2)}}}$ | 391 |
| From which we deduce $(v_0)^{(2)}\le v^{(2)}(t)\le(\bar{v}_1)^{(2)}$ | 392 |
| If $0<(v_1)^{(2)}<(v_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}<(\bar{v}_1)^{(2)}$ we find like in the previous case,<br><br>$(v_1)^{(2)}\le\frac{(v_1)^{(2)}+(C)^{(2)}(v_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_2)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_2)^{(2)}\right)t\right]}}\le v^{(2)}(t)\le$<br><br>$\frac{(\bar{v}_1)^{(2)}+(\bar{C})^{(2)}(\bar{v}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}\le(\bar{v}_1)^{(2)}$ | 393 |
| If $0<(v_1)^{(2)}\le(\bar{v}_1)^{(2)}\le(v_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(v_1)^{(2)}\le v^{(2)}(t)\le\frac{(\bar{v}_1)^{(2)}+(\bar{C})^{(2)}(\bar{v}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}\le(v_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $v^{(2)}(t)$ :-<br><br>$(m_2)^{(2)}\le v^{(2)}(t)\le(m_1)^{(2)}$, $\boxed{v^{(2)}(t)=\frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :- | 396 |

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| $(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| From which one obtains<br><br>For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$<br><br>$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)} + (C)^{(3)}(\nu_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}$ , $\boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\nu_2)^{(3)}}}$<br><br>it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$ | 400 |
| In the same manner , we get<br><br>$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)} + (C)^{(3)}(\bar{\nu}_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\bar{\nu}_2)^{(3)}}}$<br><br>**Definition of** $(\bar{\nu}_1)^{(3)}$ :-<br><br>From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$ | 401 |

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If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case,

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

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If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}'')^{(3)} = (a_{21}'')^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b_{20}'')^{(3)} = (b_{21}'')^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}')^{(4)} - (a_{25}')^{(4)} + (a_{24}'')^{(4)}(T_{25},t)\right) - (a_{25}'')^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_2)^{(4)}v^{(4)}-(a_{24})^{(4)}\right)\le\frac{dv^{(4)}}{dt}\le-\left((a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_4)^{(4)}v^{(4)}-(a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)},(\nu_0)^{(4)}$ :-

For $0<\boxed{(\nu_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}<(\nu_1)^{(4)}<(\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t)\ge\frac{(\nu_1)^{(4)}+(C)^{(4)}(\nu_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_0)^{(4)}\right)t\right]}}{4+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_0)^{(4)}\right)t\right]}}\quad,\quad\boxed{(C)^{(4)}=\frac{(\nu_1)^{(4)}-(\nu_0)^{(4)}}{(\nu_0)^{(4)}-(\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)}\le\nu^{(4)}(t)\le(\nu_1)^{(4)}$

In the same manner , we get 405

$$\nu^{(4)}(t)\le\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{4+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(4)}=\frac{(\bar{\nu}_1)^{(4)}-(\nu_0)^{(4)}}{(\nu_0)^{(4)}-(\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)}\le\nu^{(4)}(t)\le(\bar{\nu}_1)^{(4)}$

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If $0<(\nu_1)^{(4)}<(\nu_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}<(\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)}\le\frac{(\nu_1)^{(4)}+(C)^{(4)}(\nu_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}}{1+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}}\le\nu^{(4)}(t)\le$$

$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}\le(\bar{\nu}_1)^{(4)}$$

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If $0<(\nu_1)^{(4)}\le(\bar{\nu}_1)^{(4)}\le\boxed{(\nu_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)}\le\nu^{(4)}(t)\le\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}\le(\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$$(m_2)^{(4)}\le\nu^{(4)}(t)\le(m_1)^{(4)},\quad\boxed{\nu^{(4)}(t)=\frac{G_{24}(t)}{G_{25}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$$(\mu_2)^{(4)}\le u^{(4)}(t)\le(\mu_1)^{(4)},\quad\boxed{u^{(4)}(t)=\frac{T_{24}(t)}{T_{25}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}'')^{(5)} = (b_{29}'')^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}')^{(6)} - (a_{33}')^{(6)} + (a_{32}'')^{(6)}(T_{33}, t)\right) - (a_{33}'')^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

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In the same manner , we get

$$\nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\bar{\nu}_1)^{(6)}$

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If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

$$(\nu_1)^{(6)} \leq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_2)^{(6)}\right)t\right]}} \leq \nu^{(6)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\bar{\nu}_1)^{(6)}$$

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If $0 < (\nu_1)^{(6)} \leq (\bar{\nu}_1)^{(6)} \leq \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(\nu_1)^{(6)} \leq \nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

$$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}, \quad \boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}, \quad \boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)}G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a'_{36})^{(7)} - (a'_{37})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right) - (a''_{37})^{(7)}(T_{37},t)v^{(7)} - (a_{37})^{(7)}v^{(7)}$$

**Definition of** $v^{(7)}$ :- $\boxed{v^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)}\right) \le \frac{dv^{(7)}}{dt} \le -\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(7)}, (v_0)^{(7)}$ :-

For $0 < \boxed{(v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}} < (v_1)^{(7)} < (\bar{v}_1)^{(7)}$

$$v^{(7)}(t) \ge \frac{(v_1)^{(7)} + (C)^{(7)}(v_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(v_1)^{(7)} - (v_0)^{(7)}}{(v_0)^{(7)} - (v_2)^{(7)}}}$$

it follows $(v_0)^{(7)} \le v^{(7)}(t) \le (v_1)^{(7)}$

In the same manner , we get

$$v^{(7)}(t) \le \frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{v}_1)^{(7)} - (v_0)^{(7)}}{(v_0)^{(7)} - (\bar{v}_2)^{(7)}}}$$

From which we deduce $(v_0)^{(7)} \le v^{(7)}(t) \le (\bar{v}_1)^{(7)}$

If $0 < (v_1)^{(7)} < (v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}} < (\bar{v}_1)^{(7)}$ we find like in the previous case,

$$(v_1)^{(7)} \le \frac{(v_1)^{(7)} + (C)^{(7)}(v_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_2)^{(7)}\right)t\right]}} \le v^{(7)}(t) \le$$

$$\frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \le (\bar{v}_1)^{(7)}$$

If $0 < (v_1)^{(7)} \le (\bar{v}_1)^{(7)} \le \boxed{(v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$ , we obtain

$$(v_1)^{(7)} \le v^{(7)}(t) \le \frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \le (v_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

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**Definition of** $v^{(7)}(t)$ :-

$(m_2)^{(7)} \leq v^{(7)}(t) \leq (m_1)^{(7)}$, $\boxed{v^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :-

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$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $v^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)} G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

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$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right) - (a_{41}^{''})^{(8)}(T_{41}, t) v^{(8)} - (a_{41})^{(8)} v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)} v^{(8)} - (a_{40})^{(8)}\right) \leq \frac{dv^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)} v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$v^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)}-(\nu_0)^{(8)}}{(\nu_0)^{(8)}-(\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\bar{\nu}_8)^{(8)}$

If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case, 423

$$(\nu_1)^{(8)} \leq \frac{(\nu_1)^{(8)}+(C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}} \leq \nu^{(8)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\bar{\nu}_1)^{(8)}$$

If $0 < (\nu_1)^{(8)} \leq (\bar{\nu}_1)^{(8)} \leq \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain 424

$$(\nu_1)^{(8)} \leq \nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)}+(C)^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \leq \nu^{(8)}(t) \leq (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)}G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)}T_{41}(t)$ This is an important

consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{d\nu^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right) - (a''_{45})^{(9)}(T_{45},t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \le \frac{d\nu^{(9)}}{dt} \le -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \ge \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \le \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}} \le \nu^{(9)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \le (\bar{\nu}_1)^{(9)} \le \boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$ , we obtain

$$(\nu_1)^{(9)} \le \nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$(m_2)^{(9)} \leq \nu^{(9)}(t) \leq (m_1)^{(9)}$, $\boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}$, $\boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a''_{44})^{(9)} = (a''_{45})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b''_{44})^{(9)} = (b''_{45})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a''_i)^{(1)} and\ (b''_i)^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a'_{13})^{(1)}(a'_{14})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a'_{13})^{(1)}(a'_{14})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a'_{14})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b'_{13})^{(1)}(r_{14})^{(1)} - (b'_{14})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a''_i)^{(2)} and\ (b''_i)^{(2)}$ are independent on t , and the conditions with the notations 426

$(a'_{16})^{(2)}(a'_{17})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a'_{16})^{(2)}(a'_{17})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a'_{17})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b'_{16})^{(2)}(b'_{17})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

$(b'_{16})^{(2)}(b'_{17})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b'_{16})^{(2)}(r_{17})^{(2)} - (b'_{17})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ 430

$with\ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a''_i)^{(3)} and\ (b''_i)^{(3)}$ are independent on $t$ , and the conditions with the notations 431

$(a'_{20})^{(3)}(a'_{21})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$

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| $(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$ | 435 |

| | |
|---|---|
| $(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |

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| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |

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| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |

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| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(m) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) +$ | 488 |

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| $(a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first | 494 |

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| equations | |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a'_{16})^{(2)}+(a''_{16})^{(2)}(T^*_{17})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a'_{18})^{(2)}+(a''_{18})^{(2)}(T^*_{17})]}$ | 495 |
| **Definition and uniqueness of** $T^*_{21}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T^*_{21}$ for which $f(T^*_{21}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a'_{20})^{(3)}+(a''_{20})^{(3)}(T^*_{21})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a'_{22})^{(3)}+(a''_{22})^{(3)}(T^*_{21})]}$ | 496 |
| **Definition and uniqueness of** $T^*_{25}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T^*_{25}$ for which $f(T^*_{25}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a'_{24})^{(4)}+(a''_{24})^{(4)}(T^*_{25})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a'_{26})^{(4)}+(a''_{26})^{(4)}(T^*_{25})]}$ | 497 |
| **Definition and uniqueness of** $T^*_{29}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T^*_{29}$ for which $f(T^*_{29}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a'_{28})^{(5)}+(a''_{28})^{(5)}(T^*_{29})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a'_{30})^{(5)}+(a''_{30})^{(5)}(T^*_{29})]}$ | 498 |
| **Definition and uniqueness of** $T^*_{33}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T^*_{33}$ for which $f(T^*_{33}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a'_{32})^{(6)}+(a''_{32})^{(6)}(T^*_{33})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a'_{34})^{(6)}+(a''_{34})^{(6)}(T^*_{33})]}$ | 499 |
| **Definition and uniqueness of** $T^*_{37}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T^*_{37}$ for which $f(T^*_{37}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a'_{36})^{(7)}+(a''_{36})^{(7)}(T^*_{37})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a'_{38})^{(7)}+(a''_{38})^{(7)}(T^*_{37})]}$ | 500 |

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| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{40}=\frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}=\frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 501 |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{44}=\frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}=\frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G)=(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)}-(b_{13})^{(1)}(b_{14})^{(1)}-$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G)+(b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G)=0$<br><br>Where in $G(G_{13},G_{14},G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*})=0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19})=(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)}-(b_{16})^{(2)}(b_{17})^{(2)}-$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19})+(b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19})=0$ | 503 |
| Where in $(G_{19})(G_{16},G_{17},G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*})=0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23})=(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)}-(b_{20})^{(3)}(b_{21})^{(3)}-$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23})+(b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23})=0$<br><br>Where in $G_{23}(G_{20},G_{21},G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*})=0$ | 505 |

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| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right] + (b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br><br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br><br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br><br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br><br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$ | |

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| $\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right]+(b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0$ , $T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0$ , $T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*}) = 0$ , $T_{21}^{*}$ given by $f(T_{21}^{*}) = 0$ and<br><br>$G_{20}^{*} = \frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*} = \frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br><br>$T_{20}^{*} = \frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*} = \frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br><br>$G_{25}^{*}$ given by $\varphi(G_{27}) = 0$ , $T_{25}^{*}$ given by $f(T_{25}^{*}) = 0$ and<br><br>$G_{24}^{*} = \frac{(a_{24})^{(4)}G_{25}^{*}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*} = \frac{(a_{26})^{(4)}G_{25}^{*}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*} = \frac{(b_{24})^{(4)}T_{25}^{*}}{[(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*} = \frac{(b_{26})^{(4)}T_{25}^{*}}{[(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}((G_{27})^{*})]}$ | 517 |

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| Obviously, these values represent an equilibrium solution of global equations | |
| Finally we obtain the unique solution<br><br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*}) = 0$ , $T_{29}^{*}$ given by $f(T_{29}^{*}) = 0$ and<br><br>$G_{28}^{*} = \frac{(a_{28})^{(5)} G_{29}^{*}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*} = \frac{(a_{30})^{(5)} G_{29}^{*}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |
| $T_{28}^{*} = \frac{(b_{28})^{(5)} T_{29}^{*}}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)} T_{29}^{*}}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br><br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br><br>$G_{32}^{*} = \frac{(a_{32})^{(6)} G_{33}^{*}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)} G_{33}^{*}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)} T_{33}^{*}}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)} T_{33}^{*}}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br><br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br><br>$G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99 | 523 A |

$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and

$$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]} \quad , \quad G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$$

$$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47})^{*})]} \quad , \quad T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47})^{*})]}$$

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**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

**Proof:** Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-

$$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$$

$$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^{*}) = (q_{14})^{(1)} \quad , \frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^{*}) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*}\mathbb{T}_{14} \quad 525$$

$$\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*}\mathbb{T}_{14} \quad 526$$

$$\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*}\mathbb{T}_{14} \quad 527$$

$$\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*}\mathbb{G}_j\right) \quad 528$$

$$\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*}\mathbb{G}_j\right) \quad 529$$

$$\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*}\mathbb{G}_j\right) \quad 530$$

**ASYMPTOTIC STABILITY ANALYSIS** 531

**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-

$$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i \quad 532$$

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| $\frac{\partial(a_{17}^{''})^{(2)}}{\partial \mathrm{T}_{17}}(\mathrm{T}_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial(b_i^{''})^{(2)}}{\partial \mathrm{G}_j}((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{\mathrm{d}\mathbb{G}_{16}}{\mathrm{dt}} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}\mathrm{G}_{16}^{*}\mathbb{T}_{17}$ | 534 |
| $\frac{\mathrm{d}\mathbb{G}_{17}}{\mathrm{dt}} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}\mathrm{G}_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{\mathrm{d}\mathbb{G}_{18}}{\mathrm{dt}} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}\mathrm{G}_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{\mathrm{d}\mathbb{T}_{16}}{\mathrm{dt}} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}\mathrm{T}_{16}^{*}\mathbb{G}_j\right)$ | 537 |
| $\frac{\mathrm{d}\mathbb{T}_{17}}{\mathrm{dt}} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}\mathrm{T}_{17}^{*}\mathbb{G}_j\right)$ | 538 |
| $\frac{\mathrm{d}\mathbb{T}_{18}}{\mathrm{dt}} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}\mathrm{T}_{18}^{*}\mathbb{G}_j\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)}$ , $\frac{\partial(b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_j\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 547 |

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| **Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)}$ , $\frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^*) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ $and$ $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^*) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |

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| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^{*}\mathbb{G}_{j}\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^{*}\mathbb{G}_{j}\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^{*}\mathbb{G}_{j}\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ $and$ $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^{*}) = (q_{33})^{(6)}$ , $\frac{\partial(b_i^{''})^{(6)}}{\partial G_j}((G_{35})^{*}) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^{*}\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^{*}\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^{*}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^{*}\mathbb{G}_{j}\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^{*}\mathbb{G}_{j}\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^{*}\mathbb{G}_{j}\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ $and$ $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 572 |

| | |
|---|---|
| $\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^{*}) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^{*}\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^{*}\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^{*}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^{*}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^{*}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^{*}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^{*}) = (q_{41})^{(8)}$ , $\frac{\partial (b_i^{''})^{(8)}}{\partial G_j}\left((G_{43})^{*}\right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^{*}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^{*}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^{*}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^{*}\mathbb{G}_j\right)$ | 584 |

| | |
|---|---|
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a''_i)^{(9)}$ *and* $(b''_i)^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{45})^{(9)}}{\partial T_{45}}(T^*_{45}) = (q_{45})^{(9)} \ , \ \frac{\partial(b''_i)^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a'_{44})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G^*_{44}\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a'_{45})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G^*_{45}\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a'_{46})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G^*_{46}\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$<br><br>$\left[\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\Big]$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\Big)$<br><br>$+\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\Big)$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\Big)$<br><br>$\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$ | |

$$\left((\lambda)^{(1)}\right)^2+\left((b_{13}')^{(1)}+(b_{14}')^{(1)}-(r_{13})^{(1)}+(r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$$

$$+\left(\left((\lambda)^{(1)}\right)^2+\left((a_{13}')^{(1)}+(a_{14}')^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$$

$$+\left((\lambda)^{(1)}+(a_{13}')^{(1)}+(p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^{*}+(a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^{*}\right)$$

$$\left((\lambda)^{(1)}+(b_{13}')^{(1)}-(r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^{*}+(b_{14})^{(1)}s_{(13),(15)}T_{13}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(2)}+(b_{18}')^{(2)}-(r_{18})^{(2)}\}\{\left((\lambda)^{(2)}+(a_{18}')^{(2)}+(p_{18})^{(2)}\right)$$

$$\left[\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)\Big]$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(17)}\mathrm{T}_{17}^{*}\Big)$$

$$+\left(\left((\lambda)^{(2)}+(a_{17}')^{(2)}+(p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}_{16}^{*}+(a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}\right)$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(16)}\mathrm{T}_{16}^{*}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((b_{16}')^{(2)}+(b_{17}')^{(2)}-(r_{16})^{(2)}+(r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left(\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\}\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\Big]$$

$$\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\Big)$$

$$+\left(\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\right)$$

$$\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\Big)$$

$$\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((b_{20}^{'})^{(3)}+(b_{21}^{'})^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}^{'})^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}^{'})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left((\lambda)^{(5)}\right)^2+\left((b_{28}')^{(5)}+(b_{29}')^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$+\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}')^{(5)}+(a_{29}')^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}')^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}')^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}')^{(6)}-(r_{34})^{(6)}\{\left((\lambda)^{(6)}+(a_{34}')^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left((\lambda)^{(6)}+(a_{32}')^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\Big]$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\Big)$$

$$+\left((\lambda)^{(6)}+(a_{33}')^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\Big)$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((a_{32}')^{(6)}+(a_{33}')^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((b_{32}')^{(6)}+(b_{33}')^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$+\left(\left((\lambda)^{(6)}\right)^2+\left((a_{32}')^{(6)}+(a_{33}')^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}')^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b_{38}')^{(7)}-(r_{38})^{(7)}\{\left((\lambda)^{(7)}+(a_{38}')^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left((\lambda)^{(7)}+(a_{36}')^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\Big]$$

$$\left((\lambda)^{(7)}+(b_{36}')^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\Big)$$

$$+\left((\lambda)^{(7)}+(a_{37}')^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\Big)$$

$$\left((\lambda)^{(7)}+(b_{36}')^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a_{36}')^{(7)}+(a_{37}')^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b_{36}')^{(7)}+(b_{37}')^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left(\left((\lambda)^{(7)}\right)^2+\left((a_{36}')^{(7)}+(a_{37}')^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}')^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left((\lambda)^{(7)}+(b_{36}')^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b_{42}')^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a_{42}')^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a_{40}')^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(q_{40})^{(8)}G_{40}^{*}\right)\right]$$

$$\left((\lambda)^{(8)}+(b_{40}')^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(41)}T_{41}^{*}\Big)$$

$$+\left(\left((\lambda)^{(8)}+(a_{41}')^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G_{40}^{*}+(a_{40})^{(8)}(q_{41})^{(8)}G_{41}^{*}\right)$$

$$\left((\lambda)^{(8)}+(b_{40}')^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(40)}T_{40}^{*}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((a_{40}')^{(8)}+(a_{41}')^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b_{40}')^{(8)}+(b_{41}')^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left(\left((\lambda)^{(8)}\right)^2+\left((a_{40}')^{(8)}+(a_{41}')^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a_{40}')^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G_{40}^{*}\right)$$

$$\left((\lambda)^{(8)}+(b_{40}')^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(42)}T_{40}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b_{46}')^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a_{46}')^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a_{44}')^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G_{45}^{*}+(a_{45})^{(9)}(q_{44})^{(9)}G_{44}^{*}\right)\right]$$

$$\left((\lambda)^{(9)}+(b_{44}')^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T_{45}^{*}+(b_{45})^{(9)}s_{(44),(45)}T_{45}^{*}\Big)$$

$$+\left(\left((\lambda)^{(9)}+(a_{45}')^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G_{44}^{*}+(a_{44})^{(9)}(q_{45})^{(9)}G_{45}^{*}\right)$$

$$\left((\lambda)^{(9)}+(b_{44}')^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T_{45}^{*}+(b_{45})^{(9)}s_{(44),(44)}T_{44}^{*}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left(\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45}+(b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt}= A_i\ (x)\ xi$ where $x_i \in R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i \ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i \ X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions

are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$(13) \qquad \frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17].

In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 13. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon$ $C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \ i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega\ (U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \epsilon C :\ \omega\ (x) \neq \emptyset \text{ and } \omega\ (x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s\ (B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i\ (x)$ is continuous

**A2:** $A_i\ (x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma\ (A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population $i$. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility

assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C\backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^n_i$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\in \Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two

propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population *i* is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) In \ |B_i \ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^t_0 (A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = limsup_{t\to\infty} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni}= E_{i(x)} \ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \ \leq \ a| \ B_i(t,x)u_i(x)|+ \ |B_i(t,x)w| \ \ \leq \ | \ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)| \ w|)$. Hence, $limsup_{t\to\infty} \frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$limsup_{t\to\infty} \frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\in$ $C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \ \geq \ a|B_i(t,x)u_i(x)|-|B_i(t,x)w|$$

$$\geq | \ B_i(t,x)u_i(x)|(a- \alpha+ exp\ (-\beta t)| \ w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \ \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \ \geq \ \lambda i(x) \geq \ \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) \ = \ In|B_i(t+s,x)\ u_i(x)|= In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

| |
|---|
| $= \text{In } \lvert B_i(s,x.t)\, u_i(x.t) \rvert \lvert B_i(t,x)\, u_i(x) \rvert$ |
| $= \text{In } b_i(s,t.x)\, b_i(t,x)$ |
| |
| Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies |
| $\frac{d}{dt} b_i(t,x) = \lim_{s \to 0} \frac{bi(t+s,x-bi(t,x)}{S-}$ |
| $= \lim_{s \to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}\lvert_{s=0}\, b_i(s,x.t)$ |
| = |
| $=(A_i(x.t)u_i(x.s),u_i(x.s),$ |
| The Fundamental Theorem of Calculus implies |
| $B_i(t,x) = \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$ |
| and the second assertion follows. |
| **Proposition 2:** *For the solutions of (2) we have:* if $x_i > 0$, then $\lambda i(x) \leq 0$. If $\lambda i(x) < 0$, then $\lim_{t \to \infty} (x.t)_i = 0$ If $\limsup_{t \to \infty}$ $(x.t)_i \rvert > 0$, then $\lambda i(x) = 0$. |
| **Proof:** First, assume that x satisfies $x_i > 0$. Since the semi flow is dissipative, there exists $\gamma > 0$ such that $\lvert x.t \rvert \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply |
| $\lambda i(x) = \limsup_{t \to \infty} \frac{1}{t} \text{In } \lvert B_i(t,x)x_i \rvert = \limsup_{t \to \infty} \frac{1}{t} \text{In } \lvert (x.t)_i \rvert$ |
| $\leq \limsup_{t \to \infty} \frac{In\, \gamma}{t} = 0$ |
| To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i = 0$, then the invariance of the faces of C imply that $(x.t)_i = 0$ for all $t \geq 0$. Alternatively, if $x_i > 0$, then Proposition I and the skew product flow imply |
| $\limsup_{t \to \infty} \frac{1}{t} In \lvert (x.t)_i \rvert = \lambda i(x) < 0.$ |
| The final assertion follows from the first two assertions. 5 |
| |

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on $\boldsymbol{C}$, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:*$\lambda_i(\mathrm{x}) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle \mathrm{A}_i(x.s)\mathrm{u}_i(x.s), \mathrm{u}_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(\mathrm{x}) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $\mathrm{h}_i(x) = \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I}\boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I}\boldsymbol{C}_i^+$such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following

equivalent conditions hold For all invariant measures $\mu$ supported by $K$ (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by $K$.

then $K$ is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and $K$ is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by $K$.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x)u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$. Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large

so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \leq \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \leq \in$, and $|h(x.t) - h(x^n.t)| \leq \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$$\leq \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$$

$$\leq \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$$

$$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$$

$$\leq \quad 3 \in$$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t) d\mu(x) = \int_c h(x) d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x) u_i^n(x), u_i^n(x) \rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop. 3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop. 3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)| d\mu(x) + |\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\, d\mu_n(x)$$

$$\leq \quad 2 \in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm. 2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}$

$\int_0^t \delta_{y.s}$ ds where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$ . Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}_{n=1}^{\infty}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

| |
|---|
| Min max $\lambda i$ ($\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_{i} Pi\lambda i(\mu)$ |
| |
| Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that |
| (6) min max $\lambda i(\mu) = \min_{\mu} \max_{p\epsilon\Delta} \sum_{i} Pi\lambda i(\mu)$ |
| |
| Where the minimum is taken over invariant probability measures $\mu$ with support in *K.* Since min$\mu$ |
| $\sum_{i} P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established. |
| |
| To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta -$perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ [0,1]$be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by |
| |
| $\tilde{A}_i(x) = \{\frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i\epsilon I}{if\ i \notin I}$ |
| |
| Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of |

$\dot{x} = \tilde{A}(x)x$ *and* $\lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\} \backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). *Since* $E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently *K* is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent*. Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume x* $\mapsto$ *A(x) is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely*

*ergodic, or There exists j Є { 1,….,k} such that* $M_i$ ----- $\{ x \in C : x_l \gg 0 \text{ for } l \neq j \}$ *Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by* $U_i M_i$ .

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----{1,…,k} such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures* $\mu$ *with support in* $\partial$C, $\lambda_i(\mu) > 0$ *for all* $I \in \{1,\ldots,k\} \setminus supp(\mu)$ *(2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1, \ldots, x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda$ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0 < e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j \ (x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1^1, \ldots, E_1^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i_2(E_1{}^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1) > 0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1) > 0$ and $\max_j f_2^j(E^j_2) > 0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0, E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is

sufficiently slow (i.e. $0< e^{j}{}_{i}\ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j (E^j{}_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1-x_1^j - \beta^j\ x_2^j\ - \alpha^j\ x_3^j\ ) + \sum d_1^{jk}\ \ x_1^k\ - e_1^j\ x_1^j$$

$$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j\ x_1^j\ - x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_2^{jk}\ \ x_2^k\ - e_2^j\ x_2^j$$

$$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j\ x_1^j\ - \alpha^j x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_3^{jk}\ \ x_3^k\ - e_3^j\ x_3^j$$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all *j*. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j{}_i = \sum_l d^{l,j}_i$ for all *i, j*) , the maximal invariant set in $\partial$C consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0< e^j{}_i\ll 1$ for all *i*, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2$ $(E_1)$and $\lambda_3$ $(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1 - \beta^j$. Consider the Morse decomposition of $\partial$C given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1> 0$ for *i* = 1, 2, 3, 0 is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j (1 - \beta^j)$$

Equivalently,

$$2>\min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j (r_i - \sum_s a_{is} x_s^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j , \qquad i = 1, \dots, k;\ j = 1, \dots, m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$ . We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set H= $\{x \in C: x_i^j = x_i^l \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \qquad i = 1, \dots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \dots, k$$

It has been shown [12,ch.16.1,16.2] That for $k$ =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there

is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\, \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within $\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as

well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that

there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{matrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{matrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has

abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left( \beta(1 - y - z)\frac{P}{\epsilon + P} - r - m(1 - y) - b(N)\ y \right)$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1 = N$, $x_2 = p$, $x_3 = (y,z)'$ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} - y_i)$$

With

$$(16)\quad F(x)_i = x_i g\,(x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\quad \begin{pmatrix} x_i \\ y_i\ dot \end{pmatrix} = \begin{pmatrix} -\beta & \beta \\ ag\,(x_i, x_i) - 1 & \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon

plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational

consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical

propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$ to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.37) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.38) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.39) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the

model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_{1,}$ $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_{i,}$ $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as* $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 13. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^{+}\text{x}\ k_{\ 2}\text{x}\ k_{\ 1}, R^{-})$,

$A_3$. $\varphi_i\ (\cdot\ , s) \in L^{\ 2}\ (R^{+}, R^{+})\ \cap C(R^{+}, R^{+})$ for each $s \in\ [-\tau, 0]$,

$i,\ = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i$: $R^{+}$ ▯$[-\tau, T] \rightarrow R^{+}$ $(i,\ = 1,\ 2,)$ With the following properties:

$S_1$. $P_i(\cdot\ , t) \in L^{\ 1}\ (R^{+}, R^{+})$ for each $t\ \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot\ ) \in C\ ([-\tau, T], R^{+})$ for each $a\ \in R^{+}$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^{+}$ ▯$\{R^{+} \cap\ [0, T]\}$ And is continuous $(i,\ = 1,\ 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,)\ \in R^{+}$ ▯$[-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$P_i(a + x, t + x) =\ \ P_i(x), (i,\ = 1,\ 2,)$ ,

Then (1.3) considered at $(1.3)$ considered at $(a + x,\ t + x)$ becomes

$$\frac{dp_{\ i}(x)}{dx} + f_i(a + x,\ P(t + x))\ \ P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0)\ \exp\left[-\int_0^{\infty} f_{\ i}(a + s, P(t + S))ds\right]$$

And hence

(2.1) $p_{\ 1}\ (a + x, t + x) = p_{\ 1}\ (a, t)\ exp\left[-\int_o^{x} f_{\ 1}\ (a + s, p(t + s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_{\ 1}\ (a, t) =\ \varphi_{\ 1}\ (a - t, 0)\exp$▯$\left[-\int_o^{t} f_{\ 1}\left(a - t + s, p_{\ 1}\ (s)\right) ds\right]$

$O \leq t \leq a, i = 1,2,$

With

$P_1(s)=(p_1(s),p_2(s-t)),$

$P_2(s)=(p_1(s-t),p_2(s)),$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$t > a \geq, i = 1,2,$

Where

(2.4) $B_i(t) = p_i(0, t), i = 1,2, t > 0.$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$

(2.5) $L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(a+s, p_i(s))ds]$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da + \int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$t > 0, i = 1,2,$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.(a, t, P_i(t))da$

$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$

$i = 1, 2, t > 0$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t) \phi(0) d_i^*] t \quad for \ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T],R^+), \quad B_i \in C([0,T],R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau,T] = \{f \in C([-\tau,T],R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau,T]$. Then for any

$(x, y) \in C^+[-\tau,T] \times C^+[-\tau,T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau,T], \quad C_T^2 = C^+[-\tau,T] \times C^+[-\tau,T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau,T]$ *and* $t \in [0,T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Phi: C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Phi(P_t) - \Phi_t\|_{t*} \le \begin{cases} \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \\ t^* \end{cases}$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ and $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \leq \int_0^t B_1^{(1)}(a)\left\{M_1\left(a,t,P_t^{(1)}\right) - M_1\left(a,t,P_t^{(2)}\right)\right\}da$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \text{ for } x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_t ds\right)da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o)\left[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds\right]da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \|_t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$=\int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\mathrm{X}[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \, /\!/ P_t^{(1)} - P_t^{(2)} /\!/ \text{ for } a \, \varepsilon[o, t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \leq \beta_1 \Phi_1(0) ||P_t^{(1)} - P_t^{(2)}||_t^* + \Phi_1(0) \beta_1 d_1 ||P_t^{(1)} - P_t^{(2)}||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) \, \mathrm{e}^{\delta 1^a} \beta_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 \mathrm{d}_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*}(a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) \, \mathrm{e}^{\delta 1^a} /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*}\{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| \mathrm{B}_1^{(1)}(\sigma) - \mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{B}_1^{*\mathrm{d}}\sigma$$

$$+ \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \left\{ B_1 + B_2 d_1 a + B_1^* \left( B_2 a + d_1 a^2/2 \right) e^{\delta 1^a} \right\}$$

$$\leq \beta_1^* \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| do + \left( r^* + /\!/ \Phi_t /\!/_{t*} \right) \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \rightarrow \beta_1$ as $a \rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq \left( r^* + /\!/ \Phi_t /\!/_{t*} \right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{ g_1(a) + 0(a) \}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a)) - b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a)) do \right|$$

$$\left| B_1^* \left( r^* + /\!/ \Phi_t /\!/_{t*} \right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} 0(a) as\ a \rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1(P_t^{(2)}) \right| \leq ((r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2(P_t^{(2)}) \right|$ leads to

$$\| \pi(P_t^{(1)} - \pi(P_t^{(2)}) /\!/ \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^*)$$

As $t^* \rightarrow o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

(3.1) $\frac{dp\,P_2^*(a)}{da}+f_i(\mathrm{a}, P_1^*, P_2^*)\,P_i^*(a) = 0,\ \ i = 1,2, a > 0,$

(3.2) $P_i^* = \int_o^\infty P_i^*(a)da, i = 1,\ 2,$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i = 1.\ \ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s, P_1^*, P_2^*)ds\right], i = 1,\quad 2, a \geq 0,$

Which together with (3.3) lead to

$1=\int_o^\infty b_1(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_1(s, P_1^*, P_2^*)ds\right]da,$

(3.5) $1=\int_o^\infty b_2(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_2(s, P_1^*, P_2^*)ds\right]da$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1, P_2) = \int_o^\infty b_i(a, P_1, P_2)\exp\left[-\int_o^\infty f_i(s, P_1, P_2)ds\right]da,$

$i$ =1, 2, $P_1 \geq 0$, $P_2 \geq 0$,

**And suppose the following hold:**

(3.7) $A_{4.}$ $F_{i.}\ (0,\ 0) > 1,\ i = 1,2$:

$A_{5.}$ *there exists a positive constant c for which*

$F_1$(C,0)< $F_{2.}$(C,0),

(3.8) $F_1$(0.C)< $F_{2.}$(0,C),

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $F_1(P_1^*, P_2^*) = 1 = F_2(P_1^*, P_2^*)$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a) = [P_2^* \exp[-f_i(s, P_1^*, P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s, P_1^*, P_2^*)ds]da)$,

$i$ =1, 2, $a \geq 0$,

Proof Consider the elements of the surfaces defined by

Z= $F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0$,

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

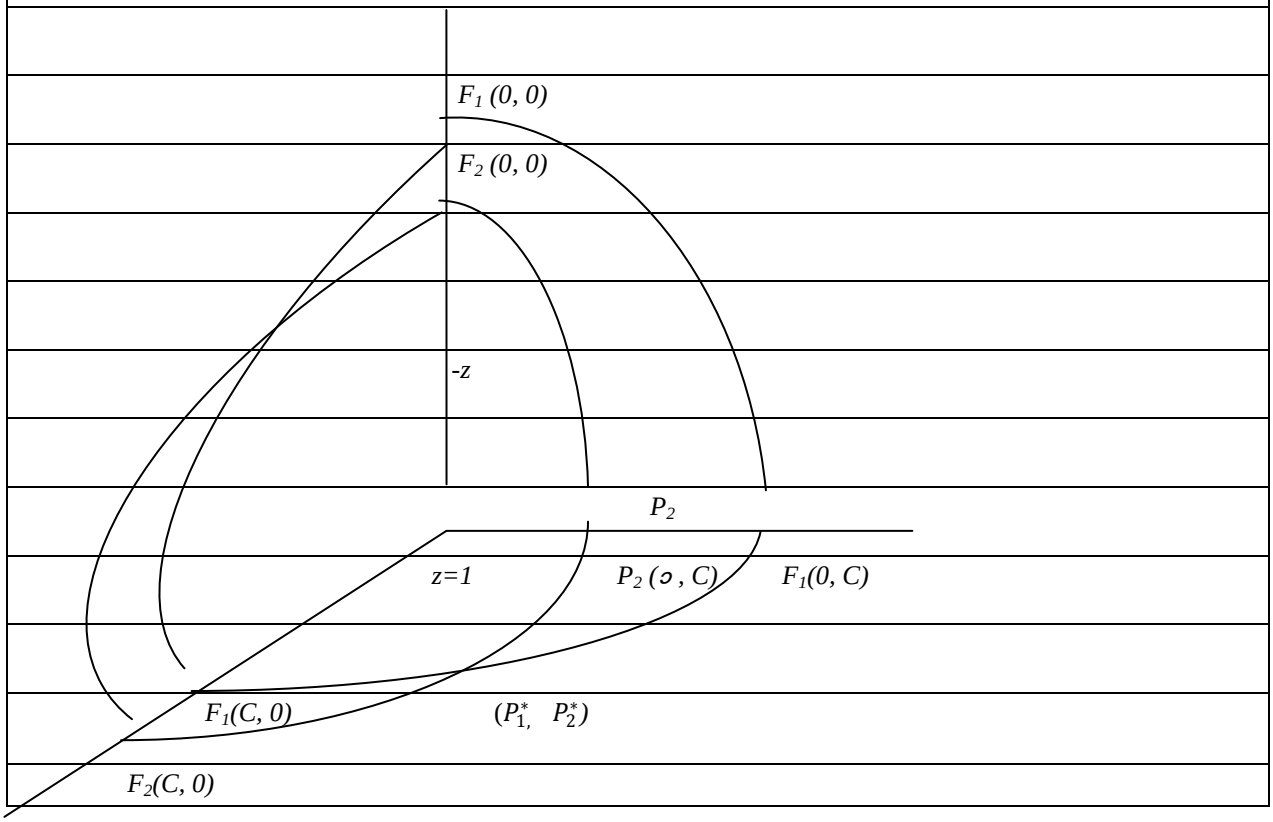
$P_1 \geq 0, P_2 \geq 0$, $i$ =1, 2,

$j$ =1, 2, Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, F_1(P_1, P_2 \geq 0$,

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.



(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11)
$$\begin{cases} p_i(a,t) = p^*_i(a) + u_i(a,t) \\ p_i(t) = p^*_i + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_i} p_i(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial b_i}{\partial p^*_i} p_i(t)\right\} da \end{cases}$$

Where $i = 1,2$ and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_i(a) \exp[\lambda t], \quad u_i(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty u_i(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{du_i}{da} + \lambda u_i = -f_i\left(a, p^*_1, p^*_2\right) u_i(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j} p^*_i\right\},$$

(3.15)

$$\phi_i(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\phi_i(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial b_i}{\partial p_j^*}\ p_i^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*,\ \exp\{-\lambda\tau\},\ p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\phi_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1\left(a, p_1^*, p_2^*\right) - f_1\left(a, p_1^*, p_2^*\right)\right]\phi_i(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2\left(a, p_1^*, p_2^*\right) - f_2\left(a, p_1^*, p_2^*\right)\right]\phi_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\phi_i(a) = \left\{\phi_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\right\}$$

$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\phi_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\,e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)\,e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\Phi_1(a)$ and $\Phi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$\left(\lambda - A_{11}\right) p_1^* - A_{12} p_2^* \exp\left(-\lambda\tau\right) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21} p_1^* \exp\left(-\lambda\tau\right) + \left(\lambda - A_{22}\right) p_2^* = g_{21}(\lambda) p_1^* \exp\left(-\lambda\tau\right) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $$A_{i\Phi} = \int_0^\infty p_i^*(a) \left[\frac{\partial b_i}{\partial p_\Phi^*} - \frac{\partial f_i}{\partial p_\Phi^*}\right] da,$$

(3.23) $$g_{i\Phi}(\lambda) = \left[p_i^*(0) c_i\right]^{-1} \left\{\int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} da\right\} \Phi \left\{\int_0^\infty \left[p_i^*(a) \frac{\partial b_i}{\partial p_\Phi^*} - b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_\Phi^*} e^{\lambda s} ds\right)\right] da\right\}$$

$$- \int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_\Phi^*} e^{\lambda s} ds\right) da, \quad i, \Phi = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i\left(a, p_1^*, p_2^*\right) - f_i\left(a, p_1^*, p_2^*\right), \quad i, = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $\left(p_1^*, p_2^*\right)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et \begin{bmatrix} \lambda - \left(A_{11} + g_{11}(\lambda)\right) & -\left(A_{12} + g_{12}\right) e^{-\lambda\tau} \\ -\left(A_{21} + g_{21}\right) e^{-\lambda\tau} & \lambda - \left(A_{22} + g_{22}(\lambda)\right) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda\left(A_{11} + A_{22}\right) + A_{11}A_{22} - A_{12}A_{21} \exp\left[-2\lambda\tau\right] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = \left[A_{21} g_{12}(\lambda) + A_{12} g_{21}(\lambda) + g_{12}(\lambda) g_{21}(\lambda)\right] \exp\left[-2\lambda\tau\right]$

$$-A_{11} g_{22}(\lambda) - A_{22} g_{11}(\lambda) - g_{11}(\lambda) g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ =1,\ 2.)\ \ (see(A_1)\ and\ (A_2))$;

$A_7.\ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = |A_{11}\, g_{12}(0)| + |A_{12}\, g_{21}(0)| + |g_{12}(0)\, g_{21}(0)|$$

$$+ |A_{11}\, g_{22}(0)| + |A_{22}\, g_{11}(0)| + |g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$$F_1(\lambda, \tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau];$$

$$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau),$$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$, $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \Big|$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2,$$

$$q = \left\{ 4 |(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| | - |A_{11} + A_{22} + \| S_1(0) \| |^2 \right\}/2,$$

It will now follow, from (A) and (3.26), $F_1(\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions

of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends,

assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner

wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the

controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## SECTION A(8)-2

## CME And Related Quantum Phenomena

### INTRODUCTION—VARIABLES USED

**Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

(1) As a result, the CME and related (e&eb) quantum phenomena affect (e&eb) the hydrodynamical and (e&eb) transport behavior of strongly coupled (e&eb) quark–gluon plasma, and can be studied in relativistic heavy ion collisions where (e) strong magnetic fields are created by (e) the colliding

ions.

(2) Evidence for the CME and related phenomena has been reported by (e) the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN.

**(3)** The goal of the present review is to provide (eb) an elementary introduction into (e&eb) the physics of anomalous chiral effects, to describe (eb) the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate (e) the existing uncertainties in (eb) the interpretation of the data.

**(4)** Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

**NOTATION**

**Module One**

As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled (e&eb) quark–gluon plasma, and can be studied in relativistic heavy ion collisions where (e) strong magnetic fields are created by (e) the colliding ions

$G_{13}$ : Category one of **CME and related quantum phenomena affect the hydrodynamical and transport behavior** quark–gluon plasma, and can be studied in relativistic heavy ion collisions where (e) strong magnetic fields are created by (e) the colliding ions**;**

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of quark–gluon plasma, and can be studied in relativistic heavy ion collisions where (e) strong magnetic fields are created by (e) the colliding ions; **CME and related quantum phenomena affect the hydrodynamical and transport behavior**

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where (e) strong magnetic fields are created by (e) the colliding ions

$G_{16}$ : Category one of strong magnetic fields are created by (e) the colliding ions

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

$T_{16}$ : Category one of CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions

$T_{17}$ : Category two of SAS

| | |
|---|---|
| $T_{18}$ : Category three of SAS | |
| **Module three**<br><br>As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by (e) the colliding ions | |
| $G_{20}$ : Category one of colliding ions<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>The goal of the present review is to provide (eb) an elementary introduction into the physics of anomalous chiral effects, to describe (eb) the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate (e) the existing uncertainties in (eb) the interpretation of the data. | |
| $G_{24}$ : Category one of physics of anomalous chiral effects,<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate (e) the existing uncertainties in (eb) the interpretation of the data<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate (e) the existing uncertainties in (eb) the interpretation of the data. | |
| $G_{28}$ : Category one of existing uncertainties in (eb) the interpretation of the data.<br><br>$G_{29}$ : Category two of SAS | |

| | |
|---|---|
| $G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>Keywords Chiral magnetic effect; **Chiral vortical effect**; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved | |
| $G_{32}$ : Category one of Chiral magnetic effect; **Chiral vortical effect**<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of **Chiral vortical effect** ;Chiral magnetic effect<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven** | |
| $G_{36}$ : Category one of **Chiral vortical effect;** Chiral anomaly<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of Chiral anomaly **;Chiral vortical effect**<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight** | |
| $G_{40}$ : Category one of Chiral anomaly; **Chiral magnetic effect**<br>$G_{41}$ : Category two of SAS<br>$G_{42}$ : Category three of SAS | |

| | |
|---|---|
| $T_{40}$ : Category one of **Chiral magnetic effect** ;Chiral anomaly<br>$T_{41}$ : Category two of SAS<br>$T_{42}$ : Category three of SAS | |
| **Module Nine** | |
| $G_{44}$ : Category one of **Chiral magnetic effect;** Quark-gluon plasma<br>$G_{45}$ : Category two of SAS<br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of Quark-gluon plasma **;Chiral magnetic effect**<br>$T_{45}$ : Category two of SAS<br>$T_{46}$ : Category three of SAS | |

## SECTION A(9)

## Fermi Gases In One-Dimension

**INTRODUCTION—VARIABLES USED**

**Universal Properties of Fermi Gases in One-dimension Wen-Bin He, Yang-Yang Chen, Shizhong Zhang, Xi-Wen Guan (Submitted on 23 Mar 2016)**

(1) In this Letter, we investigate the universal properties of (e) a spin-polarized two-component Fermi gas in one dimension (1D) using Bethe ansatz. **Wen-Bin He, Yang-Yang Chen, Shizhong Zhang, Xi-Wen Guan** discuss the quantum phases and (e&eb) phase transitions by obtaining exact results for (e) the equation of states, the contact, the magnetic susceptibility and (e&eb) also the contact susceptibility throughout (e&eb) the one-dimensional analogue of the Bose-Einstein condensate and (e&eb) Bardeen-Cooper-Schrieffer crossover.

(2) In particular, authors obtain the exact form of (e) the magnetic susceptibility $\chi \sim 1/T\sqrt{\exp}(-\Delta/T)$ at low temperatures, where (e) $\Delta$ is the energy gap and T is the temperature.

(3) Moreover, we establish exact upper and (e&eb) lower bounds for (e) the relation between polarization P and (e&eb) the contact C for both repulsive and (e&eb) attractive Fermi gases.

**(4)** Findings shed (eb) new light on the role of (e) the pair fluctuations in strongly interacting (e&eb) one-dimensional fermion system that can have implications for (e&eb) higher dimensions. Subjects: Quantum Gases (cond-mat.quant-gas) Cite as:arXiv: 1603.07207 [cond-mat.quant-gas] (or arXiv: 1603.07207v1 [cond-mat.quant-gas] for this version)

**NOTATION**

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| **Module One**<br>In this Letter, we investigate the universal properties of (e) a spin-polarized two-component Fermi gas in one dimension (1D) using Bethe ansatz. **Wen-Bin He, Yang-Yang Chen, Shizhong Zhang, Xi-Wen Guan** discuss the quantum phases and (e&eb) phase transitions by obtaining exact results for (e) the equation of states, the contact, the magnetic susceptibility and (e&eb) also the contact susceptibility throughout (e&eb) the one-dimensional analogue of the Bose-Einstein condensate and (e&eb) Bardeen-Cooper-Schrieffer crossover | |
| $G_{13}$ : Category one of spin-polarized two-component Fermi gas in one dimension (1D) using Bethe ansatz. **Wen-Bin He, Yang-Yang Chen, Shizhong Zhang, Xi-Wen Guan** discuss the quantum phases and (e&eb) phase transitions by obtaining exact results for (e) the equation of states, the contact, the magnetic susceptibility and (e&eb) also the contact susceptibility throughout (e&eb) the one-dimensional analogue of the Bose-Einstein condensate and (e&eb) Bardeen-Cooper-Schrieffer crossover<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of universal properties<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>In this Letter, we investigate the universal properties of a spin-polarized two-component Fermi gas in one dimension (1D) using Bethe ansatz.<br>**Wen-Bin He, Yang-Yang Chen, Shizhong Zhang, Xi-Wen Guan** discuss the quantum phases and (e&eb) phase transitions by obtaining exact results for (e) the equation of states, the contact, the magnetic susceptibility and (e&eb) also the contact susceptibility throughout (e&eb) the one-dimensional analogue of the Bose-Einstein condensate and (e&eb) Bardeen-Cooper-Schrieffer crossover | |
| $G_{16}$ : Category one of **quantum phases;** phase transitions by obtaining exact results for (e) the equation of states, the contact, the magnetic susceptibility and (e&eb) also the contact susceptibility throughout (e&eb) the one-dimensional analogue of the Bose-Einstein condensate and (e&eb) Bardeen-Cooper-Schrieffer crossover<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of phase transitions by obtaining exact results for (e) the equation of states, the contact, the magnetic susceptibility and (e&eb) also the contact susceptibility throughout (e&eb) the one-dimensional analogue of the Bose-Einstein condensate and (e&eb) Bardeen-Cooper-Schrieffer crossover; **quantum phases**<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>**Wen-Bin He, Yang-Yang Chen, Shizhong Zhang, Xi-Wen Guan** discuss the quantum phases and phase | |

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| transitions by obtaining exact results for (e) the equation of states, the contact, the magnetic susceptibility and (e&eb) also the contact susceptibility throughout (e&eb) the one-dimensional analogue of the Bose-Einstein condensate and (e&eb) Bardeen-Cooper-Schrieffer crossover | |
| $G_{20}$ : Category one of equation of states, the contact, the magnetic susceptibility and (e&eb) also the contact susceptibility throughout (e&eb) the one-dimensional analogue of the Bose-Einstein condensate and (e&eb) Bardeen-Cooper-Schrieffer crossover<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of quantum phases and phase transitions by obtaining exact results<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>**Wen-Bin He, Yang-Yang Chen, Shizhong Zhang, Xi-Wen Guan** discuss the quantum phases and phase transitions by obtaining exact results for the equation of states, the contact, the magnetic susceptibility and (e&eb) also the contact susceptibility throughout (e&eb) the one-dimensional analogue of the Bose-Einstein condensate and (e&eb) Bardeen-Cooper-Schrieffer crossover | |
| $G_{24}$ : Category one of **quantum phases and phase transitions by obtaining exact results for the equation of states, the contact, the magnetic susceptibility;** contact susceptibility throughout (e&eb) the one-dimensional analogue of the Bose-Einstein condensate and (e&eb) Bardeen-Cooper-Schrieffer crossover<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of contact susceptibility throughout (e&eb) the one-dimensional analogue of the Bose-Einstein condensate and (e&eb) Bardeen-Cooper-Schrieffer crossover **;quantum phases and phase transitions by obtaining exact results for the equation of states, the contact, the magnetic susceptibility**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>**Wen-Bin He, Yang-Yang Chen, Shizhong Zhang, Xi-Wen Guan** discuss the quantum phases and phase transitions by obtaining exact results for the equation of states, the contact, the magnetic susceptibility and also the contact susceptibility throughout (e&eb) the one-dimensional analogue of the Bose-Einstein condensate and (e&eb) Bardeen-Cooper-Schrieffer crossover | |
| $G_{28}$ : Category one of **quantum phases and phase transitions by obtaining exact results for the equation of states, the contact, the magnetic susceptibility and also the contact susceptibility** one-dimensional analogue of the Bose-Einstein condensate and (e&eb) Bardeen-Cooper-Schrieffer crossover**;** | |

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| $G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of one-dimensional analogue of the Bose-Einstein condensate and (e&eb) Bardeen-Cooper-Schrieffer crossover **;quantum phases and phase transitions by obtaining exact results for the equation of states, the contact, the magnetic susceptibility and also the contact susceptibility**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>**Wen-Bin He, Yang-Yang Chen, Shizhong Zhang, Xi-Wen Guan** discuss the quantum phases and phase transitions by obtaining exact results for the equation of states, the contact, the magnetic susceptibility and also the contact susceptibility throughout the one-dimensional analogue of the Bose-Einstein condensate and (e&eb) **Bardeen-Cooper-Schrieffer crossover** | |
| $G_{32}$ : Category one of quantum phases and phase transitions by obtaining exact results for the equation of states, the contact, the magnetic susceptibility and also the contact susceptibility throughout the one-dimensional analogue of the Bose-Einstein condensate; **Bardeen-Cooper-Schrieffer crossover**<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of **Bardeen-Cooper-Schrieffer crossover;** quantum phases and phase transitions by obtaining exact results for the equation of states, the contact, the magnetic susceptibility and also the contact susceptibility throughout the one-dimensional analogue of the Bose-Einstein condensate<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>In particular, authors obtain the exact form of the magnetic susceptibility $\chi \sim 1/T\sqrt{\exp}(-\Delta/T)$ at low temperatures, where (e) $\Delta$ is the energy gap and T is the temperature | |
| $G_{36}$ : Category one of $\Delta$ is the energy gap and T is the temperature<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of exact form of the magnetic susceptibility $\chi \sim 1/T\sqrt{\exp}(-\Delta/T)$ at low temperatures,<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |

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| **Module eight**<br><br>$\chi \sim 1/T\sqrt{\exp}\,(-\Delta/T)$ at low temperatures, | |
| $G_{40}$ : Category one of LHS of the above equation<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of RHS of the above equation<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Moreover, we establish exact upper and lower bounds for (e) the relation between polarization P and (e&eb) the contact C for both repulsive and (e&eb) attractive Fermi gases | |
| $G_{44}$ : Category one of **exact upper and lower bounds;** relation between polarization P and (e&eb) the contact C for both repulsive and (e&eb) attractive Fermi gases<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of relation between polarization P and (e&eb) the contact C for both repulsive and (e&eb) attractive Fermi gases **;exact upper and lower bounds**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION A(10)**<br><br>**Correlated Ultra-Cold Bosonic Many-Body Systems** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Chapter High Performance Computing in Science and Engineering ´15 pp 23-49 Date: 06 February 2016MCTDHB Physics and Technologies: Excitations and Vorticity, Single-Shot Detection, Measurement of Fragmentation, and Optimal Control in Correlated Ultra-Cold Bosonic Many-Body Systems Ofir E. Alon, Vanderlei S. Bagnato, Raphael Beinke, Ioannis Brouzos, Tommaso Calarco, Tommaso Caneva, Lorenz S. Cederbaum, Mark A. Kasevich, Shachar Klaiman and 11 more**<br><br>(1) **Ofir E. Alon, Vanderlei S. Bagnato, Raphael Beinke, Ioannis Brouzos, Tommaso Calarco, Tommaso Caneva, Lorenz S. Cederbaum, Mark A. Kasevich, Shachar Klaiman and 11 more** | |

report on further applications, developments, expansion, and proliferation of the Multi-Configurational Time-Dependent Hartree for (e) Bosons (MCTDHB) method in the context of (e) ultra-cold atomic systems.

(2) In this year authors put our main efforts to (e) understanding and (e&eb) generalizing vortices—two-dimensional (2D) and (e&eb) three-dimensional (3D) quantum objects carrying (e&eb) angular momentum—from (e) the perspective of the many-body physics.

(3) Authors have studied static properties and (e&eb) quantum dynamics of vortices confined in (eb) simple parabolic traps and (e&eb) in circular traps.

(4) Particular emphasis has been put on the loss of (e) coherence and build-up of (e) the fragmentation.

**(5)** Complimentary, authors continue to develop (eb) the MCTDHB method spanning (e&eb) several directions of (e) the theoretical and computational physics as well as optimal-control theory: (a) the linear-response on-top of (e) the MCTDHB method (LR-MCTDHB) has been reformulated in (eb) a compact block form, expanded for (e) general inter-particle interactions, and benchmarked against (e&eb) the exactly-solvable harmonic-interaction model; (b) a new analysis tool capable of (e) simulating the outcomes of typical shots generated in (eb) the experimental detection of (e) ultra-cold atomic systems has been invented, tested, and applied; (c) a novel algorithm offering (eb) a direct quantitative measurement of the possible fragmentation in bosonic systems has been proposed and applied; (d) the optimal-control Chopped RAndom Basis (CRAB) algorithm has been merged with (e&eb) the MCTDHB package and applied to manipulate (e&eb) quantum systems. Implications and further perspectives and (e&eb) future research plans are briefly discussed and addressed.

**NOTATION**

**Module One**

Moreover, we establish exact upper and lower bounds for the relation between polarization P and (e&eb) the contact C for both repulsive and (e&eb) attractive Fermi gases

$G_{13}$ : Category one of **exact upper and lower bounds for the relation between polarization P;** contact C for both repulsive and (e&eb) attractive Fermi gases

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of contact C for both repulsive and (e&eb) attractive Fermi gases; **exact upper and lower bounds for the relation between polarization P**

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

Findings shed new light on the role of the pair fluctuations in strongly interacting (e&eb) one-dimensional fermion system that can have implications for (e&eb) higher dimensions.

Subjects: Quantum Gases (cond-mat.quant-gas) Cite as:arXiv: 1603.07207 [cond-mat.quant-gas] (or arXiv: 1603.07207v1 [cond-mat.quant-gas] for this version)

$G_{16}$ : Category one of **Findings shed new light on the role of the pair fluctuations in strongly interacting;** one-dimensional fermion system that can have implications for (e&eb) higher dimensions.

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| $G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of one-dimensional fermion system that can have implications for (e&eb) higher dimensions. ;**Findings shed new light on the role of the pair fluctuations in strongly interacting**<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>Findings shed new light on the role of the pair fluctuations in strongly interacting one-dimensional fermion system that can have implications for (e&eb) higher dimensions.<br><br>Subjects: Quantum Gases (cond-mat.quant-gas) Cite as:arXiv: 1603.07207 [cond-mat.quant-gas] (or arXiv: 1603.07207v1 [cond-mat.quant-gas] for this version) | |
| $G_{20}$ : Category one of **Findings shed new light on the role of the pair fluctuations in strongly interacting one-dimensional fermion system that can have implications;** higher dimensions<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of higher dimensions **;Findings shed new light on the role of the pair fluctuations in strongly interacting one-dimensional fermion system that can have implications**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>In this year authors put our main efforts to understanding and generalizing vortices—two-dimensional (2D) and (e&eb) three-dimensional (3D) quantum objects carrying (e&eb) angular momentum—from (e) the perspective of the many-body physics | |
| $G_{24}$ : Category one of **generalizing vortices—two-dimensional (2D);** three-dimensional (3D) quantum objects carrying (e&eb) angular momentum—from (e) the perspective of the many-body physics<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of three-dimensional (3D) quantum objects carrying (e&eb) angular momentum—from (e) the perspective of the many-body physics **;generalizing vortices—two-dimensional (2D)**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>In this year authors put our main efforts to understanding and generalizing vortices—two-dimensional (2D) | |

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| and three-dimensional (3D) quantum objects carrying (e&eb) angular momentum—from (e) the perspective of the many-body physics | |
| $G_{28}$ : Category one of **generalizing vortices—two-dimensional (2D) and three-dimensional (3D) quantum objects;** three-dimensional (3D) quantum objects carrying (e&eb) angular momentum—from (e) the perspective of the many-body physics<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of three-dimensional (3D) quantum objects carrying (e&eb) angular momentum—from (e) the perspective of the many-body physics; **generalizing vortices—two-dimensional (2D) and three-dimensional (3D) quantum objects**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>In this year authors put our main efforts to understanding and generalizing vortices—two-dimensional (2D) and three-dimensional (3D) quantum objects carrying angular momentum—from (e) the perspective of the many-body physics | |
| $G_{32}$ : Category one of perspective of the many-body physics<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of generalizing vortices—two-dimensional (2D) and three-dimensional (3D) quantum objects carrying angular momentum—<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Authors have studied static properties and (e&eb) quantum dynamics of vortices confined in (eb) simple parabolic traps and (e&eb) in circular traps | |
| $G_{36}$ : Category one of **static properties;** quantum dynamics of vortices confined in (eb) simple parabolic traps and (e&eb) in circular traps<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |

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| $T_{36}$ : Category one of quantum dynamics of vortices confined in (eb) simple parabolic traps and (e&eb) in circular traps **;static properties**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Authors have studied static properties and quantum dynamics of vortices confined in (eb) simple parabolic traps and (e&eb) in circular traps | |
| $G_{40}$ : Category one of static properties and quantum dynamics of vortices confined<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of simple parabolic traps and (e&eb) in circular traps<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Authors have studied **static properties and quantum dynamics of vortices confined in (eb) simple parabolic traps** and (e&eb) in circular traps | |
| $G_{44}$ : Category one of **static properties and quantum dynamics of vortices confined in simple parabolic traps;** circular traps<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of circular traps; **static properties and quantum dynamics of vortices confined in simple parabolic traps**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)},$<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$ | |

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| $(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a_{13}^{'})^{(1)}, (a_{14}^{'})^{(1)}, (a_{15}^{'})^{(1)}, (b_{13}^{'})^{(1)}, (b_{14}^{'})^{(1)}, (b_{15}^{'})^{(1)}, (a_{16}^{'})^{(2)}, (a_{17}^{'})^{(2)}, (a_{18}^{'})^{(2)}, \ (b_{16}^{'})^{(2)}, (b_{17}^{'})^{(2)}, (b_{18}^{'})^{(2)}$<br>$, (a_{20}^{'})^{(3)}, (a_{21}^{'})^{(3)}, (a_{22}^{'})^{(3)}, (b_{20}^{'})^{(3)}, (b_{21}^{'})^{(3)}, (b_{22}^{'})^{(3)}$<br>$(a_{24}^{'})^{(4)}, (a_{25}^{'})^{(4)}, (a_{26}^{'})^{(4)}, (b_{24}^{'})^{(4)}, (b_{25}^{'})^{(4)}, (b_{26}^{'})^{(4)}, (b_{28}^{'})^{(5)}, (b_{29}^{'})^{(5)}, (b_{30}^{'})^{(5)} \ (a_{28}^{'})^{(5)}, (a_{29}^{'})^{(5)}, (a_{30}^{'})^{(5)}$ ,<br>$(a_{32}^{'})^{(6)}, (a_{33}^{'})^{(6)}, (a_{34}^{'})^{(6)}, (b_{32}^{'})^{(6)}, (b_{33}^{'})^{(6)}, (b_{34}^{'})^{(6)}$<br>$(a_{36}^{'})^{(7)}, (a_{37}^{'})^{(7)}, (a_{38}^{'})^{(7)}, (b_{36}^{'})^{(7)}, (b_{37}^{'})^{(7)}, (b_{38}^{'})^{(7)}$,<br>$(a_{40}^{'})^{(8)}, (a_{41}^{'})^{(8)}, (a_{42}^{'})^{(8)}, (b_{40}^{'})^{(8)}, (b_{41}^{'})^{(8)}, (b_{42}^{'})^{(8)}$,<br>$(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |

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| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}\left((G_{27}),t\right) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}\left((G_{31}),t\right) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}\left((G_{35}),t\right)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}\left((G_{35}),t\right)\right]T_{33}$ | 35 |

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| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{13}$ | *55* |

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| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)}\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)}\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3<br>$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3<br>$\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)}\boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)}\boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)}\boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)}\boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}\boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)}\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)}\boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)}\boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)}\boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)}\boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}\boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)}\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\boxed{-(b_{15}^{''})^{(1)}(G,t)}\boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)}\boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)}\boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)}\boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}\boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)}\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | *60* |
| Where $\boxed{-(b_{13}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3 | |

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| $-(b_{16}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{20}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{37}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
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| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[\begin{array}{c} (a_{16}^{'})^{(2)}\,+(a_{16}^{''})^{(2)}(T_{17},t)\,+(a_{13}^{''})^{(1,1,)}(T_{14},t)\,+(a_{20}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)\,+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)\,+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t)\,+(a_{40}^{''})^{(8,8,8)}(T_{41},t)\,+(a_{44}^{''})^{(9,9)}(T_{45},t) \end{array}\right] G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[\begin{array}{c} (a_{17}^{'})^{(2)}\,+(a_{17}^{''})^{(2)}(T_{17},t)\,+(a_{14}^{''})^{(1,1,)}(T_{14},t)\,+(a_{21}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)\,+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)\,+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7)}(T_{37},t)\,+(a_{41}^{''})^{(8,8,8)}(T_{41},t)\,+(a_{45}^{''})^{(9,9)}(T_{45},t) \end{array}\right] G_{17}$ | *62* |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[\begin{array}{c} (a_{18}^{'})^{(2)}\,+(a_{18}^{''})^{(2)}(T_{17},t)\,+(a_{15}^{''})^{(1,1,)}(T_{14},t)\,+(a_{22}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)\,+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)\,+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7)}(T_{37},t)\,+(a_{42}^{''})^{(8,8,8)}(T_{41},t)\,+(a_{46}^{''})^{(9,9)}(T_{45},t) \end{array}\right] G_{18}$ | *63* |
| Where $+(a_{16}^{''})^{(2)}(T_{17},t)$, $+(a_{17}^{''})^{(2)}(T_{17},t)$, $+(a_{18}^{''})^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3 | |

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| $+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{16}$ | *64* |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)}-(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{17}$ | *65* |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{18}$ | *66* |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$, $-(b_{17}^{''})^{(2)}(G_{19},t)$, $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3<br>$-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |

$$\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[\begin{array}{c} (a_{20}^{'})^{(3)}\boxed{+(a_{20}^{''})^{(3)}(T_{21},t)}\ \boxed{+(a_{16}^{''})^{(2,2,2)}(T_{17},t)}\ \boxed{+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)}\ \boxed{+(a_{44}^{''})^{(9,9,9)}(T_{45},t)} \end{array}\right] G_{20} \quad (67)$$

$$\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[\begin{array}{c} (a_{21}^{'})^{(3)}\boxed{+(a_{21}^{''})^{(3)}(T_{21},t)}\ \boxed{+(a_{17}^{''})^{(2,2,2)}(T_{17},t)}\ \boxed{+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)}\ \boxed{+(a_{45}^{''})^{(9,9,9)}(T_{45},t)} \end{array}\right] G_{21} \quad (68)$$

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[\begin{array}{c} (a_{22}^{'})^{(3)}\boxed{+(a_{22}^{''})^{(3)}(T_{21},t)}\ \boxed{+(a_{18}^{''})^{(2,2,2)}(T_{17},t)}\ \boxed{+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)}\ \boxed{+(a_{46}^{''})^{(9,9,9)}(T_{45},t)} \end{array}\right] G_{22} \quad (69)$$

$\boxed{+(a_{20}^{''})^{(3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}^{''})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2)}(T_{17},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)}$ are third augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)}$ are eight augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{44}^{''})^{(9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1, 2 and 3

$$\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[\begin{array}{c} (b_{20}^{'})^{(3)}\boxed{-(b_{20}^{''})^{(3)}(G_{23},t)}\ \boxed{-(b_{16}^{''})^{(2,2,2)}(G_{19},t)}\ \boxed{-(b_{13}^{''})^{(1,1,1,)}(G,t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)}\ \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)}\ \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)}\ \boxed{-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)}\ \boxed{-(b_{44}^{''})^{(9,9,9)}(G_{47},t)} \end{array}\right] T_{20} \quad (70)$$

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[\begin{array}{c} (b_{21}^{'})^{(3)}\boxed{-(b_{21}^{''})^{(3)}(G_{23},t)}\ \boxed{-(b_{17}^{''})^{(2,2,2)}(G_{19},t)}\ \boxed{-(b_{14}^{''})^{(1,1,1,)}(G,t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)}\ \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)}\ \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)}\ \boxed{-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)}\ \boxed{-(b_{45}^{''})^{(9,9,9)}(G_{47},t)} \end{array}\right] T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)} & -(b''_{22})^{(3)}(G_{23},t) & -(b''_{18})^{(2,2,2)}(G_{19},t) & -(b''_{15})^{(1,1,1,)}(G,t) \\ -(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b''_{38})^{(7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{22} \quad (72)$$

$-(b''_{20})^{(3)}(G_{23},t)$, $-(b''_{21})^{(3)}(G_{23},t)$, $-(b''_{22})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*

$-(b''_{16})^{(2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*

$-(b''_{13})^{(1,1,1,)}(G,t)$, $-(b''_{14})^{(1,1,1,)}(G,t)$, $-(b''_{15})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*

$-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*

$-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*

$-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*

$-(b''_{36})^{(7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7)}(G_{39},t)$ $-(b''_{38})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b''_{40})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b''_{46})^{(9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a'_{24})^{(4)} & +(a''_{24})^{(4)}(T_{25},t) & +(a''_{28})^{(5,5,)}(T_{29},t) & +(a''_{32})^{(6,6,)}(T_{33},t) \\ +(a''_{13})^{(1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{36})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a'_{25})^{(4)} & +(a''_{25})^{(4)}(T_{25},t) & +(a''_{29})^{(5,5,)}(T_{29},t) & +(a''_{33})^{(6,6)}(T_{33},t) \\ +(a''_{14})^{(1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{37})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a'_{26})^{(4)} & +(a''_{26})^{(4)}(T_{25},t) & +(a''_{30})^{(5,5,)}(T_{29},t) & +(a''_{34})^{(6,6,)}(T_{33},t) \\ +(a''_{15})^{(1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{38})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a''_{24})^{(4)}(T_{25},t)$, $(a''_{25})^{(4)}(T_{25},t)$, $(a''_{26})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a''_{28})^{(5,5,)}(T_{29},t)$, $+(a''_{29})^{(5,5,)}(T_{29},t)$, $+(a''_{30})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3

$+(a''_{32})^{(6,6,)}(T_{33},t)$, $+(a''_{33})^{(6,6,)}(T_{33},t)$, $+(a''_{34})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3

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| $+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1, 2 and 3 $+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1, 2 and 3 $+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1, 2 and 3 $+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3 $+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1, 2 and 3 $+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b_{24}^{'})^{(4)}\left[-(b_{24}^{''})^{(4)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,)}(G_{35},t)\right] \\ \left[-(b_{13}^{''})^{(1,1,1,1)}(G,t)\right] & \left[-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)\right] & \left[-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)\right] & \left[-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{24}$ | 76 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b_{25}^{'})^{(4)}\left[-(b_{25}^{''})^{(4)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,)}(G_{35},t)\right] \\ \left[-(b_{14}^{''})^{(1,1,1,1)}(G,t)\right] & \left[-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)\right] & \left[-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)\right] & \left[-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{25}$ | 77 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b_{26}^{'})^{(4)}\left[-(b_{26}^{''})^{(4)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,)}(G_{35},t)\right] \\ \left[-(b_{15}^{''})^{(1,1,1,1)}(G,t)\right] & \left[-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)\right] & \left[-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)\right] & \left[-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{26}$ | 78 |

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|---|---|
| $Where\ \boxed{-(b_{24}^{''})^{(4)}(G_{27},t)}, \boxed{-(b_{25}^{''})^{(4)}(G_{27},t)}, \boxed{-(b_{26}^{''})^{(4)}(G_{27},t)}$ $are\ first\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{28}^{''})^{(5,5,)}(G_{31},t)}, \boxed{-(b_{29}^{''})^{(5,5,)}(G_{31},t)}, \boxed{-(b_{30}^{''})^{(5,5,)}(G_{31},t)}$ $are\ second\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{32}^{''})^{(6,6,)}(G_{35},t)}, \boxed{-(b_{33}^{''})^{(6,6,)}(G_{35},t)}, \boxed{-(b_{34}^{''})^{(6,6,)}(G_{35},t)}$ $are\ third\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{13}^{''})^{(1,1,1,1)}(G,t)}, \boxed{-(b_{14}^{''})^{(1,1,1,1)}(G,t)}, \boxed{-(b_{15}^{''})^{(1,1,1,1)}(G,t)}$ $are\ fourth\ \ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)}, \boxed{-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)}, \boxed{-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)}$ $are\ fifth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)}, \boxed{-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)}, \boxed{-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)}$ $are\ sixth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)}$ $are\ seventh\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)}$ $are\ eighth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)}, \boxed{-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)}, \boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a_{28}^{'})^{(5)}\boxed{+(a_{28}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a_{29}^{'})^{(5)}\boxed{+(a_{29}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a_{30}^{'})^{(5)}\boxed{+(a_{30}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{30}$ | *81* |
| $Where\ \boxed{+(a_{28}^{''})^{(5)}(T_{29},t)}, \boxed{+(a_{29}^{''})^{(5)}(T_{29},t)}, \boxed{+(a_{30}^{''})^{(5)}(T_{29},t)}$ $are\ first\ augmentation\ coefficients\ for\ category\ 1,2\ and\ 3$ $And\ \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)}, \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)}, \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)}$ $are\ second\ augmentation\ coefficient\ for\ category\ 1,2\ and\ 3$ $\boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)}, \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)}, \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)}$ $are\ third\ \ augmentation\ coefficient\ for\ category\ 1,2\ and\ 3$ $\boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)}, \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)}, \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3 $\boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation | |

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|---|---|
| coefficients for category 1,2,and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3<br>$+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3<br>$+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3<br>$+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[\begin{matrix}(b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) & -(b_{32}^{''})^{(6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)\end{matrix}\right]T_{28}$ | *82* |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[\begin{matrix}(b_{29}^{'})^{(5)}-(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) & -(b_{33}^{''})^{(6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)\end{matrix}\right]T_{29}$ | *83* |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[\begin{matrix}(b_{30}^{'})^{(5)}-(b_{30}^{''})^{(5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,)}(G_{27},t) & -(b_{34}^{''})^{(6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ -(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)\end{matrix}\right]T_{30}$ | *84* |
| *where* $-(b_{28}^{''})^{(5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3<br>$-(b_{24}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3<br>$-(b_{32}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3<br>$-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3<br>$-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3<br>$-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3<br>$-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3<br>$-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3<br>$-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3 | |

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)} G_{33} - \begin{bmatrix} (a'_{32})^{(6)} & +(a''_{32})^{(6)}(T_{33},t) & +(a''_{28})^{(5,5,5)}(T_{29},t) & +(a''_{24})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)} G_{32} - \begin{bmatrix} (a'_{33})^{(6)} & +(a''_{33})^{(6)}(T_{33},t) & +(a''_{29})^{(5,5,5)}(T_{29},t) & +(a''_{25})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)} G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)} T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)} T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)}\boxed{-(b''_{34})^{(6)}(G_{35},t)} & \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{34} \quad (90)$$

$\boxed{-(b''_{32})^{(6)}(G_{35},t)}, \boxed{-(b''_{33})^{(6)}(G_{35},t)}, \boxed{-(b''_{34})^{(6)}(G_{35},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)}, \boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)}, \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)}$ *are second detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)}$ *are third detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a'_{36})^{(7)}\boxed{+(a''_{36})^{(7)}(T_{37},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a'_{37})^{(7)}\boxed{+(a''_{37})^{(7)}(T_{37},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a'_{38})^{(7)}\boxed{+(a''_{38})^{(7)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad (93)$$

Where $(a_{36}^{''})^{(7)}(T_{37},t)$, $(a_{37}^{''})^{(7)}(T_{37},t)$, $(a_{38}^{''})^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3

$+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}')^{(7)}-(b_{36}^{''})^{(7)}(G_{39},t) & -(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13} \quad (94)$$

$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}')^{(7)}-(b_{37}^{''})^{(7)}(G_{39},t) & -(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}')^{(7)}-(b_{38}^{''})^{(7)}(G_{39},t) & -(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

| | |
|---|---|
| $-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3<br>$-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13}$ | *95* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}^{'})^{(8)}+(a_{41}^{''})^{(8)}(T_{41},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$ | |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$ | |
| Where $+(a_{40}^{''})^{(8)}(T_{41},t)$, $+(a_{41}^{''})^{(8)}(T_{41},t)$, $+(a_{42}^{''})^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3 | |

$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ ,$+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ ,$+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ ,$+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ ,$+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} =$$

$$(b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & -(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t) & -(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} =$$

$$(b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & -(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t) & -(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & -(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t) & -(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$ , $-(b_{37}^{''})^{(7)}(G_{39},t)$ , $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$ , $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$ , $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$ , $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$ , $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

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$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a_{44}^{'})^{(9)}\boxed{+(a_{44}^{''})^{(9)}(T_{45},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13}$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a_{45}^{'})^{(9)}\boxed{+(a_{45}^{''})^{(9)}(T_{45},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a_{46}^{'})^{(9)}\boxed{+(a_{46}^{''})^{(9)}(T_{37},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$$

Where $+(a_{44}^{''})^{(9)}(T_{45},t)$, $+(a_{45}^{''})^{(9)}(T_{45},t)$, $+(a_{46}^{''})^{(9)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth

augmentation coefficient for category 1, 2 and 3
$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ are Seventh augmentation coefficient for category 1, 2 and 3
$+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3
$+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b_{44}^{'})^{(9)}\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \begin{bmatrix} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3
$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3
$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3
$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3
$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3
$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3
$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

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| $\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| Where we suppose | |
| $(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$<br><br>The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br><br>$(a_i^{''})^{(1)}(T_{14},t) \leq (p_i)^{(1)} \leq (\hat{A}_{13})^{(1)}$<br><br>$(b_i^{''})^{(1)}(G,t) \leq (r_i)^{(1)} \leq (b_i^{'})^{(1)} \leq (\hat{B}_{13})^{(1)}$ | *97* |
| $lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$<br><br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br><br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'},t) - (a_i^{''})^{(1)}(T_{14},t)\rvert \leq (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\lvert(b_i^{''})^{(1)}(G^{'},t) - (b_i^{''})^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G^{'}\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ | 101 |

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| satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i,j = 16,17,18$ | |
| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17}, t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19}, t) \leq (r_i)^{(2)} \leq (b_i')^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty} (a_i'')^{(2)}(T_{17}, t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty} (b_i'')^{(2)}((G_{19}), t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}$ are positive constants and $i = 16,17,18$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(2)}(T_{17}', t) - (a_i'')^{(2)}(T_{17}, t)\rvert \leq (\hat{k}_{16})^{(2)}\lvert T_{17} - T_{17}'\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i'')^{(2)}((G_{19})', t) - (b_i'')^{(2)}((G_{19}), t)\rvert < (\hat{k}_{16})^{(2)}\lvert\lvert(G_{19}) - (G_{19})'\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}', t)$ and $(a_i'')^{(2)}(T_{17}, t)$ . $(T_{17}', t)$ and $(T_{17}, t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}]$ . It is to be noted that $(a_i'')^{(2)}(T_{17}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i'')^{(2)}(T_{17}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together | |

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| with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$ satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i^{'})^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i, j = 20,21,22$<br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(3)}$, $(r_i)^{(3)}$:<br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(3)}(T_{21}, t) = (p_i)^{(3)}$<br>$\lim_{G \to \infty}(b_i^{''})^{(3)}(G_{23}, t) = (r_i)^{(3)}$<br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'}, t) - (a_i^{''})^{(3)}(T_{21}, t)\rvert \leq (\hat{k}_{20})^{(3)} \lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'}, t) - (b_i^{''})^{(3)}(G_{23}, t)\rvert < (\hat{k}_{20})^{(3)} \lvert\lvert G_{23} - G_{23}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'}, t)$ and $(a_i^{''})^{(3)}(T_{21}, t)$ . $(T_{21}^{'}, t)$ And $(T_{21}, t)$ are points belonging to the interval $[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |

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| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}((G_{27}), t) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(4)}((G_{27}), t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\|(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\| \leq (\hat{k}_{24})^{(4)} \|T_{25} - T_{25}^{'}\| e^{-(\hat{M}_{24})^{(4)} t}$<br><br>$\|(b_i^{''})^{(4)}((G_{27})', t) - (b_i^{''})^{(4)}((G_{27}), t)\| < (\hat{k}_{24})^{(4)} \|(G_{27}) - (G_{27})'\| e^{-(\hat{M}_{24})^{(4)} t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants | 120 |

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| $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)} + (b_i^{'})^{(4)} + (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}, (r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}((G_{31}), t) \leq (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(5)} (T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(5)} (G_{31}, t) = (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)} \lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)} t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})', t) - (b_i^{''})^{(5)}((G_{31}), t)\rvert < (\hat{k}_{28})^{(5)} \lvert\lvert(G_{31}) - (G_{31})'\rvert\rvert e^{-(\hat{M}_{28})^{(5)} t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ : | 125 |

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| $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$ | 126 |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(6)} (T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(6)} ((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)} \lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)} t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})', t) - (b_i^{''})^{(6)}((G_{35}), t)\rvert < (\hat{k}_{32})^{(6)} \lvert\lvert(G_{35}) - (G_{35})'\rvert\rvert e^{-(\hat{M}_{32})^{(6)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and $(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |

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| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)},$ are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$ | 130 |
| Where we suppose | |
| (AAAA) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i, j = 36,37,38$<br><br>(BBBB) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(7)}$, $(r_i)^{(7)}$:<br><br>$(a_i^{''})^{(7)}(T_{37}, t) \le (p_i)^{(7)} \le (\hat{A}_{36})^{(7)}$<br><br>$(b_i^{''})^{(7)}(G_{39}, t) \le (r_i)^{(7)} \le (b_i^{'})^{(7)} \le (\hat{B}_{36})^{(7)}$ | 131 |
| *(CCCC)* $\lim_{T_2 \to \infty} (a_i^{''})^{(7)}(T_{37}, t) = (p_i)^{(7)}$<br>*(DDDD)*<br>$\lim_{G \to \infty} (b_i^{''})^{(7)}((G_{39}), t) = (r_i)^{(7)}$<br><br>**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :<br><br>Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$ | 132 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)\rvert \le (\hat{k}_{36})^{(7)} \lvert T_{37} - T_{37}^{'}\rvert e^{-(\hat{M}_{36})^{(7)}t}$<br><br>$\lvert(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}((G_{39}), t)\rvert < (\hat{k}_{36})^{(7)} \lvert\lvert(G_{39}) - (G_{39})^{'}\rvert\rvert e^{-(\hat{M}_{36})^{(7)}t}$ | 133 |

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| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'},t)$ and$(a_i^{''})^{(7)}(T_{37},t)$ . $(T_{37}^{'},t)$ and $(T_{37},t)$ are points belonging to the interval $\left[(\hat{k}_{36})^{(7)},(\widehat{M}_{36})^{(7)}\right]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{36})^{(7)}=1$ then the function $(a_i^{''})^{(7)}(T_{37},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)}$ :<br><br>(EEEE) $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)},$ are positive constants<br><br>$\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}<1$ | 134 |
| **Definition of** $(\hat{P}_{36})^{(7)},(\hat{Q}_{36})^{(7)}$ :<br><br>(FFFF) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)},(\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)},(a_i^{'})^{(7)},(b_i)^{(7)},(b_i^{'})^{(7)},(p_i)^{(7)},\ (r_i)^{(7)}, i=36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\widehat{M}_{36})^{(7)}}[(a_i)^{(7)}+(a_i^{'})^{(7)}+(\hat{A}_{36})^{(7)}+(\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}]<1$<br><br>$\frac{1}{(\widehat{M}_{36})^{(7)}}[(b_i)^{(7)}+(b_i^{'})^{(7)}+(\hat{B}_{36})^{(7)}+(\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}]<1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)},(a_i^{'})^{(8)},(a_i^{''})^{(8)},(b_i)^{(8)},(b_i^{'})^{(8)},(b_i^{''})^{(8)}>0, \quad i,j=40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)},(b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)},\ (r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41},t)\leq(p_i)^{(8)}\leq(\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}),t)\leq\ (r_i)^{(8)}\leq(b_i^{'})^{(8)}\leq(\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(8)}(T_{41},t)=(p_i)^{(8)}$ | 140 |
| $\lim_{G\to\infty}(b_i^{''})^{(8)}\left((G_{43}),t\right)=\ (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)},(\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)},(\hat{B}_{40})^{(8)},(p_i)^{(8)},\ (r_i)^{(8)}}$ are positive constants and $\boxed{i=40,41,42}$ | |

They satisfy Lipschitz condition:

$$|(a_i'')^{(8)}(T_{41}', t) - (a_i'')^{(8)}(T_{41}, t)| \leq (\hat{k}_{40})^{(8)}|T_{41} - T_{41}'|e^{-(\hat{M}_{40})^{(8)}t} \quad (142)$$

$$|(b_i'')^{(8)}((G_{43})', t) - (b_i'')^{(8)}((G_{43}), t)| < (\hat{k}_{40})^{(8)}||(G_{43}) - (G_{43})'||e^{-(\hat{M}_{40})^{(8)}t} \quad (143)$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(8)}(T_{41}', t)$ and $(a_i'')^{(8)}(T_{41}, t)$ . $(T_{41}', t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i'')^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i'')^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ :

$(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants

$$\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}, \frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1 \quad (144)$$

**Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :

There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$
Satisfy the inequalities

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1 \quad (145)$$

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1 \quad (146)$$

Where we suppose

$$(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i, j = 44,45,46 \quad (146A)$$

The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:

$$(a_i'')^{(9)}(T_{45}, t) \leq (p_i)^{(9)} \leq (\hat{A}_{44})^{(9)}$$

$$(b_i'')^{(9)}(G_{47}, t) \leq (r_i)^{(9)} \leq (b_i')^{(9)} \leq (\hat{B}_{44})^{(9)}$$

$$\lim_{T_2 \to \infty} (a_i'')^{(9)}(T_{45}, t) = (p_i)^{(9)}$$

$$\lim_{G \to \infty} (b_i'')^{(9)}(G_{47}, t) = (r_i)^{(9)}$$

**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :

Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$

They satisfy Lipschitz condition:

$$|(a_i^{''})^{(9)}(T_{45}^{'}, t) - (a_i^{''})^{(9)}(T_{45}, t)| \leq (\hat{k}_{44})^{(9)}|T_{45} - T_{45}^{'}|e^{-(\hat{M}_{44})^{(9)}t}$$

$$|(b_i^{''})^{(9)}((G_{47})^{'}, t) - (b_i^{''})^{(9)}\big((G_{47}), t\big)| < (\hat{k}_{44})^{(9)}||(G_{47}) - (G_{47})^{'}||e^{-(\hat{M}_{44})^{(9)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(9)}(T_{45}^{'}, t)$ and$(a_i^{''})^{(9)}(T_{45}, t)$ . $(T_{45}^{'}, t)$ and $(T_{45}, t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i^{''})^{(9)}(T_{45}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i^{''})^{(9)}(T_{45}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)},$ are positive constants

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}} \ , \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$$

**Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and \ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}, i = 44,45,46,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0), T_i(0)$ :

$$G_i(t) \leq (\hat{P}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t} \ , \ \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \leq (\hat{Q}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t} \ , \ \boxed{T_i(0) = T_i^0 > 0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0), T_i(0)$

$$G_i(t) \leq (\hat{P}_{16})^{(2)}e^{(\hat{M}_{16})^{(2)}t} \ , \ G_i(0) = G_i^0 > 0$$

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| $T_i(t) \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>$G_i(t) \leq (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \leq (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$ | 153 A |

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| $T_i(t) \leq (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br><br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br><br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[ (a_{18})^{(2)} G_{17}(s_{(16)}) - \left( (a_{18}')^{(2)} + (a_{18}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{18}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[ (b_{16})^{(2)} T_{17}(s_{(16)}) - \left( (b_{16}')^{(2)} - (b_{16}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{16}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[ (b_{17})^{(2)} T_{16}(s_{(16)}) - \left( (b_{17}')^{(2)} - (b_{17}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[ (b_{18})^{(2)} T_{17}(s_{(16)}) - \left( (b_{18}')^{(2)} - (b_{18}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{18}(s_{(16)}) \right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a_{20}')^{(3)} + a_{20}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a_{21}')^{(3)} + (a_{21}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a_{22}')^{(3)} + (a_{22}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b_{20}')^{(3)} - (b_{20}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b_{21}')^{(3)} - (b_{21}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b_{22}')^{(3)} - (b_{22}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |

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| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{24})^{(4)}$ , $T_i^0 \leq (\hat{Q}_{24})^{(4)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{28})^{(5)}$ , $T_i^0 \leq (\hat{Q}_{28})^{(5)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |

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| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b'_{29})^{(5)} - (b''_{29})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{\mathrm{T}}_{30}(\mathrm{t}) = \mathrm{T}_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b'_{30})^{(5)} - (b''_{30})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{32})^{(6)}$ , $T_i^0 \le (\hat{Q}_{32})^{(6)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a'_{32})^{(6)} + a''_{32})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b'_{32})^{(6)} - (b''_{32})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b'_{33})^{(6)} - (b''_{33})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{\mathrm{T}}_{34}(\mathrm{t}) = \mathrm{T}_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b'_{34})^{(6)} - (b''_{34})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{36})^{(7)}$ , $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |

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| By<br>$\bar{G}_{36}(t) = G_{36}^{0} + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a_{36}^{'})^{(7)} + a_{36}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^{0} + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^{0} + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^{0} + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}^{'})^{(8)} + a_{40}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^{0} + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^{0} + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^{0} + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^{0} + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |

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| $\overline{T}_{42}(t) = T_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br><br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166<br>A |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{44})^{(9)}$, $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br><br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\overline{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br><br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\bar{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br><br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left(e^{(\bar{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br><br>$(G_{13}(t) - G_{13}^0) e^{-(\bar{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br><br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |

The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)}\left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\bar{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$$

$$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\bar{M}_{16})^{(2)}}\left(e^{(\bar{M}_{16})^{(2)} t} - 1\right) \quad (169)$$

From which it follows that

$$(G_{16}(t) - G_{16}^0) e^{-(\bar{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\bar{M}_{16})^{(2)}}\left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right] \quad (170)$$

Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$

The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that (171)

$$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$$

$$\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left(e^{(\bar{M}_{20})^{(3)} t} - 1\right)$$

From which it follows that (172)

$$(G_{20}(t) - G_{20}^0) e^{-(\bar{M}_{20})^{(3)} t} \le \frac{(a_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left[\left((\hat{P}_{20})^{(3)} + G_{21}^0\right) e^{\left(-\frac{(\hat{P}_{20})^{(3)} + G_{21}^0}{G_{21}^0}\right)} + (\hat{P}_{20})^{(3)}\right]$$

Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$

The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that (173)

$$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0 + (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} s_{(24)}}\right)\right] ds_{(24)} =$$

$$\left(1 + (a_{24})^{(4)} t\right) G_{25}^0 + \frac{(a_{24})^{(4)} (\hat{P}_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left(e^{(\bar{M}_{24})^{(4)} t} - 1\right)$$

From which it follows that (174)

$$(G_{24}(t) - G_{24}^0) e^{-(\bar{M}_{24})^{(4)} t} \le \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[\left((\hat{P}_{24})^{(4)} + G_{25}^0\right) e^{\left(-\frac{(\hat{P}_{24})^{(4)} + G_{25}^0}{G_{25}^0}\right)} + (\hat{P}_{24})^{(4)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 4

The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0 + (\hat{P}_{28})^{(5)} e^{(\bar{M}_{28})^{(5)} s_{(28)}}\right)\right] ds_{(28)} =$$

$$\left(1 + (a_{28})^{(5)} t\right) G_{29}^0 + \frac{(a_{28})^{(5)} (\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)} t} - 1\right)$$

From which it follows that

$$(G_{28}(t)-G_{28}^{0})e^{-(\hat{M}_{28})^{(5)}t} \leq \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^{0})e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^{0}}{G_{29}^{0}}\right)}+(\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5 — 175

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that — 176

$$G_{32}(t) \leq G_{32}^{0}+\int_0^t\left[(a_{32})^{(6)}\left(G_{33}^{0}+(\hat{P}_{32})^{(6)}e^{(\hat{M}_{32})^{(6)}s_{(32)}}\right)\right]ds_{(32)} =$$

$$\left(1+(a_{32})^{(6)}t\right)G_{33}^{0}+\frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)}t}-1\right)$$

From which it follows that — 177

$$(G_{32}(t)-G_{32}^{0})e^{-(\hat{M}_{32})^{(6)}t} \leq \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^{0})e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^{0}}{G_{33}^{0}}\right)}+(\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

(n) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that — 178

$$G_{36}(t) \leq G_{36}^{0}+\int_0^t\left[(a_{36})^{(7)}\left(G_{37}^{0}+(\hat{P}_{36})^{(7)}e^{(\hat{M}_{36})^{(7)}s_{(36)}}\right)\right]ds_{(36)} =$$

$$\left(1+(a_{36})^{(7)}t\right)G_{37}^{0}+\frac{(a_{36})^{(7)}(\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)}t}-1\right)$$

From which it follows that

$$(G_{36}(t)-G_{36}^{0})e^{-(\hat{M}_{36})^{(7)}t} \leq \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)}+G_{37}^{0})e^{\left(-\frac{(\hat{P}_{36})^{(7)}+G_{37}^{0}}{G_{37}^{0}}\right)}+(\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \leq G_{40}^{0}+\int_0^t\left[(a_{40})^{(8)}\left(G_{41}^{0}+(\hat{P}_{40})^{(8)}e^{(\hat{M}_{40})^{(8)}s_{(40)}}\right)\right]ds_{(40)} =$$

$$\left(1+(a_{40})^{(8)}t\right)G_{41}^{0}+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)}t}-1\right)$$ — 180

From which it follows that

$$(G_{40}(t)-G_{40}^{0})e^{-(\hat{M}_{40})^{(8)}t} \leq \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^{0})e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^{0}}{G_{41}^{0}}\right)}+(\hat{P}_{40})^{(8)}\right]$$ — 181

$(G_i^0)$ is as defined in the statement of theorem 8
Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$

The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)}\left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$$

$$\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)} t} - 1\right)$$

From which it follows that

$$(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{P}_{44})^{(9)} + G_{45}^0\right) e^{\left(-\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0}\right)} + (\hat{P}_{44})^{(9)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 9

Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$

It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ and to choose

$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have

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$$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)} + \left((\hat{P}_{13})^{(1)} + G_j^0\right) e^{-\left(\frac{(\hat{P}_{13})^{(1)} + G_j^0}{G_j^0}\right)}\right] \le (\hat{P}_{13})^{(1)} \quad (183)$$

$$\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)} + T_j^0\right) e^{-\left(\frac{(\hat{Q}_{13})^{(1)} + T_j^0}{T_j^0}\right)} + (\hat{Q}_{13})^{(1)}\right] \le (\hat{Q}_{13})^{(1)} \quad (184)$$

In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric (185)

$$d\left(\left(G^{(1)}, T^{(1)}\right), \left(G^{(2)}, T^{(2)}\right)\right) =$$

$$\sup_i \{\max_{t \in \mathbb{R}_+} \left|G_i^{(1)}(t) - G_i^{(2)}(t)\right| e^{-(\hat{M}_{13})^{(1)} t}, \max_{t \in \mathbb{R}_+} \left|T_i^{(1)}(t) - T_i^{(2)}(t)\right| e^{-(\hat{M}_{13})^{(1)} t}\}$$

Indeed if we denote

**Definition of** $\tilde{G}, \tilde{T}$ : $\left(\tilde{G}, \tilde{T}\right) = \mathcal{A}^{(1)}(G, T)$

It results

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| $\left\|\tilde{G}_{13}^{(1)}-\tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{13})^{(1)} \left\|G_{14}^{(1)}-G_{14}^{(2)}\right\| e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\, ds_{(13)} +$<br>$\int_0^t \{(a'_{13})^{(1)} \left\|G_{13}^{(1)}-G_{13}^{(2)}\right\| e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} +$<br>$(a''_{13})^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right)\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\| e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} e^{(\widehat{M}_{13})^{(1)}s_{(13)}} +$<br>$G_{13}^{(2)} \left\|(a''_{13})^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right) - (a''_{13})^{(1)}\left(T_{14}^{(2)}, s_{(13)}\right)\right\| \; e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\} ds_{(13)}$<br>Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | |
| $\left\|G^{(1)}-G^{(2)}\right\| e^{-(\widehat{M}_{13})^{(1)}t} \leq$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right) d\left(\left(G^{(1)}, T^{(1)}; G^{(2)}, T^{(2)}\right)\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(1)}$ and $(b''_i)^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br>From 19 to 24 it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(1)} - (a''_i)^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :<br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if<br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating<br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1 / (a'_{14})^{(1)}$<br>In the same way , one can obtain<br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2 / (a'_{15})^{(1)}$ | 187 |

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| If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t),t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br><br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote | 195 |

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| **Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right) d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$ | 201 |

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| In the same way , one can obtain<br>$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | |
| **Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br>Indeed let $t_2$ be so that for $t > t_2$<br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$ | 211 |

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| $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | |
| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}},\widetilde{T_{23}}:(\widetilde{(G_{23})},\widetilde{(T_{23})})=\mathcal{A}^{(3)}((G_{23}),(T_{23}))$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{20})^{(3)}\left\|G_{21}^{(1)}-G_{21}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)}+$<br><br>$\int_0^t\{(a_{20}^{'})^{(3)}\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br><br>$(a_{20}^{''})^{(3)}(T_{21}^{(1)},s_{(20)})\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br><br>$G_{20}^{(2)}\left\|(a_{20}^{''})^{(3)}(T_{21}^{(1)},s_{(20)})-(a_{20}^{''})^{(3)}(T_{21}^{(2)},s_{(20)})\right\|\ e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)}-G_{23}{}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}t}\le$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}((a_{20})^{(3)}+(a_{20}^{'})^{(3)}+(\widehat{A}_{20})^{(3)}+(\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)})d\left(((G_{23})^{(1)},(T_{23})^{(1)};\ (G_{23})^{(2)},(T_{23})^{(2)})\right)$<br><br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ and\ (\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i=20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$<br><br>it results<br><br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(3)}-(a_i^{''})^{(3)}(T_{21}(s_{(20)}),s_{(20)})\}ds_{(20)}\right]}\ge 0$<br><br>$T_i(t)\ge T_i^0e^{(-(b_i^{'})^{(3)}t)}>0$ for t > 0 | 216 |

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| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |

In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself (224)

The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric (225)

$$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right)=$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{24})^{(4)}\left|G_{25}^{(1)}-G_{25}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$$

$$\int_0^t\{(a_{24}^{'})^{(4)}\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$G_{24}^{(2)}\left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)}-(G_{27})^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}t}\le$$
$$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$$

(226)

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

**Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$. (227)

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24{,}25{,}26$ depend only on $\mathrm{T}_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 17:** There does not exist any $t$ where $G_i\,(t)=0\ and\ T_i\,(t)=0$ (228)

it results

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| $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(4)} - (a_i'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(4)} t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ :<br><br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if<br><br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}')^{(4)} G_{25}$ and by integrating<br><br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)} \left((\widehat{M}_{24})^{(4)}\right)_1 / (a_{25}')^{(4)}$<br><br>In the same way , one can obtain<br><br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)} \left((\widehat{M}_{24})^{(4)}\right)_2 / (a_{26}')^{(4)}$<br><br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t), t)) = (b_{25}')^{(4)}$ then $T_{25} \to \infty$.<br><br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br><br>Indeed let $t_4$ be so that for $t > t_4$<br><br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t), t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}\left((G_{27})(t), t\right) = (b_{26}')^{(4)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br><br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br><br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |

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| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)}+\left((\hat{P}_{28})^{(5)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{28})^{(5)}\right]\leq(\hat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $\widetilde{(G_{31})},\widetilde{(T_{31})}$ : $\left(\widetilde{(G_{31})},\widetilde{(T_{31})}\right)=\mathcal{A}^{(5)}((G_{31}),(T_{31}))$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}ds_{(28)}+$<br>$\int_0^t\{(a_{28}')^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}'')^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\leq$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}')^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}'')^{(5)}$ and $(b_{28}'')^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}\ and\ (\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it | 238 |

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| suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 22:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(5)} - (a_i^{''})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(5)} t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2\ and\ \left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}\ and\ G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}^{'})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a_{29}^{'})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a_{30}^{'})^{(5)}$<br><br>If $G_{29}\ or\ G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}\ and\ G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t), t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t), t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t), t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |

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| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\widehat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{32})^{(6)}\right] \leq (\widehat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)}, (T_{35})^{(1)}), ((G_{35})^{(2)}, (T_{35})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}), (\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}), (\widetilde{T_{35}})\right) = \mathcal{A}^{(6)}((G_{35}), (T_{35}))$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{32})^{(6)}\left|G_{33}^{(1)} - G_{33}^{(2)}\right|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\,ds_{(32)} +$<br>$\int_0^t\{(a'_{32})^{(6)}\left|G_{32}^{(1)} - G_{32}^{(2)}\right|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)\left|G_{32}^{(1)} - G_{32}^{(2)}\right|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$G_{32}^{(2)}\left|(a''_{32})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right) - (a''_{32})^{(6)}\left(T_{33}^{(2)}, s_{(32)}\right)\right|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 247 |
| $\left|(G_{35})^{(1)} - (G_{35})^{(2)}\right|e^{-(\widehat{M}_{32})^{(6)}t} \leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)} + (a'_{32})^{(6)} + (\widehat{A}_{32})^{(6)} + (\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)}, (T_{35})^{(1)};\ (G_{35})^{(2)}, (T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | 249 |

condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i = 32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 27:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$ 250

it results

$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(6)} - (a_i^{''})^{(6)}(T_{33}(s_{(32)}),s_{(32)})\}ds_{(32)}\right]} \geq 0$

$T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(6)}t)} > 0$ for t > 0

**Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2\ and\ \left((\widehat{M}_{32})^{(6)}\right)_3$ : 251

**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}\ and\ G_{34}$ . indeed if

$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}^{'})^{(6)}G_{33}$ and by integrating

$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1/(a_{33}^{'})^{(6)}$

In the same way , one can obtain

$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2/(a_{34}^{'})^{(6)}$

If $G_{33}\ or\ G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively.

**Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}\ and\ G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. 252

**Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}\ ((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$. 253

**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :

Indeed let $t_6$ be so that for $t > t_6$

$(b_{33})^{(6)} - (b_i^{''})^{(6)}\left((G_{35})(t),t\right) < \varepsilon_6, T_{32}\,(t) > (m)^{(6)}$

Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to 254

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right),\ \ t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.
The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$

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| We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{\mathrm{P}}_{36})^{(7)}$ and $(\widehat{\mathrm{Q}}_{36})^{(7)}$ large to have | 255 |
| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)}, (T_{39})^{(1)}), ((G_{39})^{(2)}, (T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})}, \widetilde{(T_{39})} : \left(\widetilde{(G_{39})}, \widetilde{(T_{39})}\right) = \mathcal{A}^{(7)}((G_{39}), (T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{36})^{(7)}\left\|G_{37}^{(1)} - G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)} +$<br>$\int_0^t\{(a'_{36})^{(7)}\left\|G_{36}^{(1)} - G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} +$<br>$(a''_{36})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right)\left\|G_{36}^{(1)} - G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}} +$<br>$G_{36}^{(2)}\left|(a''_{36})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) - (a''_{36})^{(7)}\left(T_{37}^{(2)}, s_{(36)}\right)\right|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)} - (G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t} \leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)} + (a'_{36})^{(7)} + (\widehat{A}_{36})^{(7)} + (\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)}, (T_{39})^{(1)};\ (G_{39})^{(2)}, (T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |

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| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ $and$ $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\}ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)}t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)}G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1/(a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ $is$ bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}\,((G_{39})(t),t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |

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| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right), \quad t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t), t\right) = (b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\bar{M}_{40})^{(8)}}, \frac{(b_i)^{(8)}}{(\bar{M}_{40})^{(8)}} < 1$ and to choose<br>$(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\bar{M}_{40})^{(8)}}\left[(\hat{P}_{40})^{(8)} + \left((\hat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\bar{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{40})^{(8)}\right] \leq (\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)}, (T_{43})^{(1)}), ((G_{43})^{(2)}, (T_{43})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\bar{M}_{40})^{(8)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\bar{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})}, \widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})}, \widetilde{(T_{43})}\right) = \mathcal{A}^{(8)}((G_{43}), (T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{40})^{(8)}\left\|G_{41}^{(1)} - G_{41}^{(2)}\right\|e^{-(\bar{M}_{40})^{(8)}s_{(40)}}e^{(\bar{M}_{40})^{(8)}s_{(40)}}\,ds_{(40)} +$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\bar{M}_{40})^{(8)}s_{(40)}}e^{-(\bar{M}_{40})^{(8)}s_{(40)}} +$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\bar{M}_{40})^{(8)}s_{(40)}}e^{(\bar{M}_{40})^{(8)}s_{(40)}} +$<br>$G_{40}^{(2)}\left\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right) - (a_{40}^{''})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)\right\| e^{-(\bar{M}_{40})^{(8)}s_{(40)}}e^{(\bar{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval [0, t] | 272 |

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| From the hypotheses it follows | |
| $\left\|(G_{43})^{(1)} - (G_{43})^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)}t} \leq$ $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)} + (a'_{40})^{(8)} + (\widehat{A}_{40})^{(8)} + (\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right) d\left(((G_{43})^{(1)}, (T_{43})^{(1)};\ (G_{43})^{(2)}, (T_{43})^{(2)})\right)$ And analogous inequalities for $G_i$ $and$ $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a''_{40})^{(8)}$ and $(b''_{40})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(8)}$ and $(b''_i)^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t) = 0$ $and$ $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(8)} - (a''_i)^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ $and$ $\left((\widehat{M}_{40})^{(8)}\right)_3$ : **Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ $and$ $G_{42}$ . indeed if $G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a'_{41})^{(8)} G_{41}$ and by integrating $G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a'_{41})^{(8)}$ In the same way , one can obtain $G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a'_{42})^{(8)}$ If $G_{41}$ $or$ $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}$ $and$ $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(8)}((G_{43})(t), t)) = (b'_{41})^{(8)}$ then $T_{41} \to \infty$. | 278 |

**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :

Indeed let $t_8$ be so that for $t > t_8$

$(b_{41})^{(8)} - (b_i^{''})^{(8)}\big((G_{43})(t), t\big) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$

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Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.
The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\big((G_{43})(t), t(t), t\big) = (b_{42}^{'})^{(8)}$

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It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \big((\hat{P}_{44})^{(9)} + G_j^0\big)e^{-\left(\frac{(\hat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\big((\hat{Q}_{44})^{(9)} + T_j^0\big)e^{-\left(\frac{(\hat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{44})^{(9)}\right] \leq (\hat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\Big(\big((G_{47})^{(1)}, (T_{47})^{(1)}\big), \big((G_{47})^{(2)}, (T_{47})^{(2)}\big)\Big) =$$
$$sup_i\{max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \big(\widetilde{(G_{47})}, \widetilde{(T_{47})}\big) = \mathcal{A}^{(9)}\big((G_{47}), (T_{47})\big)$
It results

$$\left|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$(a_{44}^{''})^{(9)}\big(T_{45}^{(1)}, s_{(44)}\big)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\big(T_{45}^{(1)}, s_{(44)}\big) - (a_{44}^{''})^{(9)}\big(T_{45}^{(2)}, s_{(44)}\big)\right| \; e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)} - G^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)}t} \leq \frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a'_{44})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right) d\left(((G_{47})^{(1)}, (T_{47})^{(1)};\ (G_{47})^{(2)}, (T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a''_{44})^{(9)}$ and $(b''_{44})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)} e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)} e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(9)}$ and $(b_i'')^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$

From 99 to 44 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(9)} - (a''_i)^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b'_i)^{(9)}t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2$ *and* $\left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}$ *and* $G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a'_{45})^{(9)} G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1 / (a'_{45})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2 / (a'_{46})^{(9)}$$

If $G_{45}$ *or* $G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}$ *and* $G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(9)}\left((G_{47})(t), t\right)) = (b'_{45})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\big((G_{47})(t),t\big) < \varepsilon_9, T_{44}\,(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right), \quad t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\left((G_{47})(t),t\right) = (b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} - (a_{13}^{''})^{(1)}(T_{14}, t) + (a_{14}^{''})^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)}, , (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}, \; if\; (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\; (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)},$

and $\boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}, \; if\; (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}, \; if\; (u_0)^{(1)} < (u_1)^{(1)}$

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| $(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)},$<br>and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$<br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$<br>are defined | |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)} t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)} t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)} t}\right] + G_{15}^0 e^{-(S_2)^{(1)} t} \le G_{15}(t) \le \right.$<br>$\left.\frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a'_{15})^{(1)})}[e^{(S_1)^{(1)} t} - e^{-(a'_{15})^{(1)} t}] + G_{15}^0 e^{-(a'_{15})^{(1)} t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)} T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)} - (b'_{15})^{(1)})}\left[e^{(R_1)^{(1)} t} - e^{-(b'_{15})^{(1)} t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)} t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)} T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)} + (r_{13})^{(1)} + (R_2)^{(1)})}\left[e^{((R_1)^{(1)} + (r_{13})^{(1)})t} - e^{-(R_2)^{(1)} t}\right] + T_{15}^0 e^{-(R_2)^{(1)} t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$<br>$(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ :<br>$(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |

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| $-(\sigma_2)^{(2)} \le -(a_{16}^{'})^{(2)} + (a_{17}^{'})^{(2)} - (a_{16}^{''})^{(2)}(\mathrm{T}_{17}, t) + (a_{17}^{''})^{(2)}(\mathrm{T}_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b_{16}^{'})^{(2)} + (b_{17}^{'})^{(2)} - (b_{16}^{''})^{(2)}\big((G_{19}), t\big) - (b_{17}^{''})^{(2)}\big((G_{19}), t\big) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}$,, $(\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{\mathrm{G}_{16}^0}{\mathrm{G}_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{\mathrm{T}_{16}^0}{\mathrm{T}_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$\mathrm{G}_{16}^0 \mathrm{e}^{((\mathrm{S}_1)^{(2)} - (p_{16})^{(2)})\mathrm{t}} \le G_{16}(t) \le \mathrm{G}_{16}^0 \mathrm{e}^{(\mathrm{S}_1)^{(2)}\mathrm{t}}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |

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| $\frac{1}{(m_1)^{(2)}} \mathrm{G}_{16}^{0} \mathrm{e}^{((\mathrm{S}_1)^{(2)}-(p_{16})^{(2)})\mathrm{t}} \leq G_{17}(t) \leq \frac{1}{(m_2)^{(2)}} \mathrm{G}_{16}^{0} \mathrm{e}^{(\mathrm{S}_1)^{(2)}\mathrm{t}}$ | 311 |
| $\left(\frac{(a_{18})^{(2)}\mathrm{G}_{16}^{0}}{(m_1)^{(2)}((\mathrm{S}_1)^{(2)}-(p_{16})^{(2)}-(\mathrm{S}_2)^{(2)})}\left[\mathrm{e}^{((\mathrm{S}_1)^{(2)}-(p_{16})^{(2)})\mathrm{t}} - \mathrm{e}^{-(\mathrm{S}_2)^{(2)}\mathrm{t}}\right] + \mathrm{G}_{18}^{0}\mathrm{e}^{-(\mathrm{S}_2)^{(2)}\mathrm{t}} \leq \mathrm{G}_{18}(t) \leq \right.$ $\left.\frac{(a_{18})^{(2)}\mathrm{G}_{16}^{0}}{(m_2)^{(2)}((\mathrm{S}_1)^{(2)}-(a_{18}^{'})^{(2)})}\left[\mathrm{e}^{(\mathrm{S}_1)^{(2)}\mathrm{t}} - \mathrm{e}^{-(a_{18}^{'})^{(2)}\mathrm{t}}\right] + \mathrm{G}_{18}^{0}\mathrm{e}^{-(a_{18}^{'})^{(2)}\mathrm{t}}\right)$ | 312 |
| $\boxed{\mathrm{T}_{16}^{0}\mathrm{e}^{(\mathrm{R}_1)^{(2)}t} \leq T_{16}(t) \leq \mathrm{T}_{16}^{0}\mathrm{e}^{((\mathrm{R}_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} \mathrm{T}_{16}^{0}\mathrm{e}^{(\mathrm{R}_1)^{(2)}t} \leq T_{16}(t) \leq \frac{1}{(\mu_2)^{(2)}} \mathrm{T}_{16}^{0}\mathrm{e}^{((\mathrm{R}_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)}\mathrm{T}_{16}^{0}}{(\mu_1)^{(2)}((\mathrm{R}_1)^{(2)}-(b_{18}^{'})^{(2)})}\left[\mathrm{e}^{(\mathrm{R}_1)^{(2)}t} - \mathrm{e}^{-(b_{18}^{'})^{(2)}t}\right] + \mathrm{T}_{18}^{0}\mathrm{e}^{-(b_{18}^{'})^{(2)}t} \leq T_{18}(t) \leq$ $\frac{(a_{18})^{(2)}\mathrm{T}_{16}^{0}}{(\mu_2)^{(2)}((\mathrm{R}_1)^{(2)}+(r_{16})^{(2)}+(\mathrm{R}_2)^{(2)})}\left[\mathrm{e}^{((\mathrm{R}_1)^{(2)}+(r_{16})^{(2)})t} - \mathrm{e}^{-(\mathrm{R}_2)^{(2)}t}\right] + \mathrm{T}_{18}^{0}\mathrm{e}^{-(\mathrm{R}_2)^{(2)}t}$ | 315 |
| **Definition of** $(\mathrm{S}_1)^{(2)}, (\mathrm{S}_2)^{(2)}, (\mathrm{R}_1)^{(2)}, (\mathrm{R}_2)^{(2)}$:- | 316 |
| Where $(\mathrm{S}_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a_{16}^{'})^{(2)}$ $(\mathrm{S}_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b_{16}^{'})^{(2)}$ $(\mathrm{R}_2)^{(2)} = (b_{18}^{'})^{(2)} - (r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ : $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)} \leq -(a_{20}^{'})^{(3)} + (a_{21}^{'})^{(3)} - (a_{20}^{''})^{(3)}(T_{21}, t) + (a_{21}^{''})^{(3)}(T_{21}, t) \leq -(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)} \leq -(b_{20}^{'})^{(3)} + (b_{21}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t) - (b_{21}^{''})^{(3)}((G_{23}), t) \leq -(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(\nu_1)^{(3)}, (\nu_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ : By $(\nu_1)^{(3)} > 0$, $(\nu_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$, $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}(\nu^{(3)})^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$ and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and By $(\bar{\nu}_1)^{(3)} > 0$, $(\bar{\nu}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$, $(\bar{u}_2)^{(3)} < 0$ the | 320 |

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| roots of the equations $(a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$<br><br>and $(b_{21})^{(3)}\left(u^{(3)}\right)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ | |
| **Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :-<br><br>If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by<br>$(m_2)^{(3)} = (v_0)^{(3)}, (m_1)^{(3)} = (v_1)^{(3)}$, ***if*** $(v_0)^{(3)} < (v_1)^{(3)}$<br><br>$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (\bar{v}_1)^{(3)}$, ***if*** $(v_1)^{(3)} < (v_0)^{(3)} < (\bar{v}_1)^{(3)}$,<br><br>and $\boxed{(v_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$<br><br>$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (v_0)^{(3)}$, ***if*** $(\bar{v}_1)^{(3)} < (v_0)^{(3)}$ | 321 |
| and analogously<br><br>$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$<br><br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$<br><br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$<br><br>Then the solution of global equations satisfies the inequalities<br><br>$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)}t}$<br><br>$(p_i)^{(3)}$ is defined by equation | 322 |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \le G_{22}(t) \le \right.$<br>$\left.\frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a'_{22})^{(3)})}\left[e^{(S_1)^{(3)}t} - e^{-(a'_{22})^{(3)}t}\right] + G_{22}^0 e^{-(a'_{22})^{(3)}t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b'_{22})^{(3)})}\left[e^{(R_1)^{(3)}t} - e^{-(b'_{22})^{(3)}t}\right] + T_{22}^0 e^{-(b'_{22})^{(3)}t} \le T_{22}(t) \le$<br><br>$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$ | 327 |

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- 328

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$

$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$

$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$

**Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : 329

By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$
and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{v}_1)^{(4)} > 0$, $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$
and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (v_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}$, ***if*** $(v_0)^{(4)} < (v_1)^{(4)}$

$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}$, ***if*** $(v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)}$,
and $\boxed{(v_0)^{(4)} = \frac{G^0_{24}}{G^0_{25}}}$

$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)} < (v_0)^{(4)}$

and analogously 331

$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$

$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$, ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,

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| and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \boldsymbol{if}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)} t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)} t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\right)\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)} t}\right] + G_{26}^0 e^{-(S_2)^{(4)} t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b_{26}')^{(4)})}\left[e^{(R_1)^{(4)} t} - e^{-(b_{26}')^{(4)} t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)} t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)} + (r_{24})^{(4)} + (R_2)^{(4)})}\left[e^{((R_1)^{(4)} + (r_{24})^{(4)})t} - e^{-(R_2)^{(4)} t}\right] + T_{26}^0 e^{-(R_2)^{(4)} t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |

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| **Definition of** $(\nu_1)^{(5)}, (\nu_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, \nu^{(5)}, u^{(5)}$ :<br><br>By $(\nu_1)^{(5)} > 0$ , $(\nu_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$ , $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | 339 |
| **Definition of** $(\bar{\nu}_1)^{(5)},, (\bar{\nu}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ :<br><br>By $(\bar{\nu}_1)^{(5)} > 0$ , $(\bar{\nu}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}, (\mu_2)^{(5)}, (\nu_0)^{(5)}$ :-<br><br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}, (\mu_2)^{(5)}$ by<br><br>$(m_2)^{(5)} = (\nu_0)^{(5)}, (m_1)^{(5)} = (\nu_1)^{(5)}$, ***if*** $(\nu_0)^{(5)} < (\nu_1)^{(5)}$<br><br>$(m_2)^{(5)} = (\nu_1)^{(5)}, (m_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ , ***if*** $(\nu_1)^{(5)} < (\nu_0)^{(5)} < (\bar{\nu}_1)^{(5)}$,<br>and $\boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br><br>$(m_2)^{(5)} = (\nu_1)^{(5)}, (m_1)^{(5)} = (\nu_0)^{(5)}$, ***if*** $(\bar{\nu}_1)^{(5)} < (\nu_0)^{(5)}$ | 340 |
| and analogously<br><br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br><br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |

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| $\frac{1}{(\mu_1)^{(5)}}T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}}T_{28}^0 e^{((R_1)^{(5)}+(r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)}T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)}-(b_{30}^{'})^{(5)})}\left[e^{(R_1)^{(5)}t} - e^{-(b_{30}^{'})^{(5)}t}\right] + T_{30}^0 e^{-(b_{30}^{'})^{(5)}t} \le T_{30}(t) \le$<br>$\frac{(a_{30})^{(5)}T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t} - e^{-(R_2)^{(5)}t}\right] + T_{30}^0 e^{-(R_2)^{(5)}t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a_{28}^{'})^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b_{28}^{'})^{(5)}$<br>$(R_2)^{(5)} = (b_{30}^{'})^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :<br>$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying<br>$-(\sigma_2)^{(6)} \le -(a_{32}^{'})^{(6)} + (a_{33}^{'})^{(6)} - (a_{32}^{''})^{(6)}(T_{33}, t) + (a_{33}^{''})^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$<br>$-(\tau_2)^{(6)} \le -(b_{32}^{'})^{(6)} + (b_{33}^{'})^{(6)} - (b_{32}^{''})^{(6)}\left((G_{35}), t\right) - (b_{33}^{''})^{(6)}\left((G_{35}), t\right) \le -(\tau_1)^{(6)}$ | 349 |
| **Definition of** $(v_1)^{(6)}, (v_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, v^{(6)}, u^{(6)}$ :<br>By $(v_1)^{(6)} > 0$, $(v_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$, $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_1)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\left(u^{(6)}\right)^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{v}_1)^{(6)},, (\bar{v}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br>By $(\bar{v}_1)^{(6)} > 0$, $(\bar{v}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$, $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_2)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\left(u^{(6)}\right)^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (v_0)^{(6)}$ :-<br>If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br>$(m_2)^{(6)} = (v_0)^{(6)}, (m_1)^{(6)} = (v_1)^{(6)}$, ***if*** $(v_0)^{(6)} < (v_1)^{(6)}$<br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (\bar{v}_6)^{(6)}$, ***if*** $(v_1)^{(6)} < (v_0)^{(6)} < (\bar{v}_1)^{(6)}$, | 351 |

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| and $\boxed{(v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$<br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (v_0)^{(6)},$ ***if*** $(\bar{v}_1)^{(6)} < (v_0)^{(6)}$ | |
| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},$ ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)},$ ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\right)\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34')6e(S1)6t−e−(a34')6t+ G340e−(a34')6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b_{34}')^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b_{34}')^{(6)} t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$<br>$(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** | |

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ :

$(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying

$$-(\sigma_2)^{(7)} \le -(a_{36}^{'})^{(7)} + (a_{37}^{'})^{(7)} - (a_{36}^{''})^{(7)}(T_{37}, t) + (a_{37}^{''})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$$

$$-(\tau_2)^{(7)} \le -(b_{36}^{'})^{(7)} + (b_{37}^{'})^{(7)} - (b_{36}^{''})^{(7)}\big((G_{39}), t\big) - (b_{37}^{''})^{(7)}\big((G_{39}), t\big) \le -(\tau_1)^{(7)}$$

**Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ : 361

By $(\nu_1)^{(7)} > 0, (\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ : 362

By $(\bar{\nu}_1)^{(7)} > 0, (\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$

**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,
and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$

and analogously 363

$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,
and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

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| Then the solution of global equations satisfies the inequalities<br>$G_{36}^0 e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)}t}$<br>where $(p_i)^{(7)}$ is defined by equation | 364 |
| $\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^0 e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}')^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}')^{(7)}t}] + G_{38}^0 e^{-(a_{38}')^{(7)}t})$ | 366 |
| $\boxed{T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}')^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}')^{(7)}t}\right] + T_{38}^0 e^{-(b_{38}')^{(7)}t} \le T_{38}(t) \le$<br>$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^0 e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}')^{(7)}$<br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}')^{(7)}$<br>$(R_2)^{(7)} = (b_{38}')^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br>$-(\sigma_2)^{(8)} \le -(a_{40}')^{(8)} + (a_{41}')^{(8)} - (a_{40}'')^{(8)}(T_{41}, t) + (a_{41}'')^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br>$-(\tau_2)^{(8)} \le -(b_{40}')^{(8)} + (b_{41}')^{(8)} - (b_{40}'')^{(8)}((G_{43}), t) - (b_{41}'')^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |

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| **Definition of** $(\nu_1)^{(8)}, (\nu_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, \nu^{(8)}, u^{(8)}$ : <br><br> By $(\nu_1)^{(8)} > 0$ , $(\nu_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0$ , $(u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$ <br> and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | 372 |
| **Definition of** $(\bar{\nu}_1)^{(8)},, (\bar{\nu}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ : <br><br> By $(\bar{\nu}_1)^{(8)} > 0$ , $(\bar{\nu}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0$ , $(\bar{u}_2)^{(8)} < 0$ the <br><br> roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$ <br><br> and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ <br><br> **Definition of** $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)}, (\mu_2)^{(8)}, (\nu_0)^{(8)}$ :- <br><br> If we define $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)}, (\mu_2)^{(8)}$ by <br><br> $(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)}$, ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$ <br><br> $(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)}$, <br> and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ <br><br> $(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)}$, ***if*** $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously <br><br> $(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)}$, ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$ <br><br> $(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)}$, <br> and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$ <br><br> $(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}$, ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities <br><br> $G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{40}(t) \leq G_{40}^0 e^{(S_1)^{(8)}t}$ <br><br> where $(p_i)^{(8)}$ is defined by equation | 375 |

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| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)}-(p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)}G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t} - e^{-(S_2)^{(8)}t}\right] + G_{42}^0 e^{-(S_2)^{(8)}t} \le G_{42}(t) \le \right.$ $\left.\frac{(a_{42})^{(8)}G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)}-(a'_{42})^{(8)})}[e^{(S_1)^{(8)}t} - e^{-(a'_{42})^{(8)}t}] + G_{42}^0 e^{-(a'_{42})^{(8)}t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)}-(b'_{42})^{(8)})}\left[e^{(R_1)^{(8)}t} - e^{-(b'_{42})^{(8)}t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)}t} \le T_{42}(t) \le$ $\frac{(a_{42})^{(8)}T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t} - e^{-(R_2)^{(8)}t}\right] + T_{42}^0 e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$<br>$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}((G_{47}), t) - (b''_{45})^{(9)}((G_{47}), t) \le -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :<br>By $(\nu_1)^{(9)} > 0$, $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$, $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$<br>and $(b_{45})^{(9)}(u^{(9)})^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(9)},, (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ : | |

By $(\bar{v}_1)^{(9)} > 0$ , $(\bar{v}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\left(u^{(9)}\right)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}$ , $(m_2)^{(9)}$ , $(\mu_1)^{(9)}$, $(\mu_2)^{(9)}$, $(v_0)^{(9)}$ :-

If we define $(m_1)^{(9)}$ , $(m_2)^{(9)}$ , $(\mu_1)^{(9)}$, $(\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (v_0)^{(9)}, (m_1)^{(9)} = (v_1)^{(9)}$, ***if*** $(v_0)^{(9)} < (v_1)^{(9)}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (\bar{v}_1)^{(9)}$ , ***if*** $(v_1)^{(9)} < (v_0)^{(9)} < (\bar{v}_1)^{(9)}$,

and $\boxed{(v_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (v_0)^{(9)}$, ***if*** $(\bar{v}_1)^{(9)} < (v_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^0 e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}} G_{44}^0 e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)} G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)} - (p_{44})^{(9)} - (S_2)^{(9)})}\left[e^{((S_1)^{(9)} - (p_{44})^{(9)})t} - e^{-(S_2)^{(9)}t}\right] + G_{46}^0 e^{-(S_2)^{(9)}t} \le G_{46}(t) \le \right.$$

$$\left. \frac{(a_{46})^{(9)} G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)} - (a_{46}')^{(9)})}\left[e^{(S_1)^{(9)}t} - e^{-(a_{46}')^{(9)}t}\right] + G_{46}^0 e^{-(a_{46}')^{(9)}t}\right)$$

$$\boxed{T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}} T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}} T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)} T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)} - (b_{46}')^{(9)})}\left[e^{(R_1)^{(9)}t} - e^{-(b_{46}')^{(9)}t}\right] + T_{46}^0 e^{-(b_{46}')^{(9)}t} \le T_{46}(t) \le$$

$\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$

**Definition of** $(S_1)^{(9)},(S_2)^{(9)},(R_1)^{(9)},(R_2)^{(9)}$:-

Where $(S_1)^{(9)}=(a_{44})^{(9)}(m_2)^{(9)}-(a_{44}^{'})^{(9)}$

$(S_2)^{(9)}=(a_{46})^{(9)}-(p_{46})^{(9)}$

$(R_1)^{(9)}=(b_{44})^{(9)}(\mu_2)^{(9)}-(b_{44}^{'})^{(9)}$

$(R_2)^{(9)}=(b_{46}^{'})^{(9)}-(r_{46})^{(9)}$

**Proof :** From global equations we obtain

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$$\frac{dv^{(1)}}{dt}=(a_{13})^{(1)}-\left((a_{13}^{'})^{(1)}-(a_{14}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14},t)\right)-(a_{14}^{''})^{(1)}(T_{14},t)v^{(1)}-(a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)}=\frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_2)^{(1)}v^{(1)}-(a_{13})^{(1)}\right)\leq\frac{dv^{(1)}}{dt}\leq-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_1)^{(1)}v^{(1)}-(a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)},(v_0)^{(1)}$ :-

For $0<\boxed{(v_0)^{(1)}=\frac{G_{13}^{0}}{G_{14}^{0}}}<(v_1)^{(1)}<(\bar{v}_1)^{(1)}$

$$v^{(1)}(t)\geq\frac{(v_1)^{(1)}+(C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)}-(v_0)^{(1)}\right)t\right]}}{1+(C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)}-(v_0)^{(1)}\right)t\right]}}\quad,\quad\boxed{(C)^{(1)}=\frac{(v_1)^{(1)}-(v_0)^{(1)}}{(v_0)^{(1)}-(v_2)^{(1)}}}$$

it follows $(v_0)^{(1)}\leq v^{(1)}(t)\leq(v_1)^{(1)}$

In the same manner , we get

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$$v^{(1)}(t)\leq\frac{(\bar{v}_1)^{(1)}+(\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)}-(\bar{v}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)}-(\bar{v}_2)^{(1)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(1)}=\frac{(\bar{v}_1)^{(1)}-(v_0)^{(1)}}{(v_0)^{(1)}-(\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)}\leq v^{(1)}(t)\leq(\bar{v}_1)^{(1)}$

If $0<(v_1)^{(1)}<(v_0)^{(1)}=\frac{G_{13}^{0}}{G_{14}^{0}}<(\bar{v}_1)^{(1)}$ we find like in the previous case,

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$$(\nu_1)^{(1)} \le \frac{(\nu_1)^{(1)}+(C)^{(1)}(\nu_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_2)^{(1)}\right)t\right]}}{1+(C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_2)^{(1)}\right)t\right]}} \le \nu^{(1)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \le (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \le (\bar{\nu}_1)^{(1)} \le \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \le \nu^{(1)}(t) \le \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \le (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \le \nu^{(1)}(t) \le (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \le u^{(1)}(t) \le (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)}G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)}T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}')^{(2)} - (a_{17}')^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

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| $-\left((a_{17})^{(2)}\left(v^{(2)}\right)^2+(\sigma_2)^{(2)}v^{(2)}-(a_{16})^{(2)}\right)\le\frac{dv^{(2)}}{dt}\le-\left((a_{17})^{(2)}\left(v^{(2)}\right)^2+(\sigma_1)^{(2)}v^{(2)}-(a_{16})^{(2)}\right)$ | |
| From which one obtains<br><br>**Definition of** $(\bar{v}_1)^{(2)},(v_0)^{(2)}$ :-<br><br>For $0<(v_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}<(v_1)^{(2)}<(\bar{v}_1)^{(2)}$<br><br>$v^{(2)}(t)\ge\frac{(v_1)^{(2)}+(C)^{(2)}(v_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_0)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_0)^{(2)}\right)t\right]}}$ , $\boxed{(C)^{(2)}=\frac{(v_1)^{(2)}-(v_0)^{(2)}}{(v_0)^{(2)}-(v_2)^{(2)}}}$<br><br>it follows $(v_0)^{(2)}\le v^{(2)}(t)\le(v_1)^{(2)}$ | 390 |
| In the same manner , we get<br><br>$v^{(2)}(t)\le\frac{(\bar{v}_1)^{(2)}+(\bar{C})^{(2)}(\bar{v}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(2)}=\frac{(\bar{v}_1)^{(2)}-(v_0)^{(2)}}{(v_0)^{(2)}-(\bar{v}_2)^{(2)}}}$ | 391 |
| From which we deduce $(v_0)^{(2)}\le v^{(2)}(t)\le(\bar{v}_1)^{(2)}$ | 392 |
| If $0<(v_1)^{(2)}<(v_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}<(\bar{v}_1)^{(2)}$ we find like in the previous case,<br><br>$(v_1)^{(2)}\le\frac{(v_1)^{(2)}+(C)^{(2)}(v_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_2)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_2)^{(2)}\right)t\right]}}\le v^{(2)}(t)\le$<br><br>$\frac{(\bar{v}_1)^{(2)}+(\bar{C})^{(2)}(\bar{v}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}\le(\bar{v}_1)^{(2)}$ | 393 |
| If $0<(v_1)^{(2)}\le(\bar{v}_1)^{(2)}\le(v_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(v_1)^{(2)}\le v^{(2)}(t)\le\frac{(\bar{v}_1)^{(2)}+(\bar{C})^{(2)}(\bar{v}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}\le(v_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $v^{(2)}(t)$ :-<br><br>$(m_2)^{(2)}\le v^{(2)}(t)\le(m_1)^{(2)}$, $\boxed{v^{(2)}(t)=\frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :- | 396 |

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| $(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| From which one obtains<br><br>For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$<br><br>$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)} + (C)^{(3)}(\nu_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}$ , $\boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\nu_2)^{(3)}}}$<br><br>it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$ | 400 |
| In the same manner , we get<br><br>$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)} + (C)^{(3)}(\bar{\nu}_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\bar{\nu}_2)^{(3)}}}$<br><br>**Definition of** $(\bar{\nu}_1)^{(3)}$ :-<br><br>From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$ | 401 |

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If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case,

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

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If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}'')^{(3)} = (a_{21}'')^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b_{20}'')^{(3)} = (b_{21}'')^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}')^{(4)} - (a_{25}')^{(4)} + (a_{24}'')^{(4)}(T_{25},t)\right) - (a_{25}'')^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_2)^{(4)}v^{(4)}-(a_{24})^{(4)}\right)\le\frac{dv^{(4)}}{dt}\le-\left((a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_4)^{(4)}v^{(4)}-(a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(4)},(v_0)^{(4)}$ :-

For $0<\boxed{(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}<(v_1)^{(4)}<(\bar{v}_1)^{(4)}$

$$v^{(4)}(t)\ge\frac{(v_1)^{(4)}+(C)^{(4)}(v_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_0)^{(4)}\right)t\right]}}{4+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_0)^{(4)}\right)t\right]}}\quad,\quad\boxed{(C)^{(4)}=\frac{(v_1)^{(4)}-(v_0)^{(4)}}{(v_0)^{(4)}-(v_2)^{(4)}}}$$

it follows $(v_0)^{(4)}\le v^{(4)}(t)\le(v_1)^{(4)}$

In the same manner , we get

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$$v^{(4)}(t)\le\frac{(\bar{v}_1)^{(4)}+(\bar{C})^{(4)}(\bar{v}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}{4+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(4)}=\frac{(\bar{v}_1)^{(4)}-(v_0)^{(4)}}{(v_0)^{(4)}-(\bar{v}_2)^{(4)}}}$$

From which we deduce $(v_0)^{(4)}\le v^{(4)}(t)\le(\bar{v}_1)^{(4)}$

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If $0<(v_1)^{(4)}<(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}<(\bar{v}_1)^{(4)}$ we find like in the previous case,

$$(v_1)^{(4)}\le\frac{(v_1)^{(4)}+(C)^{(4)}(v_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_2)^{(4)}\right)t\right]}}{1+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_2)^{(4)}\right)t\right]}}\le v^{(4)}(t)\le$$

$$\frac{(\bar{v}_1)^{(4)}+(\bar{C})^{(4)}(\bar{v}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}\le(\bar{v}_1)^{(4)}$$

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If $0<(v_1)^{(4)}\le(\bar{v}_1)^{(4)}\le\boxed{(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(v_1)^{(4)}\le v^{(4)}(t)\le\frac{(\bar{v}_1)^{(4)}+(\bar{C})^{(4)}(\bar{v}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}\le(v_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(4)}(t)$ :-

$$(m_2)^{(4)}\le v^{(4)}(t)\le(m_1)^{(4)},\quad\boxed{v^{(4)}(t)=\frac{G_{24}(t)}{G_{25}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$$(\mu_2)^{(4)}\le u^{(4)}(t)\le(\mu_1)^{(4)},\quad\boxed{u^{(4)}(t)=\frac{T_{24}(t)}{T_{25}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}'')^{(5)} = (b_{29}'')^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}')^{(6)} - (a_{33}')^{(6)} + (a_{32}'')^{(6)}(T_{33}, t)\right) - (a_{33}'')^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

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In the same manner , we get

$$\nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\bar{\nu}_1)^{(6)}$

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If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

$$(\nu_1)^{(6)} \leq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_2)^{(6)}\right)t\right]}} \leq \nu^{(6)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\bar{\nu}_1)^{(6)}$$

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If $0 < (\nu_1)^{(6)} \leq (\bar{\nu}_1)^{(6)} \leq \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(\nu_1)^{(6)} \leq \nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

$$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}, \quad \boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}, \quad \boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)}G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a'_{36})^{(7)} - (a'_{37})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right) - (a''_{37})^{(7)}(T_{37},t)v^{(7)} - (a_{37})^{(7)}v^{(7)}$$

**Definition of** $v^{(7)}$ :- $\boxed{v^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)}\right) \le \frac{dv^{(7)}}{dt} \le -\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(7)}, (v_0)^{(7)}$ :-

For $0 < \boxed{(v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}} < (v_1)^{(7)} < (\bar{v}_1)^{(7)}$

$$v^{(7)}(t) \ge \frac{(v_1)^{(7)} + (C)^{(7)}(v_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(v_1)^{(7)} - (v_0)^{(7)}}{(v_0)^{(7)} - (v_2)^{(7)}}}$$

it follows $(v_0)^{(7)} \le v^{(7)}(t) \le (v_1)^{(7)}$

In the same manner , we get

$$v^{(7)}(t) \le \frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{v}_1)^{(7)} - (v_0)^{(7)}}{(v_0)^{(7)} - (\bar{v}_2)^{(7)}}}$$

From which we deduce $(v_0)^{(7)} \le v^{(7)}(t) \le (\bar{v}_1)^{(7)}$

If $0 < (v_1)^{(7)} < (v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}} < (\bar{v}_1)^{(7)}$ we find like in the previous case,

$$(v_1)^{(7)} \le \frac{(v_1)^{(7)} + (C)^{(7)}(v_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_2)^{(7)}\right)t\right]}} \le v^{(7)}(t) \le$$

$$\frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \le (\bar{v}_1)^{(7)}$$

If $0 < (v_1)^{(7)} \le (\bar{v}_1)^{(7)} \le \boxed{(v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$ , we obtain

$$(v_1)^{(7)} \le v^{(7)}(t) \le \frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \le (v_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

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**Definition of** $v^{(7)}(t)$ :-

$(m_2)^{(7)} \leq v^{(7)}(t) \leq (m_1)^{(7)}$, $\boxed{v^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :-

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$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $v^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)} G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

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$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right) - (a_{41}^{''})^{(8)}(T_{41}, t) v^{(8)} - (a_{41})^{(8)} v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)} v^{(8)} - (a_{40})^{(8)}\right) \leq \frac{dv^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)} v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$v^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)}-(\nu_0)^{(8)}}{(\nu_0)^{(8)}-(\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\bar{\nu}_8)^{(8)}$

If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case, 423

$$(\nu_1)^{(8)} \leq \frac{(\nu_1)^{(8)}+(C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}} \leq \nu^{(8)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\bar{\nu}_1)^{(8)}$$

If $0 < (\nu_1)^{(8)} \leq (\bar{\nu}_1)^{(8)} \leq \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain 424

$$(\nu_1)^{(8)} \leq \nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)}+(C)^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \leq \nu^{(8)}(t) \leq (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)}G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)}T_{41}(t)$ This is an important

consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{d\nu^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right) - (a''_{45})^{(9)}(T_{45},t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \leq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}} \leq \nu^{(9)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \leq (\bar{\nu}_1)^{(9)} \leq \boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$ , we obtain

$$(\nu_1)^{(9)} \leq \nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$(m_2)^{(9)} \leq \nu^{(9)}(t) \leq (m_1)^{(9)}$, $\boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}$, $\boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ 430

$with\ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations 431

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$

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| $(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$ | 435 |

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| $(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |

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| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |

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| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |

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| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(n) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) +$ | 488 |

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| $(a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first | 494 |

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| equations | |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |

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| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{40}=\frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}=\frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 501 |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{44}=\frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}=\frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G)=(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)}-(b_{13})^{(1)}(b_{14})^{(1)}-$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G)+(b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G)=0$<br><br>Where in $G(G_{13},G_{14},G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*})=0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19})=(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)}-(b_{16})^{(2)}(b_{17})^{(2)}-$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19})+(b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19})=0$ | 503 |
| Where in $(G_{19})(G_{16},G_{17},G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*})=0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23})=(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)}-(b_{20})^{(3)}(b_{21})^{(3)}-$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23})+(b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23})=0$<br><br>Where in $G_{23}(G_{20},G_{21},G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*})=0$ | 505 |

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| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right] + (b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br><br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br><br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br><br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br><br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$ | |

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| $\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right]+(b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0$ , $T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0$ , $T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*}) = 0$ , $T_{21}^{*}$ given by $f(T_{21}^{*}) = 0$ and<br><br>$G_{20}^{*} = \frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*} = \frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br><br>$T_{20}^{*} = \frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*} = \frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br><br>$G_{25}^{*}$ given by $\varphi(G_{27}) = 0$ , $T_{25}^{*}$ given by $f(T_{25}^{*}) = 0$ and<br><br>$G_{24}^{*} = \frac{(a_{24})^{(4)}G_{25}^{*}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*} = \frac{(a_{26})^{(4)}G_{25}^{*}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*} = \frac{(b_{24})^{(4)}T_{25}^{*}}{[(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*} = \frac{(b_{26})^{(4)}T_{25}^{*}}{[(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}((G_{27})^{*})]}$ | 517 |

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| Obviously, these values represent an equilibrium solution of global equations | |
| Finally we obtain the unique solution<br><br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*}) = 0$ , $T_{29}^{*}$ given by $f(T_{29}^{*}) = 0$ and<br><br>$G_{28}^{*} = \frac{(a_{28})^{(5)} G_{29}^{*}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*} = \frac{(a_{30})^{(5)} G_{29}^{*}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |
| $T_{28}^{*} = \frac{(b_{28})^{(5)} T_{29}^{*}}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)} T_{29}^{*}}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br><br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br><br>$G_{32}^{*} = \frac{(a_{32})^{(6)} G_{33}^{*}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)} G_{33}^{*}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)} T_{33}^{*}}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)} T_{33}^{*}}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br><br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br><br>$G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99 | 523 A |

$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and

$$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]} \quad , \quad G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$$

$$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47})^{*})]} \quad , \quad T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47})^{*})]}$$

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**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

**Proof:** Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-

$$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$$

$$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^{*}) = (q_{14})^{(1)} \quad , \frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^{*}) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*}\mathbb{T}_{14} \quad 525$$

$$\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*}\mathbb{T}_{14} \quad 526$$

$$\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*}\mathbb{T}_{14} \quad 527$$

$$\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*}\mathbb{G}_j\right) \quad 528$$

$$\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*}\mathbb{G}_j\right) \quad 529$$

$$\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*}\mathbb{G}_j\right) \quad 530$$

**ASYMPTOTIC STABILITY ANALYSIS** 531

**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-

$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ 532

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| $\frac{\partial(a_{17}^{''})^{(2)}}{\partial \mathrm{T}_{17}}(\mathrm{T}_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial(b_i^{''})^{(2)}}{\partial \mathrm{G}_j}((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{\mathrm{d}\mathbb{G}_{16}}{\mathrm{dt}} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}\mathrm{G}_{16}^{*}\mathbb{T}_{17}$ | 534 |
| $\frac{\mathrm{d}\mathbb{G}_{17}}{\mathrm{dt}} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}\mathrm{G}_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{\mathrm{d}\mathbb{G}_{18}}{\mathrm{dt}} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}\mathrm{G}_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{\mathrm{d}\mathbb{T}_{16}}{\mathrm{dt}} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}\mathrm{T}_{16}^{*}\mathbb{G}_j\right)$ | 537 |
| $\frac{\mathrm{d}\mathbb{T}_{17}}{\mathrm{dt}} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}\mathrm{T}_{17}^{*}\mathbb{G}_j\right)$ | 538 |
| $\frac{\mathrm{d}\mathbb{T}_{18}}{\mathrm{dt}} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}\mathrm{T}_{18}^{*}\mathbb{G}_j\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)}$ , $\frac{\partial(b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_j\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 547 |

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| **Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)}$ , $\frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^*) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ $and$ $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^*) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |

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| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b'_{28})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T^*_{28}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b'_{29})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T^*_{29}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b'_{30})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T^*_{30}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{33})^{(6)}}{\partial T_{33}}(T^*_{33}) = (q_{33})^{(6)}$ , $\frac{\partial(b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a'_{32})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G^*_{32}\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a'_{33})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G^*_{33}\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 572 |

| | |
|---|---|
| $\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^{*}) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^{*}\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^{*}\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^{*}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^{*}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^{*}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^{*}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^{*}) = (q_{41})^{(8)}$ , $\frac{\partial (b_i^{''})^{(8)}}{\partial G_j}\left((G_{43})^{*}\right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^{*}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^{*}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^{*}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^{*}\mathbb{G}_j\right)$ | 584 |

| | |
|---|---|
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i'')^{(9)}$ *and* $(b_i'')^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{45})^{(9)}}{\partial T_{45}}(T^*_{45}) = (q_{45})^{(9)} \ , \ \frac{\partial(b_i'')^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a'_{44})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G^*_{44}\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a'_{45})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G^*_{45}\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a'_{46})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G^*_{46}\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$<br><br>$\left[\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\Big]$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\Big)$<br><br>$+\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\Big)$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\Big)$<br><br>$\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$ | |

$$\left((\lambda)^{(1)}\right)^2+\left((b'_{13})^{(1)}+(b'_{14})^{(1)}-(r_{13})^{(1)}+(r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$$

$$+\left(\left((\lambda)^{(1)}\right)^2+\left((a'_{13})^{(1)}+(a'_{14})^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$$

$$+\left((\lambda)^{(1)}+(a'_{13})^{(1)}+(p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G^*_{14}+(a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G^*_{13}\right)$$

$$\left((\lambda)^{(1)}+(b'_{13})^{(1)}-(r_{13})^{(1)}\right)s_{(14),(15)}T^*_{14}+(b_{14})^{(1)}s_{(13),(15)}T^*_{13}\Big)\}=0$$

$$+$$

$$(\lambda)^{(2)}+(b'_{18})^{(2)}-(r_{18})^{(2)}\}\{\left((\lambda)^{(2)}+(a'_{18})^{(2)}+(p_{18})^{(2)}\right)$$

$$\left[\left((\lambda)^{(2)}+(a'_{16})^{(2)}+(p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}^*_{17}+(a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^*_{16}\right)\Big]$$

$$\left((\lambda)^{(2)}+(b'_{16})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}^*_{17}+(b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^*_{17}\Big)$$

$$+\left(\left((\lambda)^{(2)}+(a'_{17})^{(2)}+(p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}^*_{16}+(a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^*_{17}\right)$$

$$\left((\lambda)^{(2)}+(b'_{16})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}^*_{17}+(b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^*_{16}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((a'_{16})^{(2)}+(a'_{17})^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((b'_{16})^{(2)}+(b'_{17})^{(2)}-(r_{16})^{(2)}+(r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left(\left((\lambda)^{(2)}\right)^2+\left((a'_{16})^{(2)}+(a'_{17})^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a'_{16})^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}^*_{17}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}^*_{16}\right)$$

$$\left((\lambda)^{(2)}+(b'_{16})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}^*_{17}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}^*_{16}\Big)\}=0$$

$$+$$

$$(\lambda)^{(3)}+(b'_{22})^{(3)}-(r_{22})^{(3)}\}\{\left((\lambda)^{(3)}+(a'_{22})^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left((\lambda)^{(3)}+(a'_{20})^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G^*_{21}+(a_{21})^{(3)}(q_{20})^{(3)}G^*_{20}\right)\Big]$$

$$\left((\lambda)^{(3)}+(b'_{20})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T^*_{21}+(b_{21})^{(3)}s_{(20),(21)}T^*_{21}\Big)$$

$$+\left(\left((\lambda)^{(3)}+(a'_{21})^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G^*_{20}+(a_{20})^{(3)}(q_{21})^{(1)}G^*_{21}\right)$$

$$\left((\lambda)^{(3)}+(b'_{20})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T^*_{21}+(b_{21})^{(3)}s_{(20),(20)}T^*_{20}\Big)$$

$$\left((\lambda)^{(3)}\right)^2+\left((a'_{20})^{(3)}+(a'_{21})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^2+\left((b_{20}^{'})^{(3)}+(b_{21}^{'})^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}^{'})^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left((\lambda)^{(3)}+(b_{20}^{'})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left((\lambda)^{(5)}\right)^2+\left((b_{28}')^{(5)}+(b_{29}')^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$+\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}')^{(5)}+(a_{29}')^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}')^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}')^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}')^{(6)}-(r_{34})^{(6)}\{\left((\lambda)^{(6)}+(a_{34}')^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left((\lambda)^{(6)}+(a_{32}')^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\Big]$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\Big)$$

$$+\left((\lambda)^{(6)}+(a_{33}')^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\Big)$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((a_{32}')^{(6)}+(a_{33}')^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((b_{32}')^{(6)}+(b_{33}')^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$+\left(\left((\lambda)^{(6)}\right)^2+\left((a_{32}')^{(6)}+(a_{33}')^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}')^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b_{38}')^{(7)}-(r_{38})^{(7)}\{\left((\lambda)^{(7)}+(a_{38}')^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left((\lambda)^{(7)}+(a_{36}')^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\Big]$$

$$\left((\lambda)^{(7)}+(b_{36}')^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\Big)$$

$$+\left((\lambda)^{(7)}+(a_{37}')^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\Big)$$

$$\left((\lambda)^{(7)}+(b_{36}')^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a_{36}')^{(7)}+(a_{37}')^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left(\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^*_{36}\right)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^*_{37}+(b_{37})^{(7)}s_{(36),(38)}T^*_{36}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^*_{40}\right)\right]$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^*_{41}+(b_{41})^{(8)}s_{(40),(41)}T^*_{41}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^*_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^*_{41}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^*_{41}+(b_{41})^{(8)}s_{(40),(40)}T^*_{40}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^2+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$+\left(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^*_{40}\right)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^*_{41}+(b_{41})^{(8)}s_{(40),(42)}T^*_{40}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^*_{44}\right)\right]$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^*_{45}+(b_{45})^{(9)}s_{(44),(45)}T^*_{45}\right)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^*_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\right)$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45}+(b_{45})^{(9)}s_{(44),(44)}T^*_{44}\right)$$

$$\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45}+(b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt}= A_i\ (x)\ xi$ where $x_i \in R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i \ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i \ X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions

are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$(14) \qquad \frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17].

In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 14. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i(x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \epsilon C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population $i$. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility

assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C\backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^n_i$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\in \Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two

propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population *i* is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) In \ |B_i \ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^t_0(A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = limsup_{t\to\infty}\frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni}= E_{i(x)} \ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \ \leq \ a|\ B_i(t,x)u_i(x)|+ |B_i(t,x)w| \ \leq \ |\ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)|\ w|)$. Hence, $limsup_{t\to\infty}\frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$limsup_{t\to\infty}\frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty}\frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \ \geq \ a|B_i(t,x)u_i(x)|-|B_i(t,x)w|$$

$$\geq |\ B_i(t,x)u_i(x)|(a- \alpha+ exp\ (-\beta t)|\ w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty}\frac{1}{t} \text{In}\ |B_i(t,x)v| \ \geq \ \lambda i(x) \geq \limsup_{t\to\infty}\frac{1}{t} \text{In}\ |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) \ = \ In|B_i(t+s,x)\ u_i(x)|= In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

| |
|---|
| $= \text{In } \lvert B_i(s,x.t)\, u_i(x.t) \rvert \lvert B_i(t,x)\, u_i(x) \rvert$ |
| $= \text{In } b_i(s,t.x)\, b_i(t,x)$ |
| |
| Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies |
| $\frac{d}{dt} b_i(t,x) = \lim_{s \to 0} \frac{bi(t+s,x-bi(t,x)}{S-}$ |
| $= \lim_{s \to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}\lvert_{s=0}\, b_i(s,x.t)$ |
| = |
| $=(A_i(x.t)u_i(x.s),u_i(x.s),$ |
| The Fundamental Theorem of Calculus implies |
| $B_i(t,x) = \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$ |
| and the second assertion follows. |
| **Proposition 2:** *For the solutions of (2) we have:* if $x_i > 0$, then $\lambda i(x) \leq 0$. If $\lambda i(x) < 0$, then $\lim_{t \to \infty} (x.t)_i = 0$ If $\limsup_{t \to \infty}$ $(x.t)_i \rvert > 0$, then $\lambda i(x) = 0$. |
| **Proof:** First, assume that x satisfies $x_i > 0$. Since the semi flow is dissipative, there exists $\gamma > 0$ such that $\lvert x.t \rvert \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply |
| $\lambda i(x) = \limsup_{t \to \infty} \frac{1}{t} \text{In } \lvert B_i(t,x)x_i \rvert = \limsup_{t \to \infty} \frac{1}{t} \text{In } \lvert (x.t)_i \rvert$ |
| $\leq \limsup_{t \to \infty} \frac{In\, \gamma}{t} = 0$ |
| To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i = 0$, then the invariance of the faces of C imply that $(x.t)_i = 0$ for all $t \geq 0$. Alternatively, if $x_i > 0$, then Proposition I and the skew product flow imply |
| $\limsup_{t \to \infty} \frac{1}{t} In \lvert (x.t)_i \rvert = \lambda i(x) < 0.$ |
| The final assertion follows from the first two assertions. 5 |
| |

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on $\boldsymbol{C}$, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:*$\lambda_i(\mathrm{x}) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle \mathrm{A}_i(x.s)\mathrm{u}_i(x.s), \mathrm{u}_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(\mathrm{x}) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $\mathrm{h}_i(x) = \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I}\boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I}\boldsymbol{C}_i^+$such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following

equivalent conditions hold For all invariant measures $\mu$ supported by $K$ (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by $K$.

then $K$ is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and $K$ is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by $K$.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$. Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large

so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \leq \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \leq \in$, and $|h(x.t) - h(x^n.t)| \leq \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\leq \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t) d\mu(x) = \int_c h(x) d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x) u_i^n(x), u_i^n(x) \rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop. 3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop. 3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)| d\mu(x) + |\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)|$

$+ \int_c h_i^n(x)\, d\mu_n(x)$

$\leq \quad 2 \in$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm. 2.1]$ implies that either there exists $\mathcal{Y} \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}$

$\int_0^t \delta_{y.s}$ ds where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$ . Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}_{n=1}^{\infty}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

| |
|---|
| Min max $\lambda i$ ($\mu = \min_{\mu} \max_{p \epsilon \Delta} \sum_{i} Pi\lambda i(\mu)$ |
| |
| Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that |
| (6) min max $\lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_{i} Pi\lambda i(\mu)$ |
| |
| Where the minimum is taken over invariant probability measures $\mu$ with support in *K.* Since min$\mu$ |
| $\sum_{i} P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established. |
| |
| To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\} \backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta -$perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ [0,1]$be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C \backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by |
| |
| $\tilde{A}_i(x) = \{ \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i \epsilon I}{if\ i \notin I}$ |
| |
| Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C \backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t >$ $0. Hence, \Gamma(\tilde{A}) c\ V\ c\ N\delta\ (\ (A))). We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of |

$\dot{x} = \tilde{A}(x)x$ *and* $\lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\} \backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). *Since* $E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently *K* is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent*. Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume x* $\mapsto$ *A(x) is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely*

*ergodic, or There exists j ∈ { 1,….,k} such that* $M_i$ ----- *{* $x \in C : x_l \gg 0$ *for* $l \neq j$*} Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by* $U_i M_i$.

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----{1,…,k} such that $\mu(---_i \in_I C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures* $\mu$ *with support in* $\partial$C, $\lambda_i(\mu) > 0$ *for all* $I \in \{1,\ldots,k\} \setminus supp(\mu)$ *(2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1, \ldots, x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda$ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0 < e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j \ (x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1{}^1, \ldots, E_1{}^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i_2(E_1{}^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f^j_2(E^j_1) > 0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f^j_2(E^j_1) > 0 \quad$ and $\quad \max_j f^j_2(E^j_2) > 0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0, E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is

sufficiently slow (i.e. $0< e^{j}{}_{i}\ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j (E^j{}_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1-x_1^j - \beta^j\ x_2^j\ - \alpha^j\ x_3^j\ ) + \sum d_1^{jk}\ \ x_1^k\ - e_1^j\ x_1^j$$

$$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j\ x_1^j\ - x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_2^{jk}\ \ x_2^k\ - e_2^j\ x_2^j$$

$$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j\ x_1^j\ - \alpha^j x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_3^{jk}\ \ x_3^k\ - e_3^j\ x_3^j$$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all *j*. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j{}_i = \sum_l d^{l,j}_i$ for all *i, j*) , the maximal invariant set in $\partial$C consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0< e^j{}_i\ll 1$ for all *i*, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2$ $(E_1)$and $\lambda_3$ $(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1 - \beta^j$. Consider the Morse decomposition of $\partial$C given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1> 0$ for *i* = 1, 2, 3, 0 is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j (1 - \beta^j)$$

Equivalently,

$$2>\min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j (r_i - \sum_s a_{is} x_s^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j , \qquad i = 1, \dots, k; \; j = 1, \dots, m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$ . We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set H= $\{x \in C: x_i^j = x_i^l \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \qquad i = 1, \dots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \dots, k$$

It has been shown [12,ch.16.1,16.2] That for $k$ =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there

is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\, \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within $\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as

well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that

there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has

abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left( \beta(1 - y - z)\frac{P}{\epsilon + P} - r - m(1 - y) - b(N)\ y \right)$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1 = N$, $x_2 = p$, $x_3 = (y,z)'$ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} - y_i)$$

With

$$(16)\quad F(x)_i = x_i g\,(x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\quad \begin{pmatrix} x_i \\ y_i\ dot \end{pmatrix} = \begin{pmatrix} -\beta & \beta \\ ag\,(x_i, x_i) - 1 & \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon

plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational

consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical

propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$ to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.40) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.41) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.42) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0,' \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, \end{cases}$$ (1.4)

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \epsilon [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the

model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1$, $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as* $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 14. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^{+}\text{x}\ k_{\ 2}\text{x}\ k_{\ 1}, R^{-})$,

$A_3$. $\varphi_i\ (\cdot\ , s) \in L^{\ 2}\ (R^{+}, R^{+})\ \cap C(R^{+}, R^{+})$ for each $s \in\ [-\tau, 0]$,

$i,\ = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i$: $R^{+}$ $[-\tau, T] \rightarrow R^{+}$ $(i,\ = 1,\ 2,)$ With the following properties:

$S_1$. $P_i(\cdot\ , t) \in L^{\ 1}\ (R^{+}, R^{+})$ for each $t\ \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot\ ) \in C\ ([-\tau, T], R^{+})$ for each $a\ \in R^{+}$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^{+}$ $\{R^{+} \cap\ [0, T]\}$ And is continuous $(i,\ = 1,\ 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,)\ \in R^{+}$ $[-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$P_i(a + x, t + x) =\ \ P_i(x), (i,\ = 1,\ 2,)$ ,

Then (1.3) considered at $(1.3)$ considered at $(a + x,\ t + x)$ becomes

$$\frac{dp_{\ i}(x)}{dx} + f_i(a + x,\ P(t + x))\ \ P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0)\ \exp\left[-\int_0^{\infty} f_{\ i}(a + s, P(t + S))ds\right]$$

And hence

(2.1) $p_{\ 1}\ (a + x, t + x) = p_{\ 1}\ (a, t)\ exp\left[-\int_o^{x} f_{\ 1}\ (a + s, p(t + s))ds\right]$

For $a \geq t$ *we choose* $(a,t) = (a-t,0)$ in (1.3) and $x = t$ in (2.1) so that

(2.2) $p_{\ 1}\ (a, t) =\ \varphi_{\ 1}\ (a - t, 0)\exp\left[-\int_o^{t} f_{\ 1}\left(a - t + s, p_{\ 1}\ (s)\right) ds\right]$

$O \leq t \leq a, i = 1,2,$

With

$P_1(s)=(p_1(s),p_2(s-t)),$

$P_2(s)=(p_1(s-t),p_2(s)),$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$t > a \geq, i = 1,2,$

Where

(2.4) $B_i(t) = p_i(0, t), i = 1,2, t > 0.$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$

(2.5) $L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(a+s, p_i(s))ds]$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da + \int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$t > 0, i = 1,2,$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.(a, t, P_i(t))da$

$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$

$i = 1, 2, t > 0$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$

(2.5), (2.6) and (2.10) lead to

(2.11) $|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i a - d_i^*(t-a)]\, da$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \beta(0) d_i^* t$$

$$\le [\beta_i^* \beta(0)\, exp(\beta_i^* t) \beta(0) d_i^*] t \quad for \ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T],R^+), \quad B_i \in C([0,T],R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau,T] = \{f \in C([-\tau,T],R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau,T]$. Then for any

$(x, y) \in C^+[-\tau,T] \times C^+[-\tau,T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau,T], \quad C_T^2 = C^+[-\tau,T] \times C^+[-\tau,T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau,T]$ *and* $t \in [0,T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Phi: C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Phi(P_t) - \Phi_t\|_{t*} \le \left\{ \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right\}$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ and $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \leq \int_0^t B_1^{(1)}(a)\left\{M_1\left(a,t,P_t^{(1)}\right) - M_1\left(a,t,P_t^{(2)}\right)\right\}da$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \text{ for } x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_t ds\right)da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o)\left[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds\right]da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \|_t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$=\int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\mathrm{X}[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \left\| P_t^{(1)} - P_t^{(2)} \right\| \text{ for } a \in [o, t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \leq \beta_1 \Phi_1(0) ||P_t^{(1)} - P_t^{(2)}||_t^* + \Phi_1(0) \beta_1 d_1 ||P_t^{(1)} - P_t^{(2)}||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) \mathrm{e}^{\delta 1^a} \beta_1 \left\| P_t^{(1)} - P_t^{(2)} \right\|_{t^*} + \beta_1 \mathrm{d}_1 \left\| P_t^{(1)} - P_t^{(2)} \right\|_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) \mathrm{e}^{\delta 1^a} \left\| P_t^{(1)} - P_t^{(2)} \right\|_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| \mathrm{B}_1^{(1)}(\sigma) - \mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{B}_1^{*\mathrm{d}} \sigma$$

$$+ \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \left\{ B_1 + B_2 d_1 a + B_1^* \left( B_2 a + d_1 a^2/2 \right) e^{\delta 1^a} \right\}$$

$$\leq \beta_1^* \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| do + \left( r^* + /\!/ \Phi_t /\!/_{t*} \right) \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \rightarrow \beta_1$ as $a \rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq \left( r^* + /\!/ \Phi_t /\!/_{t*} \right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{ g_1(a) + 0(a) \}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a)) - b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a)) do \right|$$

$$\left| B_1^* \left( r^* + /\!/ \Phi_t /\!/_{t*} \right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} 0(a) as\ a \rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1(P_t^{(2)}) \right| \leq ((r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2(P_t^{(2)}) \right|$ leads to

$$\| \pi (P_t^{(1)} - \pi (P_t^{(2)}) /\!/ \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^*)$$

As $t^* \rightarrow o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

(3.1) $\frac{dp\,P_2^*(a)}{da}+f_i(\mathrm{a}, P_1^*, P_2^*)\,P_i^*(a) = 0,\ \ i = 1, 2, a > 0,$

(3.2) $P_i^* = \int_o^\infty P_i^*(a)da, i = 1,\ 2,$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i = 1.\ \ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s, P_1^*, P_2^*)ds\right], i = 1,\ \ 2, a \geq 0,$

Which together with (3.3) lead to

$1=\int_o^\infty b_1(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_1(s, P_1^*, P_2^*)ds\right]da,$

(3.5) $1=\int_o^\infty b_2(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_2(s, P_1^*, P_2^*)ds\right]da$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1, P_2) = \int_o^\infty b_i(a, P_1, P_2)\exp\left[-\int_o^\infty f_i(s, P_1, P_2)ds\right]da,$

$i$ =1, 2, $P_1 \geq 0$, $P_2 \geq 0$,

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_5$. *there exists a positive constant c for which*

$F_1$(C,0)< $F_{2.}$(C,0),

(3.8) $F_1$(0.C)< $F_{2.}$(0,C),

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $F_1(P_1^*, P_2^*) = 1 = F_2(P_1^*, P_2^*)$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a) = [P_2^* \exp[-f_i(s, P_1^*, P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s, P_1^*, P_2^*)ds]da)$,

$i = 1, 2, \quad a \geq 0,$

Proof Consider the elements of the surfaces defined by

Z= $F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0,$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

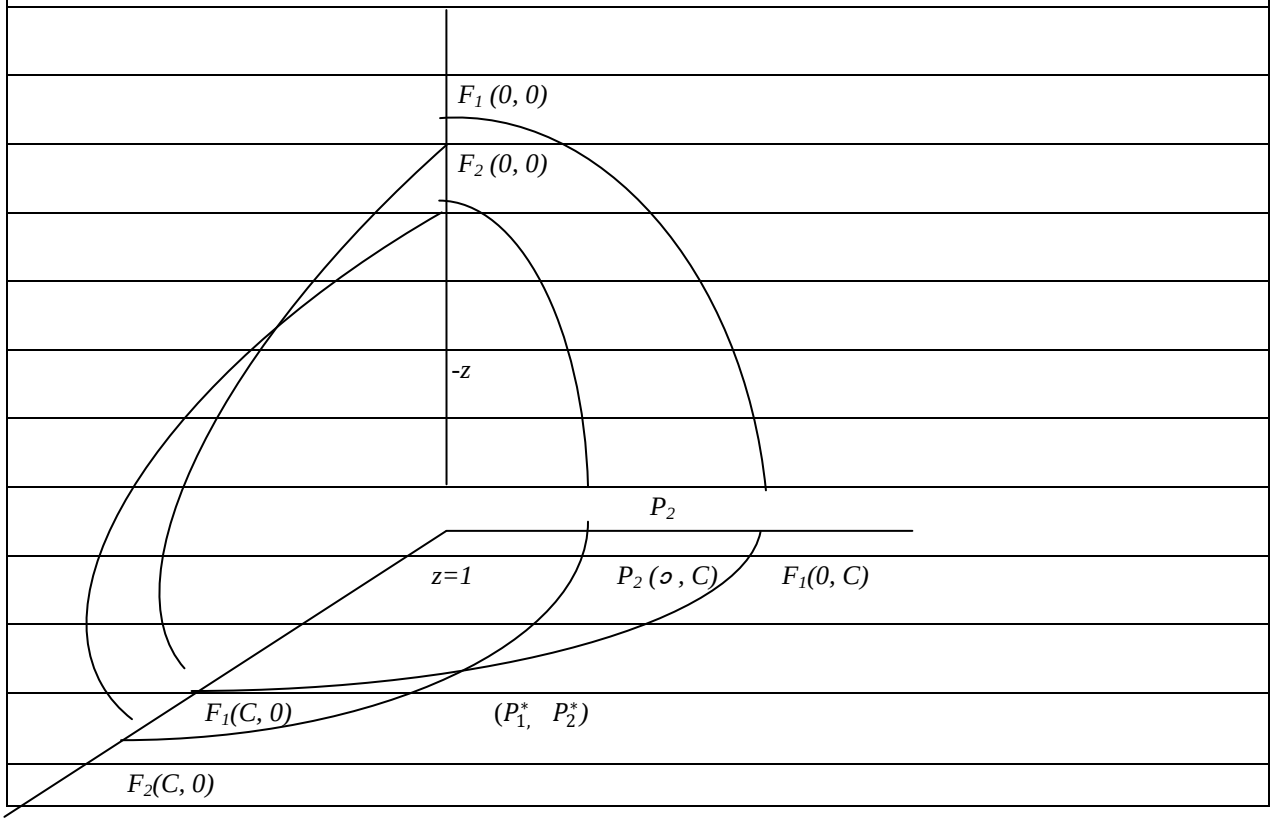
$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, F_1(P_1, P_2 \geq 0,$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.



(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11)
$$\begin{cases} p_i(a,t) = p^*_i(a) + u_i(a,t) \\ p_i(t) = p^*_i + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_i} p_i(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial b_i}{\partial p^*_i} p_i(t)\right\}da \end{cases}$$

Where $i = 1,2$ and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_i(a)\exp[\lambda t], \quad u_i(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty u_i(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{du_i}{da} + \lambda u_i = -f_i\left(a, p^*_1, p^*_2\right) u_i(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j} p^*_i\right\},$$

(3.15)

$$\lambda(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\lambda(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial b_i}{\partial p_j^*}\ p_i^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \ p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\lambda(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1\left(a, p_1^*, p_2^*\right) - f_1\left(a, p_1^*, p_2^*\right)\right]\lambda(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2\left(a, p_1^*, p_2^*\right) - f_2\left(a, p_1^*, p_2^*\right)\right]\lambda(a)da,$$

From the first of (3.15) we get

(3.18) $$\lambda(a) = \left\{\lambda(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\right\} - p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\lambda(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\,e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)\,e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\varphi_1(a)$ and $\varphi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$\left(\lambda - A_{11}\right) p_1^* - A_{12} p_2^* \exp\left(-\lambda\tau\right) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21} p_1^* \exp\left(-\lambda\tau\right) + \left(\lambda - A_{22}\right) p_2^* = g_{21}(\lambda) p_1^* \exp\left(-\lambda\tau\right) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $$A_{i\varphi} = \int_0^\infty p_i^*(a) \left[\frac{\partial b_i}{\partial p_\varphi^*} - \frac{\partial f_i}{\partial p_\varphi^*}\right] da,$$

(3.23) $$g_{i\varphi}(\lambda) = \left[p_i^*(0) c_i\right]^{-1} \left\{\int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} da\right\} \varphi \left\{\int_0^\infty \left[p_i^*(a) \frac{\partial b_i}{\partial p_\varphi^*} - b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_\varphi^*} e^{\lambda s} ds\right)\right] da\right\}$$

$$- \int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_\varphi^*} e^{\lambda s} ds\right) da, \quad i, \varphi = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i\left(a, p_1^*, p_2^*\right) - f_i\left(a, p_1^*, p_2^*\right), \quad i, = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $\left(p_1^*, p_2^*\right)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et \begin{bmatrix} \lambda - \left(A_{11} + g_{11}(\lambda)\right) & -\left(A_{12} + g_{12}\right) e^{-\lambda\tau} \\ -\left(A_{21} + g_{21}\right) e^{-\lambda\tau} & \lambda - \left(A_{22} + g_{22}(\lambda)\right) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda\left(A_{11} + A_{22}\right) + A_{11}A_{22} - A_{12}A_{21} \exp\left[-2\lambda\tau\right] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = \left[A_{21} g_{12}(\lambda) + A_{12} g_{21}(\lambda) + g_{12}(\lambda) g_{21}(\lambda)\right] \exp\left[-2\lambda\tau\right]$

$$-A_{11} g_{22}(\lambda) - A_{22} g_{11}(\lambda) - g_{11}(\lambda) g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ =1,\ 2.)\ \ (see(A_1)\ and\ (A_2));$

$A_7.\ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\},$

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = |A_{11}\, g_{12}(0)| + |A_{12}\, g_{21}(0)| + |g_{12}(0)\, g_{21}(0)|$$

$$+ |A_{11}\, g_{22}(0)| + |A_{22}\, g_{11}(0)| + |g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau),$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$, $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) | - F_2(\lambda, \tau) \right|$$

$$\geq |\lambda|^2 - |\lambda|\{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[(|\lambda| - P)^2 + (q)\right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2,$

$q = \left\{4 | (A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| | - | A_{11} + A_{22} + \| S_1(0) \| |^2\right\}/2,$

It will now follow, from (A) and (3.26), $F_1(\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions

of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends,

assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner

wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the

controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## SECTION A(10)-2

## Excitations And Vorticity

### INTRODUCTION—VARIABLES USED

**Chapter High Performance Computing in Science and Engineering ´15 pp 23-49 Date: 06 February 2016MCTDHB Physics and Technologies: Excitations and Vorticity, Single-Shot Detection, Measurement of Fragmentation, and Optimal Control in Correlated Ultra-Cold Bosonic Many-Body Systems Ofir E. Alon, Vanderlei S. Bagnato, Raphael Beinke, Ioannis Brouzos, Tommaso Calarco,**

**Tommaso Caneva, Lorenz S. Cederbaum, Mark A. Kasevich, Shachar Klaiman and 11 more**

(1) Complimentary, authors continue to develop (eb) the MCTDHB method spanning (e&eb) several directions of (e) the theoretical and computational physics as well as optimal-control theory:
(2) (a) the linear-response on-top of (e) the MCTDHB method (LR-MCTDHB) has been reformulated in (eb) a compact block form, expanded for (e) general inter-particle interactions, and benchmarked against (e&eb) the exactly-solvable harmonic-interaction model;
(3) (b) a new analysis tool capable of (e) simulating the outcomes of typical shots generated in (eb) the experimental detection of (e) ultra-cold atomic systems has been invented, tested, and applied;
(4) (c) a novel algorithm offering (eb) a direct quantitative measurement of the possible fragmentation in bosonic systems has been proposed and applied;
(5) (d) the optimal-control Chopped RAndom Basis (CRAB) algorithm has been merged with (e&eb) the MCTDHB package and applied to manipulate (e&eb) quantum systems. Implications and further perspectives and (e&eb) future research plans are briefly discussed and addressed.

## NOTATION

### Module One

Complimentary, authors continue to develop (eb) the MCTDHB method spanning (e&eb) several directions of (e) the theoretical and computational physics as well as optimal-control theory

$G_{13}$ : Category one of **MCTDHB method** ;MCTDHB method spanning (e&eb) several directions of (e) the theoretical and computational physics as well as optimal-control theory**;**

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of MCTDHB method spanning (e&eb) several directions of (e) the theoretical and computational physics as well as optimal-control theory **;MCTDHB method**

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

### Module Two

Complimentary, authors continue to develop the MCTDHB method spanning (e&eb) several directions of (e) the theoretical and computational physics as well as optimal-control theory

$G_{16}$ : Category one of **MCTDHB method spanning several directions;** theoretical and computational physics as well as optimal-control theory

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

$T_{16}$ : Category one of theoretical and computational physics as well as optimal-control theory **;MCTDHB method spanning several directions**

$T_{17}$ : Category two of SAS

$T_{18}$ : Category three of SAS

| | |
|---|---|
| **Module three**<br><br>the linear-response on-top of (e) the MCTDHB method (LR-MCTDHB) has been reformulated in (eb) a compact block form, expanded for (e) general inter-particle interactions, and benchmarked against (e&eb) the exactly-solvable harmonic-interaction model | |
| $G_{20}$ : Category one of MCTDHB method (LR-MCTDHB) has been reformulated in (eb) a compact block form, expanded for (e) general inter-particle interactions, and benchmarked against (e&eb) the exactly-solvable harmonic-interaction model<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of linear-response on-top<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>the linear-response on-top of the MCTDHB method (LR-MCTDHB) has been reformulated in (eb) a compact block form, expanded for (e) general inter-particle interactions, and benchmarked against (e&eb) the exactly-solvable harmonic-interaction model | |
| $G_{24}$ : Category one of linear-response on-top of the MCTDHB method (LR-MCTDHB) has been reformulated<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of compact block form, expanded for (e) general inter-particle interactions, and benchmarked against (e&eb) the exactly-solvable harmonic-interaction model<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>the linear-response on-top of the MCTDHB method (LR-MCTDHB) has been reformulated in a compact block form, expanded for (e) general inter-particle interactions, and benchmarked against (e&eb) the exactly-solvable harmonic-interaction model | |
| $G_{28}$ : Category one of general inter-particle interactions, and benchmarked against (e&eb) the exactly-solvable harmonic-interaction model<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of linear-response on-top of the MCTDHB method (LR-MCTDHB) has been | |

| | |
|---|---|
| reformulated in a compact block form, expanded<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>the linear-response on-top of the MCTDHB method (LR-MCTDHB) has been reformulated in a compact block form, expanded for general inter-particle interactions, and benchmarked against (e&eb) the exactly-solvable harmonic-interaction model | |
| $G_{32}$ : Category one of **linear-response on-top of the MCTDHB method (LR-MCTDHB) has been reformulated in a compact block form, expanded for general inter-particle interactions, and benchmarked against;** exactly-solvable harmonic-interaction model<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of exactly-solvable harmonic-interaction model; **linear-response on-top of the MCTDHB method (LR-MCTDHB) has been reformulated in a compact block form, expanded for general inter-particle interactions, and benchmarked against**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>new analysis tool capable of simulating the outcomes of typical shots generated in (eb) the experimental detection of (e) ultra-cold atomic systems has been invented, tested, and applied | |
| $G_{36}$ : Category one of **new analysis tool capable of simulating the outcomes of typical shots generated**<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of experimental detection of (e) ultra-cold atomic systems has been invented, tested, and applied<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>novel algorithm offering a direct quantitative measurement of the possible fragmentation in bosonic systems has been proposed and applied | |

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| $G_{40}$ : Category one of novel algorithm offering a direct quantitative measurement of the possible fragmentation<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of bosonic systems has been proposed and applied<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Optimal-control Chopped RAndom Basis (CRAB) algorithm has been merged with (e&eb) the MCTDHB package and applied to manipulate (e&eb) quantum systems.<br><br>Implications and further perspectives and (e&eb) future research plans are briefly discussed and addressed | |
| $G_{44}$ : Category one of **Optimal-control Chopped RAndom Basis (CRAB) algorithm;** MCTDHB package and applied to manipulate (e&eb) quantum systems.<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of MCTDHB package and applied to manipulate (e&eb) quantum systems. ;**Optimal-control Chopped RAndom Basis (CRAB) algorithm**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION A(11)**<br><br>**Quantum Yang-Mills Theory** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Quantum Yang-Mills theory: an overview of a programme Ashley Milsted, Tobias J. Osborne (Submitted on 7 Apr 2016)**<br><br>(1) **Ashley Milsted, Tobias J. Osborne** present an overview of a programme to understand the low-energy physics of (e) quantum Yang-Mills theory from (e) a quantum-information perspective.<br>(2) Setting is that of the Hamiltonian formulation of (e) pure Yang-Mills theory in the temporal gauges on (e&eb) the lattice. | |

(3) Firstly, inspired by (e) recent constructions for (e) Z/2Z lattice gauge theory, in particular, Kitaev's toric code, authors describe (eb) the gauge-invariant sector of (e) Hilbert space by introducing (eb) a primitive quantum gate: the quantum parallel transporter.
(4) They then develop a nonabelian generalisation of Laplace interpolation to present (eb) an ansatz for (e) the ground state of (e) pure Yang-Mills theory which interpolates between the weak- and (e&eb) strong-coupling RG fixed points.
(5) The resulting state acquires (e) the structure of a tensor network, namely, a multiscale entanglement renormalisation ansatz, and allows (eb)for the efficient computation of (e) local observables and Wilson loops.
(6) Various refinements of the tensor network are discussed leading to (eb) several generalisations. Finally, the continuum limit of ansatz as the lattice regulator is removed is then described.
**(7)** This paper is intended as an abstract for (e) an ongoing programme: there are still many open problems Subjects: Quantum Physics (quant-ph); High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1604.01979 [quant-ph] (or arXiv: 1604.01979v1 [quant-ph] for this version)

**NOTATION**

**Module One**

**Ashley Milsted, Tobias J. Osborne** present an overview of a programme to understand the low-energy physics of (e) quantum Yang-Mills theory from (e) a quantum-information perspective

$G_{13}$ : Category one of quantum Yang-Mills theory from (e) a quantum-information perspective

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of low-energy physics

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

**Ashley Milsted, Tobias J. Osborne** present an overview of a programme to understand the low-energy physics of quantum Yang-Mills theory from (e) a quantum-information perspective

$G_{16}$ : Category one of quantum-information perspective

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

$T_{16}$ : Category one of low-energy physics of quantum Yang-Mills theory

$T_{17}$ : Category two of SAS

$T_{18}$ : Category three of SAS

**Module three**

Setting is that of the Hamiltonian formulation of (e) pure Yang-Mills theory in the temporal gauges on

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| (e&eb) the lattice | |
| $G_{20}$ : Category one of pure Yang-Mills theory in the temporal gauges on (e&eb) the lattice<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of Hamiltonian formulation<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>Setting is that of the Hamiltonian formulation of pure Yang-Mills theory in the temporal gauges on (e&eb) the lattice | |
| $G_{24}$ : Category one of **Hamiltonian formulation of pure Yang-Mills theory in the temporal gauges;** lattice<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of lattice **;Hamiltonian formulation of pure Yang-Mills theory in the temporal gauges**<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>Firstly, inspired by (e) recent constructions for (e) Z/2Z lattice gauge theory, in particular, Kitaev's toric code, authors describe (eb) the gauge-invariant sector of (e) Hilbert space by introducing (eb) a primitive quantum gate: the quantum parallel transporter | |
| $G_{28}$ : Category one of Z/2Z lattice gauge theory, in particular, Kitaev's toric code, authors describe the gauge-invariant sector; **Hilbert space by introducing (eb) a primitive quantum gate: the quantum parallel transporter**<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of **Hilbert space by introducing (eb) a primitive quantum gate: the quantum parallel transporter;** Z/2Z lattice gauge theory, in particular, Kitaev's toric code, authors describe the gauge-invariant sector<br>$T_{29}$ : Category two of SAS | |

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| $T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Firstly, inspired by recent constructions for Z/2Z lattice gauge theory, in particular, Kitaev's toric code, authors describe the gauge-invariant sector of Hilbert space by introducing (eb) a primitive quantum gate: the quantum parallel transporter | |
| $G_{32}$ : Category one of Z/2Z lattice gauge theory, in particular, Kitaev's toric code, authors describe the gauge-invariant sector of Hilbert space by introducing<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of primitive quantum gate: the quantum parallel transporter<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>They then develop a nonabelian generalisation of Laplace interpolation to present (eb) an ansatz for (e) the ground state of (e) pure Yang-Mills theory which interpolates between the weak- and (e&eb) strong-coupling RG fixed points | |
| $G_{36}$ : Category one of nonabelian generalisation of Laplace interpolation<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of ansatz for (e) the ground state of (e) pure Yang-Mills theory which interpolates between the weak- and (e&eb) strong-coupling RG fixed points<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>They then develop a nonabelian generalisation of Laplace interpolation to present an ansatz for (e) the ground state of (e) pure Yang-Mills theory which interpolates between the weak- and (e&eb) strong-coupling RG fixed points | |
| $G_{40}$ : Category one of ground state of (e) pure Yang-Mills theory which interpolates between the weak- and (e&eb) strong-coupling RG fixed points<br><br>$G_{41}$ : Category two of SAS | |

$G_{42}$ : Category three of SAS

$T_{40}$ : Category one of nonabelian generalisation of Laplace interpolation to present an ansatz

$T_{41}$ : Category two of SAS

$T_{42}$ : Category three of SAS

**Module Nine**

They then develop a nonabelian generalisation of Laplace interpolation to present an ansatz for (e) the ground state of pure Yang-Mills theory which interpolates between the weak- and (e&eb) strong-coupling RG fixed points

$G_{44}$ : Category one of **nonabelian generalisation of Laplace interpolation to present an ansatz for (e) the ground state of pure Yang-Mills theory which interpolates between the weak;** strong-coupling RG fixed points

$G_{45}$ : Category two of SAS

$G_{46}$ : Category three of SAS

$T_{44}$ : Category one of strong-coupling RG fixed points **;nonabelian generalisation of Laplace interpolation to present an ansatz for (e) the ground state of pure Yang-Mills theory which interpolates between the weak**

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

**The Coefficients:**

$(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:
$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$
$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,
$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$
$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$
$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$
$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$

are Accentuation coefficients
$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$
$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$
$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}\ (a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,
$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$
$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,
$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,

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| $(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)},$ are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}, t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}, t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21}, t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23}, t) =$ First detritions factor | |

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| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25}, t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}, t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}), t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}), t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}), t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25}, t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}), t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29}, t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}, t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}), t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}), t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}), t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29}, t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}), t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33}, t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}, t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}), t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}), t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}), t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33}, t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}, t)\right]G_{37}$ | *38* |

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| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{array}{ccc} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for | |

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| category 1, 2 and 3<br>$+(a_{16}'')^{(2,2,)}(T_{17},t)$, $+(a_{17}'')^{(2,2,)}(T_{17},t)$, $+(a_{18}'')^{(2,2,)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}'')^{(3,3,)}(T_{21},t)$, $+(a_{21}'')^{(3,3,)}(T_{21},t)$, $+(a_{22}'')^{(3,3,)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}'')^{(4,4,4,4,)}(T_{25},t)$, $+(a_{25}'')^{(4,4,4,4,)}(T_{25},t)$, $+(a_{26}'')^{(4,4,4,4,)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}'')^{(5,5,5,5,)}(T_{29},t)$, $+(a_{29}'')^{(5,5,5,5,)}(T_{29},t)$, $+(a_{30}'')^{(5,5,5,5,)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}'')^{(6,6,6,6,)}(T_{33},t)$, $+(a_{33}'')^{(6,6,6,6,)}(T_{33},t)$, $+(a_{34}'')^{(6,6,6,6,)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{38}'')^{(7,7)}(T_{37},t)$ $+(a_{37}'')^{(7,7)}(T_{37},t)$ $+(a_{36}'')^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3<br>$+(a_{40}'')^{(8,8)}(T_{41},t)$ $+(a_{41}'')^{(8,8)}(T_{41},t)$ $+(a_{42}'')^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3<br>$+(a_{44}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}')^{(1)}\left[-(b_{13}'')^{(1)}(G,t)\right] & \left[-(b_{16}'')^{(2,2,)}(G_{19},t)\right] & \left[-(b_{20}'')^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{24}'')^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{28}'')^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{32}'')^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{36}'')^{(7,7,)}(G_{39},t)\right] & \left[-(b_{40}'')^{(8,8)}(G_{43},t)\right] & \left[-(b_{44}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}')^{(1)}\left[-(b_{14}'')^{(1)}(G,t)\right] & \left[-(b_{17}'')^{(2,2,)}(G_{19},t)\right] & \left[-(b_{21}'')^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{25}'')^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{29}'')^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{33}'')^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{37}'')^{(7,7,)}(G_{39},t)\right] & \left[-(b_{41}'')^{(8,8)}(G_{43},t)\right] & \left[-(b_{45}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}')^{(1)}\left[-(b_{15}'')^{(1)}(G,t)\right] & \left[-(b_{18}'')^{(2,2,)}(G_{19},t)\right] & \left[-(b_{22}'')^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{26}'')^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{30}'')^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{34}'')^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{38}'')^{(7,7,)}(G_{39},t)\right] & \left[-(b_{42}'')^{(8,8)}(G_{43},t)\right] & \left[-(b_{46}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{15}$ | *60* |
| Where $-(b_{13}'')^{(1)}(G,t)$, $-(b_{14}'')^{(1)}(G,t)$, $-(b_{15}'')^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{16}'')^{(2,2,)}(G_{19},t)$, $-(b_{17}'')^{(2,2,)}(G_{19},t)$, $-(b_{18}'')^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{20}'')^{(3,3,)}(G_{23},t)$, $-(b_{21}'')^{(3,3,)}(G_{23},t)$, $-(b_{22}'')^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}'')^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}'')^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}'')^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}'')^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}'')^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}'')^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}'')^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}'')^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}'')^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for | |

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| category 1, 2 and 3 $\boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[\begin{array}{ccc} (a_{16}^{'})^{(2)} & \boxed{+(a_{16}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)} \quad \boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)} \end{array}\right] G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[\begin{array}{ccc} (a_{17}^{'})^{(2)} & \boxed{+(a_{17}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)} \quad \boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)} \end{array}\right] G_{17}$ | *62* |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[\begin{array}{ccc} (a_{18}^{'})^{(2)} & \boxed{+(a_{18}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)} \quad \boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)} \end{array}\right] G_{18}$ | *63* |
| Where $\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)}$ are first augmentation coefficients for category 1, 2 and 3 $\boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)}$ are second augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)}$ are seventh augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)}$ are eight augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)}$ are ninth augmentation coefficient for category 1, 2 and 3 | |

$$\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b'_{16})^{(2)}\boxed{-(b''_{16})^{(2)}(G_{19},t)} & \boxed{-(b''_{13})^{(1,1,)}(G,t)} & \boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{16} \quad (64)$$

$$\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b'_{17})^{(2)}\boxed{-(b''_{17})^{(2)}(G_{19},t)} & \boxed{-(b''_{14})^{(1,1,)}(G,t)} & \boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{17} \quad (65)$$

$$\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b'_{18})^{(2)}\boxed{-(b''_{18})^{(2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,)}(G,t)} & \boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{18} \quad (66)$$

where $\boxed{-(b''_{16})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{17})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{18})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,)}(G,t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{44})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2 and 3

$$\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a'_{20})^{(3)}\boxed{+(a''_{20})^{(3)}(T_{21},t)} & \boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{20} \quad (67)$$

$$\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a'_{21})^{(3)}\boxed{+(a''_{21})^{(3)}(T_{21},t)} & \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{21} \quad (68)$$

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a'_{22})^{(3)}\boxed{+(a''_{22})^{(3)}(T_{21},t)} & \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{22} \quad (69)$$

$\boxed{+(a''_{20})^{(3)}(T_{21},t)}, \boxed{+(a''_{21})^{(3)}(T_{21},t)}, \boxed{+(a''_{22})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)}, \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)}, \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)}, \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)}, \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)}$ are third augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)}, \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)}, \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)}, \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)}, \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)}, \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)}, \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)}, \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)}, \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)}, \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)}, \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)}$ are eight augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)}, \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)}, \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1, 2 and 3

$$\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b'_{20})^{(3)}\boxed{-(b''_{20})^{(3)}(G_{23},t)} & \boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{13})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{20} \quad (70)$$

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b'_{21})^{(3)}\boxed{-(b''_{21})^{(3)}(G_{23},t)} & \boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{14})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)}\boxed{-(b''_{22})^{(3)}(G_{23},t)} & \boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{22} \quad (72)$$

$\boxed{-(b''_{20})^{(3)}(G_{23},t)}, \boxed{-(b''_{21})^{(3)}(G_{23},t)}, \boxed{-(b''_{22})^{(3)}(G_{23},t)}$ *are first detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)}, \boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)}, \boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)}$ *are second detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{13})^{(1,1,1,)}(G,t)}, \boxed{-(b''_{14})^{(1,1,1,)}(G,t)}, \boxed{-(b''_{15})^{(1,1,1,)}(G,t)}$ *are third detrition coefficients for category 1,2 and 3*

| | |
|---|---|
| $-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*<br>$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$ $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a_{24}^{'})^{(4)} & +(a_{24}^{''})^{(4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,)}(T_{29},t) & +(a_{32}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24}$ | *73* |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a_{25}^{'})^{(4)} & +(a_{25}^{''})^{(4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,)}(T_{29},t) & +(a_{33}^{''})^{(6,6)}(T_{33},t) \\ & +(a_{14}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25}$ | *74* |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a_{26}^{'})^{(4)} & +(a_{26}^{''})^{(4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,)}(T_{29},t) & +(a_{34}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26}$ | *75* |

$(a_{24}^{''})^{(4)}(T_{25},t)$, $(a_{25}^{''})^{(4)}(T_{25},t)$, $(a_{26}^{''})^{(4)}(T_{25},t)$ are first augmentation coefficients category 1, 2 3

$+(a_{28}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,)}(T_{29},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,)}(T_{33},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1, 2 and 3

$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3

$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1, 2 and 3

$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b_{24}^{'})^{(4)} & -(b_{24}^{''})^{(4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,)}(G_{35},t) \\ & -(b_{13}^{''})^{(1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3)}(G_{23},t) \\ & -(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b_{25}^{'})^{(4)} & -(b_{25}^{''})^{(4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,)}(G_{35},t) \\ & -(b_{14}^{''})^{(1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3)}(G_{23},t) \\ & -(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b_{26}^{'})^{(4)} & -(b_{26}^{''})^{(4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,)}(G_{35},t) \\ & -(b_{15}^{''})^{(1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3)}(G_{23},t) \\ & -(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{26} \quad (78)$$

Where $-(b_{24}^{''})^{(4)}(G_{27},t)$, $-(b_{25}^{''})^{(4)}(G_{27},t)$, $-(b_{26}^{''})^{(4)}(G_{27},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,)}(G_{31},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,)}(G_{35},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2 and 3

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| $-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 $-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3 $-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a_{28}^{'})^{(5)} & +(a_{28}^{''})^{(5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,)}(T_{25},t) & +(a_{32}^{''})^{(6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a_{29}^{'})^{(5)} & +(a_{29}^{''})^{(5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,)}(T_{25},t) & +(a_{33}^{''})^{(6,6,6)}(T_{33},t) \\ & +(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a_{30}^{'})^{(5)} & +(a_{30}^{''})^{(5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,)}(T_{25},t) & +(a_{34}^{''})^{(6,6,6)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{30}$ | *81* |
| *Where* $+(a_{28}^{''})^{(5)}(T_{29},t)$, $+(a_{29}^{''})^{(5)}(T_{29},t)$, $+(a_{30}^{''})^{(5)}(T_{29},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3 *And* $+(a_{24}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,)}(T_{25},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3 $+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3 $+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3 $+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3 $+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3 $+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3 $+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3 $+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b'_{28})^{(5)} & -(b''_{28})^{(5)}(G_{31},t) & -(b''_{24})^{(4,4,)}(G_{27},t) & -(b''_{32})^{(6,6,6)}(G_{35},t) \\ -(b''_{13})^{(1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b'_{29})^{(5)} & -(b''_{29})^{(5)}(G_{31},t) & -(b''_{25})^{(4,4,)}(G_{27},t) & -(b''_{33})^{(6,6,6)}(G_{35},t) \\ -(b''_{14})^{(1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b'_{30})^{(5)} & -(b''_{30})^{(5)}(G_{31},t) & -(b''_{26})^{(4,4,)}(G_{27},t) & -(b''_{34})^{(6,6,6)}(G_{35},t) \\ -(b''_{15})^{(1,1,1,1,1,)}(G,t) & -(b''_{18})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{30} \quad (84)$$

*where* $-(b''_{28})^{(5)}(G_{31},t)$ , $-(b''_{29})^{(5)}(G_{31},t)$ , $-(b''_{30})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{24})^{(4,4,)}(G_{27},t)$, $-(b''_{25})^{(4,4,)}(G_{27},t)$, $-(b''_{26})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3

$-(b''_{32})^{(6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b''_{13})^{(1,1,1,1,1)}(G,t)$, $-(b''_{14})^{(1,1,1,1,1)}(G,t)$ , $-(b''_{15})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3

$-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3

$-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)$, $-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)$, $-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3

$-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3

$-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3

$-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a'_{32})^{(6)} & +(a''_{32})^{(6)}(T_{33},t) & +(a''_{28})^{(5,5,5)}(T_{29},t) & +(a''_{24})^{(4,4,4,)}(T_{25},t) \\ +(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a'_{33})^{(6)} & +(a''_{33})^{(6)}(T_{33},t) & +(a''_{29})^{(5,5,5)}(T_{29},t) & +(a''_{25})^{(4,4,4,)}(T_{25},t) \\ +(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)} G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)} T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)} T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)} T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{28})^{(5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

| | |
|---|---|
| $\boxed{-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)}$ *are third detrition coefficients for category* 1,2 *and* 3<br>$\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2, and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2, and 3<br>$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2, and 3<br>$\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2, and 3<br>$\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2, and 3<br>$\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2, and 3 | |
| $\frac{dG_{36}}{dt}$ $= (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)}\boxed{+(a_{36}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *91* |
| $\frac{dG_{37}}{dt}$ $= (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)}\boxed{+(a_{37}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *92* |
| $\frac{dG_{38}}{dt}$ $= (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)}\boxed{+(a_{38}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *93* |
| Where $\boxed{(a_{36}^{''})^{(7)}(T_{37},t)}$, $\boxed{(a_{37}^{''})^{(7)}(T_{37},t)}$, $\boxed{(a_{38}^{''})^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation | |

coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3 $+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}(G_{39},t) & -(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$ 94

$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}-(b_{37}^{''})^{(7)}(G_{39},t) & -(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}(G_{39},t) & -(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3 $-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3 $-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3 $-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{46}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

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$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}')^{(8)}+(a_{40}'')^{(8)}(T_{41},t) & +(a_{16}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}'')^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13}$$

$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}')^{(8)}+(a_{41}'')^{(8)}(T_{41},t) & +(a_{17}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}'')^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}')^{(8)}+(a_{42}'')^{(8)}(T_{41},t) & +(a_{18}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}'')^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$$

Where $+(a_{40}'')^{(8)}(T_{41},t)$, $+(a_{41}'')^{(8)}(T_{41},t)$, $+(a_{42}'')^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & -(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t) & -(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & -(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t) & -(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & -(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t) & -(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3



$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{13}$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[\begin{array}{ccc} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$\frac{dT_{44}}{dt} =$

$$(b_{44})^{(9)}T_{45} - \left[\begin{array}{ccc} (b_{44}^{'})^{(9)}\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[\begin{array}{ccc} (b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[\begin{array}{ccc} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$$

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

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|---|---|
| **Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br>$(a_i^{''})^{(1)}(T_{14},t) \leq (p_i)^{(1)} \leq (\hat{A}_{13})^{(1)}$<br>$(b_i^{''})^{(1)}(G,t) \leq (r_i)^{(1)} \leq (b_i^{'})^{(1)} \leq (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$<br>$\lim_{G\to\infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$<br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'},t) - (a_i^{''})^{(1)}(T_{14},t)\rvert \leq (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br>$\lvert(b_i^{''})^{(1)}(G^{'},t) - (b_i^{''})^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G^{'}\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i^{'})^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i^{'})^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i^{'})^{(2)}, (a_i^{''})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (b_i^{''})^{(2)} > 0, \quad i,j = 16,17,18$ | |

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| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17}, t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19}, t) \leq (r_i)^{(2)} \leq (b_i')^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty} (a_i'')^{(2)}(T_{17}, t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty} (b_i'')^{(2)}((G_{19}), t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}$ are positive constants and $i = 16,17,18$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(2)}(T_{17}', t) - (a_i'')^{(2)}(T_{17}, t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}'\rvert e^{-(\hat{M}_{16})^{(2)} t}$ | 107 |
| $\lvert(b_i'')^{(2)}((G_{19})', t) - (b_i'')^{(2)}((G_{19}), t)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})'\rvert\rvert e^{-(\hat{M}_{16})^{(2)} t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}', t)$ and $(a_i'')^{(2)}(T_{17}, t)$ . $(T_{17}', t)$ and $(T_{17}, t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}]$ . It is to be noted that $(a_i'')^{(2)}(T_{17}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i'')^{(2)}(T_{17}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i')^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |

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|---|---|
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21},t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23},t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2\to\infty}(a_i^{''})^{(3)}(T_{21},t) = (p_i)^{(3)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'},t) - (a_i^{''})^{(3)}(T_{21},t)\rvert \leq (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'},t) - (b_i^{''})^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'},t)$ and$(a_i^{''})^{(3)}(T_{21},t)$ . $(T_{21}^{'},t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$<br>satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |

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|---|---|
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24{,}25{,}26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}\big((G_{27}), t\big) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(4)}\big((G_{27}), t\big) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24{,}25{,}26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\rvert \leq (\hat{k}_{24})^{(4)} \lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)} t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'}, t) - (b_i^{''})^{(4)}\big((G_{27}), t\big)\rvert < (\hat{k}_{24})^{(4)} \lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)} t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $\big[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\big]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24{,}25{,}26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |

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| $\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i^{'})^{(4)} +\ (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}\big((G_{31}), t\big) \leq\ (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(5)}\,(T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(5)}\,(G_{31}, t) =\ (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'}, t) - (b_i^{''})^{(5)}\big((G_{31}), t\big)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $\big[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\big]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br><br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\ , \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities | 126 |

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| $\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(6)}(T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(6)}((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)}\lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})^{'}, t) - (b_i^{''})^{(6)}((G_{35}), t)\rvert < (\hat{k}_{32})^{(6)}\lvert\lvert(G_{35}) - (G_{35})^{'}\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and$(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants | 130 |

$(a_i)^{(6)}, (a_i')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$
satisfy the inequalities

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i')^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i')^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

Where we suppose

(GGGG) $(a_i)^{(7)}, (a_i')^{(7)}, (a_i'')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (b_i'')^{(7)} > 0, \quad i, j = 36,37,38$ 131

(HHHH) The functions $(a_i'')^{(7)}, (b_i'')^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$:

$$(a_i'')^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$$

$$(b_i'')^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i')^{(7)} \leq (\hat{B}_{36})^{(7)}$$

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*(IIII)* $\lim_{T_2 \to \infty} (a_i'')^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(JJJJ)*

$\lim_{G \to \infty} (b_i'')^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, \ (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i'')^{(7)}(T_{37}', t) - (a_i'')^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)} |T_{37} - T_{37}'| e^{-(\hat{M}_{36})^{(7)} t}$$

$$|(b_i'')^{(7)}((G_{39})', t) - (b_i'')^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)} ||(G_{39}) - (G_{39})'|| e^{-(\hat{M}_{36})^{(7)} t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(7)}(T_{37}', t)$ and $(a_i'')^{(7)}(T_{37}, t)$ . $(T_{37}', t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$ . It is to be noted that $(a_i'')^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i'')^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ : 134

(KKKK) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants

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| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :<br><br>(LLLL) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}, (r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})', t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})'\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would | |

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| be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i, j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45}, t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47}, t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2 \to \infty} (a_i'')^{(9)}(T_{45}, t) = (p_i)^{(9)}$<br>$\lim_{G \to \infty} (b_i'')^{(9)}(G_{47}, t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}', t) - (a_i'')^{(9)}(T_{45}, t)\rvert \le (\hat{k}_{44})^{(9)} \lvert T_{45} - T_{45}' \rvert e^{-(\hat{M}_{44})^{(9)} t}$<br>$\lvert(b_i'')^{(9)}((G_{47})', t) - (b_i'')^{(9)}((G_{47}), t)\rvert < (\hat{k}_{44})^{(9)} \lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}', t)$ | |

and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)},(\widehat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{44})^{(9)}=1$ then the function $(a_i'')^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$ :

$(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\ ,\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}<1$$

**Definition of** $(\hat{P}_{44})^{(9)},(\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},(\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)},(a_i')^{(9)},(b_i)^{(9)},(b_i')^{(9)},(p_i)^{(9)},\ (r_i)^{(9)}, i=44,45,46,$ satisfy the inequalities

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[(a_i)^{(9)}+(a_i')^{(9)}+\ (\hat{A}_{44})^{(9)}+\ (\hat{P}_{44})^{(9)}\ (\hat{k}_{44})^{(9)}]<1$$

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[\ (b_i)^{(9)}+(b_i')^{(9)}+\ (\hat{B}_{44})^{(9)}+\ (\hat{Q}_{44})^{(9)}\ \ (\hat{k}_{44})^{(9)}]<1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0)\ ,T_i(0)$ :

$$G_i(t)\le\ \left(\hat{P}_{13}\right)^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{G_i(0)=G_i^0>0}$$

$$T_i(t)\le\ (\hat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{T_i(0)=T_i^0>0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0)\ ,T_i(0)$

$$G_i(t)\le\ (\hat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t)\le\ (\hat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

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| **Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$ | 153 B |

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| $T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |

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| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br><br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| By<br><br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} G_{21}(s_{(20)}) - \left((a_{20}')^{(3)} + a_{20}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{20}(s_{(20)})\right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[(a_{21})^{(3)} G_{20}(s_{(20)}) - \left((a_{21}')^{(3)} + (a_{21}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}')^{(3)} - (b_{21}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br><br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{28})^{(5)}$, $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[ (b_{29})^{(5)} T_{28}(s_{(28)}) - \left( (b'_{29})^{(5)} - (b''_{29})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[ (b_{30})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{30})^{(5)} - (b''_{30})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{30}(s_{(28)}) \right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** | |

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| Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[ (a_{32})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{32}(s_{(32)}) \right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[ (a_{33})^{(6)} G_{32}(s_{(32)}) - \left( (a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[ (a_{34})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{34}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[ (b_{32})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{32}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[ (b_{33})^{(6)} T_{32}(s_{(32)}) - \left( (b'_{33})^{(6)} - (b''_{33})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[ (b_{34})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{34})^{(6)} - (b''_{34})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{34}(s_{(32)}) \right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |

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| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}^{'})^{(8)} + a_{40}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^{0} + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^{0} + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^{0} + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[ (a_{44})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{44}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[ (a_{45})^{(9)} G_{44}(s_{(44)}) - \left( (a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{45}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[ (a_{46})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{46}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[ (b_{44})^{(9)} T_{45}(s_{(44)}) - \left( (b'_{44})^{(9)} - (b''_{44})^{(9)} (G_{47}(s_{(44)}), s_{(44)}) \right) T_{44}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[ (b_{45})^{(9)} T_{44}(s_{(44)}) - \left( (b'_{45})^{(9)} - (b''_{45})^{(9)} (G_{47}(s_{(44)}), s_{(44)}) \right) T_{45}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[ (b_{46})^{(9)} T_{45}(s_{(44)}) - \left( (b'_{46})^{(9)} - (b''_{46})^{(9)} (G_{47}(s_{(44)}), s_{(44)}) \right) T_{46}(s_{(44)}) \right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \leq G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} \left( G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}} \right) \right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left( e^{(\hat{M}_{13})^{(1)} t} - 1 \right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \leq \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[ \left( (\hat{P}_{13})^{(1)} + G_{14}^0 \right) e^{\left( - \frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0} \right)} + (\hat{P}_{13})^{(1)} \right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}$ , $G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \leq G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} \left( G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}} \right) \right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left( e^{(\hat{M}_{16})^{(2)} t} - 1 \right)$ | 169 |
| From which it follows that | 170 |

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| $(G_{16}(t)-G_{16}^0)e^{-(\bar{M}_{16})^{(2)}t} \le \frac{(a_{16})^{(2)}}{(\bar{M}_{16})^{(2)}}\left[((\hat{P}_{16})^{(2)}+G_{17}^0)e^{\left(-\frac{(\hat{P}_{16})^{(2)}+G_{17}^0}{G_{17}^0}\right)}+(\hat{P}_{16})^{(2)}\right]$ | |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0+(\hat{P}_{20})^{(3)}e^{(\bar{M}_{20})^{(3)}s_{(20)}}\right)\right] ds_{(20)} =$<br>$(1+(a_{20})^{(3)}t)G_{21}^0 + \frac{(a_{20})^{(3)}(\hat{P}_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left(e^{(\bar{M}_{20})^{(3)}t}-1\right)$ | 171 |
| From which it follows that<br>$(G_{20}(t)-G_{20}^0)e^{-(\bar{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that<br>$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\bar{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$<br>$(1+(a_{24})^{(4)}t)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left(e^{(\bar{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that<br>$(G_{24}(t)-G_{24}^0)e^{-(\bar{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$<br>$(1+(a_{28})^{(5)}t)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)}t}-1\right)$ | |
| From which it follows that<br>$(G_{28}(t)-G_{28}^0)e^{-(\bar{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 5 | 175 |
| The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious | 176 |

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| that<br>$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} \left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$<br>$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}} \left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$ | |
| From which it follows that<br>$(G_{32}(t) - G_{32}^0) e^{-(\hat{M}_{32})^{(6)} t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}} \left[((\hat{P}_{32})^{(6)} + G_{33}^0) e^{\left(-\frac{(\hat{P}_{32})^{(6)} + G_{33}^0}{G_{33}^0}\right)} + (\hat{P}_{32})^{(6)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 6<br>Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$ | 177 |
| (o) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} \left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$<br>$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$ | 178 |
| From which it follows that<br>$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left[((\hat{P}_{36})^{(7)} + G_{37}^0) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 7 | |
| The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} \left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$<br>$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$ | 180 |
| From which it follows that<br>$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left[((\hat{P}_{40})^{(8)} + G_{41}^0) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} \left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$ | |

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| $\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$ | |
| From which it follows that<br>$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t}\leq\frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{P}_{44})^{(9)}+G_{45}^{0}\right)e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ and to choose<br>$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{13})^{(1)}\left\|G_{14}^{(1)}-G_{14}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$<br>$\int_0^t\{(a_{13}')^{(1)}\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{-(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$G_{13}^{(2)}\left\\|(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}'')^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right\\|\ e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$ | |

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| Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | |
| $\left\|G^{(1)} - G^{(2)}\right\| e^{-(\widehat{M}_{13})^{(1)}t} \leq$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right) d\left(\left(G^{(1)}, T^{(1)};\ G^{(2)}, T^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(1)}$ and $(b''_i)^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(1)} - (a''_i)^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1 / (a'_{14})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2 / (a'_{15})^{(1)}$<br><br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(1)}(G(t), t)) = (b'_{14})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ : | 189 |

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| Indeed let $t_1$ be so that for $t > t_1$<br>$(b_{14})^{(1)} - (b_i'')^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}'')^{(1)}(G(t),t) = (b_{15}')^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ $and$ $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |

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| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$(a_{16}^{''})^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)} \left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}} \left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)} (\widehat{k}_{16})^{(2)}\right) d\left(\left((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)}\right)\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)} (T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0 \quad \text{for } t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)} \left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$<br>In the same way , one can obtain | 201 |

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| $G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$, $G_{18}$ and $G_{16}$, $G_{17}$ respectively. | |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$. The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}$, $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}} < 1$ and to choose<br><br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$, $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{20})^{(3)}t}\}$ | 211 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :\left( \widetilde{(G_{23})}, \widetilde{(T_{23})} \right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)} \left\|G_{21}^{(1)} - G_{21}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\, ds_{(20)} +$<br><br>$\int_0^t \{(a'_{20})^{(3)} \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$(a''_{20})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$G_{20}^{(2)} \left\|(a''_{20})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a''_{20})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\} ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} t} \leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a'_{20})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)} (\widehat{k}_{20})^{(3)}\right) d\left(\left((G_{23})^{(1)}, (T_{23})^{(1)};\ (G_{23})^{(2)}, (T_{23})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a''_{20})^{(3)}$ and $(b''_{20})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ *and* $(\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(3)}$ and $(b_i'')^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(3)} - (a_i'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i')^{(3)} t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a'_{21})^{(3)} G_{21}$ and by integrating | 217 |

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| $G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}')^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}')^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}\left[(\hat{P}_{24})^{(4)} + \left((\hat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}}\left[\left((\hat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{24})^{(4)}\right] \leq (\hat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$ | 225 |

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{24})^{(4)}\left|G_{25}^{(1)}-G_{25}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$$

$$\int_0^t\{(a_{24}^{'})^{(4)}\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$G_{24}^{(2)}\left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)}-(G_{27})^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}t}\le$$
$$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows

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**Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}\ and\ (\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

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**Remark 17:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

it results

$$G_i\ (t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]}\ge 0$$

$$T_i\ (t)\ge T_i^0e^{\left(-(b_i^{'})^{(4)}t\right)}>0\quad \text{for t} > 0$$

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**Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1,\left((\widehat{M}_{24})^{(4)}\right)_2\ and\ \left((\widehat{M}_{24})^{(4)}\right)_3$ :

**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if

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| $G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a'_{25})^{(4)} G_{25}$ and by integrating<br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a'_{25})^{(4)}$<br>In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a'_{26})^{(4)}$<br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t),t)) = (b_{25}^{'})^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i^{''})^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded.<br>The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + \left((\widehat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\widehat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{28})^{(5)}\right] \leq (\widehat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying | |

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| Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}((G_{31}),(T_{31}))$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)}+$<br>$\int_0^t\{(a_{28}^{'})^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\le$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}^{'})^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}^{''})^{(5)}$ and $(b_{28}^{''})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(5)}-(a_i^{''})^{(5)}(T_{29}(s_{(28)}),s_{(28)})\}ds_{(28)}\right]}\ge 0$ | 239 |

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| $T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(5)}t)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}^{'})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1/(a_{29}^{'})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2/(a_{30}^{'})^{(5)}$<br><br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t),t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |

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| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{32})^{(6)}\right]\leq(\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}ds_{(32)}+$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a''_{32})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a'_{32})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(6)}$ and $(b''_i)^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 249 |

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| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(6)} - (a_i'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(6)} t)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}')^{(6)} G_{33}$ and by integrating<br><br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}')^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}')^{(6)}$<br><br>If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(6)}((G_{35})(t), t)) = (b_{33}')^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i'')^{(6)}((G_{35})(t), t) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty} (b_{34}'')^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}')^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have | 255 |

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| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)}+\left((\hat{P}_{36})^{(7)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{36})^{(7)}\right]\leq(\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}')^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}'')^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}')^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}'')^{(7)}$ and $(b_{36}'')^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$. | 260 |

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| If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2 / (a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t), t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t), t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right),\ \ t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded. | 265 |

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| The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t),t\right)=(b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}<1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)}+\left((\hat{P}_{40})^{(8)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{40})^{(8)}\right]\leq(\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}ds_{(40)}+$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\leq$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ $and$ $T_i$. Taking into account the hypothesis the result follows | 273 |

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| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(8)} - (a_i^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2\ and\ \left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}^{'})^{(8)}$<br><br>If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t), t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t), t\right) < \varepsilon_8, T_{40}\,(t) > (m)^{(8)}$ | 278 |

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| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.<br>The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}^{'})^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$ | |
| In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying 39,35,36 into itself | |
| The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{47})^{(1)}, (T_{47})^{(1)}), ((G_{47})^{(2)}, (T_{47})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \left(\widetilde{(G_{47})}, \widetilde{(T_{47})}\right) = \mathcal{A}^{(9)}\left((G_{47}), (T_{47})\right)$<br>It results<br><br>$\left\|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$<br>$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right) - (a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right| \; e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$<br><br>Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on 45,46,47,28 and 29 it follows | |
| $\left|(G_{47})^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t} \leq$<br>$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)}, (T_{47})^{(1)}; (G_{47})^{(2)}, (T_{47})^{(2)})\right)$ | |

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44{,}45{,}46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$

From 99 to 44 it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(9)}t\right)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)}G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t), t\right) < \varepsilon_9, T_{44}\,(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded. The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b''_{46})^{(9)}\left((G_{47})(t),t\right) = (b'_{46})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a'_{13})^{(1)} + (a'_{14})^{(1)} - (a''_{13})^{(1)}(T_{14},t) + (a''_{14})^{(1)}(T_{14},t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b'_{13})^{(1)} + (b'_{14})^{(1)} - (b''_{13})^{(1)}(G,t) - (b''_{14})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,

and $\boxed{(u_0)^{(1)} = \frac{T^0_{13}}{T^0_{14}}}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$

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| are defined | |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le \right.$<br>$\left.\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a'_{15})^{(1)})}[e^{(S_1)^{(1)}t} - e^{-(a'_{15})^{(1)}t}] + G_{15}^0 e^{-(a'_{15})^{(1)}t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b'_{15})^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b'_{15})^{(1)}t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)}t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$<br>$(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ :<br>$(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a'_{16})^{(2)} + (a'_{17})^{(2)} - (a''_{16})^{(2)}(T_{17}, t) + (a''_{17})^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b'_{16})^{(2)} + (b'_{17})^{(2)} - (b''_{16})^{(2)}((G_{19}), t) - (b''_{17})^{(2)}((G_{19}), t) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(v_1)^{(2)}, (v_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |

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| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\left(u^{(2)}\right)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}$,, $(\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\left(u^{(2)}\right)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously $(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$ $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities $G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)}t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le\right.$ | 312 |

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| $\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_2)^{(2)}((S_1)^{(2)}-(a_{18}')^{(2)})}[e^{(S_1)^{(2)}t}-e^{-(a_{18}')^{(2)}t}]+G_{18}^{0}e^{-(a_{18}')^{(2)}t})$ | |
| $\boxed{T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}}T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le\frac{1}{(\mu_2)^{(2)}}T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)}T_{16}^{0}}{(\mu_1)^{(2)}((R_1)^{(2)}-(b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t}-e^{-(b_{18}')^{(2)}t}\right]+T_{18}^{0}e^{-(b_{18}')^{(2)}t}\le T_{18}(t)\le$ $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)},(S_2)^{(2)},(R_1)^{(2)},(R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)}=(a_{16})^{(2)}(m_2)^{(2)}-(a_{16}')^{(2)}$ $(S_2)^{(2)}=(a_{18})^{(2)}-(p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)}=(b_{16})^{(2)}(\mu_2)^{(1)}-(b_{16}')^{(2)}$ $(R_2)^{(2)}=(b_{18}')^{(2)}-(r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$: $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)}\le-(a_{20}')^{(3)}+(a_{21}')^{(3)}-(a_{20}'')^{(3)}(T_{21},t)+(a_{21}'')^{(3)}(T_{21},t)\le-(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)}\le-(b_{20}')^{(3)}+(b_{21}')^{(3)}-(b_{20}'')^{(3)}(G_{23},t)-(b_{21}'')^{(3)}((G_{23}),t)\le-(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)},(v_2)^{(3)},(u_1)^{(3)},(u_2)^{(3)}$: By $(v_1)^{(3)}>0,(v_2)^{(3)}<0$ and respectively $(u_1)^{(3)}>0,(u_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_1)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_1)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ and By $(\bar{v}_1)^{(3)}>0,(\bar{v}_2)^{(3)}<0$ and respectively $(\bar{u}_1)^{(3)}>0,(\bar{u}_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_2)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_2)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ | 320 |
| **Definition of** $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$:- | 321 |

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$, ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,

and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously

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$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$$

$(p_i)^{(3)}$ is defined by equation

$$\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t} \quad (323)$$

$$\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$$

$$\left. \frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right) \quad (324)$$

$$\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}} \quad (325)$$

$$\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t} \quad (326)$$

$$\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$$

$$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t} \quad (327)$$

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- (328)

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b_{20}^{'})^{(3)}$$

$$(R_2)^{(3)} = (b_{22}^{'})^{(3)} - (r_{22})^{(3)}$$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$$-(\sigma_2)^{(4)} \le -(a_{24}^{'})^{(4)} + (a_{25}^{'})^{(4)} - (a_{24}^{''})^{(4)}(T_{25}, t) + (a_{25}^{''})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$$

$$-(\tau_2)^{(4)} \le -(b_{24}^{'})^{(4)} + (b_{25}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27}), t\big) - (b_{25}^{''})^{(4)}\big((G_{27}), t\big) \le -(\tau_1)^{(4)}$$

**Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : 329

By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{v}_1)^{(4)} > 0$, $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (v_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}, \ \boldsymbol{if}\ (v_0)^{(4)} < (v_1)^{(4)}$$

$$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}, \boldsymbol{if}\ (v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)},$$

and $\boxed{(v_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$

$$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}, \ \boldsymbol{if}\ (\bar{v}_4)^{(4)} < (v_0)^{(4)}$$

and analogously 331

$$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}, \ \boldsymbol{if}\ (u_0)^{(4)} < (u_1)^{(4)}$$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}, \boldsymbol{if}\ (u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)},$$

and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \boldsymbol{if}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)} \text{ where } (u_1)^{(4)}, (\bar{u}_1)^{(4)}$$

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| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}')^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}')^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)}t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(\nu_1)^{(5)}, (\nu_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, \nu^{(5)}, u^{(5)}$ :<br>By $(\nu_1)^{(5)} > 0$, $(\nu_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}(\nu^{(5)})^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)} = 0$ | 339 |

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| and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ :<br><br>By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)},(v_0)^{(5)}$ :-<br><br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)}$ by<br><br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br><br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br><br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |

$\frac{(b_{30})^{(5)}T_{28}^{0}}{(\mu_1)^{(5)}((R_1)^{(5)}-(b_{30}^{'})^{(5)})}\left[e^{(R_1)^{(5)}t}-e^{-(b_{30}^{'})^{(5)}t}\right]+T_{30}^{0}e^{-(b_{30}^{'})^{(5)}t}\le T_{30}(t)\le$

$\frac{(a_{30})^{(5)}T_{28}^{0}}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t}-e^{-(R_2)^{(5)}t}\right]+T_{30}^{0}e^{-(R_2)^{(5)}t}$ 347

**Definition of** $(S_1)^{(5)},(S_2)^{(5)},(R_1)^{(5)},(R_2)^{(5)}$:- 348

Where $(S_1)^{(5)}=(a_{28})^{(5)}(m_2)^{(5)}-(a_{28}^{'})^{(5)}$

$(S_2)^{(5)}=(a_{30})^{(5)}-(p_{30})^{(5)}$

$(R_1)^{(5)}=(b_{28})^{(5)}(\mu_2)^{(5)}-(b_{28}^{'})^{(5)}$

$(R_2)^{(5)}=(b_{30}^{'})^{(5)}-(r_{30})^{(5)}$

**Behavior of the solutions of equation** 349

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ four constants satisfying

$-(\sigma_2)^{(6)}\le-(a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}-(a_{32}^{''})^{(6)}(T_{33},t)+(a_{33}^{''})^{(6)}(T_{33},t)\le-(\sigma_1)^{(6)}$

$-(\tau_2)^{(6)}\le-(b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(b_{32}^{''})^{(6)}\left((G_{35}),t\right)-(b_{33}^{''})^{(6)}\left((G_{35}),t\right)\le-(\tau_1)^{(6)}$

**Definition of** $(\nu_1)^{(6)},(\nu_2)^{(6)},(u_1)^{(6)},(u_2)^{(6)},\nu^{(6)},u^{(6)}$ : 350

By $(\nu_1)^{(6)}>0$ , $(\nu_2)^{(6)}<0$ and respectively $(u_1)^{(6)}>0$ , $(u_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_1)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2+(\tau_1)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$ and

**Definition of** $(\bar{\nu}_1)^{(6)},,(\bar{\nu}_2)^{(6)},(\bar{u}_1)^{(6)},(\bar{u}_2)^{(6)}$ : 351

By $(\bar{\nu}_1)^{(6)}>0$ , $(\bar{\nu}_2)^{(6)}<0$ and respectively $(\bar{u}_1)^{(6)}>0$ , $(\bar{u}_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_2)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2+(\tau_2)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$

**Definition of** $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)},(\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)}$ by

$(m_2)^{(6)}=(\nu_0)^{(6)},(m_1)^{(6)}=(\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)}<(\nu_1)^{(6)}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)}<(\nu_0)^{(6)}<(\bar{\nu}_1)^{(6)}$,

and $\boxed{(\nu_0)^{(6)}=\frac{G_{32}^{0}}{G_{33}^{0}}}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)}<(\nu_0)^{(6)}$

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| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},\ \boldsymbol{if}\ (u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}, \boldsymbol{if}\ (u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)},$<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}, \boldsymbol{if}\ (\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le\right.$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b_{34}')^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b_{34}')^{(6)} t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$<br>$(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ : | |

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| $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying<br><br>$-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$<br><br>$-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}((G_{39}), t) - (b''_{37})^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ :<br><br>By $(\nu_1)^{(7)} > 0, (\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ :<br><br>By $(\bar{\nu}_1)^{(7)} > 0, (\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$<br>**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-<br><br>If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by<br><br>$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,<br>and $\boxed{(\nu_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | 362 |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,<br>and $\boxed{(u_0)^{(7)} = \frac{T^0_{36}}{T^0_{37}}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities | 364 |

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|---|---|
| $G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^{0}e^{(S_1)^{(7)}t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | |
| $\frac{1}{(m_7)^{(7)}} G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^{0}e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^{0}e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}^{'})^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}^{'})^{(7)}t}] + G_{38}^{0}e^{-(a_{38}^{'})^{(7)}t})$ | 366 |
| $\boxed{T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)}T_{36}^{0}}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}^{'})^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}^{'})^{(7)}t}\right] + T_{38}^{0}e^{-(b_{38}^{'})^{(7)}t} \le T_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)}T_{36}^{0}}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^{0}e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}^{'})^{(7)}$<br><br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br><br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}^{'})^{(7)}$<br><br>$(R_2)^{(7)} = (b_{38}^{'})^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br><br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br><br>$-(\sigma_2)^{(8)} \le -(a_{40}^{'})^{(8)} + (a_{41}^{'})^{(8)} - (a_{40}^{''})^{(8)}(T_{41}, t) + (a_{41}^{''})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br><br>$-(\tau_2)^{(8)} \le -(b_{40}^{'})^{(8)} + (b_{41}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}), t) - (b_{41}^{''})^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |
| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : | 372 |

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| By $(\nu_1)^{(8)} > 0\,, (\nu_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0\,, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$ and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(8)},, (\bar{\nu}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :<br><br>By $(\bar{\nu}_1)^{(8)} > 0\,, (\bar{\nu}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0\,, (\bar{u}_2)^{(8)} < 0$ the<br><br>roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$<br><br>and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$<br><br>**Definition of** $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (\nu_0)^{(8)}$ :-<br><br>If we define $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by<br><br>$(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)},$ ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$<br><br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)},$<br>and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br><br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)},$ ***if*** $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br><br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},$ ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)},$ ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)}t}$<br><br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$ | 376 |

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| $\left(\frac{(a_{42})^{(8)}G_{40}^{0}}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t}-e^{-(S_2)^{(8)}t}\right]+G_{42}^{0}e^{-(S_2)^{(8)}t}\leq G_{42}(t)\leq \frac{(a_{42})^{(8)}G_{40}^{0}}{(m_2)^{(8)}((S_1)^{(8)}-(a_{42}^{'})^{(8)})}[e^{(S_1)^{(8)}t}-e^{-(a_{42}^{'})^{(8)}t}]+G_{42}^{0}e^{-(a_{42}^{'})^{(8)}t}\right)$ | 377 |
| $\boxed{T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}}T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq \frac{1}{(\mu_2)^{(8)}}T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^{0}}{(\mu_1)^{(8)}((R_1)^{(8)}-(b_{42}^{'})^{(8)})}\left[e^{(R_1)^{(8)}t}-e^{-(b_{42}^{'})^{(8)}t}\right]+T_{42}^{0}e^{-(b_{42}^{'})^{(8)}t}\leq T_{42}(t)\leq$ $\frac{(a_{42})^{(8)}T_{40}^{0}}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t}-e^{-(R_2)^{(8)}t}\right]+T_{42}^{0}e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)},(S_2)^{(8)},(R_1)^{(8)},(R_2)^{(8)}$:-<br>Where $(S_1)^{(8)}=(a_{40})^{(8)}(m_2)^{(8)}-(a_{40}^{'})^{(8)}$<br>$(S_2)^{(8)}=(a_{42})^{(8)}-(p_{42})^{(8)}$<br>$(R_1)^{(8)}=(b_{40})^{(8)}(\mu_2)^{(8)}-(b_{40}^{'})^{(8)}$<br>$(R_2)^{(8)}=(b_{42}^{'})^{(8)}-(r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)}\leq -(a_{44}^{'})^{(9)}+(a_{45}^{'})^{(9)}-(a_{44}^{''})^{(9)}(T_{45},t)+(a_{45}^{''})^{(9)}(T_{45},t)\leq -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)}\leq -(b_{44}^{'})^{(9)}+(b_{45}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47}),t)-(b_{45}^{''})^{(9)}((G_{47}),t)\leq -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(v_1)^{(9)},(v_2)^{(9)},(u_1)^{(9)},(u_2)^{(9)},v^{(9)},u^{(9)}$ :<br>By $(v_1)^{(9)}>0$ , $(v_2)^{(9)}<0$ and respectively $(u_1)^{(9)}>0$ , $(u_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(v^{(9)})^2+(\sigma_1)^{(9)}v^{(9)}-(a_{44})^{(9)}=0$<br>and $(b_{45})^{(9)}(u^{(9)})^2+(\tau_1)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$ and | |
| **Definition of** $(\bar{v}_1)^{(9)},,(\bar{v}_2)^{(9)},(\bar{u}_1)^{(9)},(\bar{u}_2)^{(9)}$ :<br>By $(\bar{v}_1)^{(9)}>0$ , $(\bar{v}_2)^{(9)}<0$ and respectively $(\bar{u}_1)^{(9)}>0$ , $(\bar{u}_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(v^{(9)})^2+(\sigma_2)^{(9)}v^{(9)}-(a_{44})^{(9)}=0$ | |

and $(b_{45})^{(9)}\left(u^{(9)}\right)^2+(\tau_2)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$

**Definition of** $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)},(\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)}$ by

$(m_2)^{(9)}=(\nu_0)^{(9)},(m_1)^{(9)}=(\nu_1)^{(9)},$ ***if*** $(\nu_0)^{(9)}<(\nu_1)^{(9)}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)}<(\nu_0)^{(9)}<(\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)}=\frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\nu_0)^{(9)},$ ***if*** $(\bar{\nu}_1)^{(9)}<(\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)}=(u_0)^{(9)},(\mu_1)^{(9)}=(u_1)^{(9)},$ ***if*** $(u_0)^{(9)}<(u_1)^{(9)}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)}<(u_0)^{(9)}<(\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)}=\frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(u_0)^{(9)},$ ***if*** $(\bar{u}_1)^{(9)}<(u_0)^{(9)}$ where $(u_1)^{(9)},(\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{44}(t)\le G_{44}^0e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{45}(t)\le\frac{1}{(m_2)^{(9)}}G_{44}^0e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)}G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^0e^{-(S_2)^{(9)}t}\le G_{46}(t)\le\right.$$

$$\left.\frac{(a_{46})^{(9)}G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}')^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}')^{(9)}t}]+\ G_{46}^0e^{-(a_{46}')^{(9)}t}\right)$$

$$\boxed{T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le\frac{1}{(\mu_2)^{(9)}}T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}')^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}')^{(9)}t}\right]+T_{46}^0e^{-(b_{46}')^{(9)}t}\le T_{46}(t)\le$$

$$\frac{(a_{46})^{(9)}T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^0e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain (383)

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right) - (a''_{14})^{(1)}(T_{14}, t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get (384)

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{v}_1)^{(1)}$ we find like in the previous case, (385)

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)}G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)}T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$$

If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain 394

$$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(2)}(t)$ :- 395

$$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}, \quad \boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$$

In a completely analogous way, we obtain 396

**Definition of** $u^{(2)}(t)$ :-

$$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}, \quad \boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :** 397

If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$

Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then

$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$ if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$

**Proof** : From global equations we obtain 398

$$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$$

**Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$ 399

It follows

$$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$$

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From which one obtains

For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)} + (C)^{(3)}(\nu_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

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In the same manner , we get

$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)} + (\bar{C})^{(3)}(\bar{\nu}_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, 402

$$(\nu_1)^{(3)} \le \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \le \nu^{(3)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \le (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \le (\bar{\nu}_1)^{(3)} \le (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

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$$(\nu_1)^{(3)} \le \nu^{(3)}(t) \le \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \le (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \le \nu^{(3)}(t) \le (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \le u^{(3)}(t) \le (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right) - (a_{25}^{''})^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\quad \boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \le \frac{d\nu^{(4)}}{dt} \le -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

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$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} =$

$(\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\quad \boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get 413

$$v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{v}_1)^{(6)}-(v_0)^{(6)}}{(v_0)^{(6)}-(\bar{v}_2)^{(6)}}}$$

From which we deduce $(v_0)^{(6)} \le v^{(6)}(t) \le (\bar{v}_1)^{(6)}$

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If $0 < (v_1)^{(6)} < (v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{v}_1)^{(6)}$ we find like in the previous case,

$$(v_1)^{(6)} \le \frac{(v_1)^{(6)}+(C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}} \le v^{(6)}(t) \le$$

$$\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (\bar{v}_1)^{(6)}$$

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If $0 < (v_1)^{(6)} \le (\bar{v}_1)^{(6)} \le \boxed{(v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(v_1)^{(6)} \le v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (v_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$(m_2)^{(6)} \le v^{(6)}(t) \le (m_1)^{(6)}$, $\boxed{v^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \le u^{(6)}(t) \le (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(v_1)^{(6)} = (\bar{v}_1)^{(6)}$ if in addition $(v_0)^{(6)} = (v_1)^{(6)}$ then $v^{(6)}(t) = (v_0)^{(6)}$ and as a consequence $G_{32}(t) = (v_0)^{(6)}G_{33}(t)$ **this also defines** $(v_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(v_1)^{(6)}$ and $(\bar{v}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right) - (a_{37}^{''})^{(7)}(T_{37},t)v^{(7)} - (a_{37})^{(7)}v^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get 417

$$\nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\bar{\nu}_1)^{(7)}$

If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case, 418

$$(\nu_1)^{(7)} \leq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}} \leq \nu^{(7)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\bar{\nu}_1)^{(7)}$$

If $0 < (\nu_1)^{(7)} \leq (\bar{\nu}_1)^{(7)} \leq \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain 419

$$(\nu_1)^{(7)} \leq \nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \leq \nu^{(7)}(t) \leq (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :- 420

$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)} G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain 421

$$\frac{d\nu^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right) - (a_{41}^{''})^{(8)}(T_{41}, t)\nu^{(8)} - (a_{41})^{(8)}\nu^{(8)}$$

**Definition of** $\nu^{(8)}$ :- $\boxed{\nu^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right) \leq \frac{d\nu^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)}-(\nu_0)^{(8)}}{(\nu_0)^{(8)}-(\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \le \nu^{(8)}(t) \le (\bar{\nu}_8)^{(8)}$

If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case, 423

$$(\nu_1)^{(8)} \le \frac{(\nu_1)^{(8)}+(C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}} \le \nu^{(8)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\bar{\nu}_1)^{(8)}$$

If $0 < (\nu_1)^{(8)} \le (\bar{\nu}_1)^{(8)} \le \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$, we obtain 424

$$(\nu_1)^{(8)} \le \nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have
**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \le \nu^{(8)}(t) \le (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain
**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \le u^{(8)}(t) \le (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)}G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)}T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$. 424

**Proof** : From 99,20,44,22,23,44 we obtain 424 A

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right) - (a''_{45})^{(9)}(T_{45},t)v^{(9)} - (a_{45})^{(9)}v^{(9)}$$

**Definition of** $v^{(9)}$ :- $\boxed{v^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)}\right) \le \frac{dv^{(9)}}{dt} \le -\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(9)}, (v_0)^{(9)}$ :-

For $0 < \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (v_1)^{(9)} < (\bar{v}_1)^{(9)}$

$$v^{(9)}(t) \ge \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(v_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (v_2)^{(9)}}}$$

it follows $(v_0)^{(9)} \le v^{(9)}(t) \le (v_9)^{(9)}$

In the same manner , we get

$$v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{v}_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (\bar{v}_2)^{(9)}}}$$

From which we deduce $(v_0)^{(9)} \le v^{(9)}(t) \le (\bar{v}_1)^{(9)}$

If $0 < (v_1)^{(9)} < (v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}} < (\bar{v}_1)^{(9)}$ we find like in the previous case,

$$(v_1)^{(9)} \le \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}} \le v^{(9)}(t) \le$$

$$\frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (\bar{v}_1)^{(9)}$$

If $0 < (v_1)^{(9)} \le (\bar{v}_1)^{(9)} \le \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$ , we obtain

$$(v_1)^{(9)} \le v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (v_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(9)}(t)$ :-

$$(m_2)^{(9)} \le v^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{v^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}$, $\boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ 430

$with\ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations 431

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$

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| $with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | |
| We can prove the following<br><br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br><br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br><br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br><br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$ | 435 |

| | |
|---|---|
| $with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |

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| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27})\big)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\big((G_{27})\big)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\big((G_{27})\big)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |

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| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |

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| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(o) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) +$ | 489 |

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| $(a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that | 496 |

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| there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{32}=\frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}=\frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{36}=\frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}=\frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |
| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*})=0$. With this value , we obtain from the three first equations | 501 |

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| $G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br>$\left[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G) + (b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)\right]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*}) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br><br>$\left[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19}) + (b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})\right]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br><br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ | 506 |

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| is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution | 511 |

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| $G_{14}^{*}$ given by $\varphi(G^{*})=0$ , $T_{14}^{*}$ given by $f(T_{14}^{*})=0$ and<br>$G_{13}^{*}=\frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*}=\frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br>$T_{13}^{*}=\frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*}=\frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br>Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*})=0$ , $T_{17}^{*}$ given by $f(T_{17}^{*})=0$ and | 512 |
| $G_{16}^{*}=\frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*}=\frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*}=\frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*}=\frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*})=0$ , $T_{21}^{*}$ given by $f(T_{21}^{*})=0$ and<br>$G_{20}^{*}=\frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*}=\frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br>$T_{20}^{*}=\frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*}=\frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^{*}$ given by $\varphi(G_{27})=0$ , $T_{25}^{*}$ given by $f(T_{25}^{*})=0$ and<br>$G_{24}^{*}=\frac{(a_{24})^{(4)}G_{25}^{*}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*}=\frac{(a_{26})^{(4)}G_{25}^{*}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*}=\frac{(b_{24})^{(4)}T_{25}^{*}}{[(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*}=\frac{(b_{26})^{(4)}T_{25}^{*}}{[(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}((G_{27})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*})=0$ , $T_{29}^{*}$ given by $f(T_{29}^{*})=0$ and<br>$G_{28}^{*}=\frac{(a_{28})^{(5)}G_{29}^{*}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*}=\frac{(a_{30})^{(5)}G_{29}^{*}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |

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| $T_{28}^{*} = \frac{(b_{28})^{(5)}T_{29}^{*}}{[(b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)}T_{29}^{*}}{[(b_{30}^{'})^{(5)}-(b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br>$G_{32}^{*} = \frac{(a_{32})^{(6)}G_{33}^{*}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)}G_{33}^{*}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)}T_{33}^{*}}{[(b_{32}^{'})^{(6)}-(b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)}T_{33}^{*}}{[(b_{34}^{'})^{(6)}-(b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br>$G_{36}^{*} = \frac{(a_{36})^{(7)}G_{37}^{*}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)}G_{37}^{*}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br>$T_{36}^{*} = \frac{(b_{36})^{(7)}T_{37}^{*}}{[(b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)}T_{37}^{*}}{[(b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br>$G_{40}^{*} = \frac{(a_{40})^{(8)}G_{41}^{*}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)}G_{41}^{*}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br>$T_{40}^{*} = \frac{(b_{40})^{(8)}T_{41}^{*}}{[(b_{40}^{'})^{(8)}-(b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)}T_{41}^{*}}{[(b_{42}^{'})^{(8)}-(b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)}G_{45}^{*}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)}G_{45}^{*}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)}T_{45}^{*}}{[(b_{44}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)}T_{45}^{*}}{[(b_{46}^{'})^{(9)}-(b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | 523 A |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^*) = (q_{14})^{(1)} \quad , \frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^*) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)}G_{13}^*\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^*\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^*\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^*\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^*\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^*\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)} \quad , \frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |

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| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ $and$ $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)}$ , $\frac{\partial(b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_{j}\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_{j}\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_{j}\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 548 |

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| $\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^{*}) = (q_{25})^{(4)}$ , $\frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^{*}) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^{*}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^{*}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^{*}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^{*}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^{*}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^{*}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^{*}) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^{*}) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^{*}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^{*}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^{*}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^{*}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^{*}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^{*}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 563 |

**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ $and$ $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-

$$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)} \quad , \frac{\partial (b_i^{''})^{(6)}}{\partial G_j}\left((G_{35})^*\right) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33} \quad (565)$$

$$\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33} \quad (566)$$

$$\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^*\mathbb{T}_{33} \quad (567)$$

$$\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^*\mathbb{G}_j\right) \quad (568)$$

$$\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^*\mathbb{G}_j\right) \quad (569)$$

$$\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^*\mathbb{G}_j\right) \quad (570)$$

**ASYMPTOTIC STABILITY ANALYSIS** (571)

**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ $and$ $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- (572)

$$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)} \quad , \frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain from

$$\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37} \quad (573)$$

$$\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37} \quad (574)$$

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| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{41})^{(8)}}{\partial T_{41}}(T^*_{41}) = (q_{41})^{(8)}$ , $\frac{\partial(b_i^{''})^{(8)}}{\partial G_j}((G_{43})^*) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a'_{40})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G^*_{40}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a'_{41})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G^*_{41}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a'_{42})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G^*_{42}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |

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| | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \ \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b_{44}^{'})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T_{44}^*\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b_{45}^{'})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T_{45}^*\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b_{46}^{'})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T_{46}^*\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)$<br>$\left[\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(q_{13})^{(1)}G_{13}^*\right)\Big]$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T_{14}^* + (b_{14})^{(1)}s_{(13),(14)}T_{14}^*\Big)$<br>$+\left(\left((\lambda)^{(1)} + (a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^* + (a_{13})^{(1)}(q_{14})^{(1)}G_{14}^*\right)$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^* + (b_{14})^{(1)}s_{(13),(13)}T_{13}^*\Big)$<br>$\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$<br>$\left((\lambda)^{(1)}\right)^2 + \left((b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$<br>$+\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$<br>$+\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^*\right)$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^* + (b_{14})^{(1)}s_{(13),(15)}T_{13}^*\Big)\} = 0$ | |

$$+$$

$$((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)})\{((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)})$$

$$\left[\left(((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)})(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)\right]$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(17)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^{*}_{17}\right)$$

$$+\left(((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)})(q_{16})^{(2)}\mathrm{G}^{*}_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(16)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^{*}_{16}\right)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left(\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(18)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(18)}\mathrm{T}^{*}_{16}\right)\} = 0$$

$$+$$

$$((\lambda)^{(3)} + (b'_{22})^{(3)} - (r_{22})^{(3)})\{((\lambda)^{(3)} + (a'_{22})^{(3)} + (p_{22})^{(3)})$$

$$\left[\left(((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)})(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)\right]$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(21)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(21)}T^{*}_{21}\right)$$

$$+\left(((\lambda)^{(3)} + (a'_{21})^{(3)} + (p_{21})^{(3)})(q_{20})^{(3)}G^{*}_{20} + (a_{20})^{(3)}(q_{21})^{(1)}G^{*}_{21}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(20)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(20)}T^{*}_{20}\right)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((b'_{20})^{(3)} + (b'_{21})^{(3)} - (r_{20})^{(3)} + (r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left(\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(22)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(22)}T^{*}_{20}\right)\} = 0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\right)\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\right)\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\right)\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\right)\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)\right]$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(41)}T^{*}_{41}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^{*}_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^{*}_{41}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(40)}T^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$+\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(42)}T^{*}_{40}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\right)\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)\right]$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(45)}T^{*}_{45}\right)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^{*}_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^{*}_{45}\right)$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(44)}T^{*}_{44}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^{2}+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$+\left(\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)$$

$\left(\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\right)\} = 0$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i \in R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i\ X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor

bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k \tag{15}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 15. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,...x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \in I, x \in K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow *x.t restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\,\delta\,(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\mathcal{A}(\tilde{A}), C\backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population *i* is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int_0^t (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \epsilon R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v = au_i(x) + w$ with $a \epsilon R$ and $w \epsilon F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |B_i(t,x)u_i(x)|(a + \alpha + exp(-\beta t)|w|)$. Hence, $\limsup_{t\to\infty} \frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\limsup_{t\to\infty} \frac{1}{t} In\ B_i(t,x)\ u_i(x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In } b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x - bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

=

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \le 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \le \gamma$ for all $t \ge 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \operatorname{In} |B_i(t,x)x_i| = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \operatorname{In} |\, (x.t)_i\, |$$

$$\le \operatorname{limsup}_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0. If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\ge 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\operatorname{limsup}_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \ge 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure*. $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$ , $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x) \rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1 \le j \le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as $n \to \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |\mathrm{h}(x.t) - h(x^n.t)|d\mu_n(x)| + |\int_c h(x^n.t)|d\mu_n(x) - \int_c \mathrm{h}(x)d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x)d\mu_n(x) - \int_c h(x)d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $\mathrm{h}_i^n(x) = \langle \mathrm{A}_i^n(x)\mathrm{u}_i^n(x), \mathrm{u}_i^n(x)\rangle$ where $\mathrm{u}_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $\mathrm{A}_i^n(x) \to \mathrm{A}_i(x)$ uniformly for $x \in N_1((\mathrm{A}))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $\mathrm{u}_i^n(x) \to \mathrm{u}_i(x)$ converges uniformly for $x \in N_1(\Lambda(\mathrm{A}))$. Given $\in > 0$, choose $n$ sufficiently large so that $|\mathrm{h}_i^n(x) - \mathrm{h}_i(x)| \leq \in$ for all $x \in N_1(\Lambda(\mathrm{A}))$ and $|\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$\lambda_i(\mu) \leq \int_c |\mathrm{h}_i(x) - \mathrm{h}_i^n(x)|d\mu(x) + |\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)|$

$+ \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)$

$\leq \quad 2\in$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\, \mathrm{ds}$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument

implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i \, (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

(6) $\min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$ $\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_{1,}\ldots.,\tilde{A}_k)$ by

$$\tilde{A}_i(x) = \{ \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i \epsilon I}{if\ i \notin I}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that \quad \tilde{V}.t \subset intV\ for\ all\ t >$ $0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c\ II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set $K$ if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$C then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$C and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume $x \mapsto A(x)$ is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists $j \in \{1,\ldots,k\}$ such that $M_i$ ----- $\{x \in C : x_l \gg 0 \text{ for } l \neq j\}$ Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_i M_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup $(\mu)$ to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(\text{---}_i \in_I C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures μ with support in ∂C, $\lambda_i(\mu)>0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $$\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j \qquad i = 1,..,k \qquad j=1,\ldots,m$$

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge

to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j$ ($x^j_i$) = 0 i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product ($E_1^1$ ,…,$E_1^m$), and the matrix $A^2$ ($E_1$) is close to the diagonal matrix diag($f^i_2(E_1{}^j)$)$^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f_2^j (E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j (E^j_1)>0$ and $\max_j f_2^j (E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial$C is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j (E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j\ x_2^j\ - \alpha^j\ x_3^j\ ) + \sum d_1^{jk}\ \ x_1^k\ - e_1^j\ x_1^j$

$\frac{dx_2^j}{dt} = x_2^j(1 - \alpha^j\ x_1^j\ - x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_2^{jk}\ \ x_2^k\ - e_2^j\ x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1 - \beta^j\ x_1^j\ - \alpha^j x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_3^{jk}\ \ x_3^k\ - e_3^j\ x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2\ (E_1)$and $\lambda_3\ (E_1)$are close to $\max_j\ 1 - a^j$ and $\max_j\ 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j\ 1 - a^j$ and $\max_j\ 1 - \beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1 - a^j) > -\max_j\ (1 - \beta^j)$$

Equivalently,

$$2 > \min_j\ a^j\ +\ \min_j\ \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i\left(r_i - \sum_s a_{is}x_s\right), \qquad i = 1,\dots,k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$$

It has been shown [12,ch.16.1,16.2] That for $k = 3$,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \ \ with\ \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing

holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & O \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left[ \beta(1 - y - z)\frac{P}{\epsilon + P} - r - m(1 - y) - b(N)\ y \right]$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*′ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\Lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\Lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\Lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-}\ y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + Kp\, x_i x_i - 1}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords

Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation

of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1$ *(t)* and $P_2$ *(t)* denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial

alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.43) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.44) $$D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.45) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_{1,}$ $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_{i,}$ $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic ($as$ $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 15. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2)) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ x\ k_1 x\ k_2, R^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ x\ k_2 x\ k_1, R^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1, 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) \quad P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^\infty f_i(a + s, P(t + S)) ds \right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[ - \int_o^x f_1(a + s, p(t + s)) ds \right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0) \exp\left[ - \int_o^t f_1(a - t + s, p_1(s)) ds \right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(\text{a+s}, p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da+\int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$$t> 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in[-\tau, 0], i = 1,2,$

(2.8) $\beta_i(\text{t}) = \int_0^t \beta_i(\text{a}) b_i(\text{t-a}, P_i(t)) M_i.(a, t, P_i(t))da$

$$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$$

$$i = 1, 2, \text{t} > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t) \left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t)\, \phi(0) d_i^*]t \quad for\ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T],R^+), \quad B_i \in C([0,T],R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau,T] = \{f \in C([-\tau,T],R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau,T]$. Then for any

$(x, y) \in C^+[-\tau,T] \times C^+[-\tau,T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau,T], \quad C_T^2 = C^+[-\tau,T] \times C^+[-\tau,T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau,T]$ *and* $t \in [0,T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$B: C_T^2 \to C_T^2$ *and* $\Pi C_T^2 \to C_T^2$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = P_t \left\{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right)$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Pi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Pi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Pi(P_t^{(1)}) - \Pi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Pi(P^{(1)}(t)) - \Pi(P^{(2)}(t)| \le \int_0^t B_1^{(1)}(a) \left\{ M_1(a, t, P_t^{(1)}) - M_1(a, t, P_t^{(2)}) \right\} da$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \le |x-y| \ for\ x,y \le 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\le \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\le d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \le \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} (\int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_* ds) da$$

$$\le d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \le \int_o^\infty \varphi(a,o)[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds] da$$

$$\le d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \|_t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$=\int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\mathrm{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \, /\!/ P_t^{(1)} - P_t^{(2)} /\!/ \text{ for } a \, \varepsilon [o, t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \leq \beta_1 \Phi_1(0) || P_t^{(1)} - P_t^{(2)} ||_t^* + \Phi_1(0) \beta_1 d_1 || P_t^{(1)} - P_t^{(2)} ||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) \, e^{\delta 1^a} \beta_1 \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) \, e^{\delta 1^a} \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| B_1^{*d}\sigma$$

$$+ \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \left\{ B_1 + B_2 d_1 a + B_1^* \left(B_2 a + d_1 a^2/2\right) e^{\delta 1^a} \right\}$$

$$\leq \beta_1^* \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| do + \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \to \beta_1$ as $a \to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{g_1(a) + 0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a)) - b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a)) do \right|$$

$$\left| B_1^* \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi\left(P_t^{(1)} - \pi\left(P_t^{(2)}\right) /\!/ \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^*)$$

As $t^* \to o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp P_2^*(a)}{da} + f_i(a, P_1^*, P_2^*) P_i^*(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^* = \int_o^\infty P_i^*(a) da, i = 1,\ 2,$$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i=1.\ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s,P_1^*,P_2^*)ds\right], i=1,\quad 2, a\geq 0,$

Which together with (3.3) lead to

$$1=\int_o^\infty b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^\infty b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1,P_2) = \int_o^\infty b_i(a,P_1,P_2)\exp\left[-\int_o^\infty f_i(s,P_1,P_2)ds\right]da,$

$$i=1,\ 2,\quad P_1\geq 0,\quad P_2\geq 0,$$

**And suppose the following hold:**

(3.7) $A_{4.}\quad F_{i.}\ (0,\ 0)>1,\ i=1,2$:

$A_{5.}$ *there exists a positive constant c for which*

$$F_1(C,0)< F_{2.}(C,0),$$

(3.8) $$F_1(0.C)< F_{2.}(0,C),$$

Then there exists a unique pair $(P_1^*,P_2^*)$ of real numbers $P_1^*>o$,

$P_2^*>0$ such that

(3,9) $F_1(P_1^*,P_2^*)=1=F_2(P_1^*,P_2^*)$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a)=[P_2^*\exp[-f_i(s,P_1^*,P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s,P_1^*,P_2^*)ds]da),$

$$i=1,2,\quad a\geq 0,$$

Proof Consider the elements of the surfaces defined by

Z= $F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0,$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

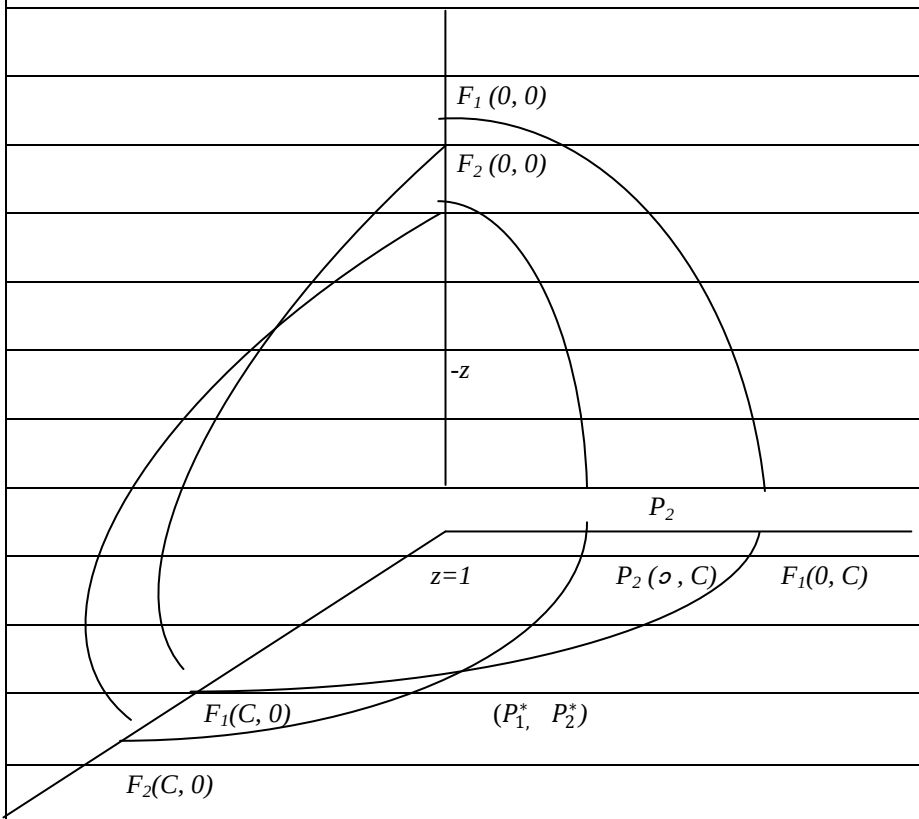
$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0,$ These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, F_1(P_1, P_2 \geq 0,$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11) $$\begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12) $$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square=1}^{2} \frac{\partial b_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}da \end{cases}$$

Where $i$ = 1,2 and (3.13) $$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14) $$\begin{cases} u_i(a,t) = u\square(a)\exp[\lambda t], \quad \square(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty \square(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d\square}{da} + \lambda\square = -f_i\left(a, p_1^*, p_2^*\right)\square(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_{i\square}^*\right\},$$

(3.15)

$$\square(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\square(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*} p_{i\square}^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^{*} = p_{1}^{*}, \quad p_{12}^{*} = p_{2}^{*} \exp\{-\lambda\tau\},$$

$$p_{21}^{*} = p_{1}^{*}, \exp\{-\lambda\tau\}, \ p_{22}^{*} = p_{2}^{*}.$$

If we integrate the first of (3.15) and use the fact that $\Omega_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\Omega_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\Omega_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\Omega_i(a) = \{\Omega_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$

$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\Omega_i\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\Omega_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\Omega_i\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\Omega_i\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\Omega_i\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)}\int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}da, \quad i = 1,\ 2,$$

Now supplying $\Omega_1(a)$ and $\Omega_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^*\exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^*\exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^* \exp(-\lambda\tau) + (\lambda - A_{22})\, p_2^* = g_{21}(\lambda)\, p_1^* \exp(-\lambda\tau) + g_{22}(\lambda)\, p_2^*$$

Where

(3.22) $$A_{i} = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_{\square}^*} - \frac{\partial f_i}{\partial p_{\square}^*}\right]da,$$

(3.23) $$g_{i}(\lambda) = \left[p_i^*(0)c_i\right]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\, da\right\}$$
$$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_{\square}^*} - b_i(a, p_1^*, p_2^*)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{\square}^*}\, e^{\lambda s}\, ds\right)\right] da\right\}$$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{\square}^*}\, e^{\lambda s}\, ds\right) da, \quad i, = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*), \quad i, = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})\, e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$$S_2(\lambda, \tau) = \left[A_{21}\, g_{12}(\lambda) + A_{12}\, g_{21}(\lambda) + g_{12}(\lambda)\, g_{21}(\lambda)\right]\exp[-2\lambda\tau]$$

$$-A_{11}\, g_{22}(\lambda) - A_{22}\, g_{11}(\lambda) - g_{11}(\lambda)\, g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6$. $0 < \beta_i^* < d_i^*$ $(i, = 1, 2.)$ $(see(A_1)\ and\ (A_2))$;

$A_7$. $\{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \square(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \left|\right.$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \,| - |\, A_{11} + A_{22} + \| S_1(0) \| \,|^2 \right\}/2,$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational

ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the

Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have

obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is:

individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## SECTION A(11)-2

## Nonabelian Generalisation Of Laplace Interpolation

### INTRODUCTION—VARIABLES USED

**Quantum Yang-Mills theory: an overview of a programme Ashley Milsted, Tobias J. Osborne (Submitted on 7 Apr 2016)**

(1) The resulting state acquires (e) the structure of a tensor network, namely, a multiscale entanglement renormalisation ansatz, and allows (eb)for the efficient computation of (e) local observables and Wilson loops.

**(2)** Various refinements of the tensor network are discussed leading to (eb) several generalisations. Finally, the continuum limit of ansatz as the lattice regulator is removed is then described.

**(3)** This paper is intended as an abstract for (e) an ongoing programme: there are still many open problems Subjects: Quantum Physics (quant-ph); High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1604.01979 [quant-ph] (or arXiv: 1604.01979v1 [quant-ph] for this version)

**Entropy production from chaoticity in Yang-Mills field theory with use of the Husimi function Hidekazu Tsukiji, Hideaki Iida, Teiji Kunihiro, Akira Ohnishi, Toru T. Takahashi (Submitted on 15**

**Mar 2016)**

(1) Authors investigate possible entropy production in Yang-Mills (YM) field theory by using (e) a quantum distribution function called Husimi function fH(A,E,t) for YM field, which is given by (=)a coarse graining of Wigner function and non-negative.
(2) They calculate the Husimi-Wehrl (HW) entropy SHW(t)=−TrfHlogfH defined as an integral over (e&eb) the phase-space, for which two adaptations of (e) the test-particle method are used combined with (e&eb) Monte-Carlo method.
(3) Authors utilize the semiclassical approximation to obtain (eb) the time evolution of the distribution functions of the YM field, which is known to show (eb) a chaotic behavior in the classical limit.
(4) They also make a simplification of the multi-dimensional phase-space integrals by making a product ansatz for (e) the Husimi function, which is found to give a 10-20 per cent over estimate of the HW entropy for a quantum system with a few degrees of freedom.
**(5)** Authors show that the quantum YM theory does exhibit(eb)the entropy production, and that the entropy production rate agrees with (=) the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting that (eb) the chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory. Subjects: High Energy Physics - Phenomenology (hep-ph); Statistical Mechanics (cond-mat.stat-mech); Nuclear Theory (nucl-th) Report number: KUNS-2616, YITP-16-40 Cite as: arXiv:1603.04622 [hep-ph] (or arXiv:1603.04622v1 [hep-ph] for this version)

**NOTATION**

**Module One**

The resulting state acquires (e) the structure of a tensor network, namely, a multiscale entanglement renormalisation ansatz, and allows (eb)for the efficient computation of (e) local observables and Wilson loops

$G_{13}$ : Category one of structure of a tensor network, namely, a multiscale entanglement renormalisation ansatz, and allows (eb)for the efficient computation of (e) local observables and Wilson loops

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of resulting state

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

The resulting state acquires the structure of a tensor network, namely, a multiscale entanglement renormalisation ansatz, and allows (eb)for the efficient computation of (e) local observables and Wilson loops

$G_{16}$ : Category one of resulting state acquires the structure of a tensor network, namely, a multiscale entanglement renormalisation ansatz; **efficient computation of (e) local observables and Wilson loops**

$G_{17}$ : Category two of SAS

| | |
|---|---|
| $G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of **efficient computation of (e) local observables and Wilson loops** ;resulting state acquires the structure of a tensor network, namely, a multiscale entanglement renormalisation ansatz,<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>The resulting state acquires the structure of a tensor network, namely, a multiscale entanglement renormalisation ansatz, and allows for the efficient computation of (e) local observables and Wilson loops | |
| $G_{20}$ : Category one of **resulting state acquires the structure of a tensor network, namely, a multiscale entanglement renormalisation ansatz, and allows for the efficient computation;** local observables and Wilson loops<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of local observables and Wilson loops **;resulting state acquires the structure of a tensor network, namely, a multiscale entanglement renormalisation ansatz, and allows for the efficient computation**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Various refinements of the tensor network are discussed leading to (eb) several generalisations.<br><br>Finally, the continuum limit of ansatz as the lattice regulator is removed is then described. | |
| $G_{24}$ : Category one of continuum limit of ansatz<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of lattice regulator is removed<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Authors investigate possible entropy production in Yang-Mills (YM) field theory by using (e) a quantum distribution function called Husimi function fH(A,E,t) for YM field, which is given by (=)a coarse graining of Wigner function and non-negative | |
| $G_{28}$ : Category one of quantum distribution function called Husimi function fH(A,E,t) for YM field, which | |

| | |
|---|---|
| is given by (=)a coarse graining of Wigner function and non-negative<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of entropy production in Yang-Mills (YM) field theory<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Authors investigate possible entropy production in Yang-Mills (YM) field theory by using a quantum distribution function called Husimi function fH(A,E,t) for YM field, which is given by (=)a coarse graining of Wigner function and non-negative | |
| $G_{32}$ : Category one of entropy production in Yang-Mills (YM) field theory by using a quantum distribution function called Husimi function fH(A,E,t) for YM field<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of coarse graining of Wigner function and non-negative<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>They calculate the Husimi-Wehrl (HW) entropy SHW(t)=−TrfHlogfH defined as an integral over (e&eb) the phase-space, for which two adaptations of (e) the test-particle method are used combined with (e&eb) Monte-Carlo method. | |
| $G_{36}$ : Category one of **Husimi-Wehrl (HW) entropy SHW (t)=−TrfHlogfH defined as an integral;** phase-space, for which two adaptations of (e) the test-particle method are used combined with (e&eb) Monte-Carlo method.<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of phase-space, for which two adaptations of (e) the test-particle method are used combined with (e&eb) Monte-Carlo method.;**Husimi-Wehrl (HW) entropy SHW(t)=−TrfHlogfH defined as an integral**<br><br>$T_{37}$ : Category two of SAS | |

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| $T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>They calculate the Husimi-Wehrl (HW) entropy SHW(t)=−TrfHlogfH defined as an integral over the phase-space, for which two adaptations of (e) the test-particle method are used combined with (e&eb) Monte-Carlo method. | |
| $G_{40}$ : Category one of test-particle method are used combined with (e&eb) Monte-Carlo method<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of Husimi-Wehrl (HW) entropy SHW(t)=−TrfHlogfH defined as an integral over the phase-space, for which two adaptations<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>They calculate the Husimi-Wehrl (HW) entropy SHW (t)=−TrfHlogfH defined as an integral over the phase-space, for which two adaptations of the test-particle method are used combined with (e&eb) Monte-Carlo method. | |
| $G_{44}$ : Category one of **Husimi-Wehrl (HW) entropy SHW(t)=−TrfHlogfH defined as an integral over the phase-space, for which two adaptations of the test-particle method;** Monte-Carlo method.<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of Monte-Carlo method.; **Husimi-Wehrl (HW) entropy SHW(t)=−TrfHlogfH defined as an integral over the phase-space, for which two adaptations of the test-particle method**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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|---|---|
| **SECTION A(11)-3**<br><br>**Entropy Production From Chaoticity In Yang-Mills Field Theory** | |

**INTRODUCTION—VARIABLES USED**

**Entropy production from chaoticity in Yang-Mills field theory with use of the Husimi function Hidekazu Tsukiji, Hideaki Iida, Teiji Kunihiro, Akira Ohnishi, Toru T. Takahashi (Submitted on 15 Mar 2016)**

(1) Authors utilize the semiclassical approximation to obtain (eb) the time evolution of the distribution functions of the YM field, which is known to show (eb) a chaotic behavior in the classical limit.

**(2)** They also make a simplification of the multi-dimensional phase-space integrals by making a product ansatz for (e) the Husimi function, which is found to give a 10-20 per cent over estimate of the HW entropy for a quantum system with a few degrees of freedom.

**(3)** Authors show that the quantum YM theory does exhibit(eb)the entropy production, and that the entropy production rate agrees with (=) the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting that (eb) the chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory. Subjects: High Energy Physics - Phenomenology (hep-ph); Statistical Mechanics (cond-mat.stat-mech); Nuclear Theory (nucl-th) Report number: KUNS-2616, YITP-16-40 Cite as: arXiv:1603.04622 [hep-ph] (or arXiv:1603.04622v1 [hep-ph] for this version)

**NOTATION**

**Module One**

Authors utilize the semiclassical approximation to obtain (eb) the time evolution of the distribution functions of the YM field, which is known to show (eb) a chaotic behavior in the classical limit

$G_{13}$ : Category one of semiclassical approximation

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of time evolution of the distribution functions of the YM field, which is known to show (eb) a chaotic behavior in the classical limit

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

Authors utilize the semiclassical approximation to obtain the time evolution of the distribution functions of the YM field, which is known to show (eb) a chaotic behavior in the classical limit

$G_{16}$ : Category one of semiclassical approximation to obtain the time evolution of the distribution functions of the YM field

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

$T_{16}$ : Category one of chaotic behavior in the classical limit

$T_{17}$ : Category two of SAS

| | |
|---|---|
| $T_{18}$ : Category three of SAS | |
| **Module three**<br><br>They also make a simplification of the multi-dimensional phase-space integrals by making a product ansatz for (e) the Husimi function, which is found to give a 10-20 per cent over estimate of the HW entropy for a quantum system with a few degrees of freedom | |
| $G_{20}$ : Category one of making a product ansatz for (e) the Husimi function, which is found to give a 10-20 per cent over estimate of the HW entropy for a quantum system with a few degrees of freedom<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of simplification of the multi-dimensional phase-space integrals<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>They also make a simplification of the multi-dimensional phase-space integrals by making a product ansatz for the Husimi function, which is found to give a 10-20 per cent over estimate of the HW entropy for a quantum system with a few degrees of freedom | |
| $G_{24}$ : Category one of quantum system with a few degrees of freedom<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of simplification of the multi-dimensional phase-space integrals by making a product ansatz for the Husimi function, which is found to give a 10-20 per cent over estimate of the HW entropy<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Authors show that the quantum YM theory does exhibit(eb)the entropy production, and that the entropy production rate agrees with (=) the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting that (eb) the chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory.<br><br>Subjects: High Energy Physics - Phenomenology (hep-ph); Statistical Mechanics (cond-mat.stat-mech); Nuclear Theory (nucl-th) Report number: KUNS-2616, YITP-16-40 Cite as: arXiv:1603.04622 [hep-ph] (or arXiv:1603.04622v1 [hep-ph] for this version) | |
| $G_{28}$ : Category one of quantum YM theory<br><br>$G_{29}$ : Category two of SAS | |

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| $G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of entropy production, and that the entropy production rate agrees with (=) the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting that (eb) the chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory.<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Authors show that the quantum YM theory does exhibit the entropy production, and that the entropy production rate agrees with (=) the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting that (eb) the chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory.<br><br>Subjects: High Energy Physics - Phenomenology (hep-ph); Statistical Mechanics (cond-mat.stat-mech); Nuclear Theory (nucl-th) Report number: KUNS-2616, YITP-16-40 Cite as: arXiv:1603.04622 [hep-ph] (or arXiv:1603.04622v1 [hep-ph] for this version) | |
| $G_{32}$ : Category one of quantum YM theory does exhibit the entropy production, and that the entropy production rate<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting that (eb) the chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory.<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Authors show that the quantum YM theory does exhibit the entropy production, and that the entropy production rate agrees with the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting that (eb) the chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory.<br><br>Subjects: High Energy Physics - Phenomenology (hep-ph); Statistical Mechanics (cond-mat.stat-mech); Nuclear Theory (nucl-th) Report number: KUNS-2616, YITP-16-40 Cite as: arXiv:1603.04622 [hep-ph] (or arXiv:1603.04622v1 [hep-ph] for this version) | |
| $G_{36}$ : Category one of quantum YM theory does exhibit the entropy production, and that the entropy production rate agrees with the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy<br><br>$G_{37}$ : Category two of SAS | |

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|---|---|
| $G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory.<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Authors show that the quantum YM theory does exhibit the entropy production, and that the entropy production rate agrees with the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting that the chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory | |
| $G_{40}$ : Category one of quantum YM theory does exhibit the entropy production, and that the entropy production rate agrees with the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting that the chaoticity of the classical YM field<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of entropy production in the quantum YM theory<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine** | |
| $G_{44}$ : Category one of entropy production in the quantum YM theory; **positive Lyapunov exponents or the Kolmogorov-Sinai entropy**<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of **positive Lyapunov exponents or the Kolmogorov-Sinai entropy** ;entropy production in the quantum YM theory<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$ | |

| | |
|---|---|
| $(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}, \ (b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$<br>$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$<br>$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)} \ (a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,<br>$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$<br>$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,<br>$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,<br>$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b'_{13})^{(1)} - (b''_{13})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b'_{14})^{(1)} - (b''_{14})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b'_{15})^{(1)} - (b''_{15})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a''_{13})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b''_{13})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b'_{16})^{(2)} - (b''_{16})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b'_{17})^{(2)} - (b''_{17})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b'_{18})^{(2)} - (b''_{18})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a''_{16})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b''_{16})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |

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|---|---|
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}\left((G_{27}),t\right) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}\left((G_{31}),t\right) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |

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| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b'_{32})^{(6)} - (b''_{32})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b'_{33})^{(6)} - (b''_{33})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b'_{34})^{(6)} - (b''_{34})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a''_{32})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b'_{36})^{(7)} - (b''_{36})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b'_{37})^{(7)} - (b''_{37})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b'_{38})^{(7)} - (b''_{38})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a''_{36})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a'_{41})^{(8)} + (a''_{41})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b'_{40})^{(8)} - (b''_{40})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b'_{41})^{(8)} - (b''_{41})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b'_{42})^{(8)} - (b''_{42})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b'_{44})^{(9)} - (b''_{44})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b'_{45})^{(9)} - (b''_{45})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b'_{46})^{(9)} - (b''_{46})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a''_{44})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b''_{44})^{(9)}((G_{47}),t) =$ **First detrition factor** | |

$$\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{13} \quad (55)$$

$$\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{array}{ccc} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{14} \quad (56)$$

$$\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{15} \quad (57)$$

Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3

$\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[\begin{array}{ccc} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{array}\right] T_{13} \quad (58)$$

$$\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[\begin{array}{ccc} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{array}\right] T_{14} \quad (59)$$

$$\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)} & -(b_{15}^{''})^{(1)}(G,t) & -(b_{18}^{''})^{(2,2,)}(G_{19},t) & -(b_{22}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,)}(G_{39},t) & -(b_{42}^{''})^{(8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15} \quad (60)$$

Where $-(b_{13}^{''})^{(1)}(G,t)$, $-(b_{14}^{''})^{(1)}(G,t)$, $-(b_{15}^{''})^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{37}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}^{'})^{(2)} & +(a_{16}^{''})^{(2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,)}(T_{14},t) & +(a_{20}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{16} \quad (61)$$

$$\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}^{'})^{(2)} & +(a_{17}^{''})^{(2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,)}(T_{14},t) & +(a_{21}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{17} \quad (62)$$

$$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}^{'})^{(2)} & +(a_{18}^{''})^{(2)}(T_{17},t) & +(a_{15}^{''})^{(1,1,)}(T_{14},t) & +(a_{22}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{18} \quad (63)$$

Where $+(a_{16}^{''})^{(2)}(T_{17},t)$, $+(a_{17}^{''})^{(2)}(T_{17},t)$, $+(a_{18}^{''})^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for

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| category 1, 2 and 3 $+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3 $+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3 $+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)}[-(b_{16}^{''})^{(2)}(G_{19},t)] & [-(b_{13}^{''})^{(1,1,)}(G,t)] & [-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)] \\ [-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)] & [-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)] & [-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)] \\ [-(b_{36}^{''})^{(7,7,7)}(G_{39},t)] & [-(b_{40}^{''})^{(8,8,8)}(G_{43},t)] & [-(b_{44}^{''})^{(9,9)}(G_{47},t)] \end{bmatrix} T_{16}$ | *64* |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)}[-(b_{17}^{''})^{(2)}(G_{19},t)] & [-(b_{14}^{''})^{(1,1,)}(G,t)] & [-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)] \\ [-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)] & [-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)] & [-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)] \\ [-(b_{37}^{''})^{(7,7,7)}(G_{39},t)] & [-(b_{41}^{''})^{(8,8,8)}(G_{43},t)] & [-(b_{45}^{''})^{(9,9)}(G_{47},t)] \end{bmatrix} T_{17}$ | *65* |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)}[-(b_{18}^{''})^{(2)}(G_{19},t)] & [-(b_{15}^{''})^{(1,1,)}(G,t)] & [-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)] \\ [-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)] & [-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)] & [-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)] \\ [-(b_{38}^{''})^{(7,7,7)}(G_{39},t)] & [-(b_{42}^{''})^{(8,8,8)}(G_{43},t)] & [-(b_{46}^{''})^{(9,9)}(G_{47},t)] \end{bmatrix} T_{18}$ | *66* |
| where $[-(b_{16}^{''})^{(2)}(G_{19},t)]$, $[-(b_{17}^{''})^{(2)}(G_{19},t)]$, $[-(b_{18}^{''})^{(2)}(G_{19},t)]$ are first detrition coefficients for category 1, 2 and 3 $[-(b_{13}^{''})^{(1,1,)}(G,t)]$, $[-(b_{14}^{''})^{(1,1,)}(G,t)]$, $[-(b_{15}^{''})^{(1,1,)}(G,t)]$ are second detrition coefficients for category 1,2 and 3 $[-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)]$, $[-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)]$, $[-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)]$ are third detrition coefficients for category 1,2 and 3 $[-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)]$, $[-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)]$, $[-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)]$ are fourth detrition coefficients for category 1,2 and 3 $[-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)]$, $[-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)]$, $[-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)]$ are fifth detrition coefficients for category 1,2 and 3 $[-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)]$, $[-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)]$, $[-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)]$ are sixth detrition coefficients for category 1,2 and 3 $[-(b_{36}^{''})^{(7,7,7)}(G_{39},t)]$, $[-(b_{37}^{''})^{(7,7,7)}(G_{39},t)]$, $[-(b_{38}^{''})^{(7,7,7)}(G_{39},t)]$ are seventh detrition coefficients for category 1,2 and 3 | |

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| $-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[\begin{array}{c}(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21},t)+(a_{16}^{''})^{(2,2,2)}(T_{17},t)+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)\\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)\\ +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{44}^{''})^{(9,9,9)}(T_{45},t)\end{array}\right]G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[\begin{array}{c}(a_{21}^{'})^{(3)}+(a_{21}^{''})^{(3)}(T_{21},t)+(a_{17}^{''})^{(2,2,2)}(T_{17},t)+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)\\ +(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)\\ +(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{45}^{''})^{(9,9,9)}(T_{45},t)\end{array}\right]G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[\begin{array}{c}(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21},t)+(a_{18}^{''})^{(2,2,2)}(T_{17},t)+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)\\ +(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)\\ +(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{46}^{''})^{(9,9,9)}(T_{45},t)\end{array}\right]G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[\begin{array}{c}(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23},t)-(b_{16}^{''})^{(2,2,2)}(G_{19},t)-(b_{13}^{''})^{(1,1,1,)}(G,t)\\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)\\ -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)-(b_{44}^{''})^{(9,9,9)}(G_{47},t)\end{array}\right]T_{20}$ | 70 |

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b'_{21})^{(3)} & -(b''_{21})^{(3)}(G_{23},t) & -(b''_{17})^{(2,2,2)}(G_{19},t) & -(b''_{14})^{(1,1,1,)}(G,t) \\ -(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b''_{37})^{(7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)} & -(b''_{22})^{(3)}(G_{23},t) & -(b''_{18})^{(2,2,2)}(G_{19},t) & -(b''_{15})^{(1,1,1,)}(G,t) \\ -(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b''_{38})^{(7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{22} \quad (72)$$

$-(b''_{20})^{(3)}(G_{23},t)$, $-(b''_{21})^{(3)}(G_{23},t)$, $-(b''_{22})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*

$-(b''_{16})^{(2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*

$-(b''_{13})^{(1,1,1,)}(G,t)$, $-(b''_{14})^{(1,1,1,)}(G,t)$, $-(b''_{15})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*

$-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*

$-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*

$-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*

$-(b''_{36})^{(7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7)}(G_{39},t)$ $-(b''_{38})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b''_{40})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b''_{46})^{(9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a'_{24})^{(4)} & +(a''_{24})^{(4)}(T_{25},t) & +(a''_{28})^{(5,5,)}(T_{29},t) & +(a''_{32})^{(6,6,)}(T_{33},t) \\ +(a''_{13})^{(1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{36})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a'_{25})^{(4)} & +(a''_{25})^{(4)}(T_{25},t) & +(a''_{29})^{(5,5,)}(T_{29},t) & +(a''_{33})^{(6,6)}(T_{33},t) \\ +(a''_{14})^{(1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{37})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a'_{26})^{(4)} & +(a''_{26})^{(4)}(T_{25},t) & +(a''_{30})^{(5,5,)}(T_{29},t) & +(a''_{34})^{(6,6,)}(T_{33},t) \\ +(a''_{15})^{(1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{38})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a''_{24})^{(4)}(T_{25},t)$, $(a''_{25})^{(4)}(T_{25},t)$, $(a''_{26})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a_{28}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* $1,2$ *and* $3$

$+(a_{32}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* $1,2$ *and* $3$

$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$ *are fourth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$ *are fifth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$ *are sixth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$ *are seventh augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$ *are eighth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)}&-(b_{24}^{''})^{(4)}(G_{27},t)&-(b_{28}^{''})^{(5,5,)}(G_{31},t)&-(b_{32}^{''})^{(6,6,)}(G_{35},t)\\&-(b_{13}^{''})^{(1,1,1,1)}(G,t)&-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)&-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)\\&-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)&-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)&-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt}=(b_{25})^{(4)}T_{24}-\begin{bmatrix}(b_{25}^{'})^{(4)}&-(b_{25}^{''})^{(4)}(G_{27},t)&-(b_{29}^{''})^{(5,5,)}(G_{31},t)&-(b_{33}^{''})^{(6,6,)}(G_{35},t)\\&-(b_{14}^{''})^{(1,1,1,1)}(G,t)&-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)&-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)\\&-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)&-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)&-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt}=(b_{26})^{(4)}T_{25}-\begin{bmatrix}(b_{26}^{'})^{(4)}&-(b_{26}^{''})^{(4)}(G_{27},t)&-(b_{30}^{''})^{(5,5,)}(G_{31},t)&-(b_{34}^{''})^{(6,6,)}(G_{35},t)\\&-(b_{15}^{''})^{(1,1,1,1)}(G,t)&-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)&-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)\\&-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)&-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)&-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{26} \quad (78)$$

*Where* $-(b_{24}^{''})^{(4)}(G_{27},t)$, $-(b_{25}^{''})^{(4)}(G_{27},t)$, $-(b_{26}^{''})^{(4)}(G_{27},t)$ *are first detrition coefficients for category* $1,2$ *and* $3$

$-(b_{28}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,)}(G_{31},t)$ *are second detrition coefficients for category* $1,2$ *and* $3$

$-(b_{32}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,)}(G_{35},t)$ *are third detrition coefficients for category* $1,2$ *and* $3$

$-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$ *are fourth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$ *are fifth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$

| | |
|---|---|
| *are sixth detrition coefficients for category* 1, 2 *and* 3 $\boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)}$ *are seventh detrition coefficients for category* 1, 2 *and* 3 $\boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)}$ *are eighth detrition coefficients for category* 1, 2 *and* 3 $\boxed{-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[\begin{array}{ccc} (a_{28}^{'})^{(5)}\boxed{+(a_{28}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[\begin{array}{ccc} (a_{29}^{'})^{(5)}\boxed{+(a_{29}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[\begin{array}{ccc} (a_{30}^{'})^{(5)}\boxed{+(a_{30}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{30}$ | *81* |
| *Where* $\boxed{+(a_{28}^{''})^{(5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5)}(T_{29},t)}$ *are first augmentation coefficients for category* 1, 2 *and* 3 *And* $\boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3 $\boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3 $\boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3 $\boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation coefficients for category 1,2,and 3 $\boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)}$ are sixth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1,2, 3 $\boxed{+(a_{40}^{''})^{(8,8\ ,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$ are eighth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1,2, 3 | |

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b'_{28})^{(5)} & -(b''_{28})^{(5)}(G_{31},t) & -(b''_{24})^{(4,4,)}(G_{27},t) & -(b''_{32})^{(6,6,6)}(G_{35},t) \\ -(b''_{13})^{(1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b'_{29})^{(5)} & -(b''_{29})^{(5)}(G_{31},t) & -(b''_{25})^{(4,4,)}(G_{27},t) & -(b''_{33})^{(6,6,6)}(G_{35},t) \\ -(b''_{14})^{(1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b'_{30})^{(5)} & -(b''_{30})^{(5)}(G_{31},t) & -(b''_{26})^{(4,4,)}(G_{27},t) & -(b''_{34})^{(6,6,6)}(G_{35},t) \\ -(b''_{15})^{(1,1,1,1,1,)}(G,t) & -(b''_{18})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{30} \quad (84)$$

*where* $-(b''_{28})^{(5)}(G_{31},t)$ , $-(b''_{29})^{(5)}(G_{31},t)$ , $-(b''_{30})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{24})^{(4,4,)}(G_{27},t)$, $-(b''_{25})^{(4,4,)}(G_{27},t)$, $-(b''_{26})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3

$-(b''_{32})^{(6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b''_{13})^{(1,1,1,1,1)}(G,t)$, $-(b''_{14})^{(1,1,1,1,1)}(G,t)$ , $-(b''_{15})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3

$-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3

$-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)$, $-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)$, $-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3

$-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3

$-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3

$-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a'_{32})^{(6)} & +(a''_{32})^{(6)}(T_{33},t) & +(a''_{28})^{(5,5,5)}(T_{29},t) & +(a''_{24})^{(4,4,4,)}(T_{25},t) \\ +(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a'_{33})^{(6)} & +(a''_{33})^{(6)}(T_{33},t) & +(a''_{29})^{(5,5,5)}(T_{29},t) & +(a''_{25})^{(4,4,4,)}(T_{25},t) \\ +(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)} G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)} T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)} T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)} T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{28})^{(5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)}+(a_{37}^{''})^{(7)}(T_{37},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15} \quad (93)$$

Where $(a_{36}^{''})^{(7)}(T_{37},t)$, $(a_{37}^{''})^{(7)}(T_{37},t)$, $(a_{38}^{''})^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation

coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3 $+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} =$$

$$(b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}\left[-(b_{36}^{''})^{(7)}(G_{39},t)\right] & \left[-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)\right] & \left[-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{13}$$

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$$\frac{dT_{37}}{dt} =$$

$$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}\left[-(b_{37}^{''})^{(7)}(G_{39},t)\right] & \left[-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)\right] & \left[-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} =$$

$$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}\left[-(b_{38}^{''})^{(7)}(G_{39},t)\right] & \left[-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)\right] & \left[-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3 $-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3 $-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3 $-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{46}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

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$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}')^{(8)}+(a_{40}'')^{(8)}(T_{41},t) & +(a_{16}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}'')^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13}$$

$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}')^{(8)}+(a_{41}'')^{(8)}(T_{41},t) & +(a_{17}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}'')^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}')^{(8)}+(a_{42}'')^{(8)}(T_{41},t) & +(a_{18}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}'')^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$$

Where $+(a_{40}'')^{(8)}(T_{41},t)$, $+(a_{41}'')^{(8)}(T_{41},t)$, $+(a_{42}'')^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & -(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & -(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & -(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3



$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{13}$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[\begin{array}{ccc} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$\frac{dT_{44}}{dt} =$

$$(b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \begin{bmatrix} (b'_{46})^{(9)}\boxed{-(b''_{46})^{(9)}(G_{47},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{44})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a'_i)^{(1)}, (a''_i)^{(1)}, (b_i)^{(1)}, (b'_i)^{(1)}, (b''_i)^{(1)} > 0, \quad i,j = 13,14,15$

The functions $(a''_i)^{(1)}, (b''_i)^{(1)}$ are positive continuous increasing and bounded.

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| **Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br>$(a_i^{''})^{(1)}(T_{14},t) \leq (p_i)^{(1)} \leq (\hat{A}_{13})^{(1)}$<br>$(b_i^{''})^{(1)}(G,t) \leq (r_i)^{(1)} \leq (b_i^{'})^{(1)} \leq (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$<br>$\lim_{G\to\infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$<br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'},t) - (a_i^{''})^{(1)}(T_{14},t)\rvert \leq (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br>$\lvert(b_i^{''})^{(1)}(G^{'},t) - (b_i^{''})^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G^{'}\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i^{'})^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i^{'})^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i^{'})^{(2)}, (a_i^{''})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (b_i^{''})^{(2)} > 0, \quad i,j = 16,17,18$ | |

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| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17},t) \le (p_i)^{(2)} \le (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19},t) \le (r_i)^{(2)} \le (b_i')^{(2)} \le (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2\to\infty} (a_i'')^{(2)}(T_{17},t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty} (b_i'')^{(2)}((G_{19}),t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}$ are positive constants and $i = 16,17,18$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(2)}(T_{17}',t) - (a_i'')^{(2)}(T_{17},t)\rvert \le (\hat{k}_{16})^{(2)}\lvert T_{17} - T_{17}'\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i'')^{(2)}((G_{19})',t) - (b_i'')^{(2)}((G_{19}),t)\rvert < (\hat{k}_{16})^{(2)}\lvert\lvert(G_{19}) - (G_{19})'\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}',t)$ and $(a_i'')^{(2)}(T_{17},t)$ . $(T_{17}',t)$ and $(T_{17},t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}]$ . It is to be noted that $(a_i'')^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i'')^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i')^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ | 111 |

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| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21},t) \le (p_i)^{(3)} \le (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23},t) \le (r_i)^{(3)} \le (b_i^{'})^{(3)} \le (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2\to\infty}(a_i^{''})^{(3)}(T_{21},t) = (p_i)^{(3)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'},t) - (a_i^{''})^{(3)}(T_{21},t)\rvert \le (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'},t) - (b_i^{''})^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lVert G_{23} - G_{23}^{'}\rVert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'},t)$ and$(a_i^{''})^{(3)}(T_{21},t)$ . $(T_{21}^{'},t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$<br>satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |

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| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24{,}25{,}26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}\big((G_{27}), t\big) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(4)}\big((G_{27}), t\big) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24{,}25{,}26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\rvert \leq (\hat{k}_{24})^{(4)} \lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)} t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'}, t) - (b_i^{''})^{(4)}\big((G_{27}), t\big)\rvert < (\hat{k}_{24})^{(4)} \lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)} t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $\big[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\big]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24{,}25{,}26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |

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| $\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i^{'})^{(4)} +\ (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}\big((G_{31}), t\big) \leq\ (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(5)}\,(T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(5)}\,(G_{31}, t) =\ (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'}, t) - (b_i^{''})^{(5)}\big((G_{31}), t\big)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $\big[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\big]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br><br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities | 126 |

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| $\frac{1}{(\widehat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\widehat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32{,}33{,}34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(6)}(T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(6)}((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32{,}33{,}34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)}\lvert T_{33} - T_{33}^{'}\rvert e^{-(\widehat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})^{'}, t) - (b_i^{''})^{(6)}((G_{35}), t)\rvert < (\hat{k}_{32})^{(6)}\lvert\lvert(G_{35}) - (G_{35})^{'}\rvert\rvert e^{-(\widehat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and$(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\widehat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\widehat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\widehat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\widehat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants | 130 |

$(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$
satisfy the inequalities

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

Where we suppose

(MMMM) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i, j = 36,37,38$ 131

(NNNN) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$:

$$(a_i^{''})^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$$

$$(b_i^{''})^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$$

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*(OOOO)* $\lim_{T_2 \to \infty} (a_i^{''})^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(PPPP)*

$\lim_{G \to \infty} (b_i^{''})^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, \ (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)} |T_{37} - T_{37}^{'}| e^{-(\hat{M}_{36})^{(7)} t}$$

$$|(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)} ||(G_{39}) - (G_{39})^{'}|| e^{-(\hat{M}_{36})^{(7)} t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'}, t)$ and $(a_i^{''})^{(7)}(T_{37}, t)$ . $(T_{37}^{'}, t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$ . It is to be noted that $(a_i^{''})^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ : 134

(QQQQ) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants

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| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :<br><br>(RRRR) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}, (r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})', t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})'\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would | |

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| be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i, j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45}, t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47}, t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2 \to \infty} (a_i'')^{(9)}(T_{45}, t) = (p_i)^{(9)}$<br>$\lim_{G \to \infty} (b_i'')^{(9)}(G_{47}, t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}', t) - (a_i'')^{(9)}(T_{45}, t)\rvert \le (\hat{k}_{44})^{(9)} \lvert T_{45} - T_{45}' \rvert e^{-(\hat{M}_{44})^{(9)} t}$<br>$\lvert(b_i'')^{(9)}((G_{47})', t) - (b_i'')^{(9)}((G_{47}), t)\rvert < (\hat{k}_{44})^{(9)} \lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}', t)$ | |

and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)},(\widehat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{44})^{(9)}=1$ then the function $(a_i'')^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$ :

$(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\ ,\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}<1$$

**Definition of** $(\hat{P}_{44})^{(9)},(\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},(\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)},(a_i')^{(9)},(b_i)^{(9)},(b_i')^{(9)},(p_i)^{(9)},\ (r_i)^{(9)}, i=44,45,46,$ satisfy the inequalities

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[(a_i)^{(9)}+(a_i')^{(9)}+\ (\hat{A}_{44})^{(9)}+\ (\hat{P}_{44})^{(9)}\ (\hat{k}_{44})^{(9)}]<1$$

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[\ (b_i)^{(9)}+(b_i')^{(9)}+\ (\hat{B}_{44})^{(9)}+\ (\hat{Q}_{44})^{(9)}\ \ (\hat{k}_{44})^{(9)}]<1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0)\ ,T_i(0)$ :

$$G_i(t)\le\ \left(\hat{P}_{13}\right)^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{G_i(0)=G_i^0>0}$$

$$T_i(t)\le\ (\hat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{T_i(0)=T_i^0>0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0)\ ,T_i(0)$

$$G_i(t)\le\ (\hat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t)\le\ (\hat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

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| **Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$ | 153 B |

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| $T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |

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| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} G_{21}(s_{(20)}) - \left((a_{20}')^{(3)} + a_{20}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{20}(s_{(20)})\right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[(a_{21})^{(3)} G_{20}(s_{(20)}) - \left((a_{21}')^{(3)} + (a_{21}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}')^{(3)} - (b_{21}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{28})^{(5)}$, $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[ (b_{29})^{(5)} T_{28}(s_{(28)}) - \left( (b'_{29})^{(5)} - (b''_{29})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[ (b_{30})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{30})^{(5)} - (b''_{30})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{30}(s_{(28)}) \right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** | |

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| Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[ (a_{32})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{32}(s_{(32)}) \right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[ (a_{33})^{(6)} G_{32}(s_{(32)}) - \left( (a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[ (a_{34})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{34}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[ (b_{32})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{32}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[ (b_{33})^{(6)} T_{32}(s_{(32)}) - \left( (b'_{33})^{(6)} - (b''_{33})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[ (b_{34})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{34})^{(6)} - (b''_{34})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{34}(s_{(32)}) \right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |

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| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}^{'})^{(8)} + a_{40}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^{0} + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^{0} + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^{0} + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{13}(t) \leq G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \leq \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}$ , $G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \leq G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that | 170 |

$$(G_{16}(t)-G_{16}^0)e^{-(\bar{M}_{16})^{(2)}t} \le \frac{(a_{16})^{(2)}}{(\bar{M}_{16})^{(2)}}\left[((\hat{P}_{16})^{(2)}+G_{17}^0)e^{\left(-\frac{(\hat{P}_{16})^{(2)}+G_{17}^0}{G_{17}^0}\right)}+(\hat{P}_{16})^{(2)}\right]$$

Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$

The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 171

$$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0+(\hat{P}_{20})^{(3)}e^{(\bar{M}_{20})^{(3)}s_{(20)}}\right)\right] ds_{(20)} =$$

$$\left(1+(a_{20})^{(3)}t\right)G_{21}^0 + \frac{(a_{20})^{(3)}(\hat{P}_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left(e^{(\bar{M}_{20})^{(3)}t}-1\right)$$

From which it follows that 172

$$(G_{20}(t)-G_{20}^0)e^{-(\bar{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$$

Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$

The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that 173

$$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\bar{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$$

$$\left(1+(a_{24})^{(4)}t\right)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left(e^{(\bar{M}_{24})^{(4)}t}-1\right)$$

From which it follows that 174

$$(G_{24}(t)-G_{24}^0)e^{-(\bar{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 4

The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$$

$$\left(1+(a_{28})^{(5)}t\right)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)}t}-1\right)$$

From which it follows that 175

$$(G_{28}(t)-G_{28}^0)e^{-(\bar{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious 176

that

$$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$$

$$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$$

From which it follows that 177

$$(G_{32}(t) - G_{32}^0) e^{-(\hat{M}_{32})^{(6)} t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)} + G_{33}^0) e^{\left(-\frac{(\hat{P}_{32})^{(6)} + G_{33}^0}{G_{33}^0}\right)} + (\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

(p) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 178

$$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)}\left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$$

$$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$$

From which it follows that

$$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)} + G_{37}^0) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)}\left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} = 180$$

$$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$$

From which it follows that 181

$$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)} + G_{41}^0) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 8

Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$

The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)}\left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$$

$$\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$$

From which it follows that

$$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t}\leq\frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{P}_{44})^{(9)}+G_{45}^{0}\right)e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 9

Analogous inequalities hold also for $G_{45}$ , $G_{46}, T_{44}, T_{45}, T_{46}$

It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ and to choose

$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have

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$$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$$

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$$\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$$

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In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric

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$$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$$

$$\sup_i\left\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\hat{M}_{13})^{(1)}t}\right\}$$

Indeed if we denote

**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$

It results

$$\left|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right|\leq\int_0^t(a_{13})^{(1)}\left|G_{14}^{(1)}-G_{14}^{(2)}\right|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}\,ds_{(13)}+$$

$$\int_0^t\{(a_{13}')^{(1)}\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{-(\hat{M}_{13})^{(1)}s_{(13)}}+$$

$$(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}+$$

$$G_{13}^{(2)}\left|(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}'')^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right|\ e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$$

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| Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | |
| $\left\|G^{(1)} - G^{(2)}\right\| e^{-(\widehat{M}_{13})^{(1)}t} \leq$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right) d\left(\left(G^{(1)}, T^{(1)};\ G^{(2)}, T^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(1)}$ and $(b''_i)^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(1)} - (a''_i)^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1 / (a'_{14})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2 / (a'_{15})^{(1)}$<br><br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(1)}(G(t), t)) = (b'_{14})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ : | 189 |

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| Indeed let $t_1$ be so that for $t > t_1$<br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |

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| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br>$\int_0^t \{(a_{16}')^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$(a_{16}'')^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$G_{16}^{(2)} \left\|(a_{16}'')^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}'')^{(2)} \left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}} \left((a_{16})^{(2)} + (a_{16}')^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)} (\widehat{k}_{16})^{(2)}\right) d\left(\left((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)}\right)\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}'')^{(2)}$ and $(b_{16}'')^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(2)}$ and $(b_i'')^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(2)} - (a_i'')^{(2)} (T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(2)} t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}')^{(2)} G_{17}$ and by integrating<br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)} \left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}')^{(2)}$<br>In the same way , one can obtain | 201 |

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| $G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$, $G_{18}$ and $G_{16}$, $G_{17}$ respectively. | |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$. The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}$, $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}} < 1$ and to choose<br><br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$, $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{20})^{(3)}t}\}$ | 211 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}}$ :$\left( \widetilde{(G_{23})}, \widetilde{(T_{23})} \right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)} \left\|G_{21}^{(1)} - G_{21}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\, ds_{(20)} +$<br><br>$\int_0^t \{(a_{20}^{'})^{(3)} \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right)\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$G_{20}^{(2)} \left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}^{''})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\} ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} t} \leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a_{20}^{'})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)} (\widehat{k}_{20})^{(3)}\right) d\left(\left((G_{23})^{(1)}, (T_{23})^{(1)}; (G_{23})^{(2)}, (T_{23})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ *and* $(\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)} (T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)} t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating | 217 |

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| $G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}')^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}')^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$ | 225 |

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{24})^{(4)}\left|G_{25}^{(1)}-G_{25}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$$

$$\int_0^t\{(a_{24}^{'})^{(4)}\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$G_{24}^{(2)}\left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)}-(G_{27})^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}t}\le$$
$$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows

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**Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}\ and\ (\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

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**Remark 17:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

it results

$$G_i\ (t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]}\ge 0$$

$$T_i\ (t)\ge T_i^0e^{\left(-(b_i^{'})^{(4)}t\right)}>0\quad \text{for t} > 0$$

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**Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1,\left((\widehat{M}_{24})^{(4)}\right)_2\ and\ \left((\widehat{M}_{24})^{(4)}\right)_3$ :

**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if

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| $G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a'_{25})^{(4)} G_{25}$ and by integrating<br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1 / (a'_{25})^{(4)}$<br>In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2 / (a'_{26})^{(4)}$<br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t),t)) = (b_{25}^{'})^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i^{''})^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded.<br>The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + \left((\widehat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{28})^{(5)} + G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\widehat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{28})^{(5)} + T_j^0}{T_j^0}\right)} + (\widehat{Q}_{28})^{(5)}\right] \leq (\widehat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying | |

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| Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}((G_{31}),(T_{31}))$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)}+$<br>$\int_0^t\{(a_{28}^{'})^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\le$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}^{'})^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}^{''})^{(5)}$ and $(b_{28}^{''})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(5)}-(a_i^{''})^{(5)}(T_{29}(s_{(28)}),s_{(28)})\}ds_{(28)}\right]}\ge 0$ | 239 |

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| $T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(5)}t)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}^{'})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1/(a_{29}^{'})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2/(a_{30}^{'})^{(5)}$<br><br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t),t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right)$, $t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |

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| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{32})^{(6)}\right]\leq(\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}ds_{(32)}+$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a''_{32})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a'_{32})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(6)}$ and $(b_i'')^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 249 |

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| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(6)} - (a_i'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(6)} t)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}')^{(6)} G_{33}$ and by integrating<br><br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}')^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}')^{(6)}$<br><br>If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(6)}((G_{35})(t), t)) = (b_{33}')^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i'')^{(6)}((G_{35})(t), t) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty} (b_{34}'')^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}')^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have | 255 |

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| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)}+\left((\hat{P}_{36})^{(7)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{36})^{(7)}\right]\leq(\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}')^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}'')^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}')^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}'')^{(7)}$ and $(b_{36}'')^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$. | 260 |

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| If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2 / (a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t), t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t), t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right),\ \ t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded. | 265 |

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| The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t),t\right)=(b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}<1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)}+\left((\hat{P}_{40})^{(8)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{40})^{(8)}\right]\leq(\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}ds_{(40)}+$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\leq$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ $and$ $T_i$. Taking into account the hypothesis the result follows | 273 |

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| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(8)} - (a_i^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2\ and\ \left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}^{'})^{(8)}$<br><br>If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t), t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t), t\right) < \varepsilon_8, T_{40}\,(t) > (m)^{(8)}$ | 278 |

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| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.<br>The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}^{'})^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$ | |
| In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying 39,35,36 into itself | |
| The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{47})^{(1)}, (T_{47})^{(1)}), ((G_{47})^{(2)}, (T_{47})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \left(\widetilde{(G_{47})}, \widetilde{(T_{47})}\right) = \mathcal{A}^{(9)}\left((G_{47}), (T_{47})\right)$<br>It results<br><br>$\left\|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$<br>$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right) - (a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$<br><br>Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on 45,46,47,28 and 29 it follows | |
| $\left|(G_{47})^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t} \leq$<br>$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)}, (T_{47})^{(1)};\ (G_{47})^{(2)}, (T_{47})^{(2)})\right)$ | |

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44{,}45{,}46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$

From 99 to 44 it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(9)}t\right)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)}G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t), t\right) < \varepsilon_9, T_{44}\,(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded. The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b''_{46})^{(9)}\left((G_{47})(t),t\right) = (b'_{46})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a'_{13})^{(1)} + (a'_{14})^{(1)} - (a''_{13})^{(1)}(T_{14},t) + (a''_{14})^{(1)}(T_{14},t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b'_{13})^{(1)} + (b'_{14})^{(1)} - (b''_{13})^{(1)}(G,t) - (b''_{14})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,

and $\boxed{(u_0)^{(1)} = \frac{T^0_{13}}{T^0_{14}}}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$

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| are defined | |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le$<br>$\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a'_{15})^{(1)})}[e^{(S_1)^{(1)}t} - e^{-(a'_{15})^{(1)}t}] + G_{15}^0 e^{-(a'_{15})^{(1)}t})$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b'_{15})^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b'_{15})^{(1)}t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)}t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$<br>$(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ :<br>$(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a'_{16})^{(2)} + (a'_{17})^{(2)} - (a''_{16})^{(2)}(T_{17}, t) + (a''_{17})^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b'_{16})^{(2)} + (b'_{17})^{(2)} - (b''_{16})^{(2)}((G_{19}), t) - (b''_{17})^{(2)}((G_{19}), t) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(v_1)^{(2)}, (v_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |

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| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\left(u^{(2)}\right)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}$,, $(\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\left(u^{(2)}\right)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$,<br>and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)}t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le\right.$ | 312 |

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| $\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_2)^{(2)}((S_1)^{(2)}-(a_{18}')^{(2)})}[e^{(S_1)^{(2)}t}-e^{-(a_{18}')^{(2)}t}]+G_{18}^{0}e^{-(a_{18}')^{(2)}t})$ | |
| $\boxed{T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}}T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le\frac{1}{(\mu_2)^{(2)}}T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)}T_{16}^{0}}{(\mu_1)^{(2)}((R_1)^{(2)}-(b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t}-e^{-(b_{18}')^{(2)}t}\right]+T_{18}^{0}e^{-(b_{18}')^{(2)}t}\le T_{18}(t)\le$ $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)},(S_2)^{(2)},(R_1)^{(2)},(R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)}=(a_{16})^{(2)}(m_2)^{(2)}-(a_{16}')^{(2)}$ $(S_2)^{(2)}=(a_{18})^{(2)}-(p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)}=(b_{16})^{(2)}(\mu_2)^{(1)}-(b_{16}')^{(2)}$ $(R_2)^{(2)}=(b_{18}')^{(2)}-(r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$: $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)}\le-(a_{20}')^{(3)}+(a_{21}')^{(3)}-(a_{20}'')^{(3)}(T_{21},t)+(a_{21}'')^{(3)}(T_{21},t)\le-(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)}\le-(b_{20}')^{(3)}+(b_{21}')^{(3)}-(b_{20}'')^{(3)}(G_{23},t)-(b_{21}'')^{(3)}((G_{23}),t)\le-(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)},(v_2)^{(3)},(u_1)^{(3)},(u_2)^{(3)}$: By $(v_1)^{(3)}>0,(v_2)^{(3)}<0$ and respectively $(u_1)^{(3)}>0,(u_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_1)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_1)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ and By $(\bar{v}_1)^{(3)}>0,(\bar{v}_2)^{(3)}<0$ and respectively $(\bar{u}_1)^{(3)}>0,(\bar{u}_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_2)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_2)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ | 320 |
| **Definition of** $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$:- | 321 |

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$, ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,

and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously

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$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$$

$(p_i)^{(3)}$ is defined by equation

$$\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t} \quad (323)$$

$$\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$$

$$\left. \frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right) \quad (324)$$

$$\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}} \quad (325)$$

$$\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t} \quad (326)$$

$$\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$$

$$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t} \quad (327)$$

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- (328)

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$$

$$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$$

$$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}\big((G_{27}), t\big) - (b''_{25})^{(4)}\big((G_{27}), t\big) \le -(\tau_1)^{(4)}$$

**Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : 329

By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{v}_1)^{(4)} > 0$, $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (v_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}, \ \textbf{\textit{if}}\ (v_0)^{(4)} < (v_1)^{(4)}$$

$$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}, \textbf{\textit{if}}\ (v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)},$$

and $\boxed{(v_0)^{(4)} = \frac{G^0_{24}}{G^0_{25}}}$

$$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}, \ \textbf{\textit{if}}\ (\bar{v}_4)^{(4)} < (v_0)^{(4)}$$

and analogously 331

$$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}, \ \textbf{\textit{if}}\ (u_0)^{(4)} < (u_1)^{(4)}$$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}, \textbf{\textit{if}}\ (u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)},$$

and $\boxed{(u_0)^{(4)} = \frac{T^0_{24}}{T^0_{25}}}$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \textbf{\textit{if}}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)} \text{ where } (u_1)^{(4)}, (\bar{u}_1)^{(4)}$$

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| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}')^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}')^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)}t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(\nu_1)^{(5)}, (\nu_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, \nu^{(5)}, u^{(5)}$ :<br>By $(\nu_1)^{(5)} > 0$ , $(\nu_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$ , $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}(\nu^{(5)})^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)} = 0$ | 339 |

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|---|---|
| and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ :<br><br>By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)},(v_0)^{(5)}$ :-<br><br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)}$ by<br><br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br><br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br><br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |

$\frac{(b_{30})^{(5)}T_{28}^{0}}{(\mu_1)^{(5)}((R_1)^{(5)}-(b_{30}^{'})^{(5)})}\left[e^{(R_1)^{(5)}t}-e^{-(b_{30}^{'})^{(5)}t}\right]+T_{30}^{0}e^{-(b_{30}^{'})^{(5)}t}\le T_{30}(t)\le$

$\frac{(a_{30})^{(5)}T_{28}^{0}}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t}-e^{-(R_2)^{(5)}t}\right]+T_{30}^{0}e^{-(R_2)^{(5)}t}$ 347

**Definition of** $(S_1)^{(5)},(S_2)^{(5)},(R_1)^{(5)},(R_2)^{(5)}$:- 348

Where $(S_1)^{(5)}=(a_{28})^{(5)}(m_2)^{(5)}-(a_{28}^{'})^{(5)}$

$(S_2)^{(5)}=(a_{30})^{(5)}-(p_{30})^{(5)}$

$(R_1)^{(5)}=(b_{28})^{(5)}(\mu_2)^{(5)}-(b_{28}^{'})^{(5)}$

$(R_2)^{(5)}=(b_{30}^{'})^{(5)}-(r_{30})^{(5)}$

**Behavior of the solutions of equation** 349

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ four constants satisfying

$-(\sigma_2)^{(6)}\le-(a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}-(a_{32}^{''})^{(6)}(T_{33},t)+(a_{33}^{''})^{(6)}(T_{33},t)\le-(\sigma_1)^{(6)}$

$-(\tau_2)^{(6)}\le-(b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(b_{32}^{''})^{(6)}\left((G_{35}),t\right)-(b_{33}^{''})^{(6)}\left((G_{35}),t\right)\le-(\tau_1)^{(6)}$

**Definition of** $(\nu_1)^{(6)},(\nu_2)^{(6)},(u_1)^{(6)},(u_2)^{(6)},\nu^{(6)},u^{(6)}$ : 350

By $(\nu_1)^{(6)}>0$ , $(\nu_2)^{(6)}<0$ and respectively $(u_1)^{(6)}>0$ , $(u_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_1)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2+(\tau_1)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$ and

**Definition of** $(\bar{\nu}_1)^{(6)},,(\bar{\nu}_2)^{(6)},(\bar{u}_1)^{(6)},(\bar{u}_2)^{(6)}$ : 351

By $(\bar{\nu}_1)^{(6)}>0$ , $(\bar{\nu}_2)^{(6)}<0$ and respectively $(\bar{u}_1)^{(6)}>0$ , $(\bar{u}_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_2)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2+(\tau_2)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$

**Definition of** $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)},(\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)}$ by

$(m_2)^{(6)}=(\nu_0)^{(6)},(m_1)^{(6)}=(\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)}<(\nu_1)^{(6)}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)}<(\nu_0)^{(6)}<(\bar{\nu}_1)^{(6)}$,

and $\boxed{(\nu_0)^{(6)}=\frac{G_{32}^{0}}{G_{33}^{0}}}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)}<(\nu_0)^{(6)}$

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|---|---|
| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},\ \boldsymbol{if}\ (u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}, \boldsymbol{if}\ (u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)},$<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}, \boldsymbol{if}\ (\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le\right.$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b_{34}')^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b_{34}')^{(6)} t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$<br>$(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ : | |

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| $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying<br><br>$-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$<br><br>$-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}((G_{39}), t) - (b''_{37})^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)}, (v_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, v^{(7)}, u^{(7)}$ :<br><br>By $(v_1)^{(7)} > 0, (v_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(v^{(7)})^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)}, , (\bar{v}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ :<br><br>By $(\bar{v}_1)^{(7)} > 0, (\bar{v}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(v^{(7)})^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$<br>**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (v_0)^{(7)}$ :-<br><br>If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by<br><br>$(m_2)^{(7)} = (v_0)^{(7)}, (m_1)^{(7)} = (v_1)^{(7)}$, ***if*** $(v_0)^{(7)} < (v_1)^{(7)}$<br><br>$(m_2)^{(7)} = (v_1)^{(7)}, (m_1)^{(7)} = (\bar{v}_1)^{(7)}$, ***if*** $(v_1)^{(7)} < (v_0)^{(7)} < (\bar{v}_1)^{(7)}$,<br>and $\boxed{(v_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$<br><br>$(m_2)^{(7)} = (v_1)^{(7)}, (m_1)^{(7)} = (v_0)^{(7)}$, ***if*** $(\bar{v}_1)^{(7)} < (v_0)^{(7)}$ | 362 |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,<br>and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities | 364 |

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|---|---|
| $G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^{0}e^{(S_1)^{(7)}t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | |
| $\frac{1}{(m_7)^{(7)}} G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^{0}e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^{0}e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}^{'})^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}^{'})^{(7)}t}] + G_{38}^{0}e^{-(a_{38}^{'})^{(7)}t})$ | 366 |
| $\boxed{T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)}T_{36}^{0}}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}^{'})^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}^{'})^{(7)}t}\right] + T_{38}^{0}e^{-(b_{38}^{'})^{(7)}t} \le T_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)}T_{36}^{0}}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^{0}e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}^{'})^{(7)}$<br><br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br><br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}^{'})^{(7)}$<br><br>$(R_2)^{(7)} = (b_{38}^{'})^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br><br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br><br>$-(\sigma_2)^{(8)} \le -(a_{40}^{'})^{(8)} + (a_{41}^{'})^{(8)} - (a_{40}^{''})^{(8)}(T_{41}, t) + (a_{41}^{''})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br><br>$-(\tau_2)^{(8)} \le -(b_{40}^{'})^{(8)} + (b_{41}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}), t) - (b_{41}^{''})^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |
| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : | 372 |

By $(v_1)^{(8)} > 0\,, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0\,, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$
and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and

**Definition of** $(\bar{v}_1)^{(8)},, (\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :

By $(\bar{v}_1)^{(8)} > 0\,, (\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0\,, (\bar{u}_2)^{(8)} < 0$ the

roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$

and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$

**Definition of** $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :-

If we define $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by

$(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)},$ ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}\,,$ ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)},$
and $\boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (v_0)^{(8)},$ ***if*** $(\bar{v}_1)^{(8)} < (v_0)^{(8)}$

and analogously

$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},$ ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$

$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}\,,$ ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$
and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$

$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)},$ ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$

Then the solution of global equations satisfies the inequalities

$$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)}t}$$

where $(p_i)^{(8)}$ is defined by equation

$$\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$$

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| $\left(\frac{(a_{42})^{(8)}G_{40}^{0}}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t}-e^{-(S_2)^{(8)}t}\right]+G_{42}^{0}e^{-(S_2)^{(8)}t}\leq G_{42}(t)\leq \frac{(a_{42})^{(8)}G_{40}^{0}}{(m_2)^{(8)}((S_1)^{(8)}-(a_{42}^{'})^{(8)})}[e^{(S_1)^{(8)}t}-e^{-(a_{42}^{'})^{(8)}t}]+G_{42}^{0}e^{-(a_{42}^{'})^{(8)}t}\right)$ | 377 |
| $\boxed{T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}}T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq \frac{1}{(\mu_2)^{(8)}}T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^{0}}{(\mu_1)^{(8)}((R_1)^{(8)}-(b_{42}^{'})^{(8)})}\left[e^{(R_1)^{(8)}t}-e^{-(b_{42}^{'})^{(8)}t}\right]+T_{42}^{0}e^{-(b_{42}^{'})^{(8)}t}\leq T_{42}(t)\leq$ $\frac{(a_{42})^{(8)}T_{40}^{0}}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t}-e^{-(R_2)^{(8)}t}\right]+T_{42}^{0}e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)},(S_2)^{(8)},(R_1)^{(8)},(R_2)^{(8)}$:-<br>Where $(S_1)^{(8)}=(a_{40})^{(8)}(m_2)^{(8)}-(a_{40}^{'})^{(8)}$<br>$(S_2)^{(8)}=(a_{42})^{(8)}-(p_{42})^{(8)}$<br>$(R_1)^{(8)}=(b_{40})^{(8)}(\mu_2)^{(8)}-(b_{40}^{'})^{(8)}$<br>$(R_2)^{(8)}=(b_{42}^{'})^{(8)}-(r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)}\leq -(a_{44}^{'})^{(9)}+(a_{45}^{'})^{(9)}-(a_{44}^{''})^{(9)}(T_{45},t)+(a_{45}^{''})^{(9)}(T_{45},t)\leq -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)}\leq -(b_{44}^{'})^{(9)}+(b_{45}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47}),t)-(b_{45}^{''})^{(9)}((G_{47}),t)\leq -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(\nu_1)^{(9)},(\nu_2)^{(9)},(u_1)^{(9)},(u_2)^{(9)},\nu^{(9)},u^{(9)}$ :<br>By $(\nu_1)^{(9)}>0$ , $(\nu_2)^{(9)}<0$ and respectively $(u_1)^{(9)}>0$ , $(u_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_1)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$<br>and $(b_{45})^{(9)}(u^{(9)})^2+(\tau_1)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(9)},,(\bar{\nu}_2)^{(9)},(\bar{u}_1)^{(9)},(\bar{u}_2)^{(9)}$ :<br>By $(\bar{\nu}_1)^{(9)}>0$ , $(\bar{\nu}_2)^{(9)}<0$ and respectively $(\bar{u}_1)^{(9)}>0$ , $(\bar{u}_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_2)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$ | |

and $(b_{45})^{(9)}\left(u^{(9)}\right)^2+(\tau_2)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$

**Definition of** $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)},(\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)}$ by

$(m_2)^{(9)}=(\nu_0)^{(9)},(m_1)^{(9)}=(\nu_1)^{(9)},$ ***if*** $(\nu_0)^{(9)}<(\nu_1)^{(9)}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)}<(\nu_0)^{(9)}<(\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)}=\frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\nu_0)^{(9)},$ ***if*** $(\bar{\nu}_1)^{(9)}<(\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)}=(u_0)^{(9)},(\mu_1)^{(9)}=(u_1)^{(9)},$ ***if*** $(u_0)^{(9)}<(u_1)^{(9)}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)}<(u_0)^{(9)}<(\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)}=\frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(u_0)^{(9)},$ ***if*** $(\bar{u}_1)^{(9)}<(u_0)^{(9)}$ where $(u_1)^{(9)},(\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{44}(t)\le G_{44}^0e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{45}(t)\le\frac{1}{(m_2)^{(9)}}G_{44}^0e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)}G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^0e^{-(S_2)^{(9)}t}\le G_{46}(t)\le\right.$$

$$\left.\frac{(a_{46})^{(9)}G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}')^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}')^{(9)}t}]+\ G_{46}^0e^{-(a_{46}')^{(9)}t}\right)$$

$$\boxed{T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le\frac{1}{(\mu_2)^{(9)}}T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}')^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}')^{(9)}t}\right]+T_{46}^0e^{-(b_{46}')^{(9)}t}\le T_{46}(t)\le$$

$$\frac{(a_{46})^{(9)}T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^0e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain (383)

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right) - (a''_{14})^{(1)}(T_{14}, t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get (384)

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{v}_1)^{(1)}$ we find like in the previous case, (385)

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)}G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)}T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$$

If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain 394

$$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(2)}(t)$ :- 395

$$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}, \quad \boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$$

In a completely analogous way, we obtain 396

**Definition of** $u^{(2)}(t)$ :-

$$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}, \quad \boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$$

| | |
|---|---|
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$ if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| From which one obtains<br><br>For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$<br><br>$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)} + (C)^{(3)}(\nu_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}$ , $\boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\nu_2)^{(3)}}}$<br><br>it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$ | 400 |
| In the same manner , we get<br><br>$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)} + (\bar{C})^{(3)}(\bar{\nu}_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\bar{\nu}_2)^{(3)}}}$<br><br>**Definition of** $(\bar{\nu}_1)^{(3)}$ :-<br><br>From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$ | 401 |
| If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, | 402 |

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

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$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right) - (a_{25}^{''})^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\quad \boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

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$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} =$

$(\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\quad \boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get 413

$$v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{v}_1)^{(6)}-(v_0)^{(6)}}{(v_0)^{(6)}-(\bar{v}_2)^{(6)}}}$$

From which we deduce $(v_0)^{(6)} \le v^{(6)}(t) \le (\bar{v}_1)^{(6)}$

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If $0 < (v_1)^{(6)} < (v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{v}_1)^{(6)}$ we find like in the previous case,

$$(v_1)^{(6)} \le \frac{(v_1)^{(6)}+(C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}} \le v^{(6)}(t) \le$$

$$\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (\bar{v}_1)^{(6)}$$

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If $0 < (v_1)^{(6)} \le (\bar{v}_1)^{(6)} \le \boxed{(v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(v_1)^{(6)} \le v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (v_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$(m_2)^{(6)} \le v^{(6)}(t) \le (m_1)^{(6)}$, $\boxed{v^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \le u^{(6)}(t) \le (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(v_1)^{(6)} = (\bar{v}_1)^{(6)}$ if in addition $(v_0)^{(6)} = (v_1)^{(6)}$ then $v^{(6)}(t) = (v_0)^{(6)}$ and as a consequence $G_{32}(t) = (v_0)^{(6)}G_{33}(t)$ **this also defines** $(v_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(v_1)^{(6)}$ and $(\bar{v}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right) - (a_{37}^{''})^{(7)}(T_{37},t)v^{(7)} - (a_{37})^{(7)}v^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get 417

$$\nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\bar{\nu}_1)^{(7)}$

If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case, 418

$$(\nu_1)^{(7)} \leq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}} \leq \nu^{(7)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\bar{\nu}_1)^{(7)}$$

If $0 < (\nu_1)^{(7)} \leq (\bar{\nu}_1)^{(7)} \leq \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain 419

$$(\nu_1)^{(7)} \leq \nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \leq \nu^{(7)}(t) \leq (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :- 420

$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)} G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain 421

$$\frac{d\nu^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right) - (a_{41}^{''})^{(8)}(T_{41}, t)\nu^{(8)} - (a_{41})^{(8)}\nu^{(8)}$$

**Definition of** $\nu^{(8)}$ :- $\boxed{\nu^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right) \leq \frac{d\nu^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)}-(\nu_0)^{(8)}}{(\nu_0)^{(8)}-(\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \le \nu^{(8)}(t) \le (\bar{\nu}_8)^{(8)}$

If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case, 423

$$(\nu_1)^{(8)} \le \frac{(\nu_1)^{(8)}+(C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}} \le \nu^{(8)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\bar{\nu}_1)^{(8)}$$

If $0 < (\nu_1)^{(8)} \le (\bar{\nu}_1)^{(8)} \le \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$, we obtain 424

$$(\nu_1)^{(8)} \le \nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have
**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \le \nu^{(8)}(t) \le (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain
**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \le u^{(8)}(t) \le (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)}G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)}T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$. 424

**Proof** : From 99,20,44,22,23,44 we obtain 424 A

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right) - (a''_{45})^{(9)}(T_{45},t)v^{(9)} - (a_{45})^{(9)}v^{(9)}$$

**Definition of** $v^{(9)}$ :- $\boxed{v^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)}\right) \le \frac{dv^{(9)}}{dt} \le -\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(9)}, (v_0)^{(9)}$ :-

For $0 < \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (v_1)^{(9)} < (\bar{v}_1)^{(9)}$

$$v^{(9)}(t) \ge \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(v_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (v_2)^{(9)}}}$$

it follows $(v_0)^{(9)} \le v^{(9)}(t) \le (v_9)^{(9)}$

In the same manner , we get

$$v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{v}_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (\bar{v}_2)^{(9)}}}$$

From which we deduce $(v_0)^{(9)} \le v^{(9)}(t) \le (\bar{v}_1)^{(9)}$

If $0 < (v_1)^{(9)} < (v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}} < (\bar{v}_1)^{(9)}$ we find like in the previous case,

$$(v_1)^{(9)} \le \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}} \le v^{(9)}(t) \le$$

$$\frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (\bar{v}_1)^{(9)}$$

If $0 < (v_1)^{(9)} \le (\bar{v}_1)^{(9)} \le \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$ , we obtain

$$(v_1)^{(9)} \le v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (v_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(9)}(t)$ :-

$$(m_2)^{(9)} \le v^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{v^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}$, $\boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ 430

$with\ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations 431

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$

| | |
|---|---|
| $with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | |
| We can prove the following<br><br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br><br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br><br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br><br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$ | 435 |

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|---|---|
| $with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |

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| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27})\big)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\big((G_{27})\big)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\big((G_{27})\big)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |

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| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |

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| $(b_{42})^{(8)}T_{41} - [(b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br>$F(T) = (a'_{13})^{(1)}(a'_{14})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a'_{13})^{(1)}(a''_{14})^{(1)}(T_{14}) + (a'_{14})^{(1)}(a''_{13})^{(1)}(T_{14}) + (a''_{13})^{(1)}(T_{14})(a''_{14})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br>(p) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br>$\mathrm{F}(T_{19}) = (a'_{16})^{(2)}(a'_{17})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a'_{16})^{(2)}(a''_{17})^{(2)}(T_{17}) + (a'_{17})^{(2)}(a''_{16})^{(2)}(T_{17}) + (a''_{16})^{(2)}(T_{17})(a''_{17})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br>$F(T_{23}) = (a'_{20})^{(3)}(a'_{21})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a'_{20})^{(3)}(a''_{21})^{(3)}(T_{21}) + (a'_{21})^{(3)}(a''_{20})^{(3)}(T_{21}) + (a''_{20})^{(3)}(T_{21})(a''_{21})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br>$F(T_{27}) = (a'_{24})^{(4)}(a'_{25})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a'_{24})^{(4)}(a''_{25})^{(4)}(T_{25}) + (a'_{25})^{(4)}(a''_{24})^{(4)}(T_{25}) + (a''_{24})^{(4)}(T_{25})(a''_{25})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br>$F(T_{31}) = (a'_{28})^{(5)}(a'_{29})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a'_{28})^{(5)}(a''_{29})^{(5)}(T_{29}) + (a'_{29})^{(5)}(a''_{28})^{(5)}(T_{29}) +$ | 489 |

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| $(a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that | 496 |

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| there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{32}=\frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}=\frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{36}=\frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}=\frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |
| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*})=0$. With this value , we obtain from the three first equations | 501 |

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| $G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G) + (b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*}) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19}) + (b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br>$[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ | 506 |

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| is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if $\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$ $\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$ Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if $\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$ $\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$ Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if $\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$ $\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$ Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if $\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$ $\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$ Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if $\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$ $\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$ Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution | 511 |

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| $G_{14}^{*}$ given by $\varphi(G^{*})=0$ , $T_{14}^{*}$ given by $f(T_{14}^{*})=0$ and<br>$G_{13}^{*}=\frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*}=\frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br>$T_{13}^{*}=\frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*}=\frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br>Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*})=0$ , $T_{17}^{*}$ given by $f(T_{17}^{*})=0$ and | 512 |
| $G_{16}^{*}=\frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*}=\frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*}=\frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*}=\frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*})=0$ , $T_{21}^{*}$ given by $f(T_{21}^{*})=0$ and<br>$G_{20}^{*}=\frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*}=\frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br>$T_{20}^{*}=\frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*}=\frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^{*}$ given by $\varphi(G_{27})=0$ , $T_{25}^{*}$ given by $f(T_{25}^{*})=0$ and<br>$G_{24}^{*}=\frac{(a_{24})^{(4)}G_{25}^{*}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*}=\frac{(a_{26})^{(4)}G_{25}^{*}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*}=\frac{(b_{24})^{(4)}T_{25}^{*}}{[(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*}=\frac{(b_{26})^{(4)}T_{25}^{*}}{[(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}((G_{27})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*})=0$ , $T_{29}^{*}$ given by $f(T_{29}^{*})=0$ and<br>$G_{28}^{*}=\frac{(a_{28})^{(5)}G_{29}^{*}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*}=\frac{(a_{30})^{(5)}G_{29}^{*}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |

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| $T_{28}^{*} = \frac{(b_{28})^{(5)}T_{29}^{*}}{[(b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)}T_{29}^{*}}{[(b_{30}^{'})^{(5)}-(b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br>$G_{32}^{*} = \frac{(a_{32})^{(6)}G_{33}^{*}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)}G_{33}^{*}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)}T_{33}^{*}}{[(b_{32}^{'})^{(6)}-(b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)}T_{33}^{*}}{[(b_{34}^{'})^{(6)}-(b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br>$G_{36}^{*} = \frac{(a_{36})^{(7)}G_{37}^{*}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)}G_{37}^{*}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br>$T_{36}^{*} = \frac{(b_{36})^{(7)}T_{37}^{*}}{[(b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)}T_{37}^{*}}{[(b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br>$G_{40}^{*} = \frac{(a_{40})^{(8)}G_{41}^{*}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)}G_{41}^{*}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br>$T_{40}^{*} = \frac{(b_{40})^{(8)}T_{41}^{*}}{[(b_{40}^{'})^{(8)}-(b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)}T_{41}^{*}}{[(b_{42}^{'})^{(8)}-(b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)}G_{45}^{*}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)}G_{45}^{*}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)}T_{45}^{*}}{[(b_{44}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)}T_{45}^{*}}{[(b_{46}^{'})^{(9)}-(b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | 523 A |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^*) = (q_{14})^{(1)} \quad , \frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^*) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)}G_{13}^*\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^*\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^*\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^*\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^*\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^*\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)} \quad , \frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |

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| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ $and$ $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)}$ , $\frac{\partial(b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_{j}\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_{j}\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_{j}\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 548 |

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| $\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^{*}) = (q_{25})^{(4)}$ , $\frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^{*}) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^{*}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^{*}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^{*}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^{*}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^{*}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^{*}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^{*}) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^{*}) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^{*}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^{*}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^{*}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^{*}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^{*}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^{*}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 563 |

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| **Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ $and$ $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^*\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^*\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^*\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^*\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ $and$ $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}((G_{39})^{**}) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |

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| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^{*}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^{*}\mathbb{G}_{j}\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^{*}\mathbb{G}_{j}\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^{*}\mathbb{G}_{j}\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^{*}) = (q_{41})^{(8)} \quad , \frac{\partial(b_i^{''})^{(8)}}{\partial G_j}\left((G_{43})^{*}\right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^{*}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^{*}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^{*}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^{*}\mathbb{G}_{j}\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b_{41}^{'})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T_{41}^{*}\mathbb{G}_{j}\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b_{42}^{'})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T_{42}^{*}\mathbb{G}_{j}\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |

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| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \ \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b_{44}^{'})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T_{44}^*\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b_{45}^{'})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T_{45}^*\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b_{46}^{'})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T_{46}^*\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)$<br>$\left[\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(q_{13})^{(1)}G_{13}^*\right)\Big]$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T_{14}^* + (b_{14})^{(1)}s_{(13),(14)}T_{14}^*\Big)$<br>$+\left(\left((\lambda)^{(1)} + (a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^* + (a_{13})^{(1)}(q_{14})^{(1)}G_{14}^*\right)$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^* + (b_{14})^{(1)}s_{(13),(13)}T_{13}^*\Big)$<br>$\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$<br>$\left((\lambda)^{(1)}\right)^2 + \left((b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$<br>$+\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$<br>$+\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^*\right)$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^* + (b_{14})^{(1)}s_{(13),(15)}T_{13}^*\Big)\} = 0$ | |

$$+$$

$$((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)})\{((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)})$$

$$\left[\left(((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)})(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)\right]$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(17)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^{*}_{17}\right)$$

$$+\left(((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)})(q_{16})^{(2)}\mathrm{G}^{*}_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(16)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^{*}_{16}\right)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left(\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(18)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(18)}\mathrm{T}^{*}_{16}\right)\} = 0$$

$$+$$

$$((\lambda)^{(3)} + (b'_{22})^{(3)} - (r_{22})^{(3)})\{((\lambda)^{(3)} + (a'_{22})^{(3)} + (p_{22})^{(3)})$$

$$\left[\left(((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)})(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)\right]$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(21)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(21)}T^{*}_{21}\right)$$

$$+\left(((\lambda)^{(3)} + (a'_{21})^{(3)} + (p_{21})^{(3)})(q_{20})^{(3)}G^{*}_{20} + (a_{20})^{(3)}(q_{21})^{(1)}G^{*}_{21}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(20)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(20)}T^{*}_{20}\right)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((b'_{20})^{(3)} + (b'_{21})^{(3)} - (r_{20})^{(3)} + (r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left(\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(22)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(22)}T^{*}_{20}\right)\} = 0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\right)\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\right)\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\right)\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(8)} + (b'_{42})^{(8)} - (r_{42})^{(8)})\{((\lambda)^{(8)} + (a'_{42})^{(8)} + (p_{42})^{(8)})$$

$$\left[\left(((\lambda)^{(8)} + (a'_{40})^{(8)} + (p_{40})^{(8)})(q_{41})^{(8)}G^*_{41} + (a_{41})^{(8)}(q_{40})^{(8)}G^*_{40}\right)\right]$$

$$\left(((\lambda)^{(8)} + (b'_{40})^{(8)} - (r_{40})^{(8)})s_{(41),(41)}T^*_{41} + (b_{41})^{(8)}s_{(40),(41)}T^*_{41}\right)$$

$$+\left(((\lambda)^{(8)} + (a'_{41})^{(8)} + (p_{41})^{(8)})(q_{40})^{(8)}G^*_{40} + (a_{40})^{(8)}(q_{41})^{(8)}G^*_{41}\right)$$

$$\left(((\lambda)^{(8)} + (b'_{40})^{(8)} - (r_{40})^{(8)})s_{(41),(40)}T^*_{41} + (b_{41})^{(8)}s_{(40),(40)}T^*_{40}\right)$$

$$\left(((\lambda)^{(8)})^2 + (\,(a'_{40})^{(8)} + (a'_{41})^{(8)} + (p_{40})^{(8)} + (p_{41})^{(8)})\,(\lambda)^{(8)}\right)$$

$$\left(((\lambda)^{(8)})^2 + (\,(b'_{40})^{(8)} + (b'_{41})^{(8)} - (r_{40})^{(8)} + (r_{41})^{(8)})\,(\lambda)^{(8)}\right)$$

$$+\left(((\lambda)^{(8)})^2 + (\,(a'_{40})^{(8)} + (a'_{41})^{(8)} + (p_{40})^{(8)} + (p_{41})^{(8)})\,(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+((\lambda)^{(8)} + (a'_{40})^{(8)} + (p_{40})^{(8)})\,((a_{42})^{(8)}(q_{41})^{(8)}G^*_{41} + (a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^*_{40})$$

$$\left(((\lambda)^{(8)} + (b'_{40})^{(8)} - (r_{40})^{(8)})s_{(41),(42)}T^*_{41} + (b_{41})^{(8)}s_{(40),(42)}T^*_{40}\right)\} = 0$$

$$+$$

$$(\lambda)^{(9)} + (b'_{46})^{(9)} - (r_{46})^{(9)})\{((\lambda)^{(9)} + (a'_{46})^{(9)} + (p_{46})^{(9)})$$

$$\left[\left(((\lambda)^{(9)} + (a'_{44})^{(9)} + (p_{44})^{(9)})(q_{45})^{(9)}G^*_{45} + (a_{45})^{(9)}(q_{44})^{(9)}G^*_{44}\right)\right]$$

$$\left(((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)})s_{(45),(45)}T^*_{45} + (b_{45})^{(9)}s_{(44),(45)}T^*_{45}\right)$$

$$+\left(((\lambda)^{(9)} + (a'_{45})^{(9)} + (p_{45})^{(9)})(q_{44})^{(9)}G^*_{44} + (a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\right)$$

$$\left(((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)})s_{(45),(44)}T^*_{45} + (b_{45})^{(9)}s_{(44),(44)}T^*_{44}\right)$$

$$\left(((\lambda)^{(9)})^2 + (\,(a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)})\,(\lambda)^{(9)}\right)$$

$$\left(((\lambda)^{(9)})^2 + (\,(b'_{44})^{(9)} + (b'_{45})^{(9)} - (r_{44})^{(9)} + (r_{45})^{(9)})\,(\lambda)^{(9)}\right)$$

$$+\left(((\lambda)^{(9)})^2 + (\,(a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)})\,(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+((\lambda)^{(9)} + (a'_{44})^{(9)} + (p_{44})^{(9)})\,((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45} + (a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44})$$

$\left(\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\right)\} = 0$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i \in R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i\ X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor

bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k \tag{16}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 16. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,...x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone $\{ x_i \epsilon C_i: \prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i(x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \in I, x \in K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow *x.t restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\,\delta\,(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\mathcal{A}(\tilde{A}), C\backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population *i* is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int_0^t (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \epsilon R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v = au_i(x) + w$ with $a \epsilon R$ and $w \epsilon F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |B_i(t,x)u_i(x)|(a + \alpha + exp(-\beta t)|w|)$. Hence, $\limsup_{t\to\infty} \frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\limsup_{t\to\infty} \frac{1}{t} In\ B_i(t,x)\ u_i(x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In } b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x - bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

=

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \le 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma>0$ such that $|x.t| \le \gamma$ for all $t \ge 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \operatorname{In} |B_i(t,x)x_i| = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \operatorname{In} |\, (x.t)_i\, |$$

$$\le \operatorname{limsup}_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0. If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\ge 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\operatorname{limsup}_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \ge 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure*. $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x) \rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1 \le j \le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$. Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |\mathrm{h}(x.t) - h(x^n.t)|d\mu_n(x)| + |\int_c h(x^n.t)|d\mu_n(x) - \int_c \mathrm{h}(x)d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x)d\mu_n(x) - \int_c h(x)d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $\mathrm{h}_i^n(x) = \langle \mathrm{A}_i^n(x)\mathrm{u}_i^n(x), \mathrm{u}_i^n(x)\rangle$ where $\mathrm{u}_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $\mathrm{A}_i^n(x) \to \mathrm{A}_i(x)$ uniformly for $x \in N_1((\mathrm{A}))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $\mathrm{u}_i^n(x) \to \mathrm{u}_i(x)$ converges uniformly for $x \in N_1(\Lambda(\mathrm{A}))$. Given $\in > 0$, choose $n$ sufficiently large so that $|\mathrm{h}_i^n(x) - \mathrm{h}_i(x)| \leq \in$ for all $x \in N_1(\Lambda(\mathrm{A}))$ and $|\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$\lambda_i(\mu) \leq \int_c |\mathrm{h}_i(x) - \mathrm{h}_i^n(x)|d\mu(x) + |\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)|$

$+ \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)$

$\leq \quad 2\in$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\, \mathrm{ds}$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument

implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i \, (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

(6) $\min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$ $\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \begin{cases} \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} & if\ i \epsilon I \\ if\ i \notin I \end{cases}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set $K$ if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume x* $\mapsto$ *A(x) is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists j* $\in$ { 1,….,k} *such that $M_i$ ----- { $x \in C : x_l \gg 0$ for $l \neq j$} Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_iM_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----{1,…,k} such that $\mu(---_i \in_I C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures μ with support in ∂C, $\lambda_i(\mu)>0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $$\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j \qquad i = 1,..,k \qquad j=1,\ldots,m$$

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge

to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j$ ($x^j_i$) = 0 i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product ($E_1^1$ ,…,$E_1^m$), and the matrix $A^2$ ($E_1$) is close to the diagonal matrix diag($f^i_2(E_1{}^j)$)$^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f_2^j (E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j (E^j_1)>0$ and $\max_j f_2^j (E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial$C is given by two equilibria {0,$E_1$}, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j (E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$

$\frac{dx_2^j}{dt} = x_2^j(1 - \alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1 - \beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > 1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria $E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$ and $\lambda_3(E_1)$ are close to $\max_j 1 - a^j$ and $\max_j 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1 - a^j$ and $\max_j 1 - \beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1 - a^j) > -\max_j(1 - \beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i\left(r_i - \sum_s a_{is}x_s\right), \qquad i = 1,\dots,k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$$

It has been shown [12,ch.16.1,16.2] That for *k* =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \ \ with\ \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing

holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & O \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left[ \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1 - y) - b(N)\ y \right]$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*′ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\curlywedge_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\curlywedge_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d\ and$

$$\curlywedge_3(\mu) = \beta \int \frac{x_2}{\epsilon+x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords

Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation

of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1$ *(t)* and $P_2$ *(t)* denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial

alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.46) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.47) $$D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.48) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_{1,}$ $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_{i,}$ $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic ($as$ $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 16. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, \text{R}^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1, 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) \quad P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^\infty f_i(a + s, P(t + S)) ds \right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[ - \int_o^x f_1(a + s, p(t + s)) ds \right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0) \exp\left[ - \int_o^t f_1\left(a - t + s, p_1(s)\right) ds \right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(\text{a+s}, p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da+\int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$$t> 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in[-\tau, 0], i = 1,2,$

(2.8) $\beta_i(\text{t}) = \int_0^t \beta_i(\text{a}) b_i(\text{t-a}, P_i(t)) M_i.(a, t, P_i(t))da$

$$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$$

$$i = 1, 2, \text{t} > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t)\, \phi(0) d_i^*]t \quad for\ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T],R^+), \quad B_i \in C([0,T],R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau,T] = \{f \in C([-\tau,T],R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau,T]$. Then for any

$(x, y) \in C^+[-\tau,T] \times C^+[-\tau,T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau,T], \quad C_T^2 = C^+[-\tau,T] \times C^+[-\tau,T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau,T]$ *and* $t \in [0,T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$B: C_T^2 \to C_T^2$ *and* $\Phi C_T^2 \to C_T^2$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = P_t \left\{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Phi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right)$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \le \int_0^t B_1^{(1)}(a) \left\{ M_1\left(a, t, P_t^{(1)}\right) - M_1\left(a, t, P_t^{(2)}\right) \right\} da$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left( \int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_* ds \right) da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o) \left[ \int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds \right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \|_t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$=\int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\mathrm{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \, /\!/ P_t^{(1)} - P_t^{(2)} /\!/ \text{ for } a \, \varepsilon [o, t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \leq \beta_1 \Phi_1(0) || P_t^{(1)} - P_t^{(2)} ||_t^* + \Phi_1(0) \beta_1 d_1 || P_t^{(1)} - P_t^{(2)} ||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) \, e^{\delta 1^a} \beta_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) \, e^{\delta 1^a} /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| B_1^{*d}\sigma$$

$$+ \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \left\{ B_1 + B_2 d_1 a + B_1^* \left(B_2 a + d_1 a^2/2\right) e^{\delta 1^a} \right\}$$

$$\leq \beta_1^* \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| do + \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \rightarrow \beta_1$ as $a \rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{g_1(a) + 0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a)) - b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a)) do \right|$$

$$\left| B_1^* \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} 0(a) as\ a \rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi (P_t^{(1)} - \pi \left(P_t^{(2)}\right) /\!/ \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^*)$$

As $t^* \rightarrow o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp P_2^*(a)}{da} + f_i(a, P_1^*, P_2^*) P_i^*(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^* = \int_o^\infty P_i^*(a) da, i = 1,\ 2,$$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i=1.\ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s,P_1^*,P_2^*)ds\right], i=1,\quad 2, a\geq 0,$

Which together with (3.3) lead to

$$1=\int_o^\infty b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^\infty b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1,P_2) = \int_o^\infty b_i(a,P_1,P_2)\exp\left[-\int_o^\infty f_i(s,P_1,P_2)ds\right]da,$

$i=1,\ 2,\quad P_1\geq 0,\quad P_2\geq 0,$

**And suppose the following hold:**

(3.7) $A_4.\quad F_{i.}\ (0,\ 0)>1,\ i=1,2$:

$A_5.$ *there exists a positive constant c for which*

$$F_1(C,0)< F_{2.}(C,0),$$

(3.8) $$F_1(0.C)< F_{2.}(0,C),$$

Then there exists a unique pair $(P_1^*,P_2^*)$ of real numbers $P_1^*>o,$

$P_2^*>0$ such that

(3,9) $F_1(P_1^*,P_2^*)=1=F_2(P_1^*,P_2^*)$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a)=[P_2^*\exp[-f_i(s,P_1^*,P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s,P_1^*,P_2^*)ds]da),$

$i=1,2,\quad a\geq 0,$

Proof Consider the elements of the surfaces defined by

$$Z= F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0,$$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

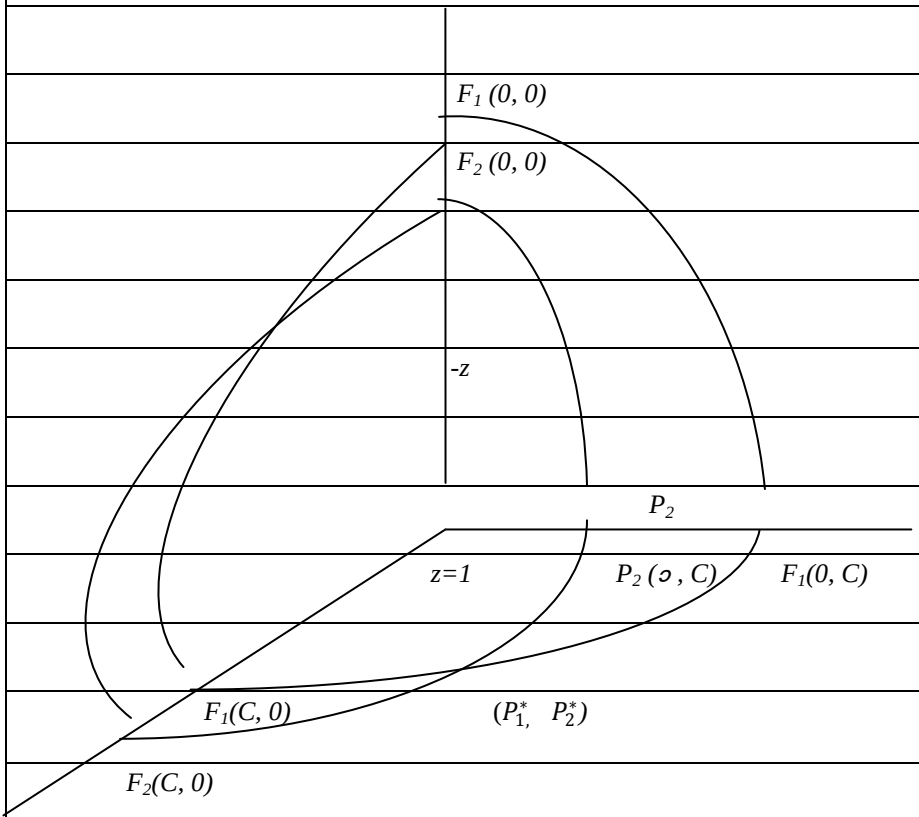
$$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0,$ These two curves are defined by

$$F_1(P_1, P_2 = 1 \; and \; F_2(P_1, P_2) = 1, \; F_1(P_1, P_2 \geq 0,$$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11) $$\begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12) $$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square=1}^{2} \frac{\partial b_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}da \end{cases}$$

Where $i$ = 1,2 and (3.13) $$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14) $$\begin{cases} u_i(a,t) = u\square(a)\exp[\lambda t], \quad \square(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty \square(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d\square}{da} + \lambda\square = -f_i\left(a, p_1^*, p_2^*\right)\square(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_{i\square}^*\right\},$$

(3.15)

$$\square(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\square(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*} p_{i\square}^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^{*} = p_{1}^{*}, \quad p_{12}^{*} = p_{2}^{*} \exp\{-\lambda\tau\},$$

$$p_{21}^{*} = p_{1}^{*}, \exp\{-\lambda\tau\}, \ p_{22}^{*} = p_{2}^{*}.$$

If we integrate the first of (3.15) and use the fact that $\Omega_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\Omega_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\Omega_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\Omega_i(a) = \{\Omega_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\Omega_i\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\Omega_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\Omega_i\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\Omega_i\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\Omega_i\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)}\int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}da, \quad i = 1,\ 2,$$

Now supplying $\Omega_1(a)$ and $\Omega_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^*\exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^*\exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^* \exp(-\lambda\tau) + (\lambda - A_{22})\, p_2^* = g_{21}(\lambda)\, p_1^* \exp(-\lambda\tau) + g_{22}(\lambda)\, p_2^*$$

Where

(3.22) $$A_i = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_i^*} - \frac{\partial f_i}{\partial p_i^*}\right]da,$$

(3.23) $$g_i(\lambda) = [p_i^*(0)c_i]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\, da\right\}$$
$$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_i^*} - b_i(a, p_1^*, p_2^*)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_i^*}\, e^{\lambda s}\, ds\right)\right]da\right\}$$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_i^*}\, e^{\lambda s}\, ds\right)da, \quad i = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*), \quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})\, e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$$S_2(\lambda, \tau) = [A_{21}\, g_{12}(\lambda) + A_{12}\, g_{21}(\lambda) + g_{12}(\lambda)\, g_{21}(\lambda)]\exp[-2\lambda\tau]$$

$$-A_{11}\, g_{22}(\lambda) - A_{22}\, g_{11}(\lambda) - g_{11}(\lambda)\, g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6$. $0 < \beta_i^* < d_i^*$ $(i = 1, 2.)$ $(see(A_1) and (A_2))$;

$A_7$. $\{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - \left| F_2(\lambda, \tau) \right|$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \,| - |A_{11} + A_{22} + \| S_1(0) \| \,|^2 \right\}/2,$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational

ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the

Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have

obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is:

individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## SECTION A(12)

## Makeenko-Migdal Equation For Yang-Mills Theory

### INTRODUCTION—VARIABLES USED

**Three proofs of the Makeenko-Migdal equation for Yang-Mills theory on the plane Bruce K. Driver, Brian C. Hall, Todd Kemp (Submitted on 23 Jan 2016 (v1), last revised 30 Mar 2016 (this version, v3))**

(1) **Bruce K. Driver, Brian C. Hall, Todd Kemp** give three short proofs of the Makeenko-Migdal equation for (e) the Yang-Mills measure on the plane, two using the edge variables and one using the loop or lasso variables.
(2) Proofs are (=) significantly simpler than the earlier pioneering rigorous proofs given by (e) T. L\'evy and by A. Dahlqvist.
(3) In particular, our proofs are (=) "local" in nature, in that they involve only derivatives with respect to variables adjacent to the crossing in question.
**(4)** In an accompanying paper with F. Gabriel, authors will show that two of our proofs can be adapted to (e&eb) the case of Yang-Mills theory on a compact surface. Subjects: Mathematical Physics (math-ph); High Energy Physics - Theory (hep-th) MSC classes: 81T13 (Primary), 81T40 Cite as: arXiv: 1601.06283 [math-ph] (or arXiv: 1601.06283v3 [math-ph] for this

version)

**Study of entropy production in Yang-Mills theory with use of Husimi function Hidekazu Tsukiji, Hideaki Iida, Teiji Kunihiro, Akira Ohnishi, Toru T. Takahashi (Submitted on 12 Nov 2015)**

(1) Understanding the thermalization process in a pure quantum system is a challenge in theoretical physics. In this work, authors explore possible thermalization mechanism in (eb) Yang-Mills (YM) theory by using (e) a positive semi-definite quantum distribution function called a (=) Husimi function which is given by (=) a coarse graining of the Wigner function within (eb) the minimum uncertainty.
(2) Then entropy is defined in terms of (e&eb) the Husimi function, which is called the Husimi-Wehrl (HW) entropy.
(3) Authors propose two numerical methods to calculate the HW entropy. They find that it is feasible to apply the semi-classical approximation with (e&eb) the use of classical YM equation.
(4) It should be noted that the semi-classical approximation is valid in the systems of (e) physical interest including the early stage of heavy-ion collisions.
(5) Using a product ansatz for the Husimi function, which is confirmed to reproduce (eb) the HW entropy within (eb) 20% error (overestimate) for (e) a few-body quantum system, they succeed in a numerical evaluation of (e) HW entropy of YM fields and show that it surely has (e)a finite value and increases in (e&eb) time.
(6) Subjects: High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th); Quantum Physics (quant-ph) Report number:YITP-15-93, KUNS-2592 Cite as: arXiv:1511.03804 [hep-ph] (or arXiv:1511.03804v1 [hep-ph] for this version)

**NOTATION**

**Module One**

**Bruce K. Driver, Brian C. Hall, Todd Kemp** give three short proofs of the Makeenko-Migdal equation for (e) the Yang-Mills measure on the plane, two using the edge variables and one using the loop or lasso variables

$G_{13}$ : Category one of Makeenko-Migdal equation; **Yang-Mills measure on the plane, two using the edge variables and one using the loop or lasso variables**

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of **Yang-Mills measure on the plane, two using the edge variables and one using the loop or lasso variables** ;Makeenko-Migdal equation

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

In particular proofs are (=) "local" in nature, in that they involve only derivatives with respect to variables adjacent to the crossing in question

$G_{16}$ : Category one of proofs

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|---|---|
| $G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of "local" in nature, in that they involve only derivatives with respect to variables adjacent to the crossing in question<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>In particular proofs are "local" in nature, in that they involve only derivatives with respect to variables adjacent to the crossing in question | |
| $G_{20}$ : Category one of **proofs are "local" in nature;** derivatives with respect to variables adjacent to the crossing in question<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of derivatives with respect to variables adjacent to the crossing in question **;proofs are "local" in nature**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>In particular proofs are "local" in nature, in that they involve only derivatives with respect to variables adjacent to the crossing in question | |
| $G_{24}$ : Category one of **proofs are "local" in nature, in that they involve only derivatives;** variables adjacent to the crossing in question<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of variables adjacent to the crossing in question **;proofs are "local" in nature, in that they involve only derivatives**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>In an accompanying paper with F. Gabriel, authors will show that two of proofs can be adapted to (e&eb) the case of Yang-Mills theory on a compact surface.<br><br>Subjects: Mathematical Physics (math-ph); High Energy Physics - Theory (hep-th) MSC classes: | |

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|---|---|
| 81T13 (Primary), 81T40 Cite as: arXiv: 1601.06283 [math-ph] (or arXiv: 1601.06283v3 [math-ph] for this version) | |
| $G_{28}$ : Category one of **two of proofs;** Yang-Mills theory on a compact surface.<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of Yang-Mills theory on a compact surface. ; **two of proofs**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Understanding the thermalization process in a pure quantum system is a challenge in theoretical physics<br><br>. In this work, authors explore possible thermalization mechanism in (eb) Yang-Mills (YM) theory by using (e) a positive semi-definite quantum distribution function called a (=) Husimi function which is given by (=) a coarse graining of the Wigner function within (eb) the minimum uncertainty | |
| $G_{32}$ : Category one of thermalization mechanism<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of Yang-Mills (YM) theory by using (e) a positive semi-definite quantum distribution function called a (=) Husimi function which is given by (=) a coarse graining of the Wigner function within (eb) the minimum uncertainty<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>In this work, authors explore possible thermalization mechanism in Yang-Mills (YM) theory by using (e) a positive semi-definite quantum distribution function called a (=) Husimi function which is given by (=) a coarse graining of the Wigner function within (eb) the minimum uncertainty | |
| $G_{36}$ : Category one of positive semi-definite quantum distribution function called a (=) Husimi function which is given by (=) a coarse graining of the Wigner function within (eb) the minimum uncertainty<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |

| | |
|---|---|
| $T_{36}$ : Category one of possible thermalization mechanism in Yang-Mills (YM) theory<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>In this work, authors explore possible thermalization mechanism in Yang-Mills (YM) theory by using a positive semi-definite quantum distribution function called a (=) Husimi function which is given by (=) a coarse graining of the Wigner function within (eb) the minimum uncertainty | |
| $G_{40}$ : Category one of thermalization mechanism in Yang-Mills (YM) theory by using a positive semi-definite quantum distribution function<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of Husimi function which is given by (=) a coarse graining of the Wigner function within (eb) the minimum uncertainty<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>In this work, authors explore possible thermalization mechanism in Yang-Mills (YM) theory by using a positive semi-definite quantum distribution function called a Husimi function which is given by (=) a coarse graining of the Wigner function within (eb) the minimum uncertainty | |
| $G_{44}$ : Category one of thermalization mechanism in Yang-Mills (YM) theory by using a positive semi-definite quantum distribution function called a Husimi function<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of coarse graining of the Wigner function within the minimum uncertainty<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION A(13)** | |

## Super-Yang-Mills In LHC Superspace

### INTRODUCTION—VARIABLES USED

**N=4 super-Yang-Mills in LHC superspace. Part I: Classical and quantum theory Dmitry Chicherin, Emery Sokatchev (Submitted on 25 Jan 2016)**

(1) **Dmitry Chicherin, Emery Sokatchev** present a formulation of (e) the maximally supersymmetric N=4 gauge theory in (eb) Lorentz harmonic chiral (LHC) superspace.
(2) It is closely related to (e&eb) the twistor formulation of the theory but employs (e) the simpler notion of Lorentz harmonic variables.
(3) They parametrize a two-sphere and allow *(eeb) us to handle efficiently infinite towers of (e) higher-spin auxiliary fields defined on (e&eb) ordinary space-time.
(4) In this approach the chiral half of N=4 supersymmetry is (=) manifest.
(5) The other half is realized non-linearly and (e&eb) the algebra closes on shell.
(6) Authors give a straightforward derivation of (e) the Feynman rules in coordinate space.
(7) They show that the LHC formulation of the N=4 super-Yang-Mills theory is remarkably similar to (=) the harmonic superspace formulation of (e) the N=2 gauge and hypermultiplet matter theories.
**(8)** In the twin paper \cite{twin} authors apply the LHC formalism to (e) the study of the non-chiral multipoint correlation functions of (e) the N=4 stress-tensor supermultiplet. Subjects: High Energy Physics - Theory (hep-th) Cite as: arXiv: 1601.06803 [hep-th] (or arXiv: 1601.06803v1 [hep-th] for this version)

### NOTATION

### Module One

Then entropy is defined in terms of (e&eb) the Husimi function, which is called the Husimi-Wehrl (HW) entropy

$G_{13}$ : Category one of **entropy is defined;** Husimi function, which is called the Husimi-Wehrl (HW) entropy

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of Husimi function, which is called the Husimi-Wehrl (HW) entropy **;entropy is defined**

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

### Module Two

Authors propose two numerical methods to calculate the HW entropy.

They find that it is feasible to apply the semi-classical approximation with (e&eb) the use of classical YM equation

$G_{16}$ : Category one of **semi-classical approximation;** use of classical YM equation

| | |
|---|---|
| $G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of use of classical YM equation **;semi-classical approximation**<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>It should be noted that the semi-classical approximation is valid in the systems of (e) physical interest including the early stage of heavy-ion collisions. | |
| $G_{20}$ : Category one of semi-classical approximation is valid in the systems; early stage of heavy-ion collisions<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of early stage of heavy-ion collisions ;semi-classical approximation is valid in the systems<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Using a product ansatz for the Husimi function, which is confirmed to reproduce (eb) the HW entropy within (eb) 20% error (overestimate) for (e) a few-body quantum system, they succeed in a numerical evaluation of (e) HW entropy of YM fields and show that it surely has (e)a finite value and increases in (e&eb) time | |
| $G_{24}$ : Category one of product ansatz for the Husimi function,<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one ofHW entropy within (eb) 20% error (overestimate) for (e) a few-body quantum system, they succeed in a numerical evaluation of (e) HW entropy of YM fields and show that it surely has (e)a finite value and increases in (e&eb) time<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Using a product ansatz for the Husimi function, which is confirmed to reproduce the HW entropy within 20% error (overestimate) for (e) a few-body quantum system, they succeed in a numerical evaluation of HW entropy of YM fields and show that it surely has (e) a finite value and increases in time | |

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| $G_{28}$ : Category one of few-body quantum system, they succeed in a numerical evaluation of (e) HW entropy of YM fields and show that it surely has (e)a finite value and increases in (e&eb) time; finite value and increases in time<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of finite value and increases in time ;product ansatz for the Husimi function, which is confirmed to reproduce the HW entropy within 20% error (overestimate)<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six** | |
| $G_{32}$ : Category one of<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of numerical evaluation of HW entropy of YM fields<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>**Dmitry Chicherin, Emery Sokatchev** present a formulation of the maximally supersymmetric N=4 gauge theory in Lorentz harmonic chiral (LHC) superspace | |
| $G_{36}$ : Category one of maximally supersymmetric N=4 gauge theory<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of Lorentz harmonic chiral (LHC) superspace<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>It is closely related to (e&eb) the twistor formulation of the theory but employs (e) the simpler notion of Lorentz harmonic variables | |

$G_{40}$ : Category one of maximally supersymmetric N=4 gauge theory in Lorentz harmonic chiral (LHC) superspace; twistor formulation of the theory but employs (e) the simpler notion of Lorentz harmonic variables

$G_{41}$ : Category two of SAS

$G_{42}$ : Category three of SAS

$T_{40}$ : Category one of twistor formulation of the theory but employs (e) the simpler notion of Lorentz harmonic variables ;maximally supersymmetric N=4 gauge theory in Lorentz harmonic chiral (LHC) superspace

$T_{41}$ : Category two of SAS

$T_{42}$ : Category three of SAS

**Module Nine**

They parametrize a two-sphere and allow *(eeb) us to handle efficiently infinite towers of (e) higher-spin auxiliary fields defined on (e&eb) ordinary space-time

$G_{44}$ : Category one of parametrize a two-sphere; handle efficiently infinite towers of (e) higher-spin auxiliary fields defined on (e&eb) ordinary space-time

$G_{45}$ : Category two of SAS

$G_{46}$ : Category three of SAS

$T_{44}$ : Category one of handle efficiently infinite towers of (e) higher-spin auxiliary fields defined on (e&eb) ordinary space-time; parametrize a two-sphere

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

**The Coefficients:**

$(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:
$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$
$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,
$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$
$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$
$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$
$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$

are Accentuation coefficients
$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}, \ (b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$
$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$
$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}\ (a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,

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| $(a_{32}^{'})^{(6)}, (a_{33}^{'})^{(6)}, (a_{34}^{'})^{(6)}, (b_{32}^{'})^{(6)}, (b_{33}^{'})^{(6)}, (b_{34}^{'})^{(6)}$<br>$(a_{36}^{'})^{(7)}, (a_{37}^{'})^{(7)}, (a_{38}^{'})^{(7)}, (b_{36}^{'})^{(7)}, (b_{37}^{'})^{(7)}, (b_{38}^{'})^{(7)},$<br>$(a_{40}^{'})^{(8)}, (a_{41}^{'})^{(8)}, (a_{42}^{'})^{(8)}, (b_{40}^{'})^{(8)}, (b_{41}^{'})^{(8)}, (b_{42}^{'})^{(8)},$<br>$(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)},$<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}, t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}, t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21}, t) =$ First augmentation factor | |

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| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |

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| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{array}{ccc} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{15}$ | *57* |

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| Where $(a_{13}^{''})^{(1)}(T_{14},t)$, $(a_{14}^{''})^{(1)}(T_{14},t)$, $(a_{15}^{''})^{(1)}(T_{14},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{38}^{''})^{(7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3<br>$+(a_{40}^{''})^{(8,8)}(T_{41},t)$ $+(a_{41}^{''})^{(8,8)}(T_{41},t)$ $+(a_{42}^{''})^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3<br>$+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\left[-(b_{13}^{''})^{(1)}(G,t)\right] & \left[-(b_{16}^{''})^{(2,2,)}(G_{19},t)\right] & \left[-(b_{20}^{''})^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{36}^{''})^{(7,7,)}(G_{39},t)\right] & \left[-(b_{40}^{''})^{(8,8)}(G_{43},t)\right] & \left[-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\left[-(b_{14}^{''})^{(1)}(G,t)\right] & \left[-(b_{17}^{''})^{(2,2,)}(G_{19},t)\right] & \left[-(b_{21}^{''})^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{37}^{''})^{(7,7,)}(G_{39},t)\right] & \left[-(b_{41}^{''})^{(8,8)}(G_{43},t)\right] & \left[-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\left[-(b_{15}^{''})^{(1)}(G,t)\right] & \left[-(b_{18}^{''})^{(2,2,)}(G_{19},t)\right] & \left[-(b_{22}^{''})^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{38}^{''})^{(7,7,)}(G_{39},t)\right] & \left[-(b_{42}^{''})^{(8,8)}(G_{43},t)\right] & \left[-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{15}$ | *60* |
| Where $-(b_{13}^{''})^{(1)}(G,t)$, $-(b_{14}^{''})^{(1)}(G,t)$, $-(b_{15}^{''})^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{16}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{20}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 | |

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| $-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$ , $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$ , $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{37}^{''})^{(7,7,)}(G_{39},t)$ , $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$ , $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ , $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ , $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[ \begin{matrix} (a_{16}^{'})^{(2)} +(a_{16}^{''})^{(2)}(T_{17},t) +(a_{13}^{''})^{(1,1,)}(T_{14},t) +(a_{20}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t) +(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t) +(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t) +(a_{40}^{''})^{(8,8,8)}(T_{41},t) +(a_{44}^{''})^{(9,9)}(T_{45},t) \end{matrix} \right] G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[ \begin{matrix} (a_{17}^{'})^{(2)} +(a_{17}^{''})^{(2)}(T_{17},t) +(a_{14}^{''})^{(1,1,)}(T_{14},t) +(a_{21}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t) +(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t) +(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7)}(T_{37},t) +(a_{41}^{''})^{(8,8,8)}(T_{41},t) +(a_{45}^{''})^{(9,9)}(T_{45},t) \end{matrix} \right] G_{17}$ | *62* |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[ \begin{matrix} (a_{18}^{'})^{(2)} +(a_{18}^{''})^{(2)}(T_{17},t) +(a_{15}^{''})^{(1,1,)}(T_{14},t) +(a_{22}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t) +(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t) +(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7)}(T_{37},t) +(a_{42}^{''})^{(8,8,8)}(T_{41},t) +(a_{46}^{''})^{(9,9)}(T_{45},t) \end{matrix} \right] G_{18}$ | *63* |
| Where $+(a_{16}^{''})^{(2)}(T_{17},t)$ , $+(a_{17}^{''})^{(2)}(T_{17},t)$ , $+(a_{18}^{''})^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$ , $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$ , $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9)}(T_{45},t)$ , $+(a_{45}^{''})^{(9,9)}(T_{45},t)$ , $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |

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| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b'_{16})^{(2)}\boxed{-(b''_{16})^{(2)}(G_{19},t)} & \boxed{-(b''_{13})^{(1,1,)}(G,t)} & \boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{16}$ | 64 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b'_{17})^{(2)}\boxed{-(b''_{17})^{(2)}(G_{19},t)} & \boxed{-(b''_{14})^{(1,1,)}(G,t)} & \boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{17}$ | 65 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b'_{18})^{(2)}\boxed{-(b''_{18})^{(2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,)}(G,t)} & \boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{18}$ | 66 |
| where $\boxed{-(b''_{16})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{17})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{18})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b''_{13})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,)}(G,t)}$ are second detrition coefficients for category 1,2 and 3<br>$\boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1,2 and 3<br>$\boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1,2 and 3<br>$\boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1,2 and 3<br>$\boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1,2 and 3<br>$\boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2 and 3<br>$\boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1,2 and 3<br>$\boxed{-(b''_{44})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a'_{20})^{(3)}\boxed{+(a''_{20})^{(3)}(T_{21},t)} & \boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a'_{21})^{(3)}\boxed{+(a''_{21})^{(3)}(T_{21},t)} & \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{21}$ | 68 |

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a'_{22})^{(3)}\boxed{+(a''_{22})^{(3)}(T_{21},t)} & \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{22} \quad (69)$$

$\boxed{+(a''_{20})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)}$ are third augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)}$ are eight augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1, 2 and 3

$$\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b'_{20})^{(3)}\boxed{-(b''_{20})^{(3)}(G_{23},t)} & \boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{13})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{20} \quad (70)$$

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b'_{21})^{(3)}\boxed{-(b''_{21})^{(3)}(G_{23},t)} & \boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{14})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)}\boxed{-(b''_{22})^{(3)}(G_{23},t)} & \boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{22} \quad (72)$$

$\boxed{-(b''_{20})^{(3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3)}(G_{23},t)}$ *are first detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)}$ *are second detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{13})^{(1,1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,)}(G,t)}$ *are third detrition coefficients for category 1,2 and 3*

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| $-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*<br>$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$ $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a_{24}^{'})^{(4)} & +(a_{24}^{''})^{(4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,)}(T_{29},t) & +(a_{32}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24}$ | *73* |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a_{25}^{'})^{(4)} & +(a_{25}^{''})^{(4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,)}(T_{29},t) & +(a_{33}^{''})^{(6,6)}(T_{33},t) \\ & +(a_{14}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25}$ | *74* |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a_{26}^{'})^{(4)} & +(a_{26}^{''})^{(4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,)}(T_{29},t) & +(a_{34}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26}$ | *75* |

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| $(a_{24}^{''})^{(4)}(T_{25},t)$, $(a_{25}^{''})^{(4)}(T_{25},t)$, $(a_{26}^{''})^{(4)}(T_{25},t)$ are first augmentation coefficients category 1, 2 3<br>$+(a_{28}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,)}(T_{29},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,)}(T_{33},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$<br>are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$<br>are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$<br>are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$<br>are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$<br>are eighth augmentation coefficients for category 1, 2 and 3<br>$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b_{24}^{'})^{(4)} & -(b_{24}^{''})^{(4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,)}(G_{35},t) \\ & -(b_{13}^{''})^{(1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3)}(G_{23},t) \\ & -(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{24}$ | 76 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b_{25}^{'})^{(4)} & -(b_{25}^{''})^{(4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,)}(G_{35},t) \\ & -(b_{14}^{''})^{(1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3)}(G_{23},t) \\ & -(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{25}$ | 77 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b_{26}^{'})^{(4)} & -(b_{26}^{''})^{(4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,)}(G_{35},t) \\ & -(b_{15}^{''})^{(1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3)}(G_{23},t) \\ & -(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{26}$ | 78 |
| Where $-(b_{24}^{''})^{(4)}(G_{27},t)$, $-(b_{25}^{''})^{(4)}(G_{27},t)$, $-(b_{26}^{''})^{(4)}(G_{27},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,)}(G_{31},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{32}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,)}(G_{35},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$<br>are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$<br>are fifth detrition coefficients for category 1, 2 and 3 | |

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| $-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 $-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3 $-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a_{28}^{'})^{(5)} & +(a_{28}^{''})^{(5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,)}(T_{25},t) & +(a_{32}^{''})^{(6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a_{29}^{'})^{(5)} & +(a_{29}^{''})^{(5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,)}(T_{25},t) & +(a_{33}^{''})^{(6,6,6)}(T_{33},t) \\ & +(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a_{30}^{'})^{(5)} & +(a_{30}^{''})^{(5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,)}(T_{25},t) & +(a_{34}^{''})^{(6,6,6)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{30}$ | *81* |
| *Where* $+(a_{28}^{''})^{(5)}(T_{29},t)$, $+(a_{29}^{''})^{(5)}(T_{29},t)$, $+(a_{30}^{''})^{(5)}(T_{29},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3 *And* $+(a_{24}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,)}(T_{25},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3 $+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3 $+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3 $+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3 $+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3 $+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3 $+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3 $+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b'_{28})^{(5)} & -(b''_{28})^{(5)}(G_{31},t) & -(b''_{24})^{(4,4,)}(G_{27},t) & -(b''_{32})^{(6,6,6)}(G_{35},t) \\ -(b''_{13})^{(1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b'_{29})^{(5)} & -(b''_{29})^{(5)}(G_{31},t) & -(b''_{25})^{(4,4,)}(G_{27},t) & -(b''_{33})^{(6,6,6)}(G_{35},t) \\ -(b''_{14})^{(1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b'_{30})^{(5)} & -(b''_{30})^{(5)}(G_{31},t) & -(b''_{26})^{(4,4,)}(G_{27},t) & -(b''_{34})^{(6,6,6)}(G_{35},t) \\ -(b''_{15})^{(1,1,1,1,1,)}(G,t) & -(b''_{18})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{30} \quad (84)$$

*where* $-(b''_{28})^{(5)}(G_{31},t)$ , $-(b''_{29})^{(5)}(G_{31},t)$ , $-(b''_{30})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{24})^{(4,4,)}(G_{27},t)$, $-(b''_{25})^{(4,4,)}(G_{27},t)$, $-(b''_{26})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3

$-(b''_{32})^{(6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b''_{13})^{(1,1,1,1,1)}(G,t)$, $-(b''_{14})^{(1,1,1,1,1)}(G,t)$ , $-(b''_{15})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3

$-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3

$-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)$, $-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)$, $-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3

$-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3

$-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3

$-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a'_{32})^{(6)} & +(a''_{32})^{(6)}(T_{33},t) & +(a''_{28})^{(5,5,5)}(T_{29},t) & +(a''_{24})^{(4,4,4,)}(T_{25},t) \\ +(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a'_{33})^{(6)} & +(a''_{33})^{(6)}(T_{33},t) & +(a''_{29})^{(5,5,5)}(T_{29},t) & +(a''_{25})^{(4,4,4,)}(T_{25},t) \\ +(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)} G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)} T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)} T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)} T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{28})^{(5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)} & +(a_{36}^{''})^{(7)}(T_{37},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)} & +(a_{37}^{''})^{(7)}(T_{37},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)} & +(a_{38}^{''})^{(7)}(T_{37},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15} \quad (93)$$

Where $(a_{36}^{''})^{(7)}(T_{37},t)$, $(a_{37}^{''})^{(7)}(T_{37},t)$, $(a_{38}^{''})^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation

coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3 $+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} =$$

$$(b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}\left[-(b_{36}^{''})^{(7)}(G_{39},t)\right] & \left[-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)\right] & \left[-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{13}$$

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$$\frac{dT_{37}}{dt} =$$

$$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}\left[-(b_{37}^{''})^{(7)}(G_{39},t)\right] & \left[-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)\right] & \left[-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} =$$

$$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}\left[-(b_{38}^{''})^{(7)}(G_{39},t)\right] & \left[-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)\right] & \left[-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3 $-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3 $-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3 $-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}'')^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}'')^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}'')^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{46}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

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$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}')^{(8)}+(a_{40}'')^{(8)}(T_{41},t) & +(a_{16}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13}$$

$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}')^{(8)}+(a_{41}'')^{(8)}(T_{41},t) & +(a_{17}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}')^{(8)}+(a_{42}'')^{(8)}(T_{41},t) & +(a_{18}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$$

Where $+(a_{40}'')^{(8)}(T_{41},t)$, $+(a_{41}'')^{(8)}(T_{41},t)$, $+(a_{42}'')^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & -(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & -(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & -(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3



$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{13}$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[\begin{array}{ccc} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$\frac{dT_{44}}{dt} =$

$$(b_{44})^{(9)}T_{45} - \left[\begin{array}{ccc} (b_{44}^{'})^{(9)}\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[\begin{array}{ccc} (b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[\begin{array}{ccc} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

| | |
|---|---|
| **Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br>$(a_i'')^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$<br>$(b_i'')^{(1)}(G,t) \le (r_i)^{(1)} \le (b_i')^{(1)} \le (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2\to\infty}(a_i'')^{(1)}(T_{14},t) = (p_i)^{(1)}$<br>$\lim_{G\to\infty}(b_i'')^{(1)}(G,t) = (r_i)^{(1)}$<br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(1)}(T_{14}',t) - (a_i'')^{(1)}(T_{14},t)\rvert \le (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}'\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br>$\lvert(b_i'')^{(1)}(G',t) - (b_i'')^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G'\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(1)}(T_{14}',t)$ and $(a_i'')^{(1)}(T_{14},t)$ . $(T_{14}',t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i'')^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i'')^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i')^{(1)}, (b_i)^{(1)}, (b_i')^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i,j = 16,17,18$ | |

| | |
|---|---|
| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17},t) \le (p_i)^{(2)} \le (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19},t) \le (r_i)^{(2)} \le (b_i')^{(2)} \le (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2\to\infty} (a_i'')^{(2)}(T_{17},t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty} (b_i'')^{(2)}((G_{19}),t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $\boxed{(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}}$ are positive constants and $\boxed{i = 16,17,18}$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(2)}(T_{17}',t) - (a_i'')^{(2)}(T_{17},t)\rvert \le (\hat{k}_{16})^{(2)}\lvert T_{17} - T_{17}'\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i'')^{(2)}((G_{19})',t) - (b_i'')^{(2)}((G_{19}),t)\rvert < (\hat{k}_{16})^{(2)}\lvert\lvert(G_{19}) - (G_{19})'\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}',t)$ and $(a_i'')^{(2)}(T_{17},t)$ . $(T_{17}',t)$ and $(T_{17},t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}]$ . It is to be noted that $(a_i'')^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i'')^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i')^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ | 111 |

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|---|---|
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21},t) \le (p_i)^{(3)} \le (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23},t) \le (r_i)^{(3)} \le (b_i^{'})^{(3)} \le (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2\to\infty}(a_i^{''})^{(3)}(T_{21},t) = (p_i)^{(3)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'},t) - (a_i^{''})^{(3)}(T_{21},t)\rvert \le (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'},t) - (b_i^{''})^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lVert G_{23} - G_{23}^{'}\rVert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'},t)$ and$(a_i^{''})^{(3)}(T_{21},t)$ . $(T_{21}^{'},t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$<br>satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |

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|---|---|
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24{,}25{,}26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}\big((G_{27}), t\big) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(4)}\big((G_{27}), t\big) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24{,}25{,}26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\rvert \leq (\hat{k}_{24})^{(4)} \lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)} t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'}, t) - (b_i^{''})^{(4)}\big((G_{27}), t\big)\rvert < (\hat{k}_{24})^{(4)} \lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)} t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $\big[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\big]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24{,}25{,}26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |

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| $\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i^{'})^{(4)} +\ (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}\big((G_{31}), t\big) \leq\ (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(5)}\,(T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(5)}\,(G_{31}, t) =\ (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'}, t) - (b_i^{''})^{(5)}\big((G_{31}), t\big)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $\big[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\big]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br><br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\ , \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities | 126 |

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| $\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(6)}(T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(6)}((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)}\lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})^{'}, t) - (b_i^{''})^{(6)}((G_{35}), t)\rvert < (\hat{k}_{32})^{(6)}\lvert\lvert(G_{35}) - (G_{35})^{'}\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and$(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants | 130 |

$(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$
satisfy the inequalities

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

Where we suppose

(SSSS) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i, j = 36,37,38$ 131

(TTTT) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$:

$$(a_i^{''})^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$$

$$(b_i^{''})^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$$

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*(UUUU)* $\lim_{T_2 \to \infty} (a_i^{''})^{(7)}(T_{37}, t) = (p_i)^{(7)}$
*(VVVV)*
$\lim_{G \to \infty} (b_i^{''})^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, \ (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)} |T_{37} - T_{37}^{'}| e^{-(\hat{M}_{36})^{(7)} t}$$

$$|(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)} ||(G_{39}) - (G_{39})^{'}|| e^{-(\hat{M}_{36})^{(7)} t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'}, t)$ and $(a_i^{''})^{(7)}(T_{37}, t)$ . $(T_{37}^{'}, t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ : 134

(WWWW) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants

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| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :<br><br>(XXXX) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}, (r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})', t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})'\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would | |

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| be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i, j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45}, t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47}, t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2 \to \infty} (a_i'')^{(9)}(T_{45}, t) = (p_i)^{(9)}$<br>$\lim_{G \to \infty} (b_i'')^{(9)}(G_{47}, t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}', t) - (a_i'')^{(9)}(T_{45}, t)\rvert \le (\hat{k}_{44})^{(9)} \lvert T_{45} - T_{45}' \rvert e^{-(\hat{M}_{44})^{(9)} t}$<br>$\lvert(b_i'')^{(9)}((G_{47})', t) - (b_i'')^{(9)}((G_{47}), t)\rvert < (\hat{k}_{44})^{(9)} \lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}', t)$ | |

and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)},(\widehat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{44})^{(9)}=1$ then the function $(a_i'')^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$ :

$(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\ ,\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}<1$$

**Definition of** $(\hat{P}_{44})^{(9)},(\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},(\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)},(a_i')^{(9)},(b_i)^{(9)},(b_i')^{(9)},(p_i)^{(9)},\ (r_i)^{(9)},i=44,45,46,$ satisfy the inequalities

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[(a_i)^{(9)}+(a_i')^{(9)}+\ (\hat{A}_{44})^{(9)}+\ (\hat{P}_{44})^{(9)}\ (\hat{k}_{44})^{(9)}]<1$$

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[\ (b_i)^{(9)}+(b_i')^{(9)}+\ (\hat{B}_{44})^{(9)}+\ (\hat{Q}_{44})^{(9)}\ \ (\hat{k}_{44})^{(9)}]<1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0)\ ,T_i(0)$ :

$$G_i(t)\le\ (\hat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{G_i(0)=G_i^0>0}$$

$$T_i(t)\le\ (\hat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{T_i(0)=T_i^0>0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0)\ ,T_i(0)$

$$G_i(t)\le\ (\hat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t)\le\ (\hat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

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| **Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$ | 153 B |

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| $T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |

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| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} G_{21}(s_{(20)}) - \left((a_{20}')^{(3)} + a_{20}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{20}(s_{(20)})\right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[(a_{21})^{(3)} G_{20}(s_{(20)}) - \left((a_{21}')^{(3)} + (a_{21}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}')^{(3)} - (b_{21}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{28})^{(5)}$, $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[ (b_{29})^{(5)} T_{28}(s_{(28)}) - \left( (b'_{29})^{(5)} - (b''_{29})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[ (b_{30})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{30})^{(5)} - (b''_{30})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{30}(s_{(28)}) \right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** | |

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| Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[ (a_{32})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{32}(s_{(32)}) \right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[ (a_{33})^{(6)} G_{32}(s_{(32)}) - \left( (a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[ (a_{34})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{34}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[ (b_{32})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{32}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[ (b_{33})^{(6)} T_{32}(s_{(32)}) - \left( (b'_{33})^{(6)} - (b''_{33})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[ (b_{34})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{34})^{(6)} - (b''_{34})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{34}(s_{(32)}) \right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |

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| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}^{'})^{(8)} + a_{40}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^{0} + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^{0} + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^{0} + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \leq G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \leq \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}$ , $G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \leq G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that | 170 |

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| $(G_{16}(t)-G_{16}^0)e^{-(\bar{M}_{16})^{(2)}t} \le \frac{(a_{16})^{(2)}}{(\bar{M}_{16})^{(2)}}\left[((\hat{P}_{16})^{(2)}+G_{17}^0)e^{\left(-\frac{(\hat{P}_{16})^{(2)}+G_{17}^0}{G_{17}^0}\right)}+(\hat{P}_{16})^{(2)}\right]$ | |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0+(\hat{P}_{20})^{(3)}e^{(\bar{M}_{20})^{(3)}s_{(20)}}\right)\right] ds_{(20)} =$<br>$(1+(a_{20})^{(3)}t)G_{21}^0 + \frac{(a_{20})^{(3)}(\hat{P}_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left(e^{(\bar{M}_{20})^{(3)}t}-1\right)$ | 171 |
| From which it follows that<br>$(G_{20}(t)-G_{20}^0)e^{-(\bar{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that<br>$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\bar{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$<br>$(1+(a_{24})^{(4)}t)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left(e^{(\bar{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that<br>$(G_{24}(t)-G_{24}^0)e^{-(\bar{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$<br>$(1+(a_{28})^{(5)}t)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)}t}-1\right)$ | |
| From which it follows that<br>$(G_{28}(t)-G_{28}^0)e^{-(\bar{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 5 | 175 |
| The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious | 176 |

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| that<br>$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$<br>$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$ | |
| From which it follows that<br>$(G_{32}(t) - G_{32}^0) e^{-(\hat{M}_{32})^{(6)} t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[\left((\hat{P}_{32})^{(6)} + G_{33}^0\right) e^{\left(-\frac{(\hat{P}_{32})^{(6)} + G_{33}^0}{G_{33}^0}\right)} + (\hat{P}_{32})^{(6)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 6<br>Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$ | 177 |
| (q) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)}\left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$<br>$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$ | 178 |
| From which it follows that<br>$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{P}_{36})^{(7)} + G_{37}^0\right) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 7 | |
| The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)}\left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$<br>$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$ | 180 |
| From which it follows that<br>$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[\left((\hat{P}_{40})^{(8)} + G_{41}^0\right) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)}\left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$ | |

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| $\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$ | |
| From which it follows that<br>$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t}\leq\frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{P}_{44})^{(9)}+G_{45}^{0}\right)e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ and to choose<br>$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{13})^{(1)}\left\|G_{14}^{(1)}-G_{14}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$<br>$\int_0^t\{(a_{13}')^{(1)}\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{-(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$G_{13}^{(2)}\left\\|(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}'')^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right\\|\ e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$ | |

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| Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | |
| $\left\|G^{(1)} - G^{(2)}\right\| e^{-(\widehat{M}_{13})^{(1)}t} \leq$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right) d\left(\left(G^{(1)}, T^{(1)};\ G^{(2)}, T^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(1)}$ and $(b''_i)^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(1)} - (a''_i)^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1 / (a'_{14})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2 / (a'_{15})^{(1)}$<br><br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(1)}(G(t), t)) = (b'_{14})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ : | 189 |

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| Indeed let $t_1$ be so that for $t > t_1$<br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ $and$ $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |

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| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br>$\int_0^t \{(a_{16}')^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$(a_{16}'')^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$G_{16}^{(2)} \left\|(a_{16}'')^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}'')^{(2)} \left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}} \left((a_{16})^{(2)} + (a_{16}')^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)} (\widehat{k}_{16})^{(2)}\right) d\left(\left((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)}\right)\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}'')^{(2)}$ and $(b_{16}'')^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(2)}$ and $(b_i'')^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(2)} - (a_i'')^{(2)} (T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(2)} t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}')^{(2)} G_{17}$ and by integrating<br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)} \left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}')^{(2)}$<br>In the same way , one can obtain | 201 |

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| $G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$, $G_{18}$ and $G_{16}$, $G_{17}$ respectively. | |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$. The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}$, $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}} < 1$ and to choose<br><br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$, $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\bar{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\bar{M}_{20})^{(3)}t}\}$ | 211 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}}$ :$\left( \widetilde{(G_{23})}, \widetilde{(T_{23})} \right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)} \left\|G_{21}^{(1)} - G_{21}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\, ds_{(20)} +$<br><br>$\int_0^t \{(a_{20}^{'})^{(3)} \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right)\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$G_{20}^{(2)} \left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}^{''})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\} ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} t} \leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a_{20}^{'})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)} (\widehat{k}_{20})^{(3)}\right) d\left(\left((G_{23})^{(1)}, (T_{23})^{(1)}; (G_{23})^{(2)}, (T_{23})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ *and* $(\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)} t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating | 217 |

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| $G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}')^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}')^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(3)}\left((G_{23})(t),t\right) = (b_{21}')^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i'')^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}'')^{(3)}\left((G_{23})(t),t\right) = (b_{22}')^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}\left[(\hat{P}_{24})^{(4)} + \left((\hat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}}\left[\left((\hat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{24})^{(4)}\right] \leq (\hat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$ | 225 |

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{24})^{(4)}\left|G_{25}^{(1)}-G_{25}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$$

$$\int_0^t\{(a_{24}^{'})^{(4)}\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$G_{24}^{(2)}\left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)}-(G_{27})^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}t}\le$$
$$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows

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**Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}\ and\ (\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

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**Remark 17:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

it results

$$G_i\ (t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]}\ge 0$$

$$T_i\ (t)\ge T_i^0e^{\left(-(b_i^{'})^{(4)}t\right)}>0\quad \text{for t} > 0$$

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**Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1,\left((\widehat{M}_{24})^{(4)}\right)_2\ and\ \left((\widehat{M}_{24})^{(4)}\right)_3$ :

**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if

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| $G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}')^{(4)} G_{25}$ and by integrating<br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a_{25}')^{(4)}$<br>In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a_{26}')^{(4)}$<br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t),t)) = (b_{25}')^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded.<br>The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}\left((G_{27})(t),t\right) = (b_{26}')^{(4)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + \left((\widehat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\widehat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{28})^{(5)}\right] \leq (\widehat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying | |

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| Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}((G_{31}),(T_{31}))$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)}+$<br>$\int_0^t\{(a_{28}^{'})^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\le$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}^{'})^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}^{''})^{(5)}$ and $(b_{28}^{''})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(5)}-(a_i^{''})^{(5)}(T_{29}(s_{(28)}),s_{(28)})\}ds_{(28)}\right]}\ge 0$ | 239 |

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| $T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(5)}t)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}^{'})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1/(a_{29}^{'})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2/(a_{30}^{'})^{(5)}$<br><br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t),t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right)$, $t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |

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| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{32})^{(6)}\right]\leq(\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}ds_{(32)}+$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a''_{32})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a'_{32})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(6)}$ and $(b_i'')^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 249 |

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| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(6)} - (a_i'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(6)} t)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}')^{(6)} G_{33}$ and by integrating<br><br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}')^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}')^{(6)}$<br><br>If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(6)}((G_{35})(t), t)) = (b_{33}')^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i'')^{(6)}((G_{35})(t), t) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty} (b_{34}'')^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}')^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have | 255 |

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| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)}+\left((\hat{P}_{36})^{(7)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{36})^{(7)}\right]\leq(\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}')^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}'')^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}')^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}'')^{(7)}$ and $(b_{36}'')^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$. | 260 |

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| If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2 / (a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t), t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t), t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right),\ \ t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded. | 265 |

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| The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}'')^{(7)}\left((G_{39})(t),t\right)=(b_{38}')^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}<1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)}+\left((\hat{P}_{40})^{(8)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{40})^{(8)}\right]\leq(\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}ds_{(40)}+$<br>$\int_0^t\{(a_{40}')^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}'')^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}'')^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}'')^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\leq$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}')^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 273 |

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| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(8)} - (a_i^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2\ and\ \left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}^{'})^{(8)}$<br><br>If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}\ is$ bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t), t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t), t\right) < \varepsilon_8, T_{40}\,(t) > (m)^{(8)}$ | 278 |

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| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.<br>The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}^{'})^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$ | |
| In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying 39,35,36 into itself | |
| The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{47})^{(1)}, (T_{47})^{(1)}), ((G_{47})^{(2)}, (T_{47})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \left(\widetilde{(G_{47})}, \widetilde{(T_{47})}\right) = \mathcal{A}^{(9)}\left((G_{47}), (T_{47})\right)$<br>It results<br><br>$\left\|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$<br>$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right) - (a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$<br><br>Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on 45,46,47,28 and 29 it follows | |
| $\left|(G_{47})^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t} \leq$<br>$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)}, (T_{47})^{(1)};\ (G_{47})^{(2)}, (T_{47})^{(2)})\right)$ | |

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$

From 99 to 44 it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(9)}t\right)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)}G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t), t\right) < \varepsilon_9, T_{44}\,(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded. The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b''_{46})^{(9)}\left((G_{47})(t),t\right) = (b'_{46})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a'_{13})^{(1)} + (a'_{14})^{(1)} - (a''_{13})^{(1)}(T_{14}, t) + (a''_{14})^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b'_{13})^{(1)} + (b'_{14})^{(1)} - (b''_{13})^{(1)}(G,t) - (b''_{14})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(v_1)^{(1)}, (v_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, v^{(1)}, u^{(1)}$ : 281

By $(v_1)^{(1)} > 0$, $(v_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{v}_1)^{(1)},, (\bar{v}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{v}_1)^{(1)} > 0$, $(\bar{v}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (v_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (v_0)^{(1)}, (m_1)^{(1)} = (v_1)^{(1)}$, $if\ (v_0)^{(1)} < (v_1)^{(1)}$

$(m_2)^{(1)} = (v_1)^{(1)}, (m_1)^{(1)} = (\bar{v}_1)^{(1)}, if\ (v_1)^{(1)} < (v_0)^{(1)} < (\bar{v}_1)^{(1)}$,

and $\boxed{(v_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$

$(m_2)^{(1)} = (v_1)^{(1)}, (m_1)^{(1)} = (v_0)^{(1)}$, $if\ (\bar{v}_1)^{(1)} < (v_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,

and $\boxed{(u_0)^{(1)} = \frac{T^0_{13}}{T^0_{14}}}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$

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| are defined | |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le \right.$<br>$\left.\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a'_{15})^{(1)})}[e^{(S_1)^{(1)}t} - e^{-(a'_{15})^{(1)}t}] + G_{15}^0 e^{-(a'_{15})^{(1)}t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b'_{15})^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b'_{15})^{(1)}t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)}t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$<br>$(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ :<br>$(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a'_{16})^{(2)} + (a'_{17})^{(2)} - (a''_{16})^{(2)}(T_{17}, t) + (a''_{17})^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b'_{16})^{(2)} + (b'_{17})^{(2)} - (b''_{16})^{(2)}((G_{19}), t) - (b''_{17})^{(2)}((G_{19}), t) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(v_1)^{(2)}, (v_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |

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| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\left(u^{(2)}\right)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}$,, $(\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\left(u^{(2)}\right)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously $(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$ $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities $G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)}t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le\right.$ | 312 |

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| $\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_2)^{(2)}((S_1)^{(2)}-(a_{18}')^{(2)})}[e^{(S_1)^{(2)}t}-e^{-(a_{18}')^{(2)}t}]+G_{18}^{0}e^{-(a_{18}')^{(2)}t})$ | |
| $\boxed{T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}}T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le\frac{1}{(\mu_2)^{(2)}}T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)}T_{16}^{0}}{(\mu_1)^{(2)}((R_1)^{(2)}-(b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t}-e^{-(b_{18}')^{(2)}t}\right]+T_{18}^{0}e^{-(b_{18}')^{(2)}t}\le T_{18}(t)\le$ $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)},(S_2)^{(2)},(R_1)^{(2)},(R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)}=(a_{16})^{(2)}(m_2)^{(2)}-(a_{16}')^{(2)}$ $(S_2)^{(2)}=(a_{18})^{(2)}-(p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)}=(b_{16})^{(2)}(\mu_2)^{(1)}-(b_{16}')^{(2)}$ $(R_2)^{(2)}=(b_{18}')^{(2)}-(r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$: $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)}\le-(a_{20}')^{(3)}+(a_{21}')^{(3)}-(a_{20}'')^{(3)}(T_{21},t)+(a_{21}'')^{(3)}(T_{21},t)\le-(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)}\le-(b_{20}')^{(3)}+(b_{21}')^{(3)}-(b_{20}'')^{(3)}(G_{23},t)-(b_{21}'')^{(3)}((G_{23}),t)\le-(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)},(v_2)^{(3)},(u_1)^{(3)},(u_2)^{(3)}$: By $(v_1)^{(3)}>0,(v_2)^{(3)}<0$ and respectively $(u_1)^{(3)}>0,(u_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_1)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_1)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ and By $(\bar{v}_1)^{(3)}>0,(\bar{v}_2)^{(3)}<0$ and respectively $(\bar{u}_1)^{(3)}>0,(\bar{u}_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_2)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_2)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ | 320 |
| **Definition of** $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$:- | 321 |

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$, ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,

and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously 322

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$$

$(p_i)^{(3)}$ is defined by equation

$$\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t} \quad (323)$$

$$\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$$

$$\left. \frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right) \quad (324)$$

$$\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}} \quad (325)$$

$$\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t} \quad (326)$$

$$\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$$

$$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t} \quad (327)$$

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- (328)

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$$

$$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$$

$$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}\big((G_{27}), t\big) - (b''_{25})^{(4)}\big((G_{27}), t\big) \le -(\tau_1)^{(4)}$$

**Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : 329

By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$
and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{v}_1)^{(4)} > 0$, $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$
and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (v_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}, \ \boldsymbol{if}\ (v_0)^{(4)} < (v_1)^{(4)}$$

$$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}, \boldsymbol{if}\ (v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)},$$

and $\boxed{(v_0)^{(4)} = \frac{G^0_{24}}{G^0_{25}}}$

$$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}, \ \boldsymbol{if}\ (\bar{v}_4)^{(4)} < (v_0)^{(4)}$$

and analogously 331

$$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}, \ \boldsymbol{if}\ (u_0)^{(4)} < (u_1)^{(4)}$$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}, \boldsymbol{if}\ (u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)},$$

and $\boxed{(u_0)^{(4)} = \frac{T^0_{24}}{T^0_{25}}}$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \boldsymbol{if}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)} \text{ where } (u_1)^{(4)}, (\bar{u}_1)^{(4)}$$

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| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le\right.$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}')^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}')^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)}t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(\nu_1)^{(5)}, (\nu_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, \nu^{(5)}, u^{(5)}$ :<br>By $(\nu_1)^{(5)} > 0$, $(\nu_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}(\nu^{(5)})^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)} = 0$ | 339 |

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| and $(b_{29})^{(5)}\left(u^{(5)}\right)^2+(\tau_1)^{(5)}u^{(5)}-(b_{28})^{(5)}=0$ and | |
| **Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ :<br><br>By $(\bar{v}_1)^{(5)}>0$ , $(\bar{v}_2)^{(5)}<0$ and respectively $(\bar{u}_1)^{(5)}>0$ , $(\bar{u}_2)^{(5)}<0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_2)^{(5)}v^{(5)}-(a_{28})^{(5)}=0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2+(\tau_2)^{(5)}u^{(5)}-(b_{28})^{(5)}=0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)},(v_0)^{(5)}$ :-<br><br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)}$ by<br><br>$(m_2)^{(5)}=(v_0)^{(5)},(m_1)^{(5)}=(v_1)^{(5)}$, ***if*** $(v_0)^{(5)}<(v_1)^{(5)}$<br><br>$(m_2)^{(5)}=(v_1)^{(5)},(m_1)^{(5)}=(\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)}<(v_0)^{(5)}<(\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)}=\frac{G_{28}^0}{G_{29}^0}}$<br><br>$(m_2)^{(5)}=(v_1)^{(5)},(m_1)^{(5)}=(v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)}<(v_0)^{(5)}$ | 340 |
| and analogously<br><br>$(\mu_2)^{(5)}=(u_0)^{(5)},(\mu_1)^{(5)}=(u_1)^{(5)}$, ***if*** $(u_0)^{(5)}<(u_1)^{(5)}$<br><br>$(\mu_2)^{(5)}=(u_1)^{(5)},(\mu_1)^{(5)}=(\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)}<(u_0)^{(5)}<(\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)}=\frac{T_{28}^0}{T_{29}^0}}$<br><br>$(\mu_2)^{(5)}=(u_1)^{(5)},(\mu_1)^{(5)}=(u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)}<(u_0)^{(5)}$ where $(u_1)^{(5)},(\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{28}^0e^{((S_1)^{(5)}-(p_{28})^{(5)})t}\le G_{28}(t)\le G_{28}^0e^{(S_1)^{(5)}t}$<br><br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}}G_{28}^0e^{((S_1)^{(5)}-(p_{28})^{(5)})t}\le G_{29}(t)\le\frac{1}{(m_2)^{(5)}}G_{28}^0e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)}G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)}-(p_{28})^{(5)}-(S_2)^{(5)})}\left[e^{((S_1)^{(5)}-(p_{28})^{(5)})t}-e^{-(S_2)^{(5)}t}\right]+G_{30}^0e^{-(S_2)^{(5)}t}\le G_{30}(t)\le\right.$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0e^{(R_1)^{(5)}t}\le T_{28}(t)\le T_{28}^0e^{((R_1)^{(5)}+(r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}}T_{28}^0e^{(R_1)^{(5)}t}\le T_{28}(t)\le\frac{1}{(\mu_2)^{(5)}}T_{28}^0e^{((R_1)^{(5)}+(r_{28})^{(5)})t}$ | 346 |

$\frac{(b_{30})^{(5)}T_{28}^{0}}{(\mu_1)^{(5)}((R_1)^{(5)}-(b_{30}^{'})^{(5)})}\left[e^{(R_1)^{(5)}t}-e^{-(b_{30}^{'})^{(5)}t}\right]+T_{30}^{0}e^{-(b_{30}^{'})^{(5)}t}\le T_{30}(t)\le$

$\frac{(a_{30})^{(5)}T_{28}^{0}}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t}-e^{-(R_2)^{(5)}t}\right]+T_{30}^{0}e^{-(R_2)^{(5)}t}$ 347

**Definition of** $(S_1)^{(5)},(S_2)^{(5)},(R_1)^{(5)},(R_2)^{(5)}$:- 348

Where $(S_1)^{(5)}=(a_{28})^{(5)}(m_2)^{(5)}-(a_{28}^{'})^{(5)}$

$(S_2)^{(5)}=(a_{30})^{(5)}-(p_{30})^{(5)}$

$(R_1)^{(5)}=(b_{28})^{(5)}(\mu_2)^{(5)}-(b_{28}^{'})^{(5)}$

$(R_2)^{(5)}=(b_{30}^{'})^{(5)}-(r_{30})^{(5)}$

**Behavior of the solutions of equation** 349

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ four constants satisfying

$-(\sigma_2)^{(6)}\le-(a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}-(a_{32}^{''})^{(6)}(T_{33},t)+(a_{33}^{''})^{(6)}(T_{33},t)\le-(\sigma_1)^{(6)}$

$-(\tau_2)^{(6)}\le-(b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(b_{32}^{''})^{(6)}\big((G_{35}),t\big)-(b_{33}^{''})^{(6)}\big((G_{35}),t\big)\le-(\tau_1)^{(6)}$

**Definition of** $(\nu_1)^{(6)},(\nu_2)^{(6)},(u_1)^{(6)},(u_2)^{(6)},\nu^{(6)},u^{(6)}$ : 350

By $(\nu_1)^{(6)}>0$ , $(\nu_2)^{(6)}<0$ and respectively $(u_1)^{(6)}>0$ , $(u_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2+(\sigma_1)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\big(u^{(6)}\big)^2+(\tau_1)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$ and

**Definition of** $(\bar{\nu}_1)^{(6)},,(\bar{\nu}_2)^{(6)},(\bar{u}_1)^{(6)},(\bar{u}_2)^{(6)}$ : 351

By $(\bar{\nu}_1)^{(6)}>0$ , $(\bar{\nu}_2)^{(6)}<0$ and respectively $(\bar{u}_1)^{(6)}>0$ , $(\bar{u}_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2+(\sigma_2)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\big(u^{(6)}\big)^2+(\tau_2)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$

**Definition of** $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)},(\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)}$ by

$(m_2)^{(6)}=(\nu_0)^{(6)},(m_1)^{(6)}=(\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)}<(\nu_1)^{(6)}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)}<(\nu_0)^{(6)}<(\bar{\nu}_1)^{(6)}$,

and $\boxed{(\nu_0)^{(6)}=\frac{G_{32}^{0}}{G_{33}^{0}}}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)}<(\nu_0)^{(6)}$

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| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},\ \boldsymbol{if}\ (u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}, \boldsymbol{if}\ (u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)},$<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}, \boldsymbol{if}\ (\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le\right.$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b_{34}')^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b_{34}')^{(6)} t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$<br>$(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ : | |

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| $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying<br><br>$-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$<br><br>$-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}((G_{39}), t) - (b''_{37})^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ :<br><br>By $(\nu_1)^{(7)} > 0, (\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ :<br><br>By $(\bar{\nu}_1)^{(7)} > 0, (\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$<br>**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-<br><br>If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by<br><br>$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,<br>and $\boxed{(\nu_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | 362 |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,<br>and $\boxed{(u_0)^{(7)} = \frac{T^0_{36}}{T^0_{37}}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities | 364 |

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|---|---|
| $G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^{0}e^{(S_1)^{(7)}t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | |
| $\frac{1}{(m_7)^{(7)}} G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^{0}e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^{0}e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}^{'})^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}^{'})^{(7)}t}] + G_{38}^{0}e^{-(a_{38}^{'})^{(7)}t})$ | 366 |
| $\boxed{T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)}T_{36}^{0}}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}^{'})^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}^{'})^{(7)}t}\right] + T_{38}^{0}e^{-(b_{38}^{'})^{(7)}t} \le T_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)}T_{36}^{0}}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^{0}e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}^{'})^{(7)}$<br><br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br><br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}^{'})^{(7)}$<br><br>$(R_2)^{(7)} = (b_{38}^{'})^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br><br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br><br>$-(\sigma_2)^{(8)} \le -(a_{40}^{'})^{(8)} + (a_{41}^{'})^{(8)} - (a_{40}^{''})^{(8)}(T_{41}, t) + (a_{41}^{''})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br><br>$-(\tau_2)^{(8)} \le -(b_{40}^{'})^{(8)} + (b_{41}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}), t) - (b_{41}^{''})^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |
| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : | 372 |

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|---|---|
| By $(\nu_1)^{(8)} > 0\,, (\nu_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0\,, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$ and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(8)},, (\bar{\nu}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :<br>By $(\bar{\nu}_1)^{(8)} > 0\,, (\bar{\nu}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0\,, (\bar{u}_2)^{(8)} < 0$ the<br>roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$<br>and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$<br>**Definition of** $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (\nu_0)^{(8)}$ :-<br>If we define $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by<br>$(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)},$ ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$<br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)}$,<br>and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)},$ ***if*** $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},$ ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)}$,<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)},$ ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)}t}$<br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$ | 376 |

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| $\left(\frac{(a_{42})^{(8)}G_{40}^{0}}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t}-e^{-(S_2)^{(8)}t}\right]+G_{42}^{0}e^{-(S_2)^{(8)}t}\leq G_{42}(t)\leq \frac{(a_{42})^{(8)}G_{40}^{0}}{(m_2)^{(8)}((S_1)^{(8)}-(a_{42}^{'})^{(8)})}[e^{(S_1)^{(8)}t}-e^{-(a_{42}^{'})^{(8)}t}]+G_{42}^{0}e^{-(a_{42}^{'})^{(8)}t}\right)$ | 377 |
| $\boxed{T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}}T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq \frac{1}{(\mu_2)^{(8)}}T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^{0}}{(\mu_1)^{(8)}((R_1)^{(8)}-(b_{42}^{'})^{(8)})}\left[e^{(R_1)^{(8)}t}-e^{-(b_{42}^{'})^{(8)}t}\right]+T_{42}^{0}e^{-(b_{42}^{'})^{(8)}t}\leq T_{42}(t)\leq$ $\frac{(a_{42})^{(8)}T_{40}^{0}}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t}-e^{-(R_2)^{(8)}t}\right]+T_{42}^{0}e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)},(S_2)^{(8)},(R_1)^{(8)},(R_2)^{(8)}$:-<br>Where $(S_1)^{(8)}=(a_{40})^{(8)}(m_2)^{(8)}-(a_{40}^{'})^{(8)}$<br>$(S_2)^{(8)}=(a_{42})^{(8)}-(p_{42})^{(8)}$<br>$(R_1)^{(8)}=(b_{40})^{(8)}(\mu_2)^{(8)}-(b_{40}^{'})^{(8)}$<br>$(R_2)^{(8)}=(b_{42}^{'})^{(8)}-(r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)}\leq -(a_{44}^{'})^{(9)}+(a_{45}^{'})^{(9)}-(a_{44}^{''})^{(9)}(T_{45},t)+(a_{45}^{''})^{(9)}(T_{45},t)\leq -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)}\leq -(b_{44}^{'})^{(9)}+(b_{45}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47}),t)-(b_{45}^{''})^{(9)}((G_{47}),t)\leq -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(\nu_1)^{(9)},(\nu_2)^{(9)},(u_1)^{(9)},(u_2)^{(9)},\nu^{(9)},u^{(9)}$ :<br>By $(\nu_1)^{(9)}>0$ , $(\nu_2)^{(9)}<0$ and respectively $(u_1)^{(9)}>0$ , $(u_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_1)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$<br>and $(b_{45})^{(9)}(u^{(9)})^2+(\tau_1)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(9)},,(\bar{\nu}_2)^{(9)},(\bar{u}_1)^{(9)},(\bar{u}_2)^{(9)}$ :<br>By $(\bar{\nu}_1)^{(9)}>0$ , $(\bar{\nu}_2)^{(9)}<0$ and respectively $(\bar{u}_1)^{(9)}>0$ , $(\bar{u}_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_2)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$ | |

and $(b_{45})^{(9)}\left(u^{(9)}\right)^2+(\tau_2)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$

**Definition of** $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)},(\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)}$ by

$(m_2)^{(9)}=(\nu_0)^{(9)},(m_1)^{(9)}=(\nu_1)^{(9)},$ ***if*** $(\nu_0)^{(9)}<(\nu_1)^{(9)}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)}<(\nu_0)^{(9)}<(\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)}=\frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\nu_0)^{(9)},$ ***if*** $(\bar{\nu}_1)^{(9)}<(\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)}=(u_0)^{(9)},(\mu_1)^{(9)}=(u_1)^{(9)},$ ***if*** $(u_0)^{(9)}<(u_1)^{(9)}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)}<(u_0)^{(9)}<(\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)}=\frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(u_0)^{(9)},$ ***if*** $(\bar{u}_1)^{(9)}<(u_0)^{(9)}$ where $(u_1)^{(9)},(\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{44}(t)\le G_{44}^0e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{45}(t)\le\frac{1}{(m_2)^{(9)}}G_{44}^0e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)}G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^0e^{-(S_2)^{(9)}t}\le G_{46}(t)\le\right.$$

$$\left.\frac{(a_{46})^{(9)}G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}')^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}')^{(9)}t}]+\ G_{46}^0e^{-(a_{46}')^{(9)}t}\right)$$

$$\boxed{T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le\frac{1}{(\mu_2)^{(9)}}T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}')^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}')^{(9)}t}\right]+T_{46}^0e^{-(b_{46}')^{(9)}t}\le T_{46}(t)\le$$

$$\frac{(a_{46})^{(9)}T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^0e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain (383)

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right) - (a''_{14})^{(1)}(T_{14}, t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get (384)

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{v}_1)^{(1)}$ we find like in the previous case, (385)

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}'')^{(1)} = (a_{14}'')^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)}G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}'')^{(1)} = (b_{14}'')^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)}T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}')^{(2)} - (a_{17}')^{(2)} + (a_{16}'')^{(2)}(T_{17}, t)\right) - (a_{17}'')^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$$

If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain 394

$$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(2)}(t)$ :- 395

$$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}, \quad \boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$$

In a completely analogous way, we obtain 396

**Definition of** $u^{(2)}(t)$ :-

$$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}, \quad \boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$$

| | |
|---|---|
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$ if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}')^{(3)} - (a_{21}')^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| From which one obtains<br><br>For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$<br><br>$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)} + (C)^{(3)}(\nu_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}$ , $\boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\nu_2)^{(3)}}}$<br><br>it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$ | 400 |
| In the same manner , we get<br><br>$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)} + (\bar{C})^{(3)}(\bar{\nu}_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\bar{\nu}_2)^{(3)}}}$<br><br>**Definition of** $(\bar{\nu}_1)^{(3)}$ :-<br><br>From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$ | 401 |
| If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, | 402 |

$$(\nu_1)^{(3)} \le \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \le \nu^{(3)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \le (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \le (\bar{\nu}_1)^{(3)} \le (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

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$$(\nu_1)^{(3)} \le \nu^{(3)}(t) \le \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \le (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \le \nu^{(3)}(t) \le (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \le u^{(3)}(t) \le (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right) - (a_{25}^{''})^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\quad \boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \le \frac{d\nu^{(4)}}{dt} \le -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

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$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} =$

$(\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b''_{24})^{(4)} = (b''_{25})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a'_{28})^{(5)} - (a'_{29})^{(5)} + (a''_{28})^{(5)}(T_{29}, t)\right) - (a''_{29})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\quad \boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get 413

$$v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{v}_1)^{(6)}-(v_0)^{(6)}}{(v_0)^{(6)}-(\bar{v}_2)^{(6)}}}$$

From which we deduce $(v_0)^{(6)} \le v^{(6)}(t) \le (\bar{v}_1)^{(6)}$

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If $0 < (v_1)^{(6)} < (v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{v}_1)^{(6)}$ we find like in the previous case,

$$(v_1)^{(6)} \le \frac{(v_1)^{(6)}+(C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}} \le v^{(6)}(t) \le$$

$$\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (\bar{v}_1)^{(6)}$$

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If $0 < (v_1)^{(6)} \le (\bar{v}_1)^{(6)} \le \boxed{(v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(v_1)^{(6)} \le v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (v_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$$(m_2)^{(6)} \le v^{(6)}(t) \le (m_1)^{(6)}, \quad \boxed{v^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)} \le u^{(6)}(t) \le (\mu_1)^{(6)}, \quad \boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(v_1)^{(6)} = (\bar{v}_1)^{(6)}$ if in addition $(v_0)^{(6)} = (v_1)^{(6)}$ then $v^{(6)}(t) = (v_0)^{(6)}$ and as a consequence $G_{32}(t) = (v_0)^{(6)}G_{33}(t)$ **this also defines** $(v_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(v_1)^{(6)}$ and $(\bar{v}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right) - (a_{37}^{''})^{(7)}(T_{37},t)v^{(7)} - (a_{37})^{(7)}v^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get 417

$$\nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\bar{\nu}_1)^{(7)}$

If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case, 418

$$(\nu_1)^{(7)} \leq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}} \leq \nu^{(7)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\bar{\nu}_1)^{(7)}$$

If $0 < (\nu_1)^{(7)} \leq (\bar{\nu}_1)^{(7)} \leq \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain 419

$$(\nu_1)^{(7)} \leq \nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \leq \nu^{(7)}(t) \leq (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :-

$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$ 420

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)} G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain 421

$$\frac{d\nu^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right) - (a_{41}^{''})^{(8)}(T_{41}, t)\nu^{(8)} - (a_{41})^{(8)}\nu^{(8)}$$

**Definition of** $\nu^{(8)}$ :- $\boxed{\nu^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right) \leq \frac{d\nu^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)}-(\nu_0)^{(8)}}{(\nu_0)^{(8)}-(\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \le \nu^{(8)}(t) \le (\bar{\nu}_8)^{(8)}$

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If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case,

$$(\nu_1)^{(8)} \le \frac{(\nu_1)^{(8)}+(C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}} \le \nu^{(8)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\bar{\nu}_1)^{(8)}$$

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If $0 < (\nu_1)^{(8)} \le (\bar{\nu}_1)^{(8)} \le \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$, we obtain

$$(\nu_1)^{(8)} \le \nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \le \nu^{(8)}(t) \le (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \le u^{(8)}(t) \le (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)}G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)}T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right) - (a''_{45})^{(9)}(T_{45},t)v^{(9)} - (a_{45})^{(9)}v^{(9)}$$

**Definition of** $v^{(9)}$ :- $\boxed{v^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)}\right) \le \frac{dv^{(9)}}{dt} \le -\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(9)}, (v_0)^{(9)}$ :-

For $0 < \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (v_1)^{(9)} < (\bar{v}_1)^{(9)}$

$$v^{(9)}(t) \ge \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(v_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (v_2)^{(9)}}}$$

it follows $(v_0)^{(9)} \le v^{(9)}(t) \le (v_9)^{(9)}$

In the same manner , we get

$$v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{v}_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (\bar{v}_2)^{(9)}}}$$

From which we deduce $(v_0)^{(9)} \le v^{(9)}(t) \le (\bar{v}_1)^{(9)}$

If $0 < (v_1)^{(9)} < (v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}} < (\bar{v}_1)^{(9)}$ we find like in the previous case,

$$(v_1)^{(9)} \le \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}} \le v^{(9)}(t) \le$$

$$\frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (\bar{v}_1)^{(9)}$$

If $0 < (v_1)^{(9)} \le (\bar{v}_1)^{(9)} \le \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$ , we obtain

$$(v_1)^{(9)} \le v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (v_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(9)}(t)$ :-

$$(m_2)^{(9)} \le v^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{v^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}$, $\boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ 430

$with\ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations 431

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$

| | |
|---|---|
| $with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | |
| We can prove the following<br><br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br><br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br><br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br><br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$ | 435 |

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|---|---|
| $with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |

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| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27})\big)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\big((G_{27})\big)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\big((G_{27})\big)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |

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| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |

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| $(b_{42})^{(8)}T_{41} - [(b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br>$F(T) = (a'_{13})^{(1)}(a'_{14})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a'_{13})^{(1)}(a''_{14})^{(1)}(T_{14}) + (a'_{14})^{(1)}(a''_{13})^{(1)}(T_{14}) + (a''_{13})^{(1)}(T_{14})(a''_{14})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br>(q) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br>$\mathrm{F}(T_{19}) = (a'_{16})^{(2)}(a'_{17})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a'_{16})^{(2)}(a''_{17})^{(2)}(T_{17}) + (a'_{17})^{(2)}(a''_{16})^{(2)}(T_{17}) + (a''_{16})^{(2)}(T_{17})(a''_{17})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br>$F(T_{23}) = (a'_{20})^{(3)}(a'_{21})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a'_{20})^{(3)}(a''_{21})^{(3)}(T_{21}) + (a'_{21})^{(3)}(a''_{20})^{(3)}(T_{21}) + (a''_{20})^{(3)}(T_{21})(a''_{21})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br>$F(T_{27}) = (a'_{24})^{(4)}(a'_{25})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a'_{24})^{(4)}(a''_{25})^{(4)}(T_{25}) + (a'_{25})^{(4)}(a''_{24})^{(4)}(T_{25}) + (a''_{24})^{(4)}(T_{25})(a''_{25})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br>$F(T_{31}) = (a'_{28})^{(5)}(a'_{29})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a'_{28})^{(5)}(a''_{29})^{(5)}(T_{29}) + (a'_{29})^{(5)}(a''_{28})^{(5)}(T_{29}) +$ | 489 |

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| $(a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that | 496 |

there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations

$$G_{20} = \frac{(a_{20})^{(3)} G_{21}}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^{*})]} \quad , \quad G_{22} = \frac{(a_{22})^{(3)} G_{21}}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^{*})]}$$

**Definition and uniqueness of** $T_{25}^{*}$ :- 497

After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations

$$G_{24} = \frac{(a_{24})^{(4)} G_{25}}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}^{*})]} \quad , \quad G_{26} = \frac{(a_{26})^{(4)} G_{25}}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}^{*})]}$$

**Definition and uniqueness of** $T_{29}^{*}$ :- 498

After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations

$$G_{28} = \frac{(a_{28})^{(5)} G_{29}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]} \quad , \quad G_{30} = \frac{(a_{30})^{(5)} G_{29}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$$

**Definition and uniqueness of** $T_{33}^{*}$ :- 499

After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*}) = 0$. With this value , we obtain from the three first equations

$$G_{32} = \frac{(a_{32})^{(6)} G_{33}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]} \quad , \quad G_{34} = \frac{(a_{34})^{(6)} G_{33}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$$

**Definition and uniqueness of** $T_{37}^{*}$ :- 500

After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*}) = 0$. With this value , we obtain from the three first equations

$$G_{36} = \frac{(a_{36})^{(7)} G_{37}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}^{*})]} \quad , \quad G_{38} = \frac{(a_{38})^{(7)} G_{37}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}^{*})]}$$

**Definition and uniqueness of** $T_{41}^{*}$ :- 501

After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*}) = 0$. With this value , we obtain from the three first equations

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| $G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G) + (b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*}) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19}) + (b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br>$[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ | 506 |

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| is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution | 511 |

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| $G_{14}^{*}$ given by $\varphi(G^{*})=0$ , $T_{14}^{*}$ given by $f(T_{14}^{*})=0$ and<br>$G_{13}^{*}=\frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*}=\frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br>$T_{13}^{*}=\frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*}=\frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br>Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*})=0$ , $T_{17}^{*}$ given by $f(T_{17}^{*})=0$ and | 512 |
| $G_{16}^{*}=\frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*}=\frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*}=\frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*}=\frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*})=0$ , $T_{21}^{*}$ given by $f(T_{21}^{*})=0$ and<br>$G_{20}^{*}=\frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*}=\frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br>$T_{20}^{*}=\frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*}=\frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^{*}$ given by $\varphi(G_{27})=0$ , $T_{25}^{*}$ given by $f(T_{25}^{*})=0$ and<br>$G_{24}^{*}=\frac{(a_{24})^{(4)}G_{25}^{*}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*}=\frac{(a_{26})^{(4)}G_{25}^{*}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*}=\frac{(b_{24})^{(4)}T_{25}^{*}}{[(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*}=\frac{(b_{26})^{(4)}T_{25}^{*}}{[(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}((G_{27})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*})=0$ , $T_{29}^{*}$ given by $f(T_{29}^{*})=0$ and<br>$G_{28}^{*}=\frac{(a_{28})^{(5)}G_{29}^{*}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*}=\frac{(a_{30})^{(5)}G_{29}^{*}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |

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| $T_{28}^{*} = \frac{(b_{28})^{(5)}T_{29}^{*}}{[(b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)}T_{29}^{*}}{[(b_{30}^{'})^{(5)}-(b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br>$G_{32}^{*} = \frac{(a_{32})^{(6)}G_{33}^{*}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)}G_{33}^{*}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)}T_{33}^{*}}{[(b_{32}^{'})^{(6)}-(b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)}T_{33}^{*}}{[(b_{34}^{'})^{(6)}-(b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br>$G_{36}^{*} = \frac{(a_{36})^{(7)}G_{37}^{*}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)}G_{37}^{*}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br>$T_{36}^{*} = \frac{(b_{36})^{(7)}T_{37}^{*}}{[(b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)}T_{37}^{*}}{[(b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br>$G_{40}^{*} = \frac{(a_{40})^{(8)}G_{41}^{*}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)}G_{41}^{*}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br>$T_{40}^{*} = \frac{(b_{40})^{(8)}T_{41}^{*}}{[(b_{40}^{'})^{(8)}-(b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)}T_{41}^{*}}{[(b_{42}^{'})^{(8)}-(b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)}G_{45}^{*}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)}G_{45}^{*}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)}T_{45}^{*}}{[(b_{44}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)}T_{45}^{*}}{[(b_{46}^{'})^{(9)}-(b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | 523 A |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^*) = (q_{14})^{(1)} \quad , \frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^*) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)}G_{13}^*\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^*\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^*\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^*\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^*\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^*\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)} \quad , \frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |

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| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)} \;\; , \frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_{j}\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_{j}\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_{j}\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ *and* $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 548 |

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| $\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^{*}) = (q_{25})^{(4)}$ , $\frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^{*}) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^{*}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^{*}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^{*}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^{*}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^{*}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^{*}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^{*}) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^{*}) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^{*}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^{*}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^{*}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^{*}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^{*}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^{*}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 563 |

**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ $and$ $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-

$$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)} \quad , \frac{\partial (b_i^{''})^{(6)}}{\partial G_j}\left((G_{35})^*\right) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33} \quad (565)$$

$$\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33} \quad (566)$$

$$\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^*\mathbb{T}_{33} \quad (567)$$

$$\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^*\mathbb{G}_j\right) \quad (568)$$

$$\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^*\mathbb{G}_j\right) \quad (569)$$

$$\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^*\mathbb{G}_j\right) \quad (570)$$

**ASYMPTOTIC STABILITY ANALYSIS** (571)

**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ $and$ $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- (572)

$$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)} \quad , \frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain from

$$\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37} \quad (573)$$

$$\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37} \quad (574)$$

| | |
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| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^{*}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^{*}\mathbb{G}_{j}\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^{*}\mathbb{G}_{j}\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^{*}\mathbb{G}_{j}\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^{*}) = (q_{41})^{(8)} \quad , \frac{\partial(b_i^{''})^{(8)}}{\partial G_j}\left((G_{43})^{*}\right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^{*}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^{*}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^{*}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^{*}\mathbb{G}_{j}\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b_{41}^{'})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T_{41}^{*}\mathbb{G}_{j}\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b_{42}^{'})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T_{42}^{*}\mathbb{G}_{j}\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |

| | |
|---|---|
| | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \ \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b_{44}^{'})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T_{44}^*\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b_{45}^{'})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T_{45}^*\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b_{46}^{'})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T_{46}^*\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)$<br>$\left[\left(\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(q_{13})^{(1)}G_{13}^*\right)\right]$<br>$\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T_{14}^* + (b_{14})^{(1)}s_{(13),(14)}T_{14}^*\right)$<br>$+\left(\left((\lambda)^{(1)} + (a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^* + (a_{13})^{(1)}(q_{14})^{(1)}G_{14}^*\right)$<br>$\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^* + (b_{14})^{(1)}s_{(13),(13)}T_{13}^*\right)$<br>$\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$<br>$\left(\left((\lambda)^{(1)}\right)^2 + \left((b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$<br>$+\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$<br>$+\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^*\right)$<br>$\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^* + (b_{14})^{(1)}s_{(13),(15)}T_{13}^*\right)\} = 0$ | |

$$+$$

$$((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)})\{((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)})$$

$$\left[\left(((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)})(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)\right]$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(17)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^{*}_{17}\right)$$

$$+\left(((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)})(q_{16})^{(2)}\mathrm{G}^{*}_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(16)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^{*}_{16}\right)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left(\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(18)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(18)}\mathrm{T}^{*}_{16}\right)\} = 0$$

$$+$$

$$((\lambda)^{(3)} + (b'_{22})^{(3)} - (r_{22})^{(3)})\{((\lambda)^{(3)} + (a'_{22})^{(3)} + (p_{22})^{(3)})$$

$$\left[\left(((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)})(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)\right]$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(21)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(21)}T^{*}_{21}\right)$$

$$+\left(((\lambda)^{(3)} + (a'_{21})^{(3)} + (p_{21})^{(3)})(q_{20})^{(3)}G^{*}_{20} + (a_{20})^{(3)}(q_{21})^{(1)}G^{*}_{21}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(20)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(20)}T^{*}_{20}\right)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((b'_{20})^{(3)} + (b'_{21})^{(3)} - (r_{20})^{(3)} + (r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left(\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(22)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(22)}T^{*}_{20}\right)\} = 0$$

$$+$$

$$\left((\lambda)^{(4)}+(b'_{26})^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a'_{26})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a'_{24})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G^{*}_{25}+(a_{25})^{(4)}(q_{24})^{(4)}G^{*}_{24}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(25)}T^{*}_{25}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a'_{25})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G^{*}_{24}+(a_{24})^{(4)}(q_{25})^{(4)}G^{*}_{25}\right)$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(24)}T^{*}_{24}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((a'_{24})^{(4)}+(a'_{25})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((b'_{24})^{(4)}+(b'_{25})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^{2}+\left((a'_{24})^{(4)}+(a'_{25})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a'_{24})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G^{*}_{25}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G^{*}_{24}\right)$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(26)}T^{*}_{24}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(5)}+(b'_{30})^{(5)}-(r_{30})^{(5)}\right)\{\left((\lambda)^{(5)}+(a'_{30})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a'_{28})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G^{*}_{29}+(a_{29})^{(5)}(q_{28})^{(5)}G^{*}_{28}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(29)}T^{*}_{29}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a'_{29})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G^{*}_{28}+(a_{28})^{(5)}(q_{29})^{(5)}G^{*}_{29}\right)$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(28)}T^{*}_{28}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((a'_{28})^{(5)}+(a'_{29})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((b'_{28})^{(5)}+(b'_{29})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^{2}+\left((a'_{28})^{(5)}+(a'_{29})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a'_{28})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G^{*}_{29}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G^{*}_{28}\right)$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(30)}T^{*}_{28}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\right)\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\right)\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\right)\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)\right]$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(41)}T^{*}_{41}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^{*}_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^{*}_{41}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(40)}T^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$+\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(42)}T^{*}_{40}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\right)\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)\right]$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(45)}T^{*}_{45}\right)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^{*}_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^{*}_{45}\right)$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(44)}T^{*}_{44}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^{2}+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$+\left(\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)$$

$\left(\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\right)\} = 0$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i \in R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i\ X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor

bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k \tag{17}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 17. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,...x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \in I, x \in K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\,\delta\,(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\mathcal{A}(\tilde{A}), C\backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) in\ ||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population $i$ persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population $i$ is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int_0^t (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\to\infty}{limsup}\frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \epsilon R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v=au_i(x)+w$ with $a \epsilon R$ and $w \epsilon F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|\ B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |\ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)|\ w|)$. Hence, $\underset{t\to\infty}{limsup}\frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\underset{t\to\infty}{limsup}\frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \underset{t\to\infty}{\text{limsup}}\frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)|-|B_i(t,x)w|$$

$$\geq |\ B_i(t,x)u_i(x)|(a- \alpha+ exp\ (-\beta t)|\ w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \underset{t\to\infty}{\text{limsup}}\frac{1}{t} \text{In}\ |B_i(t,x)v| \geq \lambda i(x) \geq \underset{t\to\infty}{\text{limsup}}\frac{1}{t} \text{In}\ |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In}\ |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In}\ b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x)$ =0 implies

$$\frac{d}{dt}b_i(t,x) = \lim_{s\to 0}\frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

=

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \le 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \le \gamma$ for all $t \ge 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \text{In } |\, (x.t)_i\, |$$

$$\le \operatorname{limsup}_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0. If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\ge 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\operatorname{limsup}_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x) =$ Integral $h(x.t)d\,\mu(x)$for all $t \ge 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure*. $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$ , $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x) \rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1 \le j \le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |\mathrm{h}(x.t) - h(x^n.t)|d\mu_n(x)| + |\int_c h(x^n.t)|d\mu_n(x) - \int_c \mathrm{h}(x)d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x)d\mu_n(x) - \int_c h(x)d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $\mathrm{h}_i^n(x) = \langle \mathrm{A}_i^n(x)\mathrm{u}_i^n(x), \mathrm{u}_i^n(x)\rangle$ where $\mathrm{u}_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $\mathrm{A}_i^n(x) \to \mathrm{A}_i(x)$ uniformly for $x \in N_1((\mathrm{A}))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $\mathrm{u}_i^n(x) \to \mathrm{u}_i(x)$ converges uniformly for $x \in N_1(\Lambda(\mathrm{A}))$. Given $\in > 0$, choose $n$ sufficiently large so that $|\mathrm{h}_i^n(x) - \mathrm{h}_i(x)| \leq \in$ for all $x \in N_1(\Lambda(\mathrm{A}))$ and $|\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$\lambda_i(\mu) \leq \int_c |\mathrm{h}_i(x) - \mathrm{h}_i^n(x)|d\mu(x) + |\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)|$

$+ \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)$

$\leq \quad 2\in$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\, \mathrm{ds}$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument

implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

(6) $\min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$ $\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_{1,}\ldots.,\tilde{A}_k)$ by

$$\tilde{A}_i(x) = \{ \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i \epsilon I}{if\ i \notin I}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that \quad \tilde{V}.t \subset intV\ for\ all\ t >$ $0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set $K$ if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$?then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$?and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume x* $\mapsto$ *A(x) is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists j* $\in$ { 1,….,k} *such that $M_i$ ----- { $x \in C : x_l \gg 0$ for $l \neq j$} Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_iM_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----{1,…,k} such that $\mu(---_i \in_I C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures μ with support in ∂C, $\lambda_i(\mu)>0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $$\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j \qquad i = 1,..,k \qquad j=1,\ldots,m$$

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge

to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j$ ($x^j_i$) = 0 i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product ($E_1^1$ ,…,$E_1^m$), and the matrix $A^2$ ($E_1$) is close to the diagonal matrix diag($f^i_2(E_1{}^j)$)$^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f_2^j(E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial$C is given by two equilibria {0,$E_1$}, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j\ x_2^j\ - \alpha^j\ x_3^j\ ) + \sum d_1^{jk}\ \ x_1^k\ - e_1^j\ x_1^j$

$\frac{dx_2^j}{dt} = x_2^j(1 - \alpha^j\ x_1^j\ - x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_2^{jk}\ \ x_2^k\ - e_2^j\ x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1 - \beta^j\ x_1^j\ - \alpha^j x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_3^{jk}\ \ x_3^k\ - e_3^j\ x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2\ (E_1)$and $\lambda_3\ (E_1)$are close to $\max_j\ 1 - a^j$ and $\max_j\ 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j\ 1 - a^j$ and $\max_j\ 1 - \beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1 - a^j) > -\max_j\ (1 - \beta^j)$$

Equivalently,

$$2 > \min_j\ a^j + \min_j\ \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i\left(r_i - \sum_s a_{is}x_s\right), \qquad i = 1,\dots,k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$$

It has been shown [12,ch.16.1,16.2] That for *k* =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \ \ with\ \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing

holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & O \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left[ \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1 - y) - b(N)\ y \right]$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*′ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon+x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords

Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation

of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1$ *(t)* and $P_2$ *(t)* denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial

alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.49) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.50) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.51) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_{1,}$ $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_{i,}$ $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic ($as$ $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 17. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, \text{R}^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1, 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) \quad P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^\infty f_i(a + s, P(t + S)) ds \right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[ - \int_o^x f_1(a + s, p(t + s)) ds \right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0) \exp\left[ - \int_o^t f_1\left(a - t + s, p_1(s)\right) ds \right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(\text{a+s}, p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da+\int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$$t> 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in[-\tau, 0], i = 1,2,$

(2.8) $\beta_i(\text{t}) = \int_0^t \beta_i(\text{a}) b_i(\text{t-a}, P_i(t)) M_i.(a, t, P_i(t))da$

$$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$$

$$i = 1, 2, \text{t} > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t) \phi(0) d_i^*]t \quad for\ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T],R^+), \quad B_i \in C([0,T],R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau,T] = \{f \in C([-\tau,T],R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau,T]$. Then for any

$(x, y) \in C^+[-\tau,T] \times C^+[-\tau,T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau,T], \quad C_T^2 = C^+[-\tau,T] \times C^+[-\tau,T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau,T]$ *and* $t \in [0,T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$B: C_T^2 \to C_T^2$ *and* $\Pi C_T^2 \to C_T^2$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = P_t \left\{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

$$(2.13) \quad \|\Pi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right)$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Pi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Pi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Pi(P_t^{(1)}) - \Pi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Pi(P^{(1)}(t)) - \Pi(P^{(2)}(t)| \le \int_0^t B_1^{(1)}(a) \left\{ M_1(a, t, P_t^{(1)}) - M_1(a, t, P_t^{(2)}) \right\} da$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left( \int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_* ds \right) da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o) \left[ \int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds \right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \|_t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$=\int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\mathrm{X}[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \, /\!/ P_t^{(1)} - P_t^{(2)} /\!/ \text{ for } a \, \varepsilon [o, t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \leq \beta_1 \Phi_1(0) || P_t^{(1)} - P_t^{(2)} ||_t^* + \Phi_1(0) \beta_1 d_1 || P_t^{(1)} - P_t^{(2)} ||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) \, \mathrm{e}^{\delta 1^a} \beta_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 \mathrm{d}_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*}(a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) \, \mathrm{e}^{\delta 1^a} /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*}\{\beta_1 a + d_1 \mathrm{a}^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| \mathrm{B}_1^{(1)}(\sigma) - \mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{B}_1^{*\mathrm{d}} \sigma$$

$$+ \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \left\{ B_1 + B_2 \, d_1 \, a + B_1^* \left( B_2 a + d_1 a^2/2 \right) e^{\delta 1^a} \right\}$$

$$\leq \beta_1^* \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| do + \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \rightarrow \beta_1$ as $a \rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{g_1(a) + 0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a)) - b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a)) do \right|$$

$$\left| B_1^* \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} 0(a) as\ a \rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi (P_t^{(1)} - \pi \left(P_t^{(2)}\right) /\!/ \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^*)$$

As $t^* \rightarrow o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp P_2^*(a)}{da} + f_i(a, P_1^*, P_2^*) P_i^*(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^* = \int_o^\infty P_i^*(a) da, i = 1,\ 2,$$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i=1.\ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s,P_1^*,P_2^*)ds\right], i=1,\quad 2, a\geq 0,$

Which together with (3.3) lead to

$1=\int_o^\infty b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$

(3.5) $1=\int_o^\infty b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1,P_2) = \int_o^\infty b_i(a,P_1,P_2)\exp\left[-\int_o^\infty f_i(s,P_1,P_2)ds\right]da,$

$i=1,\ 2,\quad P_1\geq 0,\quad P_2\geq 0,$

**And suppose the following hold:**

(3.7) $A_{4.}\quad F_{i.}\ (0,\ 0)>1,\ i=1,2$:

$A_{5.}$ *there exists a positive constant c for which*

$F_1(C,0)< F_{2.}(C,0),$

(3.8) $F_1(0.C)< F_{2.}(0,C),$

Then there exists a unique pair $(P_1^*,P_2^*)$ of real numbers $P_1^*>o,$

$P_2^*>0$ such that

(3,9) $F_1(P_1^*,P_2^*)=1=F_2(P_1^*,P_2^*)$

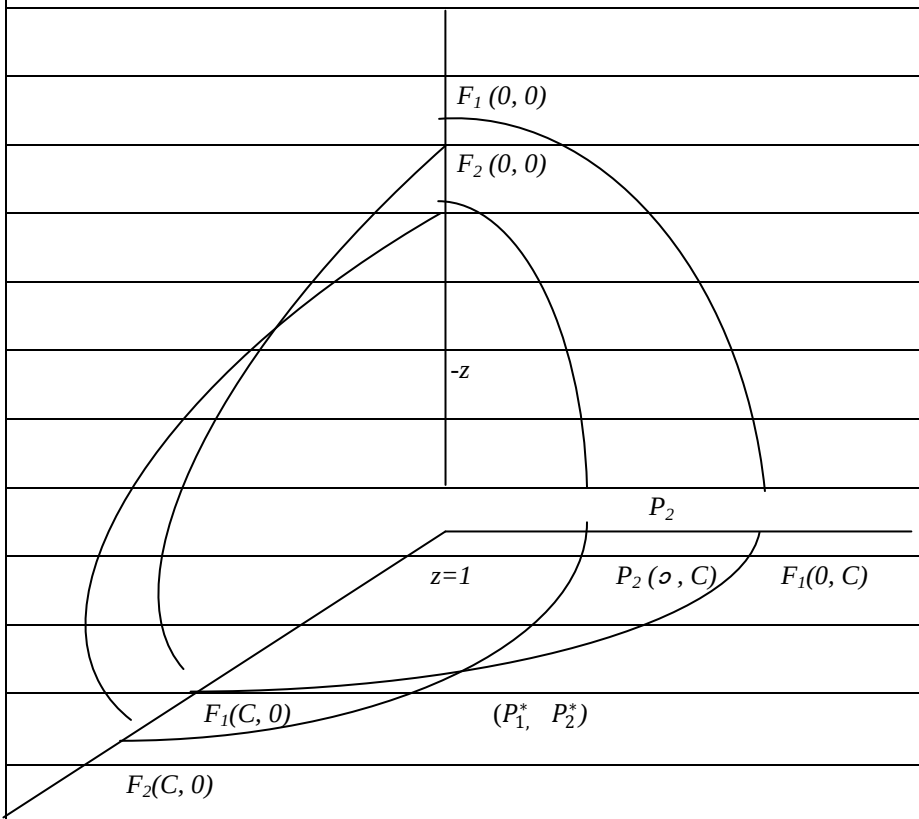
And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a)=[P_2^*\exp[-f_i(s,P_1^*,P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s,P_1^*,P_2^*)ds]da),$

$i=1,2,\quad a\geq 0,$

Proof Consider the elements of the surfaces defined by

| |
|---|
| Z= $F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0,$ |
| In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all |
| $P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$ |
| $j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0,$ These two curves are defined by |
| $F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, F_1(P_1, P_2 \geq 0,$ |
| Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$ |
| Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. |
|  |
| (The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility 1< $F_1(0,0)$ is treated similarly) It will now |
| Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10). |
| Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let |

(3.11) $$\begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12) $$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square=1}^{2} \frac{\partial b_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}da \end{cases}$$

Where $i$ = 1,2 and (3.13) $$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14) $$\begin{cases} u_i(a,t) = u\square(a) \exp[\lambda t], \quad \square(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty \square(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d\square}{da} + \lambda\square = -f_i\left(a, p_1^*, p_2^*\right)\square(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_{i\square}^*\right\},$$

(3.15)

$$\square(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\square(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*} p_{i\square}^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^{*} = p_{1}^{*}, \quad p_{12}^{*} = p_{2}^{*} \exp\{-\lambda\tau\},$$

$$p_{21}^{*} = p_{1}^{*}, \exp\{-\lambda\tau\}, \ p_{22}^{*} = p_{2}^{*}.$$

If we integrate the first of (3.15) and use the fact that $\mu_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\mu_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\mu_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\mu_i(a) = \{\mu_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\mu_i\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\mu_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\mu_i\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\mu_i\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\mu_i\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)}\int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}da, \quad i = 1,\ 2,$$

Now supplying $\mu_1(a)$ and $\mu_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^*\exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^*\exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^* \exp(-\lambda\tau) + (\lambda - A_{22})\, p_2^* = g_{21}(\lambda)\, p_1^* \exp(-\lambda\tau) + g_{22}(\lambda)\, p_2^*$$

Where

(3.22) $$A_i = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_i^*} - \frac{\partial f_i}{\partial p_i^*}\right]da,$$

(3.23) $$g_i(\lambda) = [p_i^*(0)c_i]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\, da\right\}$$
$$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_i^*} - b_i(a, p_1^*, p_2^*)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_i^*}\, e^{\lambda s}\, ds\right)\right]da\right\}$$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_i^*}\, e^{\lambda s}\, ds\right)da, \quad i = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*), \quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})\, e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$$S_2(\lambda, \tau) = [A_{21}\, g_{12}(\lambda) + A_{12}\, g_{21}(\lambda) + g_{12}(\lambda)\, g_{21}(\lambda)]\exp[-2\lambda\tau]$$

$$-A_{11}\, g_{22}(\lambda) - A_{22}\, g_{11}(\lambda) - g_{11}(\lambda)\, g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6$. $0 < \beta_i^* < d_i^*$ $(i = 1, 2.)$ $(see(A_1) and (A_2))$;

$A_7$. $\{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \text{[illegible]}(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \left|\right.$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \,| - |\, A_{11} + A_{22} + \| S_1(0) \| \,|^2 \right\}/2,$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational

ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the

Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have

obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is:

individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## SECTION A(14)

## Faddeev–Popov Path Integral Formula

### INTRODUCTION—VARIABLES USED

**Modern Physics Letters A Particles and Fields; Gravitation; Cosmology and Nuclear Physics Volume 31, Issue 04, 10 February 2016 Kazuo Fujikawa, Mod. Phys. Lett. A, 31, 1630004 (2016) [15 pages] DOI: http://dx.doi.org/10.1142/S0217732316300044 Yang–Mills theory and fermionic path integrals Kazuo Fujikawa1**

(1) The Yang–Mills gauge field theory, which was proposed 60 years ago, is extremely successful in describing (eb) the basic interactions of fundamental particles.
(2) The Yang–Mills theory in the course of its developments also stimulated (eb) much important field theoretical machinery.
(3) In this brief review authors discuss the path integral techniques, in particular, the fermionic path integrals which were developed together with (e&eb) the successful applications of quantized Yang–Mills field theory.
(4) Authors start with the **Faddeev–Popov path integral formula** with (e&eb) emphasis on the treatment of (e) fermionic ghosts as (=) an application of Grassmann numbers.
**(5)** Authors then discuss the ordinary fermionic path integrals and (e&eb) the general treatment of

| | |
|---|---|
| quantum anomalies. The contents of this review are mostly pedagogical except for a recent analysis of path integral bosonization. | |
| **NOTATION** | |
| **Module One**<br>They parametrize a two-sphere and allow us to handle efficiently infinite towers of (e) higher-spin auxiliary fields defined on (e&eb) ordinary space-time | |
| $G_{13}$ : Category one of higher-spin auxiliary fields defined on (e&eb) ordinary space-time<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of parametrize a two-sphere and allow us to handle efficiently infinite towers<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>They parametrize a two-sphere and allow us to handle efficiently infinite towers of higher-spin auxiliary fields defined on (e&eb) ordinary space-time | |
| $G_{16}$ : Category one of **parameterization a two-sphere and allow us to handle efficiently infinite towers of higher-spin auxiliary fields;** ordinary space-time<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of ordinary space-time; **parameterization a two-sphere and allow us to handle efficiently infinite towers of higher-spin auxiliary fields**<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>In this approach the chiral half of N=4 supersymmetry is (=) manifest | |
| $G_{20}$ : Category one of chiral half of N=4 supersymmetry<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of manifest<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |

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| **Module four**<br><br>The other half is realized non-linearly and (e&eb) the algebra closes on shell | |
| $G_{24}$ : Category one of **other half is realized non-linearly;** algebra closes on shell<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of algebra closes on shell **;other half is realized non-linearly**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>They show that the LHC formulation of the N=4 super-Yang-Mills theory is remarkably similar to (=) the harmonic superspace formulation of (e) the N=2 gauge and hypermultiplet matter theories<br><br>Take AC=DC=0 | |
| $G_{28}$ : Category one of **LHC formulation of the N=4 super-Yang-Mills theory is remarkably similar;** harmonic superspace formulation of (e) the N=2 gauge and hypermultiplet matter theories<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of harmonic superspace formulation of (e) the N=2 gauge and hypermultiplet matter theories; **LHC formulation of the N=4 super-Yang-Mills theory is remarkably similar**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Authors start with the **Faddeev–Popov path integral formula** with (e&eb) emphasis on the treatment of (e) fermionic ghosts as (=) an application of Grassmann numbers | |
| $G_{32}$ : Category one of **Faddeev–Popov path integral formula;** fermionic ghosts as (=) an application of Grassmann numbers<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of fermionic ghosts as (=) an application of Grassmann numbers; **Faddeev–Popov path integral formula** | |

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| $T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br>fermionic ghosts as (=) an application of Grassmann numbers | |
| $G_{36}$ : Category one of fermionic ghosts<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of application of Grassmann numbers<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>Authors then discuss the ordinary fermionic path integrals and (e&eb) the general treatment of quantum anomalies | |
| $G_{40}$ : Category one of **ordinary fermionic path integrals;** general treatment of quantum anomalies<br>$G_{41}$ : Category two of SAS<br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of general treatment of quantum anomalies; **ordinary fermionic path integrals**<br>$T_{41}$ : Category two of SAS<br>$T_{42}$ : Category three of SAS | |
| **Module Nine** | |
| $G_{44}$ : Category one of **ordinary fermionic path integrals; Faddeev–Popov path integral formula**<br>$G_{45}$ : Category two of SAS<br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of **Faddeev–Popov path integral formula; ordinary fermionic path integrals**<br>$T_{45}$ : Category two of SAS<br>$T_{46}$ : Category three of SAS | |

## SECTION A(15)

## Band Structure In Yang-Mills Theories

### INTRODUCTION—VARIABLES USED

**Band Structure in Yang-Mills Theories Constantin Bachas, Theodore Tomaras (Submitted on 29 Mar 2016 (v1), last revised 12 Apr 2016 (this version, v2))**

(1) **Constantin Bachas, Theodore Tomaras** show how Yang-Mills theory on (e&eb) S3×R can exhibit (eb) a spectrum with (e&eb) continuous bands if (e) coupled either to (e) a topological 3-form gauge field, or to a dynamical axion with (e&eb)heavy Peccei-Quinn scale.
(2) The basic mechanism consists in associating winding histories to (e&eb) a bosonic zero mode whose role is to convert a circle in configuration space into (e&eb) a helix.
(3) The zero mode is, respectively, the holonomy of (e) the 3-form field or the axion momentum.
(4) In these models different theta sectors coexist but are not (e) mixed by local operators.
**(5)** Analysis sheds light on, and extends Seiberg's proposal for modifying (e&eb) the topological sums in quantum field theories. It refutes a recent claim that B+L violation at LHC is unsuppressed. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph) Report number: LPTENS 16/01, ITCP-IPP 2016/03 Cite as: arXiv: 1603.08749 [hep-th] (or arXiv: 1603.08749v2 [hep-th] for this version)

### NOTATION

**Module One**

**Constantin Bachas, Theodore Tomaras** show how Yang-Mills theory on (e&eb) S3×R can exhibit (eb) a spectrum with (e&eb) continuous bands if (e) coupled either to (e) a topological 3-form gauge field, or to a dynamical axion with (e&eb)heavy Peccei-Quinn scale

$G_{13}$ : Category one of **Yang-Mills theory;** S3×R can exhibit (eb) a spectrum with (e&eb) continuous bands if (e) coupled either to (e) a topological 3-form gauge field, or to a dynamical axion with (e&eb)heavy Peccei-Quinn scale

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of S3×R can exhibit (eb) a spectrum with (e&eb) continuous bands if (e) coupled either to (e) a topological 3-form gauge field, or to a dynamical axion with (e&eb)heavy Peccei-Quinn scale **;Yang-Mills theory**

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

**Constantin Bachas, Theodore Tomaras** show how Yang-Mills theory on S3×R can exhibit (eb) a spectrum with (e&eb) continuous bands if (e) coupled either to (e) a topological 3-form gauge field, or to a dynamical axion with (e&eb)heavy Peccei-Quinn scale

$G_{16}$ : Category one of Yang-Mills theory on S3×R can exhibit

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| $G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of spectrum with (e&eb) continuous bands if (e) coupled either to (e) a topological 3-form gauge field, or to a dynamical axion with (e&eb)heavy Peccei-Quinn scale<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>**Constantin Bachas, Theodore Tomaras** show how Yang-Mills theory on S3×R can exhibit a spectrum with (e&eb) continuous bands if (e) coupled either to (e) a topological 3-form gauge field, or to a dynamical axion with (e&eb)heavy Peccei-Quinn scale | |
| $G_{20}$ : Category one of **Yang-Mills theory on S3×R can exhibit a spectrum;** continuous bands if (e) coupled either to (e) a topological 3-form gauge field, or to a dynamical axion with (e&eb)heavy Peccei-Quinn scale<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of continuous bands if (e) coupled either to (e) a topological 3-form gauge field, or to a dynamical axion with (e&eb)heavy Peccei-Quinn scale **;Yang-Mills theory on S3×R can exhibit a spectrum**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>**Constantin Bachas, Theodore Tomaras** show how Yang-Mills theory on S3×R can exhibit a spectrum with continuous bands if (e) coupled either to (e) a topological 3-form gauge field, or to a dynamical axion with (e&eb)heavy Peccei-Quinn scale | |
| $G_{24}$ : Category one of **coupled either to (e) a topological 3-form gauge field, or to a dynamical axion with (e&eb)heavy Peccei-Quinn scale;** Yang-Mills theory on S3×R can exhibit a spectrum with continuous bands<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of Yang-Mills theory on S3×R can exhibit a spectrum with continuous bands; **coupled either to (e) a topological 3-form gauge field, or to a dynamical axion with (e&eb)heavy Peccei-Quinn scale**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |

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| **Module five**<br><br>**Constantin Bachas, Theodore Tomaras** show how Yang-Mills theory on S3×R can exhibit a spectrum with continuous bands if coupled either to (e&eb) a topological 3-form gauge field, or to a dynamical axion with (e&eb)heavy Peccei-Quinn scale | |
| $G_{28}$ : Category one of **Yang-Mills theory on S3×R can exhibit a spectrum with continuous bands if coupled either;** topological 3-form gauge field, or to a dynamical axion with (e&eb)heavy Peccei-Quinn scale<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of topological 3-form gauge field, or to a dynamical axion with (e&eb)heavy Peccei-Quinn scale **;Yang-Mills theory on S3×R can exhibit a spectrum with continuous bands if coupled either**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>**Constantin Bachas, Theodore Tomaras** show how Yang-Mills theory on S3×R can exhibit a spectrum with continuous bands if coupled either to a topological 3-form gauge field, or to a dynamical axion with (e&eb) heavy Peccei-Quinn scale | |
| $G_{32}$ : Category one of **Yang-Mills theory on S3×R can exhibit a spectrum with continuous bands if coupled either to a topological 3-form gauge field, or to a dynamical axion;** heavy Peccei-Quinn scale<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of heavy Peccei-Quinn scale **;Yang-Mills theory on S3×R can exhibit a spectrum with continuous bands if coupled either to a topological 3-form gauge field, or to a dynamical axion**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>The basic mechanism consists in associating winding histories to (e&eb) a bosonic zero mode whose role is to convert a circle in configuration space into (e&eb) a helix.<br><br>The zero mode is, respectively, the holonomy of (e) the 3-form field or the axion momentum | |
| $G_{36}$ : Category one of **basic mechanism consists in associating winding histories;** bosonic zero mode whose role is to convert a circle in configuration space into (e&eb) a helix | |

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| $G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of bosonic zero mode whose role is to convert a circle in configuration space into (e&eb) a helix; **basic mechanism consists in associating winding histories**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The basic mechanism consists in associating winding histories to a bosonic zero mode whose role is to convert a circle in configuration space into (e&eb) a helix. | |
| $G_{40}$ : Category one of **basic mechanism consists in associating winding histories to a bosonic zero mode whose role is to convert a circle in configuration space;** helix.<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of helix. ;**basic mechanism consists in associating winding histories to a bosonic zero mode whose role is to convert a circle in configuration space**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The zero mode is, respectively, the holonomy of (e) the 3-form field or the axion momentum | |
| $G_{44}$ : Category one of3-form field or the axion momentum<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of zero mode is, respectively, the holonomy<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **The Coefficients:** | |

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| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$: $(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$ $(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$, $(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$ $(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$ $(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$ $(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$ , $(a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$ $(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}$ $(a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ , $(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$ $(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$, $(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$, $(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b'_{13})^{(1)} - (b''_{13})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b'_{14})^{(1)} - (b''_{14})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b'_{15})^{(1)} - (b''_{15})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a''_{13})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b''_{13})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b'_{16})^{(2)} - (b''_{16})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b'_{17})^{(2)} - (b''_{17})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b'_{18})^{(2)} - (b''_{18})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a''_{16})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b''_{16})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |

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| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |

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| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b'_{32})^{(6)} - (b''_{32})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b'_{33})^{(6)} - (b''_{33})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b'_{34})^{(6)} - (b''_{34})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a''_{32})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b'_{36})^{(7)} - (b''_{36})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b'_{37})^{(7)} - (b''_{37})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b'_{38})^{(7)} - (b''_{38})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a''_{36})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a'_{41})^{(8)} + (a''_{41})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b'_{40})^{(8)} - (b''_{40})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b'_{41})^{(8)} - (b''_{41})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b'_{42})^{(8)} - (b''_{42})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b'_{44})^{(9)} - (b''_{44})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b'_{45})^{(9)} - (b''_{45})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b'_{46})^{(9)} - (b''_{46})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a''_{44})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b''_{44})^{(9)}((G_{47}),t) =$ **First detrition factor** | |

$$\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{13} \quad (55)$$

$$\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{array}{ccc} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{14} \quad (56)$$

$$\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{15} \quad (57)$$

Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3

$\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[\begin{array}{ccc} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{array}\right] T_{13} \quad (58)$$

$$\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[\begin{array}{ccc} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{array}\right] T_{14} \quad (59)$$

$$\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)} & -(b_{15}^{''})^{(1)}(G,t) & -(b_{18}^{''})^{(2,2,)}(G_{19},t) & -(b_{22}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,)}(G_{39},t) & -(b_{42}^{''})^{(8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15} \quad (60)$$

Where $-(b_{13}^{''})^{(1)}(G,t)$, $-(b_{14}^{''})^{(1)}(G,t)$, $-(b_{15}^{''})^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{37}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}^{'})^{(2)} & +(a_{16}^{''})^{(2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,)}(T_{14},t) & +(a_{20}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{16} \quad (61)$$

$$\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}^{'})^{(2)} & +(a_{17}^{''})^{(2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,)}(T_{14},t) & +(a_{21}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{17} \quad (62)$$

$$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}^{'})^{(2)} & +(a_{18}^{''})^{(2)}(T_{17},t) & +(a_{15}^{''})^{(1,1,)}(T_{14},t) & +(a_{22}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{18} \quad (63)$$

Where $+(a_{16}^{''})^{(2)}(T_{17},t)$, $+(a_{17}^{''})^{(2)}(T_{17},t)$, $+(a_{18}^{''})^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for

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| category 1, 2 and 3 $+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3 $+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3 $+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)}\,-(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{16}$ | *64* |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)}\,-(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{17}$ | *65* |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)}\,-(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{18}$ | *66* |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$, $-(b_{17}^{''})^{(2)}(G_{19},t)$, $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{13}^{''})^{(1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3 $-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3 $-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3 $-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3 $-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3 $-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3 | |

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| $-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[\begin{array}{c}(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21},t)+(a_{16}^{''})^{(2,2,2)}(T_{17},t)+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)\\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)\\ +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{44}^{''})^{(9,9,9)}(T_{45},t)\end{array}\right]G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[\begin{array}{c}(a_{21}^{'})^{(3)}+(a_{21}^{''})^{(3)}(T_{21},t)+(a_{17}^{''})^{(2,2,2)}(T_{17},t)+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)\\ +(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)\\ +(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{45}^{''})^{(9,9,9)}(T_{45},t)\end{array}\right]G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[\begin{array}{c}(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21},t)+(a_{18}^{''})^{(2,2,2)}(T_{17},t)+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)\\ +(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)\\ +(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{46}^{''})^{(9,9,9)}(T_{45},t)\end{array}\right]G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[\begin{array}{c}(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23},t)-(b_{16}^{''})^{(2,2,2)}(G_{19},t)-(b_{13}^{''})^{(1,1,1,)}(G,t)\\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)\\ -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)-(b_{44}^{''})^{(9,9,9)}(G_{47},t)\end{array}\right]T_{20}$ | 70 |

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b'_{21})^{(3)} & -(b''_{21})^{(3)}(G_{23},t) & -(b''_{17})^{(2,2,2)}(G_{19},t) & -(b''_{14})^{(1,1,1,)}(G,t) \\ -(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b''_{37})^{(7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)} & -(b''_{22})^{(3)}(G_{23},t) & -(b''_{18})^{(2,2,2)}(G_{19},t) & -(b''_{15})^{(1,1,1,)}(G,t) \\ -(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b''_{38})^{(7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{22} \quad (72)$$

$-(b''_{20})^{(3)}(G_{23},t)$, $-(b''_{21})^{(3)}(G_{23},t)$, $-(b''_{22})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*

$-(b''_{16})^{(2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*

$-(b''_{13})^{(1,1,1,)}(G,t)$, $-(b''_{14})^{(1,1,1,)}(G,t)$, $-(b''_{15})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*

$-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*

$-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*

$-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*

$-(b''_{36})^{(7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7)}(G_{39},t)$ $-(b''_{38})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b''_{40})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b''_{46})^{(9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a'_{24})^{(4)} & +(a''_{24})^{(4)}(T_{25},t) & +(a''_{28})^{(5,5,)}(T_{29},t) & +(a''_{32})^{(6,6,)}(T_{33},t) \\ +(a''_{13})^{(1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{36})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a'_{25})^{(4)} & +(a''_{25})^{(4)}(T_{25},t) & +(a''_{29})^{(5,5,)}(T_{29},t) & +(a''_{33})^{(6,6)}(T_{33},t) \\ +(a''_{14})^{(1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{37})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a'_{26})^{(4)} & +(a''_{26})^{(4)}(T_{25},t) & +(a''_{30})^{(5,5,)}(T_{29},t) & +(a''_{34})^{(6,6,)}(T_{33},t) \\ +(a''_{15})^{(1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{38})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a''_{24})^{(4)}(T_{25},t)$, $(a''_{25})^{(4)}(T_{25},t)$, $(a''_{26})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a_{28}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* $1,2$ *and* $3$

$+(a_{32}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* $1,2$ *and* $3$

$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$ *are fourth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$ *are fifth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$ *are sixth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$ *are seventh augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$ *are eighth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)}&-(b_{24}^{''})^{(4)}(G_{27},t)&-(b_{28}^{''})^{(5,5,)}(G_{31},t)&-(b_{32}^{''})^{(6,6,)}(G_{35},t)\\-(b_{13}^{''})^{(1,1,1,1)}(G,t)&-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)&-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)\\-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)&-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)&-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt}=(b_{25})^{(4)}T_{24}-\begin{bmatrix}(b_{25}^{'})^{(4)}&-(b_{25}^{''})^{(4)}(G_{27},t)&-(b_{29}^{''})^{(5,5,)}(G_{31},t)&-(b_{33}^{''})^{(6,6,)}(G_{35},t)\\-(b_{14}^{''})^{(1,1,1,1)}(G,t)&-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)&-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)\\-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)&-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)&-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt}=(b_{26})^{(4)}T_{25}-\begin{bmatrix}(b_{26}^{'})^{(4)}&-(b_{26}^{''})^{(4)}(G_{27},t)&-(b_{30}^{''})^{(5,5,)}(G_{31},t)&-(b_{34}^{''})^{(6,6,)}(G_{35},t)\\-(b_{15}^{''})^{(1,1,1,1)}(G,t)&-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)&-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)\\-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)&-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)&-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{26} \quad (78)$$

*Where* $-(b_{24}^{''})^{(4)}(G_{27},t)$, $-(b_{25}^{''})^{(4)}(G_{27},t)$, $-(b_{26}^{''})^{(4)}(G_{27},t)$ *are first detrition coefficients for category* $1,2$ *and* $3$

$-(b_{28}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,)}(G_{31},t)$ *are second detrition coefficients for category* $1,2$ *and* $3$

$-(b_{32}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,)}(G_{35},t)$ *are third detrition coefficients for category* $1,2$ *and* $3$

$-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$ *are fourth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$ *are fifth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$

| | |
|---|---|
| are sixth detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[\begin{array}{c} (a_{28}^{'})^{(5)}\boxed{+(a_{28}^{''})^{(5)}(T_{29},t)}\ \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)}\ \boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)}\ \boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)}\ \boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}\ \boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[\begin{array}{c} (a_{29}^{'})^{(5)}\boxed{+(a_{29}^{''})^{(5)}(T_{29},t)}\ \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)}\ \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)}\ \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)}\ \boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}\ \boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[\begin{array}{c} (a_{30}^{'})^{(5)}\boxed{+(a_{30}^{''})^{(5)}(T_{29},t)}\ \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)}\ \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)}\ \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)}\ \boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}\ \boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{30}$ | *81* |
| Where $\boxed{+(a_{28}^{''})^{(5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5)}(T_{29},t)}$ are first augmentation coefficients for category 1, 2 and 3 And $\boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)}$ are second augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)}$ are third augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3 $\boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation coefficients for category 1,2,and 3 $\boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)}$ are sixth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1,2, 3 $\boxed{+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$ are eighth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1,2, 3 | |

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b'_{28})^{(5)} & -(b''_{28})^{(5)}(G_{31},t) & -(b''_{24})^{(4,4,)}(G_{27},t) & -(b''_{32})^{(6,6,6)}(G_{35},t) \\ -(b''_{13})^{(1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b'_{29})^{(5)} & -(b''_{29})^{(5)}(G_{31},t) & -(b''_{25})^{(4,4,)}(G_{27},t) & -(b''_{33})^{(6,6,6)}(G_{35},t) \\ -(b''_{14})^{(1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b'_{30})^{(5)} & -(b''_{30})^{(5)}(G_{31},t) & -(b''_{26})^{(4,4,)}(G_{27},t) & -(b''_{34})^{(6,6,6)}(G_{35},t) \\ -(b''_{15})^{(1,1,1,1,1,)}(G,t) & -(b''_{18})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{30} \quad (84)$$

*where* $-(b''_{28})^{(5)}(G_{31},t)$ , $-(b''_{29})^{(5)}(G_{31},t)$ , $-(b''_{30})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{24})^{(4,4,)}(G_{27},t)$, $-(b''_{25})^{(4,4,)}(G_{27},t)$, $-(b''_{26})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3

$-(b''_{32})^{(6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b''_{13})^{(1,1,1,1,1)}(G,t)$, $-(b''_{14})^{(1,1,1,1,1)}(G,t)$ , $-(b''_{15})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3

$-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3

$-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)$, $-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)$, $-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3

$-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3

$-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3

$-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a'_{32})^{(6)} & +(a''_{32})^{(6)}(T_{33},t) & +(a''_{28})^{(5,5,5)}(T_{29},t) & +(a''_{24})^{(4,4,4,)}(T_{25},t) \\ +(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a'_{33})^{(6)} & +(a''_{33})^{(6)}(T_{33},t) & +(a''_{29})^{(5,5,5)}(T_{29},t) & +(a''_{25})^{(4,4,4,)}(T_{25},t) \\ +(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)} G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)} T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)} T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)} T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{28})^{(5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)}+(a_{37}^{''})^{(7)}(T_{37},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15} \quad (93)$$

Where $(a_{36}^{''})^{(7)}(T_{37},t)$, $(a_{37}^{''})^{(7)}(T_{37},t)$, $(a_{38}^{''})^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation

coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3 $+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}(G_{39},t) & -(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

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$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}-(b_{37}^{''})^{(7)}(G_{39},t) & -(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}(G_{39},t) & -(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3 $-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3 $-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3 $-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

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$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}')^{(8)} & +(a_{40}^{''})^{(8)}(T_{41},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13}$$

$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}')^{(8)} & +(a_{41}^{''})^{(8)}(T_{41},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}')^{(8)} & +(a_{42}^{''})^{(8)}(T_{41},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$$

Where $+(a_{40}^{''})^{(8)}(T_{41},t)$, $+(a_{41}^{''})^{(8)}(T_{41},t)$, $+(a_{42}^{''})^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & -(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & -(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & -(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3



$$\frac{dG_{44}}{dt} = (a_{44})^{(9)} G_{45} - \left[ \begin{array}{l} (a'_{44})^{(9)} \boxed{+(a''_{44})^{(9)}(T_{45},t)} \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array} \right] G_{13}$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)} G_{44} - \left[ \begin{array}{l} (a'_{45})^{(9)} \boxed{+(a''_{45})^{(9)}(T_{45},t)} \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array} \right] G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)} G_{45} - \left[ \begin{array}{l} (a'_{46})^{(9)} \boxed{+(a''_{46})^{(9)}(T_{37},t)} \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array} \right] G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$\frac{dT_{44}}{dt} =$

$$(b_{44})^{(9)}T_{45} - \left[\begin{array}{ccc} (b_{44}^{'})^{(9)}\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[\begin{array}{ccc} (b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[\begin{array}{ccc} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

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| **Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br>$(a_i'')^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$<br>$(b_i'')^{(1)}(G,t) \le (r_i)^{(1)} \le (b_i')^{(1)} \le (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2\to\infty}(a_i'')^{(1)}(T_{14},t) = (p_i)^{(1)}$<br>$\lim_{G\to\infty}(b_i'')^{(1)}(G,t) = (r_i)^{(1)}$<br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(1)}(T_{14}',t) - (a_i'')^{(1)}(T_{14},t)\rvert \le (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}'\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br>$\lvert(b_i'')^{(1)}(G',t) - (b_i'')^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G'\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(1)}(T_{14}',t)$ and $(a_i'')^{(1)}(T_{14},t)$ . $(T_{14}',t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i'')^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i'')^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i')^{(1)}, (b_i)^{(1)}, (b_i')^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i,j = 16,17,18$ | |

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| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17},t) \le (p_i)^{(2)} \le (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19},t) \le (r_i)^{(2)} \le (b_i')^{(2)} \le (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2\to\infty} (a_i'')^{(2)}(T_{17},t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty} (b_i'')^{(2)}((G_{19}),t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}$ are positive constants and $i = 16,17,18$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(2)}(T_{17}',t) - (a_i'')^{(2)}(T_{17},t)\rvert \le (\hat{k}_{16})^{(2)}\lvert T_{17} - T_{17}'\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i'')^{(2)}((G_{19})',t) - (b_i'')^{(2)}((G_{19}),t)\rvert < (\hat{k}_{16})^{(2)}\lvert\lvert(G_{19}) - (G_{19})'\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}',t)$ and $(a_i'')^{(2)}(T_{17},t)$ . $(T_{17}',t)$ and $(T_{17},t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}]$ . It is to be noted that $(a_i'')^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i'')^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i')^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ | 111 |

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| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(3)} (T_{21}, t) = (p_i)^{(3)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(3)} (G_{23}, t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'}, t) - (a_i^{''})^{(3)}(T_{21}, t)\rvert \leq (\hat{k}_{20})^{(3)} \lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)} t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'}, t) - (b_i^{''})^{(3)}(G_{23}, t)\rvert < (\hat{k}_{20})^{(3)} \lvert\lvert G_{23} - G_{23}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)} t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'}, t)$ and$(a_i^{''})^{(3)}(T_{21}, t)$ . $(T_{21}^{'}, t)$ And $(T_{21}, t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$<br>satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$ | 116 |

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| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24{,}25{,}26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}\big((G_{27}), t\big) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(4)}\big((G_{27}), t\big) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24{,}25{,}26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\rvert \leq (\hat{k}_{24})^{(4)} \lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)} t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'}, t) - (b_i^{''})^{(4)}\big((G_{27}), t\big)\rvert < (\hat{k}_{24})^{(4)} \lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)} t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $\big[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\big]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24{,}25{,}26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |

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| $\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i^{'})^{(4)} +\ (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}\big((G_{31}), t\big) \leq\ (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(5)}\,(T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(5)}\,(G_{31}, t) =\ (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'}, t) - (b_i^{''})^{(5)}\big((G_{31}), t\big)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $\big[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\big]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br><br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities | 126 |

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| $\frac{1}{(\widehat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\widehat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(6)}(T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(6)}((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)}\lvert T_{33} - T_{33}^{'}\rvert e^{-(\widehat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})^{'}, t) - (b_i^{''})^{(6)}((G_{35}), t)\rvert < (\hat{k}_{32})^{(6)}\lvert\lvert(G_{35}) - (G_{35})^{'}\rvert\rvert e^{-(\widehat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and$(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\widehat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\widehat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\widehat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\widehat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants | 130 |

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| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$<br>satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$ | |
| Where we suppose | |
| (YYYY) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i, j = 36,37,38$<br><br>(ZZZZ) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$:<br><br>$(a_i^{''})^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$<br><br>$(b_i^{''})^{(7)}(G_{39}, t) \leq \ (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$ | 131 |
| *(AAAAA)* $\lim_{T_2 \to \infty} (a_i^{''})^{(7)}(T_{37}, t) = (p_i)^{(7)}$<br>*(BBBBB)*<br>$\lim_{G \to \infty} (b_i^{''})^{(7)}((G_{39}), t) = \ (r_i)^{(7)}$<br><br>**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :<br><br>Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, \ (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$ | 132 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)\rvert \leq (\hat{k}_{36})^{(7)} \lvert T_{37} - T_{37}^{'}\rvert e^{-(\hat{M}_{36})^{(7)} t}$<br><br>$\lvert(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}((G_{39}), t)\rvert < (\hat{k}_{36})^{(7)} \lvert\lvert(G_{39}) - (G_{39})^{'}\rvert\rvert e^{-(\hat{M}_{36})^{(7)} t}$ | 133 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'}, t)$ and $(a_i^{''})^{(7)}(T_{37}, t)$ . $(T_{37}^{'}, t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :<br><br>(CCCCC) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants | 134 |

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| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :<br><br>(DDDDD) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}, (r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty}(a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty}(b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)}\lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})', t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)}\lvert\lvert(G_{43}) - (G_{43})'\rvert\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would | |

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| be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i,j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45}, t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47}, t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2 \to \infty} (a_i'')^{(9)}(T_{45}, t) = (p_i)^{(9)}$<br>$\lim_{G \to \infty} (b_i'')^{(9)}(G_{47}, t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}', t) - (a_i'')^{(9)}(T_{45}, t)\rvert \le (\hat{k}_{44})^{(9)} \lvert T_{45} - T_{45}' \rvert e^{-(\hat{M}_{44})^{(9)} t}$<br>$\lvert(b_i'')^{(9)}((G_{47})', t) - (b_i'')^{(9)}((G_{47}), t)\rvert < (\hat{k}_{44})^{(9)} \lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}', t)$ | |

and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)},(\widehat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{44})^{(9)}=1$ then the function $(a_i'')^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$ :

$(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\ ,\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}<1$$

**Definition of** $(\hat{P}_{44})^{(9)},(\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},(\hat{A}_{44})^{(9)}and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)},(a_i')^{(9)},(b_i)^{(9)},(b_i')^{(9)},(p_i)^{(9)},\ (r_i)^{(9)},i=44,45,46,$ satisfy the inequalities

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[(a_i)^{(9)}+(a_i')^{(9)}+\ (\hat{A}_{44})^{(9)}+\ (\hat{P}_{44})^{(9)}\ (\hat{k}_{44})^{(9)}]<1$$

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[\ (b_i)^{(9)}+(b_i')^{(9)}+\ (\hat{B}_{44})^{(9)}+\ (\hat{Q}_{44})^{(9)}\ \ (\hat{k}_{44})^{(9)}]<1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0)\ ,T_i(0)$ :

$$G_i(t)\le\ (\hat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{G_i(0)=G_i^0>0}$$

$$T_i(t)\le\ (\hat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{T_i(0)=T_i^0>0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0)\ ,T_i(0)$

$$G_i(t)\le\ (\hat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t)\le\ (\hat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

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| **Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$ | 153 B |

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| $T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |

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| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br><br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| By<br><br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} G_{21}(s_{(20)}) - \left((a_{20}')^{(3)} + a_{20}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{20}(s_{(20)})\right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[(a_{21})^{(3)} G_{20}(s_{(20)}) - \left((a_{21}')^{(3)} + (a_{21}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}')^{(3)} - (b_{21}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br><br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{28})^{(5)}$, $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[ (b_{29})^{(5)} T_{28}(s_{(28)}) - \left( (b'_{29})^{(5)} - (b''_{29})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[ (b_{30})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{30})^{(5)} - (b''_{30})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{30}(s_{(28)}) \right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** | |

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| Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[ (a_{32})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{32}(s_{(32)}) \right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[ (a_{33})^{(6)} G_{32}(s_{(32)}) - \left( (a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[ (a_{34})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{34}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[ (b_{32})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{32}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[ (b_{33})^{(6)} T_{32}(s_{(32)}) - \left( (b'_{33})^{(6)} - (b''_{33})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[ (b_{34})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{34})^{(6)} - (b''_{34})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{34}(s_{(32)}) \right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |

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| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}^{'})^{(8)} + a_{40}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^{0} + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^{0} + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^{0} + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{13}(t) \leq G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \leq \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}$ , $G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \leq G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that | 170 |

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| $(G_{16}(t)-G_{16}^0)e^{-(\bar{M}_{16})^{(2)}t} \le \frac{(a_{16})^{(2)}}{(\bar{M}_{16})^{(2)}}\left[((\hat{P}_{16})^{(2)}+G_{17}^0)e^{\left(-\frac{(\hat{P}_{16})^{(2)}+G_{17}^0}{G_{17}^0}\right)}+(\hat{P}_{16})^{(2)}\right]$ | |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0+(\hat{P}_{20})^{(3)}e^{(\bar{M}_{20})^{(3)}s_{(20)}}\right)\right] ds_{(20)} =$<br>$(1+(a_{20})^{(3)}t)G_{21}^0 + \frac{(a_{20})^{(3)}(\hat{P}_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left(e^{(\bar{M}_{20})^{(3)}t}-1\right)$ | 171 |
| From which it follows that<br>$(G_{20}(t)-G_{20}^0)e^{-(\bar{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that<br>$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\bar{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$<br>$(1+(a_{24})^{(4)}t)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left(e^{(\bar{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that<br>$(G_{24}(t)-G_{24}^0)e^{-(\bar{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$<br>$(1+(a_{28})^{(5)}t)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)}t}-1\right)$ | |
| From which it follows that<br>$(G_{28}(t)-G_{28}^0)e^{-(\bar{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 5 | 175 |
| The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious | 176 |

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| that<br>$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$<br>$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$ | |
| From which it follows that<br>$(G_{32}(t) - G_{32}^0) e^{-(\hat{M}_{32})^{(6)} t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)} + G_{33}^0) e^{\left(-\frac{(\hat{P}_{32})^{(6)} + G_{33}^0}{G_{33}^0}\right)} + (\hat{P}_{32})^{(6)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 6<br>Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$ | 177 |
| (r) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)}\left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$<br>$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$ | 178 |
| From which it follows that<br>$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)} + G_{37}^0) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 7 | |
| The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)}\left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$<br>$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$ | 180 |
| From which it follows that<br>$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)} + G_{41}^0) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)}\left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$ | |

$$\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$$

From which it follows that

$$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t}\leq\frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{P}_{44})^{(9)}+G_{45}^{0}\right)e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 9

Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$

It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ and to choose

$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have

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$$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$$

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$$\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$$

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In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric

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$$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$$

$$\sup_i\left\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\hat{M}_{13})^{(1)}t}\right\}$$

Indeed if we denote

**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$

It results

$$\left|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right|\leq\int_0^t(a_{13})^{(1)}\left|G_{14}^{(1)}-G_{14}^{(2)}\right|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$$

$$\int_0^t\{(a_{13}')^{(1)}\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{-(\hat{M}_{13})^{(1)}s_{(13)}}+$$

$$(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}+$$

$$G_{13}^{(2)}\left|(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}'')^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right|\ e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$$

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| Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | |
| $\left\|G^{(1)} - G^{(2)}\right\| e^{-(\widehat{M}_{13})^{(1)}t} \leq$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right) d\left(\left(G^{(1)}, T^{(1)};\ G^{(2)}, T^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(1)}$ and $(b''_i)^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(1)} - (a''_i)^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1 / (a'_{14})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2 / (a'_{15})^{(1)}$<br><br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(1)}(G(t), t)) = (b'_{14})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ : | 189 |

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| Indeed let $t_1$ be so that for $t > t_1$<br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ $and$ $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |

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| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$(a_{16}^{''})^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)} \left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}} \left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)} (\widehat{k}_{16})^{(2)}\right) d\left(\left((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)}\right)\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)} (T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0 \quad \text{for } t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)} \left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$<br>In the same way , one can obtain | 201 |

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| $G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$, $G_{18}$ and $G_{16}$, $G_{17}$ respectively. | |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$. The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}$, $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}} < 1$ and to choose<br><br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$, $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{20})^{(3)}t}\}$ | 211 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}}$ :$\left( \widetilde{(G_{23})}, \widetilde{(T_{23})} \right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)} \left\|G_{21}^{(1)} - G_{21}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\, ds_{(20)} +$<br><br>$\int_0^t \{(a_{20}^{'})^{(3)} \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right)\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$G_{20}^{(2)} \left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}^{''})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\} ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} t} \leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a_{20}^{'})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)} (\widehat{k}_{20})^{(3)}\right) d\left(\left((G_{23})^{(1)}, (T_{23})^{(1)}; (G_{23})^{(2)}, (T_{23})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ *and* $(\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)} t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating | 217 |

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| $G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}')^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}')^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(3)}\left((G_{23})(t),t\right) = (b_{21}')^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i'')^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}'')^{(3)}\left((G_{23})(t),t\right) = (b_{22}')^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}\left[(\hat{P}_{24})^{(4)} + \left((\hat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}}\left[\left((\hat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{24})^{(4)}\right] \leq (\hat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$ | 225 |

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{27})},\widetilde{(T_{27})}$ : $\left(\widetilde{(G_{27})},\widetilde{(T_{27})}\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{24})^{(4)}\left|G_{25}^{(1)}-G_{25}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$$

$$\int_0^t\{(a_{24}^{'})^{(4)}\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$G_{24}^{(2)}\left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)}-(G_{27})^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}t}\le$$
$$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows

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**Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}\ and\ (\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

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**Remark 17:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

it results

$$G_i\ (t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]}\ge 0$$

$$T_i\ (t)\ge T_i^0e^{\left(-(b_i^{'})^{(4)}t\right)}>0\quad \text{for t} > 0$$

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**Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1,\left((\widehat{M}_{24})^{(4)}\right)_2\ and\ \left((\widehat{M}_{24})^{(4)}\right)_3$ :

**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if

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| $G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a'_{25})^{(4)} G_{25}$ and by integrating<br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a'_{25})^{(4)}$<br>In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a'_{26})^{(4)}$<br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t),t)) = (b'_{25})^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded.<br>The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}\left((G_{27})(t),t\right) = (b'_{26})^{(4)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + \left((\widehat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\widehat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{28})^{(5)}\right] \leq (\widehat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying | |

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| Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}((G_{31}),(T_{31}))$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)}+$<br>$\int_0^t\{(a_{28}^{'})^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\le$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}^{'})^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}^{''})^{(5)}$ and $(b_{28}^{''})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(5)}-(a_i^{''})^{(5)}(T_{29}(s_{(28)}),s_{(28)})\}ds_{(28)}\right]}\ge 0$ | 239 |

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| $T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(5)}t)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}^{'})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1/(a_{29}^{'})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2/(a_{30}^{'})^{(5)}$<br><br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t),t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\,\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |

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| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{32})^{(6)}\right]\leq(\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}ds_{(32)}+$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a''_{32})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a'_{32})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(6)}$ and $(b_i'')^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 249 |

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| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(6)} - (a_i'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(6)} t)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}')^{(6)} G_{33}$ and by integrating<br><br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}')^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}')^{(6)}$<br><br>If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(6)}((G_{35})(t), t)) = (b_{33}')^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i'')^{(6)}((G_{35})(t), t) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty} (b_{34}'')^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}')^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have | 255 |

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| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)}+\left((\hat{P}_{36})^{(7)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{36})^{(7)}\right]\leq(\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}')^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}'')^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}')^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}'')^{(7)}$ and $(b_{36}'')^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$. | 260 |

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| If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2 / (a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t), t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t), t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right),\quad t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded. | 265 |

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| The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t),t\right)=(b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}<1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)}+\left((\hat{P}_{40})^{(8)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{40})^{(8)}\right]\leq(\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}ds_{(40)}+$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\leq$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ $and$ $T_i$. Taking into account the hypothesis the result follows | 273 |

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| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(8)} - (a_i^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2\ and\ \left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}^{'})^{(8)}$<br><br>If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t), t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t), t\right) < \varepsilon_8, T_{40}\,(t) > (m)^{(8)}$ | 278 |

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| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.<br>The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}^{'})^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$ | |
| In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying 39,35,36 into itself | |
| The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{47})^{(1)}, (T_{47})^{(1)}), ((G_{47})^{(2)}, (T_{47})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \left(\widetilde{(G_{47})}, \widetilde{(T_{47})}\right) = \mathcal{A}^{(9)}\left((G_{47}), (T_{47})\right)$<br>It results<br><br>$\left\|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$<br>$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right) - (a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right| \; e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$<br><br>Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on 45,46,47,28 and 29 it follows | |
| $\left|(G_{47})^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t} \leq$<br>$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)}, (T_{47})^{(1)}; (G_{47})^{(2)}, (T_{47})^{(2)})\right)$ | |

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$

From 99 to 44 it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(9)}t\right)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)}G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t), t\right) < \varepsilon_9, T_{44}\,(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded. The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b''_{46})^{(9)}\left((G_{47})(t),t\right) = (b'_{46})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a'_{13})^{(1)} + (a'_{14})^{(1)} - (a''_{13})^{(1)}(T_{14}, t) + (a''_{14})^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b'_{13})^{(1)} + (b'_{14})^{(1)} - (b''_{13})^{(1)}(G,t) - (b''_{14})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,

and $\boxed{(u_0)^{(1)} = \frac{T^0_{13}}{T^0_{14}}}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$

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| are defined | |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le \right.$<br>$\left.\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a'_{15})^{(1)})}[e^{(S_1)^{(1)}t} - e^{-(a'_{15})^{(1)}t}] + G_{15}^0 e^{-(a'_{15})^{(1)}t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b'_{15})^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b'_{15})^{(1)}t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)}t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$<br>$(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ :<br>$(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a'_{16})^{(2)} + (a'_{17})^{(2)} - (a''_{16})^{(2)}(T_{17}, t) + (a''_{17})^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b'_{16})^{(2)} + (b'_{17})^{(2)} - (b''_{16})^{(2)}((G_{19}), t) - (b''_{17})^{(2)}((G_{19}), t) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(v_1)^{(2)}, (v_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |

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| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\left(u^{(2)}\right)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}$,, $(\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\left(u^{(2)}\right)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously $(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$ $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities $G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)}t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le\right.$ | 312 |

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| $\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_2)^{(2)}((S_1)^{(2)}-(a_{18}')^{(2)})}[e^{(S_1)^{(2)}t}-e^{-(a_{18}')^{(2)}t}]+G_{18}^{0}e^{-(a_{18}')^{(2)}t})$ | |
| $\boxed{T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}}T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le\frac{1}{(\mu_2)^{(2)}}T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)}T_{16}^{0}}{(\mu_1)^{(2)}((R_1)^{(2)}-(b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t}-e^{-(b_{18}')^{(2)}t}\right]+T_{18}^{0}e^{-(b_{18}')^{(2)}t}\le T_{18}(t)\le$ $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)},(S_2)^{(2)},(R_1)^{(2)},(R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)}=(a_{16})^{(2)}(m_2)^{(2)}-(a_{16}')^{(2)}$ $(S_2)^{(2)}=(a_{18})^{(2)}-(p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)}=(b_{16})^{(2)}(\mu_2)^{(1)}-(b_{16}')^{(2)}$ $(R_2)^{(2)}=(b_{18}')^{(2)}-(r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$: $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)}\le-(a_{20}')^{(3)}+(a_{21}')^{(3)}-(a_{20}'')^{(3)}(T_{21},t)+(a_{21}'')^{(3)}(T_{21},t)\le-(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)}\le-(b_{20}')^{(3)}+(b_{21}')^{(3)}-(b_{20}'')^{(3)}(G_{23},t)-(b_{21}'')^{(3)}((G_{23}),t)\le-(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)},(v_2)^{(3)},(u_1)^{(3)},(u_2)^{(3)}$: By $(v_1)^{(3)}>0,(v_2)^{(3)}<0$ and respectively $(u_1)^{(3)}>0,(u_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_1)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_1)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ and By $(\bar{v}_1)^{(3)}>0,(\bar{v}_2)^{(3)}<0$ and respectively $(\bar{u}_1)^{(3)}>0,(\bar{u}_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_2)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_2)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ | 320 |
| **Definition of** $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$:- | 321 |

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$, ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,

and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously

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$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$$

$(p_i)^{(3)}$ is defined by equation

$$\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t} \quad (323)$$

$$\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$$

$$\left. \frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right) \quad (324)$$

$$\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}} \quad (325)$$

$$\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t} \quad (326)$$

$$\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$$

$$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t} \quad (327)$$

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- (328)

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$$

$$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$$

$$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}\big((G_{27}), t\big) - (b''_{25})^{(4)}\big((G_{27}), t\big) \le -(\tau_1)^{(4)}$$

**Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : 329

By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{v}_1)^{(4)} > 0$, $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (v_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}, \ \textbf{\textit{if}}\ (v_0)^{(4)} < (v_1)^{(4)}$$

$$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}, \textbf{\textit{if}}\ (v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)},$$

and $\boxed{(v_0)^{(4)} = \frac{G^0_{24}}{G^0_{25}}}$

$$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}, \ \textbf{\textit{if}}\ (\bar{v}_4)^{(4)} < (v_0)^{(4)}$$

and analogously 331

$$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}, \ \textbf{\textit{if}}\ (u_0)^{(4)} < (u_1)^{(4)}$$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}, \textbf{\textit{if}}\ (u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)},$$

and $\boxed{(u_0)^{(4)} = \frac{T^0_{24}}{T^0_{25}}}$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \textbf{\textit{if}}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)} \text{ where } (u_1)^{(4)}, (\bar{u}_1)^{(4)}$$

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| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}')^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}')^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)}t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(\nu_1)^{(5)}, (\nu_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, \nu^{(5)}, u^{(5)}$ :<br>By $(\nu_1)^{(5)} > 0$, $(\nu_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}(\nu^{(5)})^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)} = 0$ | 339 |

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| and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ :<br><br>By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)},(v_0)^{(5)}$ :-<br><br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)}$ by<br><br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br><br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br><br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |

$\frac{(b_{30})^{(5)}T_{28}^{0}}{(\mu_1)^{(5)}((R_1)^{(5)}-(b_{30}^{'})^{(5)})}\left[e^{(R_1)^{(5)}t}-e^{-(b_{30}^{'})^{(5)}t}\right]+T_{30}^{0}e^{-(b_{30}^{'})^{(5)}t}\leq T_{30}(t)\leq$

$\frac{(a_{30})^{(5)}T_{28}^{0}}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t}-e^{-(R_2)^{(5)}t}\right]+T_{30}^{0}e^{-(R_2)^{(5)}t}$ 347

**Definition of** $(S_1)^{(5)},(S_2)^{(5)},(R_1)^{(5)},(R_2)^{(5)}$:- 348

Where $(S_1)^{(5)}=(a_{28})^{(5)}(m_2)^{(5)}-(a_{28}^{'})^{(5)}$

$(S_2)^{(5)}=(a_{30})^{(5)}-(p_{30})^{(5)}$

$(R_1)^{(5)}=(b_{28})^{(5)}(\mu_2)^{(5)}-(b_{28}^{'})^{(5)}$

$(R_2)^{(5)}=(b_{30}^{'})^{(5)}-(r_{30})^{(5)}$

**Behavior of the solutions of equation** 349

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ four constants satisfying

$-(\sigma_2)^{(6)}\leq-(a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}-(a_{32}^{''})^{(6)}(T_{33},t)+(a_{33}^{''})^{(6)}(T_{33},t)\leq-(\sigma_1)^{(6)}$

$-(\tau_2)^{(6)}\leq-(b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(b_{32}^{''})^{(6)}\left((G_{35}),t\right)-(b_{33}^{''})^{(6)}\left((G_{35}),t\right)\leq-(\tau_1)^{(6)}$

**Definition of** $(\nu_1)^{(6)},(\nu_2)^{(6)},(u_1)^{(6)},(u_2)^{(6)},\nu^{(6)},u^{(6)}$ : 350

By $(\nu_1)^{(6)}>0$ , $(\nu_2)^{(6)}<0$ and respectively $(u_1)^{(6)}>0$ , $(u_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_1)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2+(\tau_1)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$ and

**Definition of** $(\bar{\nu}_1)^{(6)},,(\bar{\nu}_2)^{(6)},(\bar{u}_1)^{(6)},(\bar{u}_2)^{(6)}$ : 351

By $(\bar{\nu}_1)^{(6)}>0$ , $(\bar{\nu}_2)^{(6)}<0$ and respectively $(\bar{u}_1)^{(6)}>0$ , $(\bar{u}_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_2)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2+(\tau_2)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$

**Definition of** $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)},(\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)}$ by

$(m_2)^{(6)}=(\nu_0)^{(6)},(m_1)^{(6)}=(\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)}<(\nu_1)^{(6)}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)}<(\nu_0)^{(6)}<(\bar{\nu}_1)^{(6)}$,

and $\boxed{(\nu_0)^{(6)}=\frac{G_{32}^{0}}{G_{33}^{0}}}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)}<(\nu_0)^{(6)}$

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| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},\ \boldsymbol{if}\ (u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}, \boldsymbol{if}\ (u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)},$<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}, \boldsymbol{if}\ (\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le\right.$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b_{34}')^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b_{34}')^{(6)} t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$<br>$(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ : | |

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| $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying<br><br>$-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$<br><br>$-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}((G_{39}), t) - (b''_{37})^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ :<br><br>By $(\nu_1)^{(7)} > 0, (\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ :<br><br>By $(\bar{\nu}_1)^{(7)} > 0, (\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$<br>**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-<br><br>If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by<br><br>$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,<br>and $\boxed{(\nu_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | 362 |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,<br>and $\boxed{(u_0)^{(7)} = \frac{T^0_{36}}{T^0_{37}}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities | 364 |

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| $G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^{0}e^{(S_1)^{(7)}t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | |
| $\frac{1}{(m_7)^{(7)}} G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^{0}e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^{0}e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}^{'})^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}^{'})^{(7)}t}] + G_{38}^{0}e^{-(a_{38}^{'})^{(7)}t})$ | 366 |
| $\boxed{T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)}T_{36}^{0}}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}^{'})^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}^{'})^{(7)}t}\right] + T_{38}^{0}e^{-(b_{38}^{'})^{(7)}t} \le T_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)}T_{36}^{0}}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^{0}e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}^{'})^{(7)}$<br><br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br><br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}^{'})^{(7)}$<br><br>$(R_2)^{(7)} = (b_{38}^{'})^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br><br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br><br>$-(\sigma_2)^{(8)} \le -(a_{40}^{'})^{(8)} + (a_{41}^{'})^{(8)} - (a_{40}^{''})^{(8)}(T_{41}, t) + (a_{41}^{''})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br><br>$-(\tau_2)^{(8)} \le -(b_{40}^{'})^{(8)} + (b_{41}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}), t) - (b_{41}^{''})^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |
| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : | 372 |

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| By $(\nu_1)^{(8)} > 0\,, (\nu_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0\,, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$ and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(8)},, (\bar{\nu}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :<br><br>By $(\bar{\nu}_1)^{(8)} > 0\,, (\bar{\nu}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0\,, (\bar{u}_2)^{(8)} < 0$ the<br><br>roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$<br><br>and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$<br><br>**Definition of** $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (\nu_0)^{(8)}$ :-<br><br>If we define $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by<br><br>$(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)},$ ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$<br><br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)},$<br>and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br><br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)},$ ***if*** $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br><br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},$ ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)},$ ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)}t}$<br><br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$ | 376 |

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| $\left(\frac{(a_{42})^{(8)}G_{40}^{0}}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t}-e^{-(S_2)^{(8)}t}\right]+G_{42}^{0}e^{-(S_2)^{(8)}t}\leq G_{42}(t)\leq \frac{(a_{42})^{(8)}G_{40}^{0}}{(m_2)^{(8)}((S_1)^{(8)}-(a_{42}^{'})^{(8)})}[e^{(S_1)^{(8)}t}-e^{-(a_{42}^{'})^{(8)}t}]+G_{42}^{0}e^{-(a_{42}^{'})^{(8)}t}\right)$ | 377 |
| $\boxed{T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}}T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq \frac{1}{(\mu_2)^{(8)}}T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^{0}}{(\mu_1)^{(8)}((R_1)^{(8)}-(b_{42}^{'})^{(8)})}\left[e^{(R_1)^{(8)}t}-e^{-(b_{42}^{'})^{(8)}t}\right]+T_{42}^{0}e^{-(b_{42}^{'})^{(8)}t}\leq T_{42}(t)\leq$ $\frac{(a_{42})^{(8)}T_{40}^{0}}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t}-e^{-(R_2)^{(8)}t}\right]+T_{42}^{0}e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)},(S_2)^{(8)},(R_1)^{(8)},(R_2)^{(8)}$:-<br>Where $(S_1)^{(8)}=(a_{40})^{(8)}(m_2)^{(8)}-(a_{40}^{'})^{(8)}$<br>$(S_2)^{(8)}=(a_{42})^{(8)}-(p_{42})^{(8)}$<br>$(R_1)^{(8)}=(b_{40})^{(8)}(\mu_2)^{(8)}-(b_{40}^{'})^{(8)}$<br>$(R_2)^{(8)}=(b_{42}^{'})^{(8)}-(r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)}\leq -(a_{44}^{'})^{(9)}+(a_{45}^{'})^{(9)}-(a_{44}^{''})^{(9)}(T_{45},t)+(a_{45}^{''})^{(9)}(T_{45},t)\leq -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)}\leq -(b_{44}^{'})^{(9)}+(b_{45}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47}),t)-(b_{45}^{''})^{(9)}((G_{47}),t)\leq -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(\nu_1)^{(9)},(\nu_2)^{(9)},(u_1)^{(9)},(u_2)^{(9)},\nu^{(9)},u^{(9)}$ :<br>By $(\nu_1)^{(9)}>0$ , $(\nu_2)^{(9)}<0$ and respectively $(u_1)^{(9)}>0$ , $(u_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_1)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$<br>and $(b_{45})^{(9)}(u^{(9)})^2+(\tau_1)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(9)},,(\bar{\nu}_2)^{(9)},(\bar{u}_1)^{(9)},(\bar{u}_2)^{(9)}$ :<br>By $(\bar{\nu}_1)^{(9)}>0$ , $(\bar{\nu}_2)^{(9)}<0$ and respectively $(\bar{u}_1)^{(9)}>0$ , $(\bar{u}_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_2)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$ | |

and $(b_{45})^{(9)}\left(u^{(9)}\right)^2+(\tau_2)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$

**Definition of** $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)},(\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)}$ by

$(m_2)^{(9)}=(\nu_0)^{(9)},(m_1)^{(9)}=(\nu_1)^{(9)},$ ***if*** $(\nu_0)^{(9)}<(\nu_1)^{(9)}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)}<(\nu_0)^{(9)}<(\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)}=\frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\nu_0)^{(9)},$ ***if*** $(\bar{\nu}_1)^{(9)}<(\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)}=(u_0)^{(9)},(\mu_1)^{(9)}=(u_1)^{(9)},$ ***if*** $(u_0)^{(9)}<(u_1)^{(9)}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)}<(u_0)^{(9)}<(\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)}=\frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(u_0)^{(9)},$ ***if*** $(\bar{u}_1)^{(9)}<(u_0)^{(9)}$ where $(u_1)^{(9)},(\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{44}(t)\le G_{44}^0e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{45}(t)\le\frac{1}{(m_2)^{(9)}}G_{44}^0e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)}G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^0e^{-(S_2)^{(9)}t}\le G_{46}(t)\le\right.$$

$$\left.\frac{(a_{46})^{(9)}G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}')^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}')^{(9)}t}]+\ G_{46}^0e^{-(a_{46}')^{(9)}t}\right)$$

$$\boxed{T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le\frac{1}{(\mu_2)^{(9)}}T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}')^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}')^{(9)}t}\right]+T_{46}^0e^{-(b_{46}')^{(9)}t}\le T_{46}(t)\le$$

$$\frac{(a_{46})^{(9)}T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^0e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain (383)

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right) - (a''_{14})^{(1)}(T_{14}, t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get (384)

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{v}_1)^{(1)}$ we find like in the previous case, (385)

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)}G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)}T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$$

If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain 394

$$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(2)}(t)$ :- 395

$$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}, \quad \boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$$

In a completely analogous way, we obtain 396

**Definition of** $u^{(2)}(t)$ :-

$$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}, \quad \boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :** 397

If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$

Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then

$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$ if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$

**Proof** : From global equations we obtain 398

$$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$$

**Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$ 399

It follows

$$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$$

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From which one obtains

For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)} + (C)^{(3)}(\nu_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

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In the same manner , we get

$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)} + (\bar{C})^{(3)}(\bar{\nu}_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, 402

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

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$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right) - (a_{25}^{''})^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\quad \boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

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$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} =$

$(\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\quad \boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get 413

$$v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{v}_1)^{(6)}-(v_0)^{(6)}}{(v_0)^{(6)}-(\bar{v}_2)^{(6)}}}$$

From which we deduce $(v_0)^{(6)} \le v^{(6)}(t) \le (\bar{v}_1)^{(6)}$

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If $0 < (v_1)^{(6)} < (v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{v}_1)^{(6)}$ we find like in the previous case,

$$(v_1)^{(6)} \le \frac{(v_1)^{(6)}+(C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}} \le v^{(6)}(t) \le$$

$$\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (\bar{v}_1)^{(6)}$$

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If $0 < (v_1)^{(6)} \le (\bar{v}_1)^{(6)} \le \boxed{(v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(v_1)^{(6)} \le v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (v_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$(m_2)^{(6)} \le v^{(6)}(t) \le (m_1)^{(6)}$, $\boxed{v^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \le u^{(6)}(t) \le (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(v_1)^{(6)} = (\bar{v}_1)^{(6)}$ if in addition $(v_0)^{(6)} = (v_1)^{(6)}$ then $v^{(6)}(t) = (v_0)^{(6)}$ and as a consequence $G_{32}(t) = (v_0)^{(6)}G_{33}(t)$ **this also defines** $(v_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(v_1)^{(6)}$ and $(\bar{v}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right) - (a_{37}^{''})^{(7)}(T_{37},t)v^{(7)} - (a_{37})^{(7)}v^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get 417

$$\nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\bar{\nu}_1)^{(7)}$

If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case, 418

$$(\nu_1)^{(7)} \leq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}} \leq \nu^{(7)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\bar{\nu}_1)^{(7)}$$

If $0 < (\nu_1)^{(7)} \leq (\bar{\nu}_1)^{(7)} \leq \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain 419

$$(\nu_1)^{(7)} \leq \nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \leq \nu^{(7)}(t) \leq (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :- 420

$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)} G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain 421

$$\frac{d\nu^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right) - (a_{41}^{''})^{(8)}(T_{41}, t)\nu^{(8)} - (a_{41})^{(8)}\nu^{(8)}$$

**Definition of** $\nu^{(8)}$ :- $\boxed{\nu^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right) \leq \frac{d\nu^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)}-(\nu_0)^{(8)}}{(\nu_0)^{(8)}-(\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \le \nu^{(8)}(t) \le (\bar{\nu}_8)^{(8)}$

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If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case,

$$(\nu_1)^{(8)} \le \frac{(\nu_1)^{(8)}+(C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}} \le \nu^{(8)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\bar{\nu}_1)^{(8)}$$

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If $0 < (\nu_1)^{(8)} \le (\bar{\nu}_1)^{(8)} \le \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$, we obtain

$$(\nu_1)^{(8)} \le \nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \le \nu^{(8)}(t) \le (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \le u^{(8)}(t) \le (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)}G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)}T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right) - (a''_{45})^{(9)}(T_{45},t)v^{(9)} - (a_{45})^{(9)}v^{(9)}$$

**Definition of** $v^{(9)}$ :- $\boxed{v^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)}\right) \le \frac{dv^{(9)}}{dt} \le -\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(9)}, (v_0)^{(9)}$ :-

For $0 < \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (v_1)^{(9)} < (\bar{v}_1)^{(9)}$

$$v^{(9)}(t) \ge \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(v_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (v_2)^{(9)}}}$$

it follows $(v_0)^{(9)} \le v^{(9)}(t) \le (v_9)^{(9)}$

In the same manner , we get

$$v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{v}_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (\bar{v}_2)^{(9)}}}$$

From which we deduce $(v_0)^{(9)} \le v^{(9)}(t) \le (\bar{v}_1)^{(9)}$

If $0 < (v_1)^{(9)} < (v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}} < (\bar{v}_1)^{(9)}$ we find like in the previous case,

$$(v_1)^{(9)} \le \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}} \le v^{(9)}(t) \le$$

$$\frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (\bar{v}_1)^{(9)}$$

If $0 < (v_1)^{(9)} \le (\bar{v}_1)^{(9)} \le \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$ , we obtain

$$(v_1)^{(9)} \le v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (v_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(9)}(t)$ :-

$$(m_2)^{(9)} \le v^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{v^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}$, $\boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ 430

$with\ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations 431

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$

| | |
|---|---|
| $with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | |
| We can prove the following<br><br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br><br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br><br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br><br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$ | 435 |

| | |
|---|---|
| $with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |

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|---|---|
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27})\big)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\big((G_{27})\big)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\big((G_{27})\big)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |

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|---|---|
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |

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| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(r) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) +$ | 489 |

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| $(a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that | 496 |

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| there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{32}=\frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}=\frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{36}=\frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}=\frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |
| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*})=0$. With this value , we obtain from the three first equations | 501 |

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| $G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G) + (b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*}) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19}) + (b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br>$[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ | 506 |

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| is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution | 511 |

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| $G_{14}^{*}$ given by $\varphi(G^{*})=0$ , $T_{14}^{*}$ given by $f(T_{14}^{*})=0$ and<br>$G_{13}^{*}=\frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*}=\frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br>$T_{13}^{*}=\frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*}=\frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br>Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*})=0$ , $T_{17}^{*}$ given by $f(T_{17}^{*})=0$ and | 512 |
| $G_{16}^{*}=\frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*}=\frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*}=\frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*}=\frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*})=0$ , $T_{21}^{*}$ given by $f(T_{21}^{*})=0$ and<br>$G_{20}^{*}=\frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*}=\frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br>$T_{20}^{*}=\frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*}=\frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^{*}$ given by $\varphi(G_{27})=0$ , $T_{25}^{*}$ given by $f(T_{25}^{*})=0$ and<br>$G_{24}^{*}=\frac{(a_{24})^{(4)}G_{25}^{*}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*}=\frac{(a_{26})^{(4)}G_{25}^{*}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*}=\frac{(b_{24})^{(4)}T_{25}^{*}}{[(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*}=\frac{(b_{26})^{(4)}T_{25}^{*}}{[(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}((G_{27})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*})=0$ , $T_{29}^{*}$ given by $f(T_{29}^{*})=0$ and<br>$G_{28}^{*}=\frac{(a_{28})^{(5)}G_{29}^{*}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*}=\frac{(a_{30})^{(5)}G_{29}^{*}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |

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| $T_{28}^{*} = \frac{(b_{28})^{(5)}T_{29}^{*}}{[(b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)}T_{29}^{*}}{[(b_{30}^{'})^{(5)}-(b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br>$G_{32}^{*} = \frac{(a_{32})^{(6)}G_{33}^{*}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)}G_{33}^{*}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)}T_{33}^{*}}{[(b_{32}^{'})^{(6)}-(b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)}T_{33}^{*}}{[(b_{34}^{'})^{(6)}-(b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br>$G_{36}^{*} = \frac{(a_{36})^{(7)}G_{37}^{*}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)}G_{37}^{*}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br>$T_{36}^{*} = \frac{(b_{36})^{(7)}T_{37}^{*}}{[(b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)}T_{37}^{*}}{[(b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br>$G_{40}^{*} = \frac{(a_{40})^{(8)}G_{41}^{*}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)}G_{41}^{*}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br>$T_{40}^{*} = \frac{(b_{40})^{(8)}T_{41}^{*}}{[(b_{40}^{'})^{(8)}-(b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)}T_{41}^{*}}{[(b_{42}^{'})^{(8)}-(b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)}G_{45}^{*}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)}G_{45}^{*}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)}T_{45}^{*}}{[(b_{44}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)}T_{45}^{*}}{[(b_{46}^{'})^{(9)}-(b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | 523 A |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^*) = (q_{14})^{(1)} \quad , \frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^*) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)}G_{13}^*\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^*\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^*\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^*\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^*\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^*\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)} \quad , \frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |

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| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ $and$ $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)}$ , $\frac{\partial(b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_{j}\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_{j}\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_{j}\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 548 |

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| $\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^{*}) = (q_{25})^{(4)}$ , $\frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^{*}) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^{*}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^{*}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^{*}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^{*}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^{*}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^{*}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^{*}) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^{*}) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^{*}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^{*}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^{*}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^{*}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^{*}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^{*}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 563 |

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| **Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ $and$ $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^*\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^*\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^*\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^*\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ $and$ $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}((G_{39})^{**}) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |

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| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i'')^{(8)}$ *and* $(b_i'')^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{41})^{(8)}}{\partial T_{41}}(T^*_{41}) = (q_{41})^{(8)}$ , $\frac{\partial(b_i'')^{(8)}}{\partial G_j}((G_{43})^*) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a'_{40})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G^*_{40}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a'_{41})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G^*_{41}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a'_{42})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G^*_{42}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i'')^{(9)}$ *and* $(b_i'')^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |

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| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \ \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}(\ (G_{47})^*\ ) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b_{44}^{'})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T_{44}^*\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b_{45}^{'})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T_{45}^*\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b_{46}^{'})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T_{46}^*\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)$<br>$\left[\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(q_{13})^{(1)}G_{13}^*\right)\Big]$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T_{14}^* + (b_{14})^{(1)}s_{(13),(14)}T_{14}^*\Big)$<br>$+\left(\left((\lambda)^{(1)} + (a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^* + (a_{13})^{(1)}(q_{14})^{(1)}G_{14}^*\right)$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^* + (b_{14})^{(1)}s_{(13),(13)}T_{13}^*\Big)$<br>$\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$<br>$\left(\left((\lambda)^{(1)}\right)^2 + \left((b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$<br>$+\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$<br>$+\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^*\right)$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^* + (b_{14})^{(1)}s_{(13),(15)}T_{13}^*\Big)\} = 0$ | |

$$+$$

$$((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)})\{((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)})$$

$$\left[\left(((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)})(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)\right]$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(17)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^{*}_{17}\right)$$

$$+\left(((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)})(q_{16})^{(2)}\mathrm{G}^{*}_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(16)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^{*}_{16}\right)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left(\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(18)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(18)}\mathrm{T}^{*}_{16}\right)\} = 0$$

$$+$$

$$((\lambda)^{(3)} + (b'_{22})^{(3)} - (r_{22})^{(3)})\{((\lambda)^{(3)} + (a'_{22})^{(3)} + (p_{22})^{(3)})$$

$$\left[\left(((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)})(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)\right]$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(21)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(21)}T^{*}_{21}\right)$$

$$+\left(((\lambda)^{(3)} + (a'_{21})^{(3)} + (p_{21})^{(3)})(q_{20})^{(3)}G^{*}_{20} + (a_{20})^{(3)}(q_{21})^{(1)}G^{*}_{21}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(20)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(20)}T^{*}_{20}\right)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((b'_{20})^{(3)} + (b'_{21})^{(3)} - (r_{20})^{(3)} + (r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left(\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(22)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(22)}T^{*}_{20}\right)\} = 0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\right)\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\right)\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\right)\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\right)\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)\right]$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(41)}T^{*}_{41}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^{*}_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^{*}_{41}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(40)}T^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$+\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(42)}T^{*}_{40}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\right)\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)\right]$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(45)}T^{*}_{45}\right)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^{*}_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^{*}_{45}\right)$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(44)}T^{*}_{44}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^{2}+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$+\left(\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)$$

$$\left(\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\right)\} = 0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i \in R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i\ X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor

bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k \tag{18}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 18. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,...x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \in I, x \in K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow *x.t restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\,\delta\,(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\mathcal{A}(\tilde{A}), C\backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population *i* is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int_0^t (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\to\infty}{limsup}\frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \epsilon R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v=au_i(x)+w$ with $a \epsilon R$ and $w \epsilon F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |B_i(t,x)u_i(x)|(a+ \alpha+ exp(-\beta t)|w|)$. Hence, $\underset{t\to\infty}{limsup}\frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\underset{t\to\infty}{limsup}\frac{1}{t} In\ B_i(t,x)\ u_i(x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \underset{t\to\infty}{limsup}\frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a- \alpha+ exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \underset{t\to\infty}{limsup}\frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \underset{t\to\infty}{limsup}\frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In } b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) =0$ implies

$$\frac{d}{dt}b_i(t,x) = \lim_{s\to 0}\frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

=

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \le 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \le \gamma$ for all $t \ge 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } |\, (x.t)_i\, |$$

$$\le \limsup_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0. If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\ge 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x) =$ Integral $h(x.t)d\,\mu(x)$for all $t \ge 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure*. $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$ , $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x) \rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1 \le j \le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |\mathrm{h}(x.t) - h(x^n.t)|d\mu_n(x)| + |\int_c h(x^n.t)|d\mu_n(x) - \int_c \mathrm{h}(x)d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x)d\mu_n(x) - \int_c h(x)d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $\mathrm{h}_i^n(x) = \langle \mathrm{A}_i^n(x)\mathrm{u}_i^n(x), \mathrm{u}_i^n(x)\rangle$ where $\mathrm{u}_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $\mathrm{A}_i^n(x) \to \mathrm{A}_i(x)$ uniformly for $x \in N_1((\mathrm{A}))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $\mathrm{u}_i^n(x) \to \mathrm{u}_i(x)$ converges uniformly for $x \in N_1(\Lambda(\mathrm{A}))$. Given $\in > 0$, choose $n$ sufficiently large so that $|\mathrm{h}_i^n(x) - \mathrm{h}_i(x)| \leq \in$ for all $x \in N_1(\Lambda(\mathrm{A}))$ and $|\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$\lambda_i(\mu) \leq \int_c |\mathrm{h}_i(x) - \mathrm{h}_i^n(x)|d\mu(x) + |\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)|$

$+ \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)$

$\leq \quad 2\in$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\, \mathrm{ds}$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument

implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i \, (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

(6) $\min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$ $\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_{1,}\ldots.,\tilde{A}_k)$ by

$$\tilde{A}_i(x) = \begin{cases} \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} & if\ i \epsilon I \\ if\ i \notin I \end{cases}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set $K$ if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$?then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$?and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume $x \mapsto A(x)$ is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists $j \in \{1,\ldots,k\}$ such that $M_i$ ----- $\{x \in C : x_l \gg 0 \text{ for } l \neq j\}$ Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_i M_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup $(\mu)$ to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i\in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures μ with support in ∂C, $\lambda_i(\mu)>0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $$\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j \qquad i = 1,..,k \qquad j=1,\ldots,m$$

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge

to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j$ ($x^j_i$) = 0 i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product ($E_1^1$ ,…,$E_1^m$), and the matrix $A^2$ ($E_1$) is close to the diagonal matrix diag($f^i_2(E_1{}^j)$)$^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f_2^j(E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial$C is given by two equilibria {0,$E_1$}, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j\ x_2^j\ - \alpha^j\ x_3^j\ ) + \sum d_1^{jk}\ \ x_1^k\ - e_1^j\ x_1^j$

$\frac{dx_2^j}{dt} = x_2^j(1 - \alpha^j\ x_1^j\ - x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_2^{jk}\ \ x_2^k\ - e_2^j\ x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1 - \beta^j\ x_1^j\ - \alpha^j x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_3^{jk}\ \ x_3^k\ - e_3^j\ x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2\ (E_1)$and $\lambda_3\ (E_1)$are close to $\max_j\ 1 - a^j$ and $\max_j\ 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j\ 1 - a^j$ and $\max_j\ 1 - \beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1 - a^j) > -\max_j\ (1 - \beta^j)$$

Equivalently,

$$2 > \min_j\ a^j + \min_j\ \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \ \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i\left(r_i - \sum_s a_{is}x_s\right), \qquad i = 1,\dots,k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$$

It has been shown [12,ch.16.1,16.2] That for $k = 3$,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \ \ with\ \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1 \to E_2 \to E_3 \to E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within $\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing

holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & O \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left[ \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1 - y) - b(N)\ y \right]$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*′ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon+x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords

Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation

of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1$ *(t)* and $P_2$ *(t)* denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial

alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da$ $(a \geq 0)$ at time $t$ so that we have

(1.52) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.53) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.54) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_{1,}$ $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_{i,}$ $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic ($as$ $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 18. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, \text{R}^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1, 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) \; P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^\infty f_i(a + s, P(t + S)) ds \right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) \, exp\left[ - \int_o^x f_1(a + s, p(t + s)) ds \right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0) \exp\left[ - \int_o^t f_1(a - t + s, p_1(s)) ds \right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(\text{a+s}, p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da+\int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$$t> 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \epsilon[-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.(a, t, P_i(t))da$

$$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$$

$$i = 1, 2, t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t) \left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t) \phi(0) d_i^*]t \quad for\ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T],R^+), \quad B_i \in C([0,T],R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau,T] = \{f \in C([-\tau,T],R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau,T]$. Then for any

$(x, y) \in C^+[-\tau,T] \times C^+[-\tau,T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau,T], \quad C_T^2 = C^+[-\tau,T] \times C^+[-\tau,T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau,T]$ *and* $t \in [0,T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$B: C_T^2 \to C_T^2$ *and* $\Phi C_T^2 \to C_T^2$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = P_t \left\{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

$$\|\Phi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right)$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \le \int_0^t B_1^{(1)}(a) \left\{ M_1\left(a, t, P_t^{(1)}\right) - M_1\left(a, t, P_t^{(2)}\right) \right\} da$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left( \int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_* ds \right) da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o) \left[ \int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds \right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \|_t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$=\int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\mathrm{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \left\| P_t^{(1)} - P_t^{(2)} \right\| \text{ for } a \in [o, t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \leq \beta_1 \Phi_1(0) ||P_t^{(1)} - P_t^{(2)}||_{t^*} + \Phi_1(0) \beta_1 d_1 ||P_t^{(1)} - P_t^{(2)}||_{t^*} a,$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) e^{\delta 1^a} \beta_1 \left\| P_t^{(1)} - P_t^{(2)} \right\|_{t^*} + \beta_1 d_1 \left\| P_t^{(1)} - P_t^{(2)} \right\|_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) e^{\delta 1^a} \left\| P_t^{(1)} - P_t^{(2)} \right\|_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| B_1^* d\sigma$$

$$+ \Phi_1(0) \left\| P_t^{(1)} - P_t^{(2)} \right\|_{t^*} \left\{ B_1 + B_2 d_1 a + B_1^* \left(B_2 a + d_1 a^2/2\right) e^{\delta 1^a} \right\}$$

$$\leq \beta_1^* \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| do + \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \rightarrow \beta_1$ as $a \rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{g_1(a) + 0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a)) - b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a)) do \right|$$

$$\left| B_1^* \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} 0(a) as\ a \rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi (P_t^{(1)} - \pi \left(P_t^{(2)}\right) /\!/ \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^*)$$

As $t^* \rightarrow o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp P_2^*(a)}{da} + f_i(a, P_1^*, P_2^*) P_i^*(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^* = \int_o^\infty P_i^*(a) da, i = 1,\ 2,$$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i=1.\ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s,P_1^*,P_2^*)ds\right], i=1,\quad 2, a\geq 0,$

Which together with (3.3) lead to

$$1=\int_o^\infty b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^\infty b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1,P_2) = \int_o^\infty b_i(a,P_1,P_2)\exp\left[-\int_o^\infty f_i(s,P_1,P_2)ds\right]da,$

$i=1,\ 2,\quad P_1\geq 0,\quad P_2\geq 0,$

**And suppose the following hold:**

(3.7) $A_4.\quad F_{i.}\ (0,\ 0)>1,\ i=1,2$:

$A_5.$ *there exists a positive constant c for which*

$$F_1(C,0)< F_{2.}(C,0),$$

(3.8) $$F_1(0.C)< F_{2.}(0,C),$$

Then there exists a unique pair $(P_1^*,P_2^*)$ of real numbers $P_1^*>o,$

$P_2^*>0$ such that

(3,9) $F_1(P_1^*,P_2^*)=1=F_2(P_1^*,P_2^*)$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a)=[P_2^*\exp[-f_i(s,P_1^*,P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s,P_1^*,P_2^*)ds]da),$

$i=1,2,\quad a\geq 0,$

Proof Consider the elements of the surfaces defined by

Z= $F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0,$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

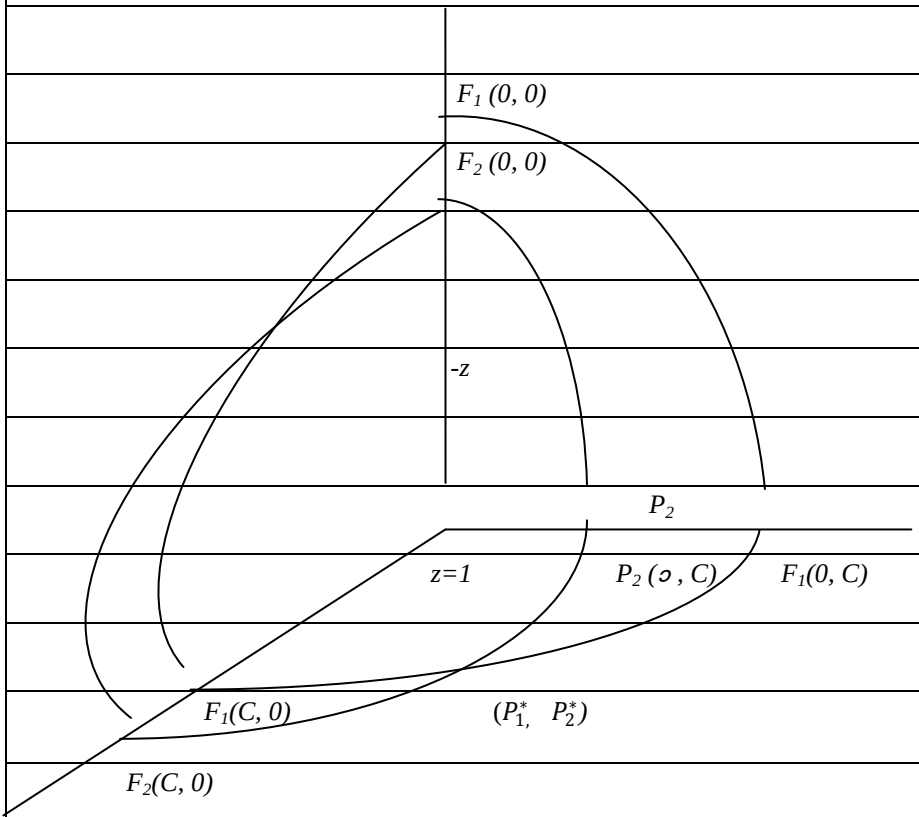
$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0,$ These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, F_1(P_1, P_2 \geq 0,$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11) $$\begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12) $$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square=1}^{2} \frac{\partial b_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}da \end{cases}$$

Where $i$ = 1,2 and (3.13) $$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14) $$\begin{cases} u_i(a,t) = u\square(a) \exp[\lambda t], \quad \square(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty \square(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d\square}{da} + \lambda\square = -f_i\left(a, p_1^*, p_2^*\right)\square(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_{i\square}^*\right\},$$

(3.15)

$$\square(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\square(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*} p_{i\square}^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^{*} = p_{1}^{*}, \quad p_{12}^{*} = p_{2}^{*} \exp\{-\lambda\tau\},$$

$$p_{21}^{*} = p_{1}^{*}, \exp\{-\lambda\tau\}, \ p_{22}^{*} = p_{2}^{*}.$$

If we integrate the first of (3.15) and use the fact that $\mu_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\mu_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\mu_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\mu_i(a) = \{\mu_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\mu_i\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\mu_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\mu_i\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\mu_i\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\mu_i\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)}\int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}da, \quad i = 1,\ 2,$$

Now supplying $\mu_1(a)$ and $\mu_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^*\exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^*\exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^* \exp(-\lambda\tau) + (\lambda - A_{22})\, p_2^* = g_{21}(\lambda)\, p_1^* \exp(-\lambda\tau) + g_{22}(\lambda)\, p_2^*$$

Where

(3.22) $$A_{i} = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_{}^*} - \frac{\partial f_i}{\partial p_{}^*}\right]da,$$

(3.23) $$g_{i}(\lambda) = \left[p_i^*(0)c_i\right]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\, da\right\}$$
$$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_{}^*} - b_i(a, p_1^*, p_2^*)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{}^*}\, e^{\lambda s}\, ds\right)\right]da\right\}$$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{}^*}\, e^{\lambda s}\, ds\right)da, \quad i = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*), \quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})\, e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$$S_2(\lambda, \tau) = \left[A_{21}\, g_{12}(\lambda) + A_{12}\, g_{21}(\lambda) + g_{12}(\lambda)\, g_{21}(\lambda)\right]\exp[-2\lambda\tau]$$

$$-A_{11}\, g_{22}(\lambda) - A_{22}\, g_{11}(\lambda) - g_{11}(\lambda)\, g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6$. $0 < \beta_i^* < d_i^*$ $(i = 1, 2.)$ $(see(A_1) and (A_2))$;

$A_7$. $\{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \text{[illegible]}(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \left|\right.$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \,| - |\, A_{11} + A_{22} + \| S_1(0) \| \,|^2 \right\}/2,$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational

ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the

Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have

obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is:

individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## SECTION A(16)

## Yang-Mills Condensate As Dark Energy

### INTRODUCTION—VARIABLES USED

**Yang-Mills condensate as dark energy: A nonperturbative approach Pietro Donà, Antonino Marcianò, Yang Zhang, and Claudia Antolini Phys. Rev. D 93, 043012 – Published 19 February 2016**

(1) Models based on the Yang-Mills condensate (YMC) have been advocated for in the literature and claimed as (=) successful candidates for explaining dark energy.
(2) Several variations on this simple idea have been considered, the most promising of which are reviewed here. Nevertheless, the previously attained results relied heavily on (e) the perturbative approach to (e) the analysis of (e) the effective Yang-Mills action, which is (=) only adequate in (eb) the asymptotically free limit, and were extended into (e&eb) a regime, the infrared limit, in which confinement is expected.
(3) Authors show that if a minimum of the effective Lagrangian in $\theta$=−Fa μνFaμν/2 exists, (eb) an YMC forms that drives (eb) the Universe toward an (e&eb) accelerated de Sitter phase.
(4) The details of the models depend weakly on (=) the specific form of (e) the effective Yang-Mills

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| Lagrangian.<br>(5) Using (e) nonperturbative techniques mutated from (e) the functional renormalization-group procedure; authors finally show that (eb) the minimum in θ of (e) the effective Lagrangian exists.<br>(6) Thus, an YMC can actually take place.<br>(7) The nonperturbative model has properties similar to (=) the ones in the perturbative model.<br>**(8)** In the early stage of the Universe, the YMC equation of state has (e) an evolution that resembles (=) the radiation component, i.e., wy→1/3. | |
| **NOTATION** | |
| **Module One**<br>Models based on the Yang-Mills condensate (YMC) have been advocated for in the literature and claimed as (=) successful candidates for explaining dark energy | |
| $G_{13}$ : Category one of Models based on the Yang-Mills condensate (YMC) have been advocated for in the literature and claimed<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of successful candidates for explaining dark energy<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>Several variations on this simple idea have been considered, the most promising of which are reviewed here.<br>Nevertheless, the previously attained results relied heavily on (e) the perturbative approach to (e) the analysis of (e) the effective Yang-Mills action, which is (=) only adequate in (eb) the asymptotically free limit, and were extended into (e&eb) a regime, the infrared limit, in which confinement is expected | |
| $G_{16}$ : Category one of perturbative approach to (e) the analysis of (e) the effective Yang-Mills action, which is (=) only adequate in (eb) the asymptotically free limit, and were extended into (e&eb) a regime, the infrared limit, in which confinement is expected<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of previously attained results<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>Nevertheless, the previously attained results relied heavily on the perturbative approach to (e) the analysis of (e) the effective Yang-Mills action, which is (=) only adequate in (eb) the asymptotically free limit, and | |

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| were extended into (e&eb) a regime, the infrared limit, in which confinement is expected | |
| $G_{20}$ : Category one of analysis of (e) the effective Yang-Mills action, which is (=) only adequate in (eb) the asymptotically free limit, and were extended into (e&eb) a regime, the infrared limit, in which confinement is expected<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of previously attained results relied heavily on the perturbative approach<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Nevertheless, the previously attained results relied heavily on the perturbative approach to the analysis of (e) the effective Yang-Mills action, which is (=) only adequate in (eb) the asymptotically free limit, and were extended into (e&eb) a regime, the infrared limit, in which confinement is expected | |
| $G_{24}$ : Category one of effective Yang-Mills action, which is (=) only adequate in (eb) the asymptotically free limit, and were extended into (e&eb) a regime, the infrared limit, in which confinement is expected<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of the previously attained results relied heavily on the perturbative approach to the analysis<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Nevertheless, the previously attained results relied heavily on the perturbative approach to the analysis of the effective Yang-Mills action, which is (=) only adequate in (eb) the asymptotically free limit, and were extended into (e&eb) a regime, the infrared limit, in which confinement is expected | |
| $G_{28}$ : Category one of previously attained results relied heavily on the perturbative approach to the analysis of the effective Yang-Mills action<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of adequate in (eb) the asymptotically free limit, and were extended into (e&eb) a regime, the infrared limit, in which confinement is expected<br><br>$T_{29}$ : Category two of SAS | |

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| $T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Nevertheless, the previously attained results relied heavily on the perturbative approach to the analysis of the effective Yang-Mills action, which is only adequate in (eb) the asymptotically free limit, and were extended into (e&eb) a regime, the infrared limit, in which confinement is expected | |
| $G_{32}$ : Category one of previously attained results relied heavily on the perturbative approach to the analysis of the effective Yang-Mills action, which is only adequate<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of asymptotically free limit, and were extended into (e&eb) a regime, the infrared limit, in which confinement is expected<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Nevertheless, the previously attained results relied heavily on the perturbative approach to the analysis of the effective Yang-Mills action, which is only adequate in the asymptotically free limit, and were extended into (e&eb) a regime, the infrared limit, in which confinement is expected | |
| $G_{36}$ : Category one of **previously attained results relied heavily on the perturbative approach to the analysis of the effective Yang-Mills action, which is only adequate in the asymptotically free limit, and were extended;** regime, the infrared limit, in which confinement is expected<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of regime, the infrared limit, in which confinement is expected; **previously attained results relied heavily on the perturbative approach to the analysis of the effective Yang-Mills action, which is only adequate in the asymptotically free limit, and were extended**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Authors show that if a minimum of the effective Lagrangian in θ=−Fa μνFaμν/2 exists, (eb) an YMC forms that drives (eb) the Universe toward an (e&eb) accelerated de Sitter phase | |

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| $G_{40}$ : Category one of minimum of the effective Lagrangian in θ=−Fa μνFaμν/2 exists<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of YMC forms that drives (eb) the Universe toward an (e&eb) accelerated de Sitter phase<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Authors show that if a minimum of the effective Lagrangian in θ=−Fa μνFaμν/2 exists, an YMC forms that drives (eb) the Universe toward an (e&eb) accelerated de Sitter phase | |
| $G_{44}$ : Category one of minimum of the effective Lagrangian in θ=−Fa μνFaμν/2 exists, an YMC forms<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of Universe toward an (e&eb) accelerated de Sitter phase<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION A(17)**<br><br>**Dark Energy Dominated Stage** | |
| **INTRODUCTION—VARIABLES USED**<br><br>(1) However, in the late stage, wy naturally runs to (e) the critical state with wy=−1, and the Universe transitions from (e) a matter-dominated into (e&eb) a dark energy dominated stage only at (eb) latest time, at (eb) a redshift whose value depends on (=) the initial conditions that are (=) chosen while (e) solving the dynamical system. Received 17 September 2015 DOI:http://dx.doi.org/10.1103/PhysRevD.93.043012 © 2016 American Physical Society<br><br>**Letters in Mathematical Physics January 2016, Volume 106, Issue 1, pp 1-27 First online: 06 October 2015 On Twisted N = 2 5D Super Yang–Mills Theory Jian Qiu, Maxim Zabzine**<br><br>(2) On a five-dimensional simply connected Sasaki–Einstein manifold, one can construct (eb) Yang–Mills theories coupled to (e&eb) matter with (e&eb) at least two supersymmetries. | |

(3) The partition function of these theories localises on (eb) the contact instantons, however, the contact instanton equations are (=) not elliptic.
(4) It turns out that these equations can be embedded into (eb) the Haydys–Witten equations (which are (=) elliptic) in the same way the 4D anti-self-dual instanton equations are embedded in (eb) the Vafa–Witten equations.
(5) Authors show that under (e&eb) some favourable circumstances, the latter equations will reduce to (e) the former by (e) proving some vanishing theorems.
(6) It was also known that the Haydys–Witten equations on (e&eb) product manifolds M5=M4×RM5=M4×R arise in (eb) the context of twisting (e&eb) the 5D maximally supersymmetric Yang–Mills theory.
(7) In this paper, authors present the construction of twisted N = 2 Yang–Mills theory on (e&eb) Sasaki–Einstein manifolds, and more generally on K-contact manifolds.
**(8)** The localisation locus of this new theory thus provides (eb) a covariant version of the Haydys–Witten equation. Keywords five-dimensional gauge theory contact geometry

**NOTATION**

**Module One**

The details of the models depend weakly on (=) the specific form of (e) the effective Yang-Mills Lagrangian

$G_{13}$ : Category one of **details of the models depend weakly on the specific form;** effective Yang-Mills Lagrangian

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of effective Yang-Mills Lagrangian; **details of the models depend weakly on the specific form**

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

However, in the late stage, wy naturally runs to (e) the critical state with wy=−1, and the Universe transitions from (e) a matter-dominated into (e&eb) a dark energy dominated stage only at (eb) latest time, at (eb) a redshift whose value depends on (=) the initial conditions that are (=) chosen while (e) solving the dynamical system.

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$G_{16}$ : Category one of **wy naturally runs;** critical state with wy=−1, and the Universe transitions from (e) a matter-dominated into (e&eb) a dark energy dominated stage only at (eb) latest time, at (eb) a redshift whose value depends on (=) the initial conditions that are (=) chosen while (e) solving the dynamical system.

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

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| $T_{16}$ : Category one of critical state with wy=−1, and the Universe transitions from (e) a matter-dominated into (e&eb) a dark energy dominated stage only at (eb) latest time, at (eb) a redshift whose value depends on (=) the initial conditions that are (=) chosen while (e) solving the dynamical system; **wy naturally runs**<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>However, in the late stage, wy naturally runs to the critical state with wy=−1, and the Universe transitions from a matter-dominated into (e&eb) a dark energy dominated stage only at (eb) latest time, at (eb) a redshift whose value depends on (=) the initial conditions that are (=) chosen while (e) solving the dynamical system.<br><br>Received 17 September 2015 DOI:http://dx.doi.org/10.1103/PhysRevD.93.043012 © 2016 American Physical Society | |
| $G_{20}$ : Category one of **wy naturally runs to the critical state with wy=−1, and the Universe transitions from a matter-dominated;** dark energy dominated stage only at (eb) latest time, at (eb) a redshift whose value depends on (=) the initial conditions that are (=) chosen while (e) solving the dynamical system.<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of dark energy dominated stage only at (eb) latest time, at (eb) a redshift whose value depends on (=) the initial conditions that are (=) chosen while (e) solving the dynamical system. ;**wy naturally runs to the critical state with wy=−1, and the Universe transitions from a matter-dominated**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>However, in the late stage, wy naturally runs to the critical state with wy=−1, and the Universe transitions from a matter-dominated into a dark energy dominated stage only at (eb) latest time, at (eb) a redshift whose value depends on (=) the initial conditions that are (=) chosen while (e) solving the dynamical system.<br><br>Received 17 September 2015 DOI:http://dx.doi.org/10.1103/PhysRevD.93.043012 © 2016 American Physical Society | |
| $G_{24}$ : Category one of wy naturally runs to the critical state with wy=−1, and the Universe transitions from a matter-dominated into a dark energy dominated stage only<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of latest time, at (eb) a redshift whose value depends on (=) the initial conditions that are (=) chosen while (e) solving the dynamical system | |

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| $T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>However, in the late stage, wy naturally runs to the critical state with wy=−1, and the Universe transitions from a matter-dominated into a dark energy dominated stage only at latest time, at (eb) a redshift whose value depends on (=) the initial conditions that are (=) chosen while (e) solving the dynamical system.<br><br>Received 17 September 2015 DOI:http://dx.doi.org/10.1103/PhysRevD.93.043012 © 2016 American Physical Society | |
| $G_{28}$ : Category one of wy naturally runs to the critical state with wy=−1, and the Universe transitions from a matter-dominated into a dark energy dominated stage only at latest time<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of redshift whose value depends on (=) the initial conditions that are (=) chosen while (e) solving the dynamical system<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>However, in the late stage, wy naturally runs to the critical state with wy=−1, and the Universe transitions from a matter-dominated into a dark energy dominated stage only at latest time, at a redshift whose value depends on (=) the initial conditions that are (=) chosen while (e) solving the dynamical system.<br><br>Received 17 September 2015 DOI:http://dx.doi.org/10.1103/PhysRevD.93.043012 © 2016 American Physical Society | |
| $G_{32}$ : Category one of wy naturally runs to the critical state with wy=−1, and the Universe transitions from a matter-dominated into a dark energy dominated stage only at latest time, at a redshift whose value depends<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of initial conditions that are (=) chosen while (e) solving the dynamical system<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven** | |

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| However, in the late stage, wy naturally runs to the critical state with wy=−1, and the Universe transitions from a matter-dominated into a dark energy dominated stage only at latest time, at a redshift whose value depends on the initial conditions that are chosen while (e) solving the dynamical system.<br><br>Received 17 September 2015 DOI:http://dx.doi.org/10.1103/PhysRevD.93.043012 © 2016 American Physical Society | |
| $G_{36}$ : Category one of solving the dynamical system<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of wy naturally runs to the critical state with wy=−1, and the Universe transitions from a matter-dominated into a dark energy dominated stage only at latest time, at a redshift whose value depends on the initial conditions that are chosen<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>On a five-dimensional simply connected Sasaki–Einstein manifold, one can construct (eb) Yang–Mills theories coupled to (e&eb) matter with (e&eb) at least two supersymmetries | |
| $G_{40}$ : Category one of **five-dimensional simply connected Sasaki–Einstein manifold;** Yang–Mills theories coupled to (e&eb) matter with (e&eb) at least two supersymmetries<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of Yang–Mills theories coupled to (e&eb) matter with (e&eb) at least two supersymmetries **;five-dimensional simply connected Sasaki–Einstein manifold**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>On a five-dimensional simply connected Sasaki–Einstein manifold, one can construct Yang–Mills theories coupled to (e&eb) matter with (e&eb) at least two supersymmetries | |
| $G_{44}$ : Category one of **five-dimensional simply connected Sasaki–Einstein manifold, one can construct Yang–Mills theories**<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |

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| $T_{44}$ : Category one of **five-dimensional simply connected Sasaki–Einstein manifold, one can construct Yang–Mills theories**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a_{13}^{'})^{(1)}, (a_{14}^{'})^{(1)}, (a_{15}^{'})^{(1)}, (b_{13}^{'})^{(1)}, (b_{14}^{'})^{(1)}, (b_{15}^{'})^{(1)}, (a_{16}^{'})^{(2)}, (a_{17}^{'})^{(2)}, (a_{18}^{'})^{(2)}$, $(b_{16}^{'})^{(2)}, (b_{17}^{'})^{(2)}, (b_{18}^{'})^{(2)}$<br>$, (a_{20}^{'})^{(3)}, (a_{21}^{'})^{(3)}, (a_{22}^{'})^{(3)}, (b_{20}^{'})^{(3)}, (b_{21}^{'})^{(3)}, (b_{22}^{'})^{(3)}$<br>$(a_{24}^{'})^{(4)}, (a_{25}^{'})^{(4)}, (a_{26}^{'})^{(4)}, (b_{24}^{'})^{(4)}, (b_{25}^{'})^{(4)}, (b_{26}^{'})^{(4)}, (b_{28}^{'})^{(5)}, (b_{29}^{'})^{(5)}, (b_{30}^{'})^{(5)}\ (a_{28}^{'})^{(5)}, (a_{29}^{'})^{(5)}, (a_{30}^{'})^{(5)}$ ,<br>$(a_{32}^{'})^{(6)}, (a_{33}^{'})^{(6)}, (a_{34}^{'})^{(6)}, (b_{32}^{'})^{(6)}, (b_{33}^{'})^{(6)}, (b_{34}^{'})^{(6)}$<br>$(a_{36}^{'})^{(7)}, (a_{37}^{'})^{(7)}, (a_{38}^{'})^{(7)}, (b_{36}^{'})^{(7)}, (b_{37}^{'})^{(7)}, (b_{38}^{'})^{(7)}$,<br>$(a_{40}^{'})^{(8)}, (a_{41}^{'})^{(8)}, (a_{42}^{'})^{(8)}, (b_{40}^{'})^{(8)}, (b_{41}^{'})^{(8)}, (b_{42}^{'})^{(8)}$,<br>$(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |

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| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19}), t)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}((G_{19}), t)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19}), t)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}((G_{19}), t) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}, t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}, t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21}, t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23}, t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25}, t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}, t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}), t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}), t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}), t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25}, t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}), t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29}, t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}, t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}), t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}), t)\right]T_{29}$ | 29 |

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| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b'_{30})^{(5)} - (b''_{30})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a''_{28})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b''_{28})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a'_{32})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b'_{32})^{(6)} - (b''_{32})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b'_{33})^{(6)} - (b''_{33})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b'_{34})^{(6)} - (b''_{34})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a''_{32})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b'_{36})^{(7)} - (b''_{36})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b'_{37})^{(7)} - (b''_{37})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b'_{38})^{(7)} - (b''_{38})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a''_{36})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a'_{41})^{(8)} + (a''_{41})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b'_{40})^{(8)} - (b''_{40})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b'_{41})^{(8)} - (b''_{41})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b'_{42})^{(8)} - (b''_{42})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |

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| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}\left((G_{47}),t\right) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}^{'})^{(1)} & \boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)} & \boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)} & \boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3<br>$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3<br>$\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)} & \boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 58 |

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| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\boxed{-(b_{15}^{''})^{(1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | *60* |
| Where $\boxed{-(b_{13}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}^{'})^{(2)}\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}^{'})^{(2)}\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{17}$ | *62* |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}^{'})^{(2)}\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{18}$ | *63* |
| Where $\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)}$ are first augmentation coefficients for | |

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| category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$ , $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$ , $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9)}(T_{45},t)$ , $+(a_{45}^{''})^{(9,9)}(T_{45},t)$ , $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)} & -(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t) & \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{16}$ | *64* |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)} & -(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t) & \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{17}$ | *65* |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)} & -(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t) & \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{18}$ | *66* |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$ , $-(b_{17}^{''})^{(2)}(G_{19},t)$ , $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,)}(G,t)$ , $-(b_{14}^{''})^{(1,1,)}(G,t)$ , $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3<br>$-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$ , $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$ , $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3 | |

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| $-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[ \begin{matrix} (a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t) + (a_{16}^{''})^{(2,2,2)}(T_{17},t) + (a_{13}^{''})^{(1,1,1,)}(T_{14},t) \\ + (a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t) + (a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t) + (a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ + (a_{36}^{''})^{(7,7,7,7)}(T_{37},t) + (a_{40}^{''})^{(8,8,8,8)}(T_{41},t) + (a_{44}^{''})^{(9,9,9)}(T_{45},t) \end{matrix} \right] G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[ \begin{matrix} (a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t) + (a_{17}^{''})^{(2,2,2)}(T_{17},t) + (a_{14}^{''})^{(1,1,1,)}(T_{14},t) \\ + (a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t) + (a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t) + (a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ + (a_{37}^{''})^{(7,7,7,7)}(T_{37},t) + (a_{41}^{''})^{(8,8,8,8)}(T_{41},t) + (a_{45}^{''})^{(9,9,9)}(T_{45},t) \end{matrix} \right] G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[ \begin{matrix} (a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t) + (a_{18}^{''})^{(2,2,2)}(T_{17},t) + (a_{15}^{''})^{(1,1,1,)}(T_{14},t) \\ + (a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t) + (a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t) + (a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ + (a_{38}^{''})^{(7,7,7,7)}(T_{37},t) + (a_{42}^{''})^{(8,8,8,8)}(T_{41},t) + (a_{46}^{''})^{(9,9,9)}(T_{45},t) \end{matrix} \right] G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |

$$\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b'_{20})^{(3)}\boxed{-(b''_{20})^{(3)}(G_{23},t)} & \boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{13})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{20} \quad (70)$$

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b'_{21})^{(3)}\boxed{-(b''_{21})^{(3)}(G_{23},t)} & \boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{14})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)}\boxed{-(b''_{22})^{(3)}(G_{23},t)} & \boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{22} \quad (72)$$

$\boxed{-(b''_{20})^{(3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3)}(G_{23},t)}$ *are first detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)}$ *are second detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{13})^{(1,1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,)}(G,t)}$ *are third detrition coefficients for category 1,2 and 3*

$\boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)}$ *are fourth detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)}$ *are fifth detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)}$ *are sixth detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{36})^{(7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7)}(G_{39},t)}$ $\boxed{-(b''_{38})^{(7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a'_{24})^{(4)}\boxed{+(a''_{24})^{(4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{16})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a'_{25})^{(4)}\boxed{+(a''_{25})^{(4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{17})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a'_{26})^{(4)} & +(a''_{26})^{(4)}(T_{25},t) & +(a''_{30})^{(5,5,)}(T_{29},t) & +(a''_{34})^{(6,6,)}(T_{33},t) \\ & +(a''_{15})^{(1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a''_{24})^{(4)}(T_{25},t)$, $(a''_{25})^{(4)}(T_{25},t)$, $(a''_{26})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a''_{28})^{(5,5,)}(T_{29},t)$, $+(a''_{29})^{(5,5,)}(T_{29},t)$, $+(a''_{30})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3

$+(a''_{32})^{(6,6,)}(T_{33},t)$, $+(a''_{33})^{(6,6,)}(T_{33},t)$, $+(a''_{34})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1)}(T_{14},t)$ *are fourth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{16})^{(2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2)}(T_{17},t)$ *are fifth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{20})^{(3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3)}(T_{21},t)$ *are sixth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{36})^{(7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7)}(T_{37},t)$ *are seventh augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{40})^{(8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8)}(T_{41},t)$ *are eighth augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{46})^{(9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9)}(T_{45},t)$, $+(a''_{44})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b'_{24})^{(4)} & -(b''_{24})^{(4)}(G_{27},t) & -(b''_{28})^{(5,5,)}(G_{31},t) & -(b''_{32})^{(6,6,)}(G_{35},t) \\ & -(b''_{13})^{(1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b'_{25})^{(4)} & -(b''_{25})^{(4)}(G_{27},t) & -(b''_{29})^{(5,5,)}(G_{31},t) & -(b''_{33})^{(6,6,)}(G_{35},t) \\ & -(b''_{14})^{(1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b'_{26})^{(4)} & -(b''_{26})^{(4)}(G_{27},t) & -(b''_{30})^{(5,5,)}(G_{31},t) & -(b''_{34})^{(6,6,)}(G_{35},t) \\ & -(b''_{15})^{(1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{26} \quad (78)$$

*Where* $-(b''_{24})^{(4)}(G_{27},t)$, $-(b''_{25})^{(4)}(G_{27},t)$, $-(b''_{26})^{(4)}(G_{27},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{28})^{(5,5,)}(G_{31},t)$, $-(b''_{29})^{(5,5,)}(G_{31},t)$, $-(b''_{30})^{(5,5,)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b''_{32})^{(6,6,)}(G_{35},t)$, $-(b''_{33})^{(6,6,)}(G_{35},t)$, $-(b''_{34})^{(6,6,)}(G_{35},t)$ *are third detrition coefficients for category* 1, 2 *and* 3

| | |
|---|---|
| $-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2 and 3 $-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 $-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3 $-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[\begin{matrix} (a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,)}(T_{25},t) & +(a_{32}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{matrix}\right] G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[\begin{matrix} (a_{29}^{'})^{(5)}+(a_{29}^{''})^{(5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,)}(T_{25},t) & +(a_{33}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{matrix}\right] G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[\begin{matrix} (a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,)}(T_{25},t) & +(a_{34}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{matrix}\right] G_{30}$ | *81* |
| *Where* $+(a_{28}^{''})^{(5)}(T_{29},t)$, $+(a_{29}^{''})^{(5)}(T_{29},t)$, $+(a_{30}^{''})^{(5)}(T_{29},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3 *And* $+(a_{24}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,)}(T_{25},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3 $+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3 $+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3 $+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3 $+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3 $+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3 $+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation | |

coefficients for category 1,2, 3
$+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3

$$\frac{dT_{28}}{dt}=(b_{28})^{(5)}T_{29}-\begin{bmatrix}(b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) & -(b_{32}^{''})^{(6,6,6)}(G_{35},t)\\ -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)\\ -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt}=(b_{29})^{(5)}T_{28}-\begin{bmatrix}(b_{29}^{'})^{(5)}-(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) & -(b_{33}^{''})^{(6,6,6)}(G_{35},t)\\ -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)\\ -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt}=(b_{30})^{(5)}T_{29}-\begin{bmatrix}(b_{30}^{'})^{(5)}-(b_{30}^{''})^{(5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,)}(G_{27},t) & -(b_{34}^{''})^{(6,6,6)}(G_{35},t)\\ -(b_{15}^{''})^{(1,1,1,1,1,)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)\\ -(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{30} \quad (84)$$

*where* $-(b_{28}^{''})^{(5)}(G_{31},t)$, $-(b_{29}^{''})^{(5)}(G_{31},t)$, $-(b_{30}^{''})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3

$-(b_{32}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3

$-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt}=(a_{32})^{(6)}G_{33}-\begin{bmatrix}(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a'_{33})^{(6)} & +(a''_{33})^{(6)}(T_{33},t) & +(a''_{29})^{(5,5,5)}(T_{29},t) & +(a''_{25})^{(4,4,4,)}(T_{25},t) \\ +(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients*

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| $for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{28}^{''})^{(5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5)}(G_{31},t)}$ $are\ second\ detrition\ coefficients$ $for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)}$ $are\ third\ detrition\ coefficients$ $for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2, and 3 $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2, and 3 | |
| $\frac{dG_{36}}{dt}$ $= (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)}\boxed{+(a_{36}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *91* |
| $\frac{dG_{37}}{dt}$ $= (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)}\boxed{+(a_{37}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *92* |
| $\frac{dG_{38}}{dt}$ $= (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)}\boxed{+(a_{38}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *93* |
| Where $\boxed{(a_{36}^{''})^{(7)}(T_{37},t)}$, $\boxed{(a_{37}^{''})^{(7)}(T_{37},t)}$, $\boxed{(a_{38}^{''})^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3 $\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3 | |

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| $+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ , $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ , $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$ , $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$ , $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3<br>$+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ , $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ , $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{36}}{dt} =$<br>$(b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 94 |
| $\frac{dT_{37}}{dt} =$<br>$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | |
| $\frac{dT_{38}}{dt} =$<br>$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | |
| Where $-(b_{36}^{''})^{(7)}(G_{39},t)$ , $-(b_{37}^{''})^{(7)}(G_{39},t)$ , $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ , $-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ , $-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ , $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ , $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 | |

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| $-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ , $-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ , $-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ , $-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ , $-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)$<br>are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ , $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ , $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3<br>$-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ , $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ , $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13}$ | *95* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}^{'})^{(8)}+(a_{41}^{''})^{(8)}(T_{41},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$ | |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$ | |
| Where $+(a_{40}^{''})^{(8)}(T_{41},t)$ , $+(a_{41}^{''})^{(8)}(T_{41},t)$ , $+(a_{42}^{''})^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ , $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ , $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ , $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ , $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ , $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ , $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 | |

$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ , $+(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$ , $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$ , $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & \left[-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & \left[-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & \left[-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$ , $-(b_{37}^{''})^{(7)}(G_{39},t)$ , $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ , $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ , $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ , $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ , $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$ , $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ , $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ , $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ , $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ , $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

| | |
|---|---|
| $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a_{44}^{'})^{(9)}\boxed{+(a_{44}^{''})^{(9)}(T_{45},t)}\ \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\ \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13}$ | *96* |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a_{45}^{'})^{(9)}\boxed{+(a_{45}^{''})^{(9)}(T_{45},t)}\ \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\ \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$ | |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a_{46}^{'})^{(9)}\boxed{+(a_{46}^{''})^{(9)}(T_{37},t)}\ \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\ \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$ | |
| Where $\boxed{+(a_{44}^{''})^{(9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3<br>$\boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth | |

augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \begin{bmatrix} (b'_{46})^{(9)}\boxed{-(b''_{46})^{(9)}(G_{47},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{44})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$$(a_i)^{(1)}, (a'_i)^{(1)}, (a''_i)^{(1)}, (b_i)^{(1)}, (b'_i)^{(1)}, (b''_i)^{(1)} > 0, \quad i,j = 13,14,15 \qquad (97)$$

| | |
|---|---|
| The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br><br>$(a_i^{''})^{(1)}(T_{14}, t) \leq (p_i)^{(1)} \leq (\hat{A}_{13})^{(1)}$<br><br>$(b_i^{''})^{(1)}(G, t) \leq (r_i)^{(1)} \leq (b_i^{'})^{(1)} \leq (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2 \to \infty}(a_i^{''})^{(1)}(T_{14}, t) = (p_i)^{(1)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(1)}(G, t) = (r_i)^{(1)}$<br><br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br><br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'}, t) - (a_i^{''})^{(1)}(T_{14}, t)\rvert \leq (\hat{k}_{13})^{(1)} \lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)} t}$<br><br>$\lvert(b_i^{''})^{(1)}(G^{'}, t) - (b_i^{''})^{(1)}(G, t)\rvert < (\hat{k}_{13})^{(1)} \lVert G - G^{'}\rVert e^{-(\hat{M}_{13})^{(1)} t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'}, t)$ and $(a_i^{''})^{(1)}(T_{14}, t)$ . $(T_{14}^{'}, t)$ and $(T_{14}, t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i^{'})^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)} (\hat{k}_{13})^{(1)}] < 1$<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i^{'})^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)} (\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |

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| $(a_i)^{(2)}, (a_i^{'})^{(2)}, (a_i^{''})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (b_i^{''})^{(2)} > 0, \quad i,j = 16,17,18$ | |
| The functions $(a_i^{''})^{(2)}, (b_i^{''})^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(\mathrm{p_i})^{(2)}$, $(\mathrm{r_i})^{(2)}$: | |
| $(a_i^{''})^{(2)}(T_{17}, t) \leq (p_i)^{(2)} \leq \left(\hat{A}_{16}\right)^{(2)}$ | 102 |
| $(b_i^{''})^{(2)}(G_{19}, t) \leq \ (r_i)^{(2)} \leq (b_i^{'})^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(2)}(T_{17}, t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty} (b_i^{''})^{(2)}\left((G_{19}), t\right) = \ (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $\boxed{(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, \ (r_i)^{(2)}}$ are positive constants and $\boxed{i = 16,17,18}$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'}, t) - (a_i^{''})^{(2)}(T_{17}, t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}^{'}\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'}, t) - (b_i^{''})^{(2)}\left((G_{19}), t\right)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})^{'}\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'}, t)$ and $(a_i^{''})^{(2)}(T_{17}, t)$ . $(T_{17}^{'}, t)$ and $(T_{17}, t)$ are points belonging to the interval $\left[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}\right]$. It is to be noted that $(a_i^{''})^{(2)}(T_{17}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i^{''})^{(2)}(T_{17}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, \ (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + \ (\hat{A}_{16})^{(2)} + \ (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |

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| $\frac{1}{(\hat{M}_{16})^{(2)}}[\ (b_i)^{(2)} + (b_i^{'})^{(2)} +\ (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)}\ (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23}, t) \leq\ (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(3)}\ (T_{21}, t) = (p_i)^{(3)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(3)}\ (G_{23}, t) =\ (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)},\ (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'}, t) - (a_i^{''})^{(3)}(T_{21}, t)\rvert \leq (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}{}^{'}, t) - (b_i^{''})^{(3)}(G_{23}, t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}{}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'}, t)$ and$(a_i^{''})^{(3)}(T_{21}, t)$ . $(T_{21}^{'}, t)$ And $(T_{21}, t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |

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| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}((G_{27}), t) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(4)}((G_{27}), t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\|(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\| \leq (\hat{k}_{24})^{(4)} \|T_{25} - T_{25}^{'}\| e^{-(\hat{M}_{24})^{(4)} t}$<br><br>$\|(b_i^{''})^{(4)}((G_{27})', t) - (b_i^{''})^{(4)}((G_{27}), t)\| < (\hat{k}_{24})^{(4)} \|(G_{27}) - (G_{27})'\| e^{-(\hat{M}_{24})^{(4)} t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants | 120 |

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| $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)} + (b_i^{'})^{(4)} + (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}, (r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}((G_{31}), t) \leq (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(5)} (T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(5)} (G_{31}, t) = (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)} \lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)} t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})', t) - (b_i^{''})^{(5)}((G_{31}), t)\rvert < (\hat{k}_{28})^{(5)} \lvert\lvert(G_{31}) - (G_{31})'\rvert\rvert e^{-(\hat{M}_{28})^{(5)} t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ : | 125 |

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| $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$ | 126 |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(6)} (T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(6)} ((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)} \lvert T_{33} - T_{33}^{'} \rvert e^{-(\hat{M}_{32})^{(6)} t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})', t) - (b_i^{''})^{(6)}((G_{35}), t)\rvert < (\hat{k}_{32})^{(6)} \lvert\lvert(G_{35}) - (G_{35})'\rvert\rvert e^{-(\hat{M}_{32})^{(6)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and $(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |

**Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ : 129

$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)},$ are positive constants

$$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$$

**Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ : 130

There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i')^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$$

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i')^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$$

Where we suppose

(EEEEE) $(a_i)^{(7)}, (a_i')^{(7)}, (a_i'')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (b_i'')^{(7)} > 0, \quad i, j = 36,37,38$ 131

(FFFFF) The functions $(a_i'')^{(7)}, (b_i'')^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}$, $(r_i)^{(7)}$:

$$(a_i'')^{(7)}(T_{37}, t) \le (p_i)^{(7)} \le (\hat{A}_{36})^{(7)}$$

$$(b_i'')^{(7)}(G_{39}, t) \le (r_i)^{(7)} \le (b_i')^{(7)} \le (\hat{B}_{36})^{(7)}$$

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*(GGGGG)* $\lim_{T_2 \to \infty} (a_i'')^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(HHHHH)*

$\lim_{G \to \infty} (b_i'')^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i'')^{(7)}(T_{37}', t) - (a_i'')^{(7)}(T_{37}, t)| \le (\hat{k}_{36})^{(7)}|T_{37} - T_{37}'|e^{-(\hat{M}_{36})^{(7)}t}$$

$$|(b_i'')^{(7)}((G_{39})', t) - (b_i'')^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)}||(G_{39}) - (G_{39})'||e^{-(\hat{M}_{36})^{(7)}t}$$

| | |
|---|---|
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'},t)$ and$(a_i^{''})^{(7)}(T_{37},t)$ . $(T_{37}^{'},t)$ and $(T_{37},t)$ are points belonging to the interval $\left[(\hat{k}_{36})^{(7)},(\widehat{M}_{36})^{(7)}\right]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{36})^{(7)}=1$ then the function $(a_i^{''})^{(7)}(T_{37},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)}$ :<br><br>(IIIII) $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)},$ are positive constants<br><br>$\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}<1$ | 134 |
| **Definition of** $(\hat{P}_{36})^{(7)},(\hat{Q}_{36})^{(7)}$ :<br><br>(JJJJJ) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\widehat{M}_{36})^{(7)},(\hat{k}_{36})^{(7)},(\hat{A}_{36})^{(7)}and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)},(a_i^{'})^{(7)},(b_i)^{(7)},(b_i^{'})^{(7)},(p_i)^{(7)},\ (r_i)^{(7)},i=36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\widehat{M}_{36})^{(7)}}[(a_i)^{(7)}+(a_i^{'})^{(7)}+(\hat{A}_{36})^{(7)}+(\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}]<1$<br><br>$\frac{1}{(\widehat{M}_{36})^{(7)}}[(b_i)^{(7)}+(b_i^{'})^{(7)}+(\hat{B}_{36})^{(7)}+(\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}]<1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)},(a_i^{'})^{(8)},(a_i^{''})^{(8)},(b_i)^{(8)},(b_i^{'})^{(8)},(b_i^{''})^{(8)}>0,\quad i,j=40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)},(b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)},\ (r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41},t)\leq(p_i)^{(8)}\leq(\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}),t)\leq\ (r_i)^{(8)}\leq(b_i^{'})^{(8)}\leq(\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(8)}(T_{41},t)=(p_i)^{(8)}$ | 140 |
| $\lim_{G\to\infty}(b_i^{''})^{(8)}\left((G_{43}),t\right)=\ (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)},(\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)},(\hat{B}_{40})^{(8)},(p_i)^{(8)},\ (r_i)^{(8)}}$ are positive constants and $\boxed{i=40,41,42}$ | |

They satisfy Lipschitz condition:

$$|(a_i'')^{(8)}(T_{41}', t) - (a_i'')^{(8)}(T_{41}, t)| \leq (\hat{k}_{40})^{(8)}|T_{41} - T_{41}'|e^{-(\hat{M}_{40})^{(8)}t} \quad (142)$$

$$|(b_i'')^{(8)}((G_{43})', t) - (b_i'')^{(8)}((G_{43}), t)| < (\hat{k}_{40})^{(8)}||(G_{43}) - (G_{43})'||e^{-(\hat{M}_{40})^{(8)}t} \quad (143)$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(8)}(T_{41}', t)$ and $(a_i'')^{(8)}(T_{41}, t)$ . $(T_{41}', t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i'')^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i'')^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ :

$(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants

$$\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}, \frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1 \quad (144)$$

**Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :

There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$
Satisfy the inequalities

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1 \quad (145)$$

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1 \quad (146)$$

Where we suppose

$$(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i, j = 44,45,46 \quad (146A)$$

The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:

$$(a_i'')^{(9)}(T_{45}, t) \leq (p_i)^{(9)} \leq (\hat{A}_{44})^{(9)}$$

$$(b_i'')^{(9)}(G_{47}, t) \leq (r_i)^{(9)} \leq (b_i')^{(9)} \leq (\hat{B}_{44})^{(9)}$$

$$\lim_{T_2 \to \infty} (a_i'')^{(9)}(T_{45}, t) = (p_i)^{(9)}$$

$$\lim_{G \to \infty} (b_i'')^{(9)}(G_{47}, t) = (r_i)^{(9)}$$

**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :

Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44{,}45{,}46}$

They satisfy Lipschitz condition:

$$|(a_i^{''})^{(9)}(T_{45}^{'}, t) - (a_i^{''})^{(9)}(T_{45}, t)| \leq (\hat{k}_{44})^{(9)}|T_{45} - T_{45}^{'}|e^{-(\hat{M}_{44})^{(9)}t}$$

$$|(b_i^{''})^{(9)}((G_{47})^{'}, t) - (b_i^{''})^{(9)}\big((G_{47}), t\big)| < (\hat{k}_{44})^{(9)}||(G_{47}) - (G_{47})^{'}||e^{-(\hat{M}_{44})^{(9)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(9)}(T_{45}^{'}, t)$ and $(a_i^{''})^{(9)}(T_{45}, t)$ . $(T_{45}^{'}, t)$ and $(T_{45}, t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i^{''})^{(9)}(T_{45}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i^{''})^{(9)}(T_{45}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}} \ , \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$$

**Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and \ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}, i = 44{,}45{,}46,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0), T_i(0)$ :

$$G_i(t) \leq (\hat{P}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t} \ , \ \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \leq (\hat{Q}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t} \ , \ \boxed{T_i(0) = T_i^0 > 0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0), T_i(0)$

$$G_i(t) \leq (\hat{P}_{16})^{(2)}e^{(\hat{M}_{16})^{(2)}t} \ , \ G_i(0) = G_i^0 > 0$$

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| $T_i(t) \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>$G_i(t) \leq (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \leq (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$ | 153 A |

| | |
|---|---|
| $T_i(t) \leq (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br><br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br><br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |

| | |
|---|---|
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[ (a_{18})^{(2)} G_{17}(s_{(16)}) - \left( (a_{18}')^{(2)} + (a_{18}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{18}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[ (b_{16})^{(2)} T_{17}(s_{(16)}) - \left( (b_{16}')^{(2)} - (b_{16}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{16}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[ (b_{17})^{(2)} T_{16}(s_{(16)}) - \left( (b_{17}')^{(2)} - (b_{17}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[ (b_{18})^{(2)} T_{17}(s_{(16)}) - \left( (b_{18}')^{(2)} - (b_{18}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{18}(s_{(16)}) \right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a_{20}')^{(3)} + a_{20}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a_{21}')^{(3)} + (a_{21}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a_{22}')^{(3)} + (a_{22}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b_{20}')^{(3)} - (b_{20}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b_{21}')^{(3)} - (b_{21}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b_{22}')^{(3)} - (b_{22}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |

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| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{24})^{(4)}$ , $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{28})^{(5)}$ , $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |

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| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b_{29}')^{(5)} - (b_{29}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{\mathrm{T}}_{30}(\mathrm{t}) = \mathrm{T}_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b_{30}')^{(5)} - (b_{30}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{32})^{(6)}$ , $T_i^0 \le (\hat{Q}_{32})^{(6)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a_{32}')^{(6)} + a_{32}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a_{33}')^{(6)} + (a_{33}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a_{34}')^{(6)} + (a_{34}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b_{32}')^{(6)} - (b_{32}'')^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b_{33}')^{(6)} - (b_{33}'')^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{\mathrm{T}}_{34}(\mathrm{t}) = \mathrm{T}_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b_{34}')^{(6)} - (b_{34}'')^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{36})^{(7)}$ , $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |

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| By<br>$\bar{G}_{36}(t) = G_{36}^{0} + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a_{36}^{'})^{(7)} + a_{36}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^{0} + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^{0} + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^{0} + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}^{'})^{(8)} + a_{40}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^{0} + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^{0} + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^{0} + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^{0} + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |

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| $\overline{T}_{42}(t) = T_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br><br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166<br>A |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{44})^{(9)}$, $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br><br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\overline{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br><br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\bar{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br><br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left(e^{(\bar{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br><br>$(G_{13}(t) - G_{13}^0) e^{-(\bar{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br><br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |

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| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)}\left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\bar{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$ $\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\bar{M}_{16})^{(2)}}\left(e^{(\bar{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that $(G_{16}(t) - G_{16}^0) e^{-(\bar{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\bar{M}_{16})^{(2)}}\left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$ $\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left(e^{(\bar{M}_{20})^{(3)} t} - 1\right)$ | 171 |
| From which it follows that $(G_{20}(t) - G_{20}^0) e^{-(\bar{M}_{20})^{(3)} t} \le \frac{(a_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left[\left((\hat{P}_{20})^{(3)} + G_{21}^0\right) e^{\left(-\frac{(\hat{P}_{20})^{(3)} + G_{21}^0}{G_{21}^0}\right)} + (\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that $G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0 + (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} s_{(24)}}\right)\right] ds_{(24)} =$ $\left(1 + (a_{24})^{(4)} t\right) G_{25}^0 + \frac{(a_{24})^{(4)} (\hat{P}_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left(e^{(\bar{M}_{24})^{(4)} t} - 1\right)$ | 173 |
| From which it follows that $(G_{24}(t) - G_{24}^0) e^{-(\bar{M}_{24})^{(4)} t} \le \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[\left((\hat{P}_{24})^{(4)} + G_{25}^0\right) e^{\left(-\frac{(\hat{P}_{24})^{(4)} + G_{25}^0}{G_{25}^0}\right)} + (\hat{P}_{24})^{(4)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0 + (\hat{P}_{28})^{(5)} e^{(\bar{M}_{28})^{(5)} s_{(28)}}\right)\right] ds_{(28)} =$ $\left(1 + (a_{28})^{(5)} t\right) G_{29}^0 + \frac{(a_{28})^{(5)} (\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)} t} - 1\right)$ | |

From which it follows that

$$(G_{28}(t)-G_{28}^{0})e^{-(\hat{M}_{28})^{(5)}t} \leq \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^{0})e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^{0}}{G_{29}^{0}}\right)}+(\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5 (175)

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that (176)

$$G_{32}(t) \leq G_{32}^{0}+\int_0^t\left[(a_{32})^{(6)}\left(G_{33}^{0}+(\hat{P}_{32})^{(6)}e^{(\hat{M}_{32})^{(6)}s_{(32)}}\right)\right]ds_{(32)} =$$

$$\left(1+(a_{32})^{(6)}t\right)G_{33}^{0}+\frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)}t}-1\right)$$

From which it follows that (177)

$$(G_{32}(t)-G_{32}^{0})e^{-(\hat{M}_{32})^{(6)}t} \leq \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^{0})e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^{0}}{G_{33}^{0}}\right)}+(\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

(s) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that (178)

$$G_{36}(t) \leq G_{36}^{0}+\int_0^t\left[(a_{36})^{(7)}\left(G_{37}^{0}+(\hat{P}_{36})^{(7)}e^{(\hat{M}_{36})^{(7)}s_{(36)}}\right)\right]ds_{(36)} =$$

$$\left(1+(a_{36})^{(7)}t\right)G_{37}^{0}+\frac{(a_{36})^{(7)}(\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)}t}-1\right)$$

From which it follows that

$$(G_{36}(t)-G_{36}^{0})e^{-(\hat{M}_{36})^{(7)}t} \leq \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)}+G_{37}^{0})e^{\left(-\frac{(\hat{P}_{36})^{(7)}+G_{37}^{0}}{G_{37}^{0}}\right)}+(\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \leq G_{40}^{0}+\int_0^t\left[(a_{40})^{(8)}\left(G_{41}^{0}+(\hat{P}_{40})^{(8)}e^{(\hat{M}_{40})^{(8)}s_{(40)}}\right)\right]ds_{(40)} =$$

$$\left(1+(a_{40})^{(8)}t\right)G_{41}^{0}+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)}t}-1\right)$$ (180)

From which it follows that

$$(G_{40}(t)-G_{40}^{0})e^{-(\hat{M}_{40})^{(8)}t} \leq \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^{0})e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^{0}}{G_{41}^{0}}\right)}+(\hat{P}_{40})^{(8)}\right]$$ (181)

$(G_i^0)$ is as defined in the statement of theorem 8
Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$

The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)}\left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$$

$$\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)} t} - 1\right)$$

From which it follows that

$$(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{P}_{44})^{(9)} + G_{45}^0\right) e^{\left(-\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0}\right)} + (\hat{P}_{44})^{(9)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 9

Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$

It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ and to choose

$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have

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$$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)} + \left((\hat{P}_{13})^{(1)} + G_j^0\right) e^{-\left(\frac{(\hat{P}_{13})^{(1)} + G_j^0}{G_j^0}\right)}\right] \le (\hat{P}_{13})^{(1)} \quad (183)$$

$$\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)} + T_j^0\right) e^{-\left(\frac{(\hat{Q}_{13})^{(1)} + T_j^0}{T_j^0}\right)} + (\hat{Q}_{13})^{(1)}\right] \le (\hat{Q}_{13})^{(1)} \quad (184)$$

In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric (185)

$$d\left(\left(G^{(1)}, T^{(1)}\right), \left(G^{(2)}, T^{(2)}\right)\right) =$$

$$\sup_i \{\max_{t \in \mathbb{R}_+} \left|G_i^{(1)}(t) - G_i^{(2)}(t)\right| e^{-(\hat{M}_{13})^{(1)} t}, \max_{t \in \mathbb{R}_+} \left|T_i^{(1)}(t) - T_i^{(2)}(t)\right| e^{-(\hat{M}_{13})^{(1)} t}\}$$

Indeed if we denote

**Definition of** $\tilde{G}, \tilde{T}$ : $\left(\tilde{G}, \tilde{T}\right) = \mathcal{A}^{(1)}(G, T)$

It results

$$\left|\tilde{G}_{13}^{(1)}-\tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{13})^{(1)} \left|G_{14}^{(1)}-G_{14}^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)} s_{(13)}} e^{(\widehat{M}_{13})^{(1)} s_{(13)}}\, ds_{(13)} +$$

$$\int_0^t \{(a'_{13})^{(1)} \left|G_{13}^{(1)}-G_{13}^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)} s_{(13)}} e^{-(\widehat{M}_{13})^{(1)} s_{(13)}} +$$

$$(a''_{13})^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right)\left|G_{13}^{(1)}-G_{13}^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)} s_{(13)}} e^{(\widehat{M}_{13})^{(1)} s_{(13)}} +$$

$$G_{13}^{(2)} \left|(a''_{13})^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right) - (a''_{13})^{(1)}\left(T_{14}^{(2)}, s_{(13)}\right)\right|\ e^{-(\widehat{M}_{13})^{(1)} s_{(13)}} e^{(\widehat{M}_{13})^{(1)} s_{(13)}}\} ds_{(13)}$$

Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses it follows

$$\left|G^{(1)}-G^{(2)}\right| e^{-(\widehat{M}_{13})^{(1)} t} \leq$$
$$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right) d\left(\left(G^{(1)}, T^{(1)};\ G^{(2)}, T^{(2)}\right)\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

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**Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)} t}$ *and* $(\widehat{Q}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(1)}$ and $(b''_i)^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 2:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$

From 19 to 24 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(1)} - (a''_i)^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(1)} t\right)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :

**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if

$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating

$$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1 / (a'_{14})^{(1)}$$

In the same way , one can obtain

$$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2 / (a'_{15})^{(1)}$$

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| If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t),t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br><br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote | 195 |

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| **Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right) d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$ | 201 |

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| In the same way , one can obtain<br>$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | |
| **Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br>Indeed let $t_2$ be so that for $t > t_2$<br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\widehat{P}_{20})^{(3)}$ and $(\widehat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[(\widehat{P}_{20})^{(3)} + \left((\widehat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[\left((\widehat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{20})^{(3)}\right] \leq (\widehat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$ | 211 |

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| $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}},\widetilde{T_{23}}:(\widetilde{(G_{23})},\widetilde{(T_{23})})=\mathcal{A}^{(3)}((G_{23}),(T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{20})^{(3)}\left\|G_{21}^{(1)}-G_{21}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)}+$<br>$\int_0^t\{(a_{20}')^{(3)}\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$(a_{20}'')^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$G_{20}^{(2)}\left\|(a_{20}'')^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)-(a_{20}'')^{(3)}\left(T_{21}^{(2)},s_{(20)}\right)\right\|\ e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)}-G_{23}{}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}t}\le$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)}+(a_{20}')^{(3)}+(\widehat{A}_{20})^{(3)}+(\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)},(T_{23})^{(1)};\ (G_{23})^{(2)},(T_{23})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}'')^{(3)}$ and $(b_{20}'')^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ *and* $(\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(3)}$ and $(b_i'')^{(3)}, i=20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i')^{(3)}-(a_i'')^{(3)}(T_{21}(s_{(20)}),s_{(20)})\}ds_{(20)}\right]}\ge 0$<br>$T_i(t)\ge T_i^0e^{(-(b_i')^{(3)}t)}>0$ for t > 0 | 216 |

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| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |

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| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{24})^{(4)}\left\|G_{25}^{(1)}-G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$<br>$\int_0^t\{(a_{24}^{'})^{(4)}\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$G_{24}^{(2)}\left\|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)}-(G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t}\le$<br>$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 226 |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})$*(and not on* $t$*)* and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results | 228 |

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| $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(4)} - (a_i'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(4)} t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ :<br><br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if<br><br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}')^{(4)} G_{25}$ and by integrating<br><br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)} \left((\widehat{M}_{24})^{(4)}\right)_1 / (a_{25}')^{(4)}$<br><br>In the same way , one can obtain<br><br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)} \left((\widehat{M}_{24})^{(4)}\right)_2 / (a_{26}')^{(4)}$<br><br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t), t)) = (b_{25}')^{(4)}$ then $T_{25} \to \infty$.<br><br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br><br>Indeed let $t_4$ be so that for $t > t_4$<br><br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t), t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}\left((G_{27})(t), t\right) = (b_{26}')^{(4)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br><br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br><br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |

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| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)}+\left((\hat{P}_{28})^{(5)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{28})^{(5)}\right]\leq(\hat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}ds_{(28)}+$<br>$\int_0^t\{(a_{28}')^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}'')^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\leq$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}')^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}'')^{(5)}$ and $(b_{28}'')^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it | 238 |

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| suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 22:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(5)} - (a_i^{''})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(5)} t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2\ and\ \left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}\ and\ G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}^{'})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a_{29}^{'})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a_{30}^{'})^{(5)}$<br><br>If $G_{29}\ or\ G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}\ and\ G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t), t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t), t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t), t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |

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| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\widehat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{32})^{(6)}\right] \leq (\widehat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)}, (T_{35})^{(1)}), ((G_{35})^{(2)}, (T_{35})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $\widetilde{(G_{35})}, \widetilde{(T_{35})}$ : $\left(\widetilde{(G_{35})}, \widetilde{(T_{35})}\right) = \mathcal{A}^{(6)}((G_{35}), (T_{35}))$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{32})^{(6)}\left\|G_{33}^{(1)} - G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\,ds_{(32)} +$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)} - G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)\left\|G_{32}^{(1)} - G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right) - (a''_{32})^{(6)}\left(T_{33}^{(2)}, s_{(32)}\right)\right\| \; e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)} - (G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t} \leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)} + (a'_{32})^{(6)} + (\widehat{A}_{32})^{(6)} + (\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)}, (T_{35})^{(1)}; (G_{35})^{(2)}, (T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | 249 |

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| condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i = 32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(6)} - (a_i^{''})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(6)}t)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2\ and\ \left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}\ and\ G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}^{'})^{(6)}G_{33}$ and by integrating<br><br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1/(a_{33}^{'})^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2/(a_{34}^{'})^{(6)}$<br><br>If $G_{33}\ or\ G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}\ and\ G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t), t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i^{''})^{(6)}((G_{35})(t), t) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}^{'})^{(6)}$ | 254 |

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| We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{\mathrm{P}}_{36})^{(7)}$ and $(\widehat{\mathrm{Q}}_{36})^{(7)}$ large to have | 255 |
| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)}, (T_{39})^{(1)}), ((G_{39})^{(2)}, (T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $(\widetilde{G_{39}}), (\widetilde{T_{39}}) : \left((\widetilde{G_{39}}), (\widetilde{T_{39}})\right) = \mathcal{A}^{(7)}((G_{39}), (T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{36})^{(7)}\left\|G_{37}^{(1)} - G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)} +$<br>$\int_0^t\{(a_{36}')^{(7)}\left\|G_{36}^{(1)} - G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} +$<br>$(a_{36}'')^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right)\left\|G_{36}^{(1)} - G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}} +$<br>$G_{36}^{(2)}\left\|(a_{36}'')^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) - (a_{36}'')^{(7)}\left(T_{37}^{(2)}, s_{(36)}\right)\right\| \; e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)} - (G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t} \leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)} + (a_{36}')^{(7)} + (\widehat{A}_{36})^{(7)} + (\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)}, (T_{39})^{(1)}; (G_{39})^{(2)}, (T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |

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| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ $and$ $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\}ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)}t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)}G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1/(a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ $is$ bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}\,((G_{39})(t),t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |

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| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right), \quad t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t), t\right) = (b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\bar{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\bar{M}_{40})^{(8)}} < 1$ and to choose<br>$(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\bar{M}_{40})^{(8)}}\left[(\hat{P}_{40})^{(8)} + \left((\hat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\bar{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{40})^{(8)}\right] \leq (\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)}, (T_{43})^{(1)}), ((G_{43})^{(2)}, (T_{43})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\bar{M}_{40})^{(8)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\bar{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})}, \widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})}, \widetilde{(T_{43})}\right) = \mathcal{A}^{(8)}((G_{43}), (T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{40})^{(8)}\left\|G_{41}^{(1)} - G_{41}^{(2)}\right\|e^{-(\bar{M}_{40})^{(8)}s_{(40)}}e^{(\bar{M}_{40})^{(8)}s_{(40)}}\,ds_{(40)} +$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\bar{M}_{40})^{(8)}s_{(40)}}e^{-(\bar{M}_{40})^{(8)}s_{(40)}} +$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\bar{M}_{40})^{(8)}s_{(40)}}e^{(\bar{M}_{40})^{(8)}s_{(40)}} +$<br>$G_{40}^{(2)}\left\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right) - (a_{40}^{''})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)\right\| e^{-(\bar{M}_{40})^{(8)}s_{(40)}}e^{(\bar{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval [0, t] | 272 |

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|---|---|
| From the hypotheses it follows | |
| $\left\|(G_{43})^{(1)} - (G_{43})^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)}t} \leq$ $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)} + (a'_{40})^{(8)} + (\widehat{A}_{40})^{(8)} + (\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right) d\left(((G_{43})^{(1)}, (T_{43})^{(1)};\ (G_{43})^{(2)}, (T_{43})^{(2)})\right)$ And analogous inequalities for $G_i$ $and$ $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a''_{40})^{(8)}$ and $(b''_{40})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(8)}$ and $(b''_i)^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t) = 0$ $and$ $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(8)} - (a''_i)^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ $and$ $\left((\widehat{M}_{40})^{(8)}\right)_3$ : **Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ $and$ $G_{42}$ . indeed if $G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a'_{41})^{(8)} G_{41}$ and by integrating $G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a'_{41})^{(8)}$ In the same way , one can obtain $G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a'_{42})^{(8)}$ If $G_{41}$ $or$ $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}$ $and$ $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(8)}((G_{43})(t), t)) = (b'_{41})^{(8)}$ then $T_{41} \to \infty$. | 278 |

**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :

Indeed let $t_8$ be so that for $t > t_8$

$(b_{41})^{(8)} - (b_i^{''})^{(8)}\big((G_{43})(t), t\big) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$

Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.
The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\big((G_{43})(t), t(t), t\big) = (b_{42}^{'})^{(8)}$

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It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have

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$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \big((\hat{P}_{44})^{(9)} + G_j^0\big)e^{-\left(\frac{(\hat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\big((\hat{Q}_{44})^{(9)} + T_j^0\big)e^{-\left(\frac{(\hat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{44})^{(9)}\right] \leq (\hat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\Big(\big((G_{47})^{(1)}, (T_{47})^{(1)}\big), \big((G_{47})^{(2)}, (T_{47})^{(2)}\big)\Big) =$$
$$\sup_i\{\max_{t\in\mathbb{R}_+}\big|G_i^{(1)}(t) - G_i^{(2)}(t)\big|e^{-(\widehat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\big|T_i^{(1)}(t) - T_i^{(2)}(t)\big|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \big(\widetilde{(G_{47})}, \widetilde{(T_{47})}\big) = \mathcal{A}^{(9)}\big((G_{47}), (T_{47})\big)$
It results

$$\big|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\big| \leq \int_0^t (a_{44})^{(9)}\big|G_{45}^{(1)} - G_{45}^{(2)}\big|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\big|G_{44}^{(1)} - G_{44}^{(2)}\big|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$(a_{44}^{''})^{(9)}\big(T_{45}^{(1)}, s_{(44)}\big)\big|G_{44}^{(1)} - G_{44}^{(2)}\big|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$G_{44}^{(2)}\big|(a_{44}^{''})^{(9)}\big(T_{45}^{(1)}, s_{(44)}\big) - (a_{44}^{''})^{(9)}\big(T_{45}^{(2)}, s_{(44)}\big)\big|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)} - G^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)}t} \leq$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a'_{44})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right) d\left(((G_{47})^{(1)}, (T_{47})^{(1)};\ (G_{47})^{(2)}, (T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a''_{44})^{(9)}$ and $(b''_{44})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(9)}$ and $(b_i'')^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$

From 99 to 44 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(9)} - (a_i'')^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b_i')^{(9)}t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2$ *and* $\left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a'_{45})^{(9)}G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a'_{45})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a'_{46})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(9)}\left((G_{47})(t), t\right)) = (b'_{45})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\big((G_{47})(t),t\big) < \varepsilon_9, T_{44}\,(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right), \quad t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\left((G_{47})(t),t\right) = (b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} - (a_{13}^{''})^{(1)}(T_{14}, t) + (a_{14}^{''})^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)}, , (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}, \; if\; (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\; (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)},$

and $\boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}, \; if\; (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}, \; if\; (u_0)^{(1)} < (u_1)^{(1)}$

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| $(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)},$ and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$ $(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$ are defined | |
| Then the solution of global equations satisfies the inequalities $G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)} t}$ where $(p_i)^{(1)}$ is defined by equation $\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)} t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)} t}\right] + G_{15}^0 e^{-(S_2)^{(1)} t} \le G_{15}(t) \le \frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a'_{15})^{(1)})}[e^{(S_1)^{(1)} t} - e^{-(a'_{15})^{(1)} t}] + G_{15}^0 e^{-(a'_{15})^{(1)} t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)} T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)} - (b'_{15})^{(1)})}\left[e^{(R_1)^{(1)} t} - e^{-(b'_{15})^{(1)} t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)} t} \le T_{15}(t) \le \frac{(a_{15})^{(1)} T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)} + (r_{13})^{(1)} + (R_2)^{(1)})}\left[e^{((R_1)^{(1)} + (r_{13})^{(1)})t} - e^{-(R_2)^{(1)} t}\right] + T_{15}^0 e^{-(R_2)^{(1)} t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:- Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$ $(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$ $(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$ $(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ : $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |

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| $-(\sigma_2)^{(2)} \le -(a_{16}^{'})^{(2)} + (a_{17}^{'})^{(2)} - (a_{16}^{''})^{(2)}(\mathrm{T}_{17}, t) + (a_{17}^{''})^{(2)}(\mathrm{T}_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b_{16}^{'})^{(2)} + (b_{17}^{'})^{(2)} - (b_{16}^{''})^{(2)}\big((G_{19}), t\big) - (b_{17}^{''})^{(2)}\big((G_{19}), t\big) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}$,, $(\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{\mathrm{G}_{16}^0}{\mathrm{G}_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{\mathrm{T}_{16}^0}{\mathrm{T}_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$\mathrm{G}_{16}^0 \mathrm{e}^{((\mathrm{S}_1)^{(2)} - (p_{16})^{(2)})\mathrm{t}} \le G_{16}(t) \le \mathrm{G}_{16}^0 \mathrm{e}^{(\mathrm{S}_1)^{(2)}\mathrm{t}}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |

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| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)}-(p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)}-(p_{16})^{(2)}-(S_2)^{(2)})}\left[e^{((S_1)^{(2)}-(p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le \right.$ $\left.\frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)}-(a_{18}')^{(2)})}\left[e^{(S_1)^{(2)}t} - e^{-(a_{18}')^{(2)}t}\right] + G_{18}^0 e^{-(a_{18}')^{(2)}t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)}t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)}t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)}-(b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t} - e^{-(b_{18}')^{(2)}t}\right] + T_{18}^0 e^{-(b_{18}')^{(2)}t} \le T_{18}(t) \le$ $\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t} - e^{-(R_2)^{(2)}t}\right] + T_{18}^0 e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a_{16}')^{(2)}$ $(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b_{16}')^{(2)}$ $(R_2)^{(2)} = (b_{18}')^{(2)} - (r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$: $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)} \le -(a_{20}')^{(3)} + (a_{21}')^{(3)} - (a_{20}'')^{(3)}(T_{21}, t) + (a_{21}'')^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)} \le -(b_{20}')^{(3)} + (b_{21}')^{(3)} - (b_{20}'')^{(3)}(G_{23}, t) - (b_{21}'')^{(3)}((G_{23}), t) \le -(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)}, (v_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$: By $(v_1)^{(3)} > 0$, $(v_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$, $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$ and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and By $(\bar{v}_1)^{(3)} > 0$, $(\bar{v}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$, $(\bar{u}_2)^{(3)} < 0$ the | 320 |

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| roots of the equations $(a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$<br><br>and $(b_{21})^{(3)}\left(u^{(3)}\right)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ | |
| **Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :-<br><br>If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by<br>$(m_2)^{(3)} = (v_0)^{(3)}, (m_1)^{(3)} = (v_1)^{(3)}$, ***if*** $(v_0)^{(3)} < (v_1)^{(3)}$<br><br>$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (\bar{v}_1)^{(3)}$, ***if*** $(v_1)^{(3)} < (v_0)^{(3)} < (\bar{v}_1)^{(3)}$,<br><br>and $\boxed{(v_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$<br><br>$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (v_0)^{(3)}$, ***if*** $(\bar{v}_1)^{(3)} < (v_0)^{(3)}$ | 321 |
| and analogously<br><br>$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$<br><br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$<br><br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$<br><br>Then the solution of global equations satisfies the inequalities<br><br>$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)}t}$<br><br>$(p_i)^{(3)}$ is defined by equation | 322 |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \le G_{22}(t) \le \right.$<br>$\left.\frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a'_{22})^{(3)})}\left[e^{(S_1)^{(3)}t} - e^{-(a'_{22})^{(3)}t}\right] + G_{22}^0 e^{-(a'_{22})^{(3)}t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b'_{22})^{(3)})}\left[e^{(R_1)^{(3)}t} - e^{-(b'_{22})^{(3)}t}\right] + T_{22}^0 e^{-(b'_{22})^{(3)}t} \le T_{22}(t) \le$<br><br>$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$ | 327 |

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- 328

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$

$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$

$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$

**Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : 329

By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{v}_1)^{(4)} > 0$, $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (v_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}$, ***if*** $(v_0)^{(4)} < (v_1)^{(4)}$

$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}$, ***if*** $(v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)}$,

and $\boxed{(v_0)^{(4)} = \frac{G^0_{24}}{G^0_{25}}}$

$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)} < (v_0)^{(4)}$

and analogously 331

$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$

$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$, ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,

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| and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \boldsymbol{if}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)} t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)} t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\right)\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)} t}\right] + G_{26}^0 e^{-(S_2)^{(4)} t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b_{26}')^{(4)})}\left[e^{(R_1)^{(4)} t} - e^{-(b_{26}')^{(4)} t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)} t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)} + (r_{24})^{(4)} + (R_2)^{(4)})}\left[e^{((R_1)^{(4)} + (r_{24})^{(4)})t} - e^{-(R_2)^{(4)} t}\right] + T_{26}^0 e^{-(R_2)^{(4)} t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |

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| **Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ :<br><br>By $(v_1)^{(5)} > 0$ , $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$ , $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | 339 |
| **Definition of** $(\bar{v}_1)^{(5)},, (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ :<br><br>By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :-<br><br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}, (\mu_2)^{(5)}$ by<br><br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br><br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br><br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |

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| $\frac{1}{(\mu_1)^{(5)}}T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}}T_{28}^0 e^{((R_1)^{(5)}+(r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)}T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)}-(b_{30}')^{(5)})}\left[e^{(R_1)^{(5)}t} - e^{-(b_{30}')^{(5)}t}\right] + T_{30}^0 e^{-(b_{30}')^{(5)}t} \le T_{30}(t) \le$<br>$\frac{(a_{30})^{(5)}T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t} - e^{-(R_2)^{(5)}t}\right] + T_{30}^0 e^{-(R_2)^{(5)}t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a_{28}')^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b_{28}')^{(5)}$<br>$(R_2)^{(5)} = (b_{30}')^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :<br>$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying<br>$-(\sigma_2)^{(6)} \le -(a_{32}')^{(6)} + (a_{33}')^{(6)} - (a_{32}'')^{(6)}(T_{33}, t) + (a_{33}'')^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$<br>$-(\tau_2)^{(6)} \le -(b_{32}')^{(6)} + (b_{33}')^{(6)} - (b_{32}'')^{(6)}((G_{35}), t) - (b_{33}'')^{(6)}((G_{35}), t) \le -(\tau_1)^{(6)}$ | 349 |
| **Definition of** $(v_1)^{(6)}, (v_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, v^{(6)}, u^{(6)}$ :<br>By $(v_1)^{(6)} > 0$, $(v_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$, $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(v^{(6)})^2 + (\sigma_1)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{v}_1)^{(6)},, (\bar{v}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br>By $(\bar{v}_1)^{(6)} > 0$, $(\bar{v}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$, $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(v^{(6)})^2 + (\sigma_2)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (v_0)^{(6)}$ :-<br>If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br>$(m_2)^{(6)} = (v_0)^{(6)}, (m_1)^{(6)} = (v_1)^{(6)}$, ***if*** $(v_0)^{(6)} < (v_1)^{(6)}$<br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (\bar{v}_6)^{(6)}$, ***if*** $(v_1)^{(6)} < (v_0)^{(6)} < (\bar{v}_1)^{(6)}$, | 351 |

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| and $\boxed{(v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$<br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (v_0)^{(6)},$ ***if*** $(\bar{v}_1)^{(6)} < (v_0)^{(6)}$ | |
| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},$ ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)},$ ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\right)\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34')6e(S1)6t−e−(a34')6t+ G340e−(a34')6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b_{34}')^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b_{34}')^{(6)} t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$<br>$(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** | |

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ :

$(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying

$$-(\sigma_2)^{(7)} \le -(a_{36}^{'})^{(7)} + (a_{37}^{'})^{(7)} - (a_{36}^{''})^{(7)}(T_{37}, t) + (a_{37}^{''})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$$

$$-(\tau_2)^{(7)} \le -(b_{36}^{'})^{(7)} + (b_{37}^{'})^{(7)} - (b_{36}^{''})^{(7)}\big((G_{39}), t\big) - (b_{37}^{''})^{(7)}\big((G_{39}), t\big) \le -(\tau_1)^{(7)}$$

**Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ : 361

By $(\nu_1)^{(7)} > 0, (\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ : 362

By $(\bar{\nu}_1)^{(7)} > 0, (\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$

**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,
and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$

and analogously 363

$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,
and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

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| Then the solution of global equations satisfies the inequalities<br>$G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^{0}e^{(S_1)^{(7)}t}$<br>where $(p_i)^{(7)}$ is defined by equation | 364 |
| $\frac{1}{(m_7)^{(7)}}G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}}G_{36}^{0}e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t}-e^{-(S_2)^{(7)}t}\right]+G_{38}^{0}e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}^{'})^{(7)})}[e^{(S_1)^{(7)}t}-e^{-(a_{38}^{'})^{(7)}t}]+G_{38}^{0}e^{-(a_{38}^{'})^{(7)}t})$ | 366 |
| $\boxed{T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}}T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}}T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)}T_{36}^{0}}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}^{'})^{(7)})}\left[e^{(R_1)^{(7)}t}-e^{-(b_{38}^{'})^{(7)}t}\right]+T_{38}^{0}e^{-(b_{38}^{'})^{(7)}t} \le T_{38}(t) \le$<br>$\frac{(a_{38})^{(7)}T_{36}^{0}}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t}-e^{-(R_2)^{(7)}t}\right]+T_{38}^{0}e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}^{'})^{(7)}$<br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}^{'})^{(7)}$<br>$(R_2)^{(7)} = (b_{38}^{'})^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br>$-(\sigma_2)^{(8)} \le -(a_{40}^{'})^{(8)} + (a_{41}^{'})^{(8)} - (a_{40}^{''})^{(8)}(T_{41}, t) + (a_{41}^{''})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br>$-(\tau_2)^{(8)} \le -(b_{40}^{'})^{(8)} + (b_{41}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}), t) - (b_{41}^{''})^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |

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| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : <br><br> By $(v_1)^{(8)} > 0$ , $(v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0$ , $(u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$ <br> and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | 372 |
| **Definition of** $(\bar{v}_1)^{(8)}$,, $(\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ : <br><br> By $(\bar{v}_1)^{(8)} > 0$ , $(\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0$ , $(\bar{u}_2)^{(8)} < 0$ the <br><br> roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$ <br><br> and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ <br><br> **Definition of** $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :- <br><br> If we define $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)}, (\mu_2)^{(8)}$ by <br><br> $(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)}$, ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$ <br><br> $(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}$ , ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)}$, <br> and $\boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ <br><br> $(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (v_0)^{(8)}$, ***if*** $(\bar{v}_1)^{(8)} < (v_0)^{(8)}$ | |
| and analogously <br><br> $(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)}$, ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$ <br><br> $(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)}$, <br> and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$ <br><br> $(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}$, ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities <br><br> $G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{40}(t) \leq G_{40}^0 e^{(S_1)^{(8)}t}$ <br><br> where $(p_i)^{(8)}$ is defined by equation | 375 |

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| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)}-(p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)}G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t} - e^{-(S_2)^{(8)}t}\right] + G_{42}^0 e^{-(S_2)^{(8)}t} \le G_{42}(t) \le \right.$ $\left.\frac{(a_{42})^{(8)}G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)}-(a'_{42})^{(8)})}[e^{(S_1)^{(8)}t} - e^{-(a'_{42})^{(8)}t}] + G_{42}^0 e^{-(a'_{42})^{(8)}t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)}-(b'_{42})^{(8)})}\left[e^{(R_1)^{(8)}t} - e^{-(b'_{42})^{(8)}t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)}t} \le T_{42}(t) \le$ $\frac{(a_{42})^{(8)}T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t} - e^{-(R_2)^{(8)}t}\right] + T_{42}^0 e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$<br>$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}((G_{47}), t) - (b''_{45})^{(9)}((G_{47}), t) \le -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :<br>By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$<br>and $(b_{45})^{(9)}(u^{(9)})^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(9)},, (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ : | |

By $(\bar{v}_1)^{(9)} > 0$ , $(\bar{v}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\left(u^{(9)}\right)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}$ , $(m_2)^{(9)}$ , $(\mu_1)^{(9)}$, $(\mu_2)^{(9)}$, $(v_0)^{(9)}$ :-

If we define $(m_1)^{(9)}$ , $(m_2)^{(9)}$ , $(\mu_1)^{(9)}$, $(\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (v_0)^{(9)}, (m_1)^{(9)} = (v_1)^{(9)}$, ***if*** $(v_0)^{(9)} < (v_1)^{(9)}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (\bar{v}_1)^{(9)}$ , ***if*** $(v_1)^{(9)} < (v_0)^{(9)} < (\bar{v}_1)^{(9)}$,

and $\boxed{(v_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (v_0)^{(9)}$, ***if*** $(\bar{v}_1)^{(9)} < (v_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^0 e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}} G_{44}^0 e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)} G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)} - (p_{44})^{(9)} - (S_2)^{(9)})}\left[e^{((S_1)^{(9)} - (p_{44})^{(9)})t} - e^{-(S_2)^{(9)}t}\right] + G_{46}^0 e^{-(S_2)^{(9)}t} \le G_{46}(t) \le \right.$$

$$\left. \frac{(a_{46})^{(9)} G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)} - (a'_{46})^{(9)})}\left[e^{(S_1)^{(9)}t} - e^{-(a'_{46})^{(9)}t}\right] + G_{46}^0 e^{-(a'_{46})^{(9)}t}\right)$$

$$\boxed{T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}} T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}} T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)} T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)} - (b'_{46})^{(9)})}\left[e^{(R_1)^{(9)}t} - e^{-(b'_{46})^{(9)}t}\right] + T_{46}^0 e^{-(b'_{46})^{(9)}t} \le T_{46}(t) \le$$

$$\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)},(S_2)^{(9)},(R_1)^{(9)},(R_2)^{(9)}$:-

Where $(S_1)^{(9)}=(a_{44})^{(9)}(m_2)^{(9)}-(a_{44}^{'})^{(9)}$

$(S_2)^{(9)}=(a_{46})^{(9)}-(p_{46})^{(9)}$

$(R_1)^{(9)}=(b_{44})^{(9)}(\mu_2)^{(9)}-(b_{44}^{'})^{(9)}$

$(R_2)^{(9)}=(b_{46}^{'})^{(9)}-(r_{46})^{(9)}$

**Proof :** From global equations we obtain

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$$\frac{dv^{(1)}}{dt}=(a_{13})^{(1)}-\left((a_{13}^{'})^{(1)}-(a_{14}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14},t)\right)-(a_{14}^{''})^{(1)}(T_{14},t)v^{(1)}-(a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)}=\frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_2)^{(1)}v^{(1)}-(a_{13})^{(1)}\right)\leq\frac{dv^{(1)}}{dt}\leq-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_1)^{(1)}v^{(1)}-(a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)},(v_0)^{(1)}$ :-

For $0<\boxed{(v_0)^{(1)}=\frac{G_{13}^{0}}{G_{14}^{0}}}<(v_1)^{(1)}<(\bar{v}_1)^{(1)}$

$$v^{(1)}(t)\geq\frac{(v_1)^{(1)}+(C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)}-(v_0)^{(1)}\right)t\right]}}{1+(C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)}-(v_0)^{(1)}\right)t\right]}}\quad,\quad\boxed{(C)^{(1)}=\frac{(v_1)^{(1)}-(v_0)^{(1)}}{(v_0)^{(1)}-(v_2)^{(1)}}}$$

it follows $(v_0)^{(1)}\leq v^{(1)}(t)\leq(v_1)^{(1)}$

In the same manner , we get

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$$v^{(1)}(t)\leq\frac{(\bar{v}_1)^{(1)}+(\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)}-(\bar{v}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)}-(\bar{v}_2)^{(1)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(1)}=\frac{(\bar{v}_1)^{(1)}-(v_0)^{(1)}}{(v_0)^{(1)}-(\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)}\leq v^{(1)}(t)\leq(\bar{v}_1)^{(1)}$

If $0<(v_1)^{(1)}<(v_0)^{(1)}=\frac{G_{13}^{0}}{G_{14}^{0}}<(\bar{v}_1)^{(1)}$ we find like in the previous case,

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$$(\nu_1)^{(1)} \le \frac{(\nu_1)^{(1)}+(C)^{(1)}(\nu_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_2)^{(1)}\right)t\right]}}{1+(C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_2)^{(1)}\right)t\right]}} \le \nu^{(1)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \le (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \le (\bar{\nu}_1)^{(1)} \le \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \le \nu^{(1)}(t) \le \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \le (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \le \nu^{(1)}(t) \le (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \le u^{(1)}(t) \le (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)}G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)}T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}')^{(2)} - (a_{17}')^{(2)} + (a_{16}'')^{(2)}(T_{17}, t)\right) - (a_{17}'')^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

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| $-\left((a_{17})^{(2)}\left(v^{(2)}\right)^2+(\sigma_2)^{(2)}v^{(2)}-(a_{16})^{(2)}\right)\le\frac{\mathrm{d}v^{(2)}}{\mathrm{dt}}\le-\left((a_{17})^{(2)}\left(v^{(2)}\right)^2+(\sigma_1)^{(2)}v^{(2)}-(a_{16})^{(2)}\right)$ | |
| From which one obtains<br><br>**Definition of** $(\bar{v}_1)^{(2)},(v_0)^{(2)}$ :-<br><br>For $0<(v_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}<(v_1)^{(2)}<(\bar{v}_1)^{(2)}$<br><br>$v^{(2)}(t)\ge\frac{(v_1)^{(2)}+(C)^{(2)}(v_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_0)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_0)^{(2)}\right)t\right]}}$ , $\boxed{(C)^{(2)}=\frac{(v_1)^{(2)}-(v_0)^{(2)}}{(v_0)^{(2)}-(v_2)^{(2)}}}$<br><br>it follows $(v_0)^{(2)}\le v^{(2)}(t)\le(v_1)^{(2)}$ | 390 |
| In the same manner , we get<br><br>$v^{(2)}(t)\le\frac{(\bar{v}_1)^{(2)}+(\bar{C})^{(2)}(\bar{v}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(2)}=\frac{(\bar{v}_1)^{(2)}-(v_0)^{(2)}}{(v_0)^{(2)}-(\bar{v}_2)^{(2)}}}$ | 391 |
| From which we deduce $(v_0)^{(2)}\le v^{(2)}(t)\le(\bar{v}_1)^{(2)}$ | 392 |
| If $0<(v_1)^{(2)}<(v_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}<(\bar{v}_1)^{(2)}$ we find like in the previous case,<br><br>$(v_1)^{(2)}\le\frac{(v_1)^{(2)}+(C)^{(2)}(v_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_2)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((v_1)^{(2)}-(v_2)^{(2)}\right)t\right]}}\le v^{(2)}(t)\le$<br><br>$\frac{(\bar{v}_1)^{(2)}+(\bar{C})^{(2)}(\bar{v}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}\le(\bar{v}_1)^{(2)}$ | 393 |
| If $0<(v_1)^{(2)}\le(\bar{v}_1)^{(2)}\le(v_0)^{(2)}=\frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(v_1)^{(2)}\le v^{(2)}(t)\le\frac{(\bar{v}_1)^{(2)}+(\bar{C})^{(2)}(\bar{v}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)}-(\bar{v}_2)^{(2)}\right)t\right]}}\le(v_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $v^{(2)}(t)$ :-<br><br>$(m_2)^{(2)}\le v^{(2)}(t)\le(m_1)^{(2)}$, $\boxed{v^{(2)}(t)=\frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :- | 396 |

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|---|---|
| $(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| From which one obtains<br><br>For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$<br><br>$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)} + (C)^{(3)}(\nu_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}$ , $\boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\nu_2)^{(3)}}}$<br><br>it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$ | 400 |
| In the same manner , we get<br><br>$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)} + (C)^{(3)}(\bar{\nu}_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\bar{\nu}_2)^{(3)}}}$<br><br>**Definition of** $(\bar{\nu}_1)^{(3)}$ :-<br><br>From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$ | 401 |

If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, 402

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain 403

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain 404

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right) - (a_{25}^{''})^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_2)^{(4)}v^{(4)}-(a_{24})^{(4)}\right)\le\frac{dv^{(4)}}{dt}\le-\left((a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_4)^{(4)}v^{(4)}-(a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(4)},(v_0)^{(4)}$ :-

For $0<\boxed{(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}<(v_1)^{(4)}<(\bar{v}_1)^{(4)}$

$$v^{(4)}(t)\ge\frac{(v_1)^{(4)}+(C)^{(4)}(v_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_0)^{(4)}\right)t\right]}}{4+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_0)^{(4)}\right)t\right]}}\quad,\quad\boxed{(C)^{(4)}=\frac{(v_1)^{(4)}-(v_0)^{(4)}}{(v_0)^{(4)}-(v_2)^{(4)}}}$$

it follows $(v_0)^{(4)}\le v^{(4)}(t)\le(v_1)^{(4)}$

In the same manner , we get

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$$v^{(4)}(t)\le\frac{(\bar{v}_1)^{(4)}+(\bar{C})^{(4)}(\bar{v}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}{4+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(4)}=\frac{(\bar{v}_1)^{(4)}-(v_0)^{(4)}}{(v_0)^{(4)}-(\bar{v}_2)^{(4)}}}$$

From which we deduce $(v_0)^{(4)}\le v^{(4)}(t)\le(\bar{v}_1)^{(4)}$

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If $0<(v_1)^{(4)}<(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}<(\bar{v}_1)^{(4)}$ we find like in the previous case,

$$(v_1)^{(4)}\le\frac{(v_1)^{(4)}+(C)^{(4)}(v_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_2)^{(4)}\right)t\right]}}{1+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((v_1)^{(4)}-(v_2)^{(4)}\right)t\right]}}\le v^{(4)}(t)\le$$

$$\frac{(\bar{v}_1)^{(4)}+(\bar{C})^{(4)}(\bar{v}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}\le(\bar{v}_1)^{(4)}$$

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If $0<(v_1)^{(4)}\le(\bar{v}_1)^{(4)}\le\boxed{(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(v_1)^{(4)}\le v^{(4)}(t)\le\frac{(\bar{v}_1)^{(4)}+(\bar{C})^{(4)}(\bar{v}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{v}_1)^{(4)}-(\bar{v}_2)^{(4)}\right)t\right]}}\le(v_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(4)}(t)$ :-

$$(m_2)^{(4)}\le v^{(4)}(t)\le(m_1)^{(4)},\quad\boxed{v^{(4)}(t)=\frac{G_{24}(t)}{G_{25}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$$(\mu_2)^{(4)}\le u^{(4)}(t)\le(\mu_1)^{(4)},\quad\boxed{u^{(4)}(t)=\frac{T_{24}(t)}{T_{25}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}$, $\boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}$, $\boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right) - (a_{33}^{''})^{(6)}(T_{33},t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)}=\frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get

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$$\nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)}=\frac{(\bar{\nu}_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\bar{\nu}_1)^{(6)}$

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If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

$$(\nu_1)^{(6)} \leq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_2)^{(6)}\right)t\right]}} \leq \nu^{(6)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\bar{\nu}_1)^{(6)}$$

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If $0 < (\nu_1)^{(6)} \leq (\bar{\nu}_1)^{(6)} \leq \boxed{(\nu_0)^{(6)}=\frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(\nu_1)^{(6)} \leq \nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

$$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}, \quad \boxed{\nu^{(6)}(t)=\frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}, \quad \boxed{u^{(6)}(t)=\frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)}=(a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)}=(\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)}=(\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)}=(\nu_1)^{(6)}$ then $\nu^{(6)}(t)=(\nu_0)^{(6)}$ and as a consequence $G_{32}(t)=(\nu_0)^{(6)}G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)}=(b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)}=(\tau_2)^{(6)}$ and then $(u_1)^{(6)}=(\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)}=(u_1)^{(6)}$ then $T_{32}(t)=(u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a'_{36})^{(7)} - (a'_{37})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right) - (a''_{37})^{(7)}(T_{37},t)v^{(7)} - (a_{37})^{(7)}v^{(7)}$$

**Definition of** $v^{(7)}$ :- $\boxed{v^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)}\right) \le \frac{dv^{(7)}}{dt} \le -\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(7)}, (v_0)^{(7)}$ :-

For $0 < \boxed{(v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}} < (v_1)^{(7)} < (\bar{v}_1)^{(7)}$

$$v^{(7)}(t) \ge \frac{(v_1)^{(7)} + (C)^{(7)}(v_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(v_1)^{(7)} - (v_0)^{(7)}}{(v_0)^{(7)} - (v_2)^{(7)}}}$$

it follows $(v_0)^{(7)} \le v^{(7)}(t) \le (v_1)^{(7)}$

In the same manner , we get 417

$$v^{(7)}(t) \le \frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{v}_1)^{(7)} - (v_0)^{(7)}}{(v_0)^{(7)} - (\bar{v}_2)^{(7)}}}$$

From which we deduce $(v_0)^{(7)} \le v^{(7)}(t) \le (\bar{v}_1)^{(7)}$

If $0 < (v_1)^{(7)} < (v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}} < (\bar{v}_1)^{(7)}$ we find like in the previous case, 418

$$(v_1)^{(7)} \le \frac{(v_1)^{(7)} + (C)^{(7)}(v_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((v_1)^{(7)} - (v_2)^{(7)}\right)t\right]}} \le v^{(7)}(t) \le$$

$$\frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \le (\bar{v}_1)^{(7)}$$

If $0 < (v_1)^{(7)} \le (\bar{v}_1)^{(7)} \le \boxed{(v_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$ , we obtain 419

$$(v_1)^{(7)} \le v^{(7)}(t) \le \frac{(\bar{v}_1)^{(7)} + (\bar{C})^{(7)}(\bar{v}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{v}_1)^{(7)} - (\bar{v}_2)^{(7)}\right)t\right]}} \le (v_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(7)}(t)$ :-

$(m_2)^{(7)} \leq v^{(7)}(t) \leq (m_1)^{(7)}$, $\boxed{v^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :-

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$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $v^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)} G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

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$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right) - (a_{41}^{''})^{(8)}(T_{41}, t) v^{(8)} - (a_{41})^{(8)} v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)} v^{(8)} - (a_{40})^{(8)}\right) \leq \frac{dv^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)} v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$v^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)}-(\nu_0)^{(8)}}{(\nu_0)^{(8)}-(\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\bar{\nu}_8)^{(8)}$

If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case, 423

$$(\nu_1)^{(8)} \leq \frac{(\nu_1)^{(8)}+(C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}} \leq \nu^{(8)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\bar{\nu}_1)^{(8)}$$

If $0 < (\nu_1)^{(8)} \leq (\bar{\nu}_1)^{(8)} \leq \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain 424

$$(\nu_1)^{(8)} \leq \nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)}+(C)^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \leq \nu^{(8)}(t) \leq (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)}G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)}T_{41}(t)$ This is an important

consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{d\nu^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right) - (a''_{45})^{(9)}(T_{45},t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \le \frac{d\nu^{(9)}}{dt} \le -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \ge \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \le \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}} \le \nu^{(9)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \le (\bar{\nu}_1)^{(9)} \le \boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$ , we obtain

$$(\nu_1)^{(9)} \le \nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$(m_2)^{(9)} \leq \nu^{(9)}(t) \leq (m_1)^{(9)}$, $\boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}$, $\boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a''_{44})^{(9)} = (a''_{45})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b''_{44})^{(9)} = (b''_{45})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a''_i)^{(1)} and\ (b''_i)^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a'_{13})^{(1)}(a'_{14})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a'_{13})^{(1)}(a'_{14})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a'_{14})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b'_{13})^{(1)}(r_{14})^{(1)} - (b'_{14})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a''_i)^{(2)} and\ (b''_i)^{(2)}$ are independent on t , and the conditions with the notations 426

$(a'_{16})^{(2)}(a'_{17})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a'_{16})^{(2)}(a'_{17})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a'_{17})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b'_{16})^{(2)}(b'_{17})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

$(b'_{16})^{(2)}(b'_{17})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b'_{16})^{(2)}(r_{17})^{(2)} - (b'_{17})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ 430

$with\ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a''_i)^{(3)} and\ (b''_i)^{(3)}$ are independent on $t$ , and the conditions with the notations 431

$(a'_{20})^{(3)}(a'_{21})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$

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| $(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$ | 435 |

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| $(a_{36}')^{(7)}(a_{37}')^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}')^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}')^{(7)}(b_{37}')^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}')^{(7)}(b_{37}')^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}')^{(7)}(r_{37})^{(7)} - (b_{37}')^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i'')^{(8)} and\ (b_i'')^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}')^{(8)}(a_{41}')^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}')^{(8)}(a_{41}')^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}')^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}')^{(8)}(b_{41}')^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}')^{(8)}(b_{41}')^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}')^{(8)}(r_{41})^{(8)} - (b_{41}')^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i'')^{(9)} and\ (b_i'')^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}')^{(9)}(a_{45}')^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}')^{(9)}(a_{45}')^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}')^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}')^{(9)}(b_{45}')^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}')^{(9)}(b_{45}')^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}')^{(9)}(r_{45})^{(9)} - (b_{45}')^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}')^{(1)} + (a_{13}'')^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}')^{(1)} - (b_{13}'')^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}')^{(1)} - (b_{14}'')^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}')^{(1)} - (b_{15}'')^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |

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| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |

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| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |

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| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(s) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) +$ | 488 |

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| $(a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first | 494 |

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| equations | |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |

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| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{40}=\frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}=\frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 501 |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{44}=\frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}=\frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G)=(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)}-(b_{13})^{(1)}(b_{14})^{(1)}-$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G)+(b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G)=0$<br><br>Where in $G(G_{13},G_{14},G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*})=0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19})=(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)}-(b_{16})^{(2)}(b_{17})^{(2)}-$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19})+(b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19})=0$ | 503 |
| Where in $(G_{19})(G_{16},G_{17},G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*})=0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23})=(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)}-(b_{20})^{(3)}(b_{21})^{(3)}-$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23})+(b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23})=0$<br><br>Where in $G_{23}(G_{20},G_{21},G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*})=0$ | 505 |

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| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right] + (b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br><br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br><br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br><br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br><br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$ | |

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| $\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right]+(b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0$ , $T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0$ , $T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*}) = 0$ , $T_{21}^{*}$ given by $f(T_{21}^{*}) = 0$ and<br><br>$G_{20}^{*} = \frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*} = \frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br><br>$T_{20}^{*} = \frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*} = \frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br><br>$G_{25}^{*}$ given by $\varphi(G_{27}) = 0$ , $T_{25}^{*}$ given by $f(T_{25}^{*}) = 0$ and<br><br>$G_{24}^{*} = \frac{(a_{24})^{(4)}G_{25}^{*}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*} = \frac{(a_{26})^{(4)}G_{25}^{*}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*} = \frac{(b_{24})^{(4)}T_{25}^{*}}{[(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*} = \frac{(b_{26})^{(4)}T_{25}^{*}}{[(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}((G_{27})^{*})]}$ | 517 |

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| Obviously, these values represent an equilibrium solution of global equations | |
| Finally we obtain the unique solution<br><br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*}) = 0$ , $T_{29}^{*}$ given by $f(T_{29}^{*}) = 0$ and<br><br>$G_{28}^{*} = \frac{(a_{28})^{(5)} G_{29}^{*}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*} = \frac{(a_{30})^{(5)} G_{29}^{*}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |
| $T_{28}^{*} = \frac{(b_{28})^{(5)} T_{29}^{*}}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)} T_{29}^{*}}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br><br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br><br>$G_{32}^{*} = \frac{(a_{32})^{(6)} G_{33}^{*}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)} G_{33}^{*}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)} T_{33}^{*}}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)} T_{33}^{*}}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br><br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br><br>$G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99 | 523 A |

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| $G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof:** Denote<br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*}\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*}\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*}\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*}\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$ | 532 |

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|---|---|
| $\frac{\partial(a_{17}^{''})^{(2)}}{\partial \mathrm{T}_{17}}(\mathrm{T}_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial(b_i^{''})^{(2)}}{\partial \mathrm{G}_j}((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{\mathrm{d}\mathbb{G}_{16}}{\mathrm{dt}} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}\mathrm{G}_{16}^{*}\mathbb{T}_{17}$ | 534 |
| $\frac{\mathrm{d}\mathbb{G}_{17}}{\mathrm{dt}} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}\mathrm{G}_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{\mathrm{d}\mathbb{G}_{18}}{\mathrm{dt}} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}\mathrm{G}_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{\mathrm{d}\mathbb{T}_{16}}{\mathrm{dt}} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}\mathrm{T}_{16}^{*}\mathbb{G}_j\right)$ | 537 |
| $\frac{\mathrm{d}\mathbb{T}_{17}}{\mathrm{dt}} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}\mathrm{T}_{17}^{*}\mathbb{G}_j\right)$ | 538 |
| $\frac{\mathrm{d}\mathbb{T}_{18}}{\mathrm{dt}} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}\mathrm{T}_{18}^{*}\mathbb{G}_j\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)}$ , $\frac{\partial(b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_j\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 547 |

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| **Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)}$ , $\frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^*) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ $and$ $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^*) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |

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| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^{*}\mathbb{G}_{j}\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^{*}\mathbb{G}_{j}\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^{*}\mathbb{G}_{j}\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^{*}) = (q_{33})^{(6)}$ , $\frac{\partial(b_i^{''})^{(6)}}{\partial G_j}((G_{35})^{*}) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^{*}\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^{*}\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^{*}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^{*}\mathbb{G}_{j}\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^{*}\mathbb{G}_{j}\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^{*}\mathbb{G}_{j}\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 572 |

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| $\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^{*}) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^{*}\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^{*}\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^{*}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^{*}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^{*}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^{*}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^{*}) = (q_{41})^{(8)}$ , $\frac{\partial (b_i^{''})^{(8)}}{\partial G_j}\left((G_{43})^{*}\right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^{*}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^{*}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^{*}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^{*}\mathbb{G}_j\right)$ | 584 |

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| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a''_i)^{(9)}$ *and* $(b''_i)^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{45})^{(9)}}{\partial T_{45}}(T^*_{45}) = (q_{45})^{(9)} \ , \ \frac{\partial(b''_i)^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a'_{44})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G^*_{44}\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a'_{45})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G^*_{45}\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a'_{46})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G^*_{46}\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$<br><br>$\left[\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\Big]$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\Big)$<br><br>$+\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\Big)$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\Big)$<br><br>$\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$ | |

$$\left((\lambda)^{(1)}\right)^2+\left((b_{13}')^{(1)}+(b_{14}')^{(1)}-(r_{13})^{(1)}+(r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$$

$$+\left(\left((\lambda)^{(1)}\right)^2+\left((a_{13}')^{(1)}+(a_{14}')^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$$

$$+\left((\lambda)^{(1)}+(a_{13}')^{(1)}+(p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^{*}+(a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^{*}\right)$$

$$\left((\lambda)^{(1)}+(b_{13}')^{(1)}-(r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^{*}+(b_{14})^{(1)}s_{(13),(15)}T_{13}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(2)}+(b_{18}')^{(2)}-(r_{18})^{(2)}\}\{\left((\lambda)^{(2)}+(a_{18}')^{(2)}+(p_{18})^{(2)}\right)$$

$$\left[\left(\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(17)}\mathrm{T}_{17}^{*}\right)$$

$$+\left(\left((\lambda)^{(2)}+(a_{17}')^{(2)}+(p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}_{16}^{*}+(a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(16)}\mathrm{T}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)$$

$$\left(\left((\lambda)^{(2)}\right)^2+\left((b_{16}')^{(2)}+(b_{17}')^{(2)}-(r_{16})^{(2)}+(r_{17})^{(2)}\right)(\lambda)^{(2)}\right)$$

$$+\left(\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\}\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left(\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\right)$$

$$+\left(\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((b_{20}^{'})^{(3)}+(b_{21}^{'})^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}^{'})^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}^{'})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left((\lambda)^{(5)}\right)^2+\left((b_{28}')^{(5)}+(b_{29}')^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$+\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}')^{(5)}+(a_{29}')^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}')^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}')^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}')^{(6)}-(r_{34})^{(6)}\{\left((\lambda)^{(6)}+(a_{34}')^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left((\lambda)^{(6)}+(a_{32}')^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\Big]$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\Big)$$

$$+\left((\lambda)^{(6)}+(a_{33}')^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\Big)$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((a_{32}')^{(6)}+(a_{33}')^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((b_{32}')^{(6)}+(b_{33}')^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$+\left(\left((\lambda)^{(6)}\right)^2+\left((a_{32}')^{(6)}+(a_{33}')^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}')^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left((\lambda)^{(6)}+(b_{32}')^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b_{38}')^{(7)}-(r_{38})^{(7)}\{\left((\lambda)^{(7)}+(a_{38}')^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left((\lambda)^{(7)}+(a_{36}')^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\Big]$$

$$\left((\lambda)^{(7)}+(b_{36}')^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\Big)$$

$$+\left((\lambda)^{(7)}+(a_{37}')^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\Big)$$

$$\left((\lambda)^{(7)}+(b_{36}')^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a_{36}')^{(7)}+(a_{37}')^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b_{36}')^{(7)}+(b_{37}')^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left(\left((\lambda)^{(7)}\right)^2+\left((a_{36}')^{(7)}+(a_{37}')^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}')^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left((\lambda)^{(7)}+(b_{36}')^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b_{42}')^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a_{42}')^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a_{40}')^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(q_{40})^{(8)}G_{40}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(8)}+(b_{40}')^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(41)}T_{41}^{*}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a_{41}')^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G_{40}^{*}+(a_{40})^{(8)}(q_{41})^{(8)}G_{41}^{*}\right)$$

$$\left(\left((\lambda)^{(8)}+(b_{40}')^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(40)}T_{40}^{*}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^2+\left((a_{40}')^{(8)}+(a_{41}')^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^2+\left((b_{40}')^{(8)}+(b_{41}')^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$+\left(\left((\lambda)^{(8)}\right)^2+\left((a_{40}')^{(8)}+(a_{41}')^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a_{40}')^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G_{40}^{*}\right)$$

$$\left((\lambda)^{(8)}+(b_{40}')^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(42)}T_{40}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b_{46}')^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a_{46}')^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a_{44}')^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G_{45}^{*}+(a_{45})^{(9)}(q_{44})^{(9)}G_{44}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(9)}+(b_{44}')^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T_{45}^{*}+(b_{45})^{(9)}s_{(44),(45)}T_{45}^{*}\right)$$

$$+\left(\left((\lambda)^{(9)}+(a_{45}')^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G_{44}^{*}+(a_{44})^{(9)}(q_{45})^{(9)}G_{45}^{*}\right)$$

$$\left(\left((\lambda)^{(9)}+(b_{44}')^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T_{45}^{*}+(b_{45})^{(9)}s_{(44),(44)}T_{44}^{*}\right)$$

$$\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45}+(b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt}= A_i\ (x)\ xi$ where $x_i \in R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i \ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i \ X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions

are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$(19) \qquad \frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17].

In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 19. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon$ $C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \ i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega\ (U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \epsilon C :\ \omega\ (x) \neq \emptyset \text{ and } \omega\ (x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s\ (B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i\ (x)$ is continuous

**A2:** $A_i\ (x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma\ (A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population $i$. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility

assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C\backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^n_i$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\in \Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two

propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population $i$ persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population $i$ is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) In \ |B_i \ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^t_0(A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = limsup_{t\to\infty}\frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni}= E_{i(x)} \ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \ \leq \ a|\ B_i(t,x)u_i(x)|+ |B_i(t,x)w| \ \leq \ |\ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)|\ w|)$. Hence, $limsup_{t\to\infty}\frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$limsup_{t\to\infty}\frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty}\frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \ \geq \ a|B_i(t,x)u_i(x)|-|B_i(t,x)w|$$

$$\geq |\ B_i(t,x)u_i(x)|(a- \alpha+ exp\ (-\beta t)|\ w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty}\frac{1}{t} \text{In}\ |B_i(t,x)v| \ \geq \ \lambda i(x) \geq \limsup_{t\to\infty}\frac{1}{t} \text{In}\ |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) \ = \ In|B_i(t+s,x)\ u_i(x)|= In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$= \text{In } |B_i(s,x.t)\, u_i(x.t)||\, B_i(t,x)\, u_i(x)|$

$= \text{In } b_i(s,t.x)\, b_i(t,x)$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

=

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \le 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma > 0$ such that $|x.t| \le \gamma$ for all $t \ge 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } |\, (x.t)_i\, |$$

$$\le \limsup_{t\to\infty} \frac{In\, \gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\ge 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on $\boldsymbol{C}$, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:*$\lambda_i(\mathrm{x}) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle \mathrm{A}_i(x.s)\mathrm{u}_i(x.s), \mathrm{u}_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(\mathrm{x}) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $\mathrm{h}_i(x) = \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I}\boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I}\boldsymbol{C}_i^+$such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following

equivalent conditions hold For all invariant measures $\mu$ supported by $K$ (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by $K$.

then $K$ is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and $K$ is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by $K$.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$. Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large

so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \leq \epsilon$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \leq \epsilon$, and $|h(x.t) - h(x^n.t)| \leq \epsilon$ for all $x \in N_1(\Gamma(A))$. Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\leq \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \epsilon + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 2\epsilon + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 3\epsilon$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\epsilon > 0$ and $t > 0$ are arbitrary, we have $\int_c h(x.t) d\mu(x) = \int_c h(x) d\mu(x)$ for all $t > 0$ and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x) u_i^n(x), u_i^n(x) \rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop. 3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop. 3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\epsilon > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \epsilon$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)| \leq \epsilon$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)| d\mu(x) + |\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\, d\mu_n(x)$$

$$\leq 2\epsilon$$

Since $\epsilon > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm. 2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}$

$\int_0^t \delta_{y.s}$ ds where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$ . Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}_{n=1}^{\infty}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

| |
|---|
| Min max $\lambda i$ ($\mu = \min_{\mu} \max_{p \epsilon \Delta} \sum_{i} Pi\lambda i(\mu)$ |
| |
| Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that |
| (6) min max $\lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_{i} Pi\lambda i(\mu)$ |
| |
| Where the minimum is taken over invariant probability measures $\mu$ with support in *K.* Since min$\mu$ |
| $\sum_{i} P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established. |
| |
| To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\} \backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta -$perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ [0,1]$be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C \backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by |
| |
| $\tilde{A}_i(x) = \{ \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i \epsilon I}{if\ i \notin I}$ |
| |
| Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C \backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t >$ 0. $Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in Kand\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of |

$\dot{x} = \tilde{A}(x)x$ *and* $\lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\} \backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). *Since* $E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently *K* is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent*. Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume x* $\mapsto$ *A(x) is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely*

*ergodic, or There exists* $j \in \{1,\ldots,k\}$ *such that* $M_i$ ----- $\{ x \in C : x_l \gg 0 \text{ for } l \neq j \}$ *Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by* $U_i M_i$.

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure $\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures* $\mu$ *with support in* $\partial C$, $\lambda_i(\mu) > 0$ *for all* $I \in \{1,\ldots,k\} \setminus supp(\mu)$ *(2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $$\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j \qquad i = 1,..,k \qquad j=1,\ldots,m$$

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,\ldots,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda$ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species,* e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0 < e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j \ (x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1{}^1, \ldots, E_1{}^m)$, and the matrix $A^2$ ($E_1$) is close to the diagonal matrix $\text{diag}(f^i_2(E_1{}^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f^j_2(E^j_1) > 0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f^j_2(E^j_1) > 0$ and $\max_j f^j_2(E^j_2) > 0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0, E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is

sufficiently slow (i.e. $0< e^{j}{}_{i}\ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j (E^j{}_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1-x_1^j - \beta^j\ x_2^j\ - \alpha^j\ x_3^j\ ) + \sum d_1^{jk}\ \ x_1^k\ - e_1^j\ x_1^j$$

$$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j\ x_1^j\ - x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_2^{jk}\ \ x_2^k\ - e_2^j\ x_2^j$$

$$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j\ x_1^j\ - \alpha^j x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_3^{jk}\ \ x_3^k\ - e_3^j\ x_3^j$$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all *j*. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j{}_i = \sum_l d^{l,j}_i$ for all *i, j*) , the maximal invariant set in $\partial$C consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0< e^j{}_i\ll 1$ for all *i*, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2$ $(E_1)$and $\lambda_3$ $(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1 - \beta^j$. Consider the Morse decomposition of $\partial$C given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1> 0$ for *i* = 1, 2, 3, 0 is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j (1 - \beta^j)$$

Equivalently,

$$2>\min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j (r_i - \sum_s a_{is} x_s^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j , \qquad i = 1, \dots, k; \; j = 1, \dots, m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$ . We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set H= $\{x \in C: x_i^j = x_i^l \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \qquad i = 1, \dots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \dots, k$$

It has been shown [12,ch.16.1,16.2] That for $k$ =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there

is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\, \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \rightarrow \mathbf{E_2} \rightarrow \mathbf{E_3} \rightarrow \mathbf{E_1}$ then the following inequality holds

12) $\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within $\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as

well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that

there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{matrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{matrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has

abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left( \beta(1 - y - z)\frac{P}{\epsilon + P} - r - m(1 - y) - b(N)\ y \right)$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1 = N$, $x_2 = p$, $x_3 = (y,z)'$ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} - y_i)$$

With

$$(16)\quad F(x)_i = x_i g\,(x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\quad \begin{pmatrix} x_i \\ y_i\ dot \end{pmatrix} = \begin{pmatrix} -\beta & \beta \\ ag\,(x_i, x_i) - 1 & \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon

plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational

consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical

propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$ to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da$ $(a \geq 0)$ at time $t$ so that we have

(1.55) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.56) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.57) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, \text{'} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, \end{cases}$$ (1.4)

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],$ $i = 1, 2,$

(1.6) $P_i(s) = \int \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the

model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1$, $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as* $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 19. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^{+}\text{x}\ k_{\ 2}\text{x}\ k_{\ 1}, R^{-})$,

$A_3$. $\varphi_i\ (\cdot\ , s) \in L^{\ 2}\ (R^{+}, R^{+})\ \cap C(R^{+}, R^{+})$ for each $s \in\ [-\tau, 0]$,

$i,\ =1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i$: $R^{+}$ ▯$[-\tau, T] \rightarrow R^{+}$ $(i,\ =1,\ 2,)$ With the following properties:

$S_1$. $P_i(\cdot\ , t) \in L^{\ 1}\ (R^{+}, R^{+})$ for each $t\ \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot\ ) \in C\ ([-\tau, T], R^{+})$ for each $a\ \in R^{+}$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^{+}$ ▯$\{R^{+} \cap\ [0, T]\}$ And is continuous $(i,\ =1,\ 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,)\ \in R^{+}$ ▯$[-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$P_i(a + x, t + x) =\ \ P_i(x), (i,\ =1,\ 2,)$ ,

Then (1.3) considered at $(1.3)$ considered at $(a + x,\ t + x)$ becomes

$\frac{dp_{\ i}(x)}{dx} + f_i(a + x,\ P(t + x))\ \ P_i(x) = 0$

Which has a solution of the form?

$$P_i(x) = P_i(0)\ \exp\left[-\int_0^{\infty} f_{\ i}(a + s, P(t + S))ds\right]$$

And hence

(2.1) $p_{\ 1}\ (a + x, t + x) = p_{\ 1}\ (a, t)\ exp\left[-\int_o^{x} f_{\ 1}\left(a + s, p(t + s)\right)ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_{\ 1}\ (a, t) =\ \varphi_{\ 1}\ (a - t, 0)\exp$▯$\left[-\int_o^{t} f_{\ 1}\left(a - t + s, p_{\ 1}\ (s)\right) ds\right]$

$O \leq t \leq a, i = 1,2,$

With

$P_1(s)=(p_1(s),p_2(s-t)),$

$P_2(s)=(p_1(s-t),p_2(s)),$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$t > a \geq, i = 1,2,$

Where

(2.4) $B_i(t) = p_i(0, t), i = 1,2, t > 0.$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$

(2.5) $L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(a+s, p_i(s))ds]$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da + \int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$t > 0, i = 1,2,$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.(a, t, P_i(t))da$

$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$

$i = 1, 2, t > 0$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$

(2.5), (2.6) and (2.10) lead to

(2.11) $|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i a - d_i^*(t-a)]\, da$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t)\, \phi(0) d_i^*]t \quad for\ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T],R^+), \quad B_i \in C([0,T],R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau,T] = \{f \in C([-\tau,T],R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau,T]$. Then for any

$(x, y) \in C^+[-\tau,T] \times C^+[-\tau,T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau,T], \quad C_T^2 = C^+[-\tau,T] \times C^+[-\tau,T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau,T]$ *and* $t \in [0,T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Phi: C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)\, da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)\, da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Phi(P_t) - \Phi_t\|_{t*} \le \left\{ \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right\}$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ and $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \leq \int_0^t B_1^{(1)}(a)\left\{M_1\left(a,t,P_t^{(1)}\right) - M_1\left(a,t,P_t^{(2)}\right)\right\}da$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_t ds\right)da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o)[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds]da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \|_t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$=\int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\mathrm{X}[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$=R_1+R_2+R_3+R_4$ *(say)*.

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \, /\!/ P_t^{(1)} - P_t^{(2)} /\!/ \text{ for } a \, \varepsilon [o, t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \leq \beta_1 \Phi_1(0) || P_t^{(1)} - P_t^{(2)} ||_t^* + \Phi_1(0) \beta_1 d_1 || P_t^{(1)} - P_t^{(2)} ||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) \, \mathrm{e}^{\delta 1^a} \beta_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 \mathrm{d}_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) \, \mathrm{e}^{\delta 1^a} /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| \mathrm{B}_1^{(1)}(\sigma) - \mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{B}_1^{*\mathrm{d}} \sigma$$

$$+ \Phi_1(0)\| P_t^{(1)}\text{-}P_t^{(2)}\|_{t^*}\left\{B_1 + B_2\, d_1\, a + B_1^{*}\ \left(B_2 a + d_1 a^{2}/2\right)e^{\delta}\, 1^{a}\right\}$$

$$\leq \beta_1^{*}\int_o^a \left|\ B_1^{(1)}(\sigma)\text{-}\ B_1^{(2)}(\sigma)\right| \text{do+}\left(\mathrm{r}^{*} + \|\Phi_t\|_{t*}\right)\|\ P_t^{(1)}\text{-}P_t^{(2)}\ \|_{t^*} g_1^{(a)}$$

Where $g_1(a) \rightarrow \beta_1$ as $a \rightarrow 0$. By Gronwall's inequality it will follow that

$$\left|B_1^{(1)}(a)\text{-}B_1^{(2)}(a)\right| \leq \left(r^{*} + \|\Phi_t\|_{t*}\right)\|\ P_t^{(1)}\text{-}P_t^{(2)}\ \|_{t^*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma)\ B_1^{(2)}\ (\sigma),]\ b_1(a))\text{-}b_1(a\text{-}\sigma,\ P_t^{(1)}(a))\ M_1(\sigma,\ a,\ P_t^{(1)}(a))do\right|$$

$$\left|B_1^{*}\left(r^{*} + \|\Phi_t\|_{t*}\right)\|\ P_t^{(1)}\text{-}P_t^{(2)}\ \|_{t^*} 0(a) as\ a \rightarrow 0$$

*Thus*

$$\left|B_1^{(1)}(a) - B_1^{(2)}(a)\right| \leq (r^{*} + \|\Phi_t\|_{t*})\|\ P_t^{(1)} - P_t^{(2)}\|_t o(1) as \quad a \rightarrow o$$

And hence

$$\left|\pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right)\right| \leq ((r^{*} + \|\Phi_t\|_{t*})\|\ P_t^{(1)} - P_t^{(2)}\|_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left|\pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right)\right|$ leads to

$$\|\ \pi\ (P_t^{(1)} - \pi\ \left(P_t^{(2)}\right)\| \leq (r^{*} + \|\Phi_t\|_{t*})\|\ P_t^{(1)} - P_t^{(2)}\|_{t*} o(t^{*})$$

As $t^{*} \rightarrow o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^{*}, P_t^{*}$ Of (1.1)-(1.6): such solutions satisfy

(3.1) $\frac{dp\,P_2^*(a)}{da}+f_i(\mathrm{a}, P_1^*, P_2^*)\,P_i^*(a) = 0,\ \ i = 1,2, a > 0,$

(3.2) $P_i^* = \int_o^\infty P_i^*(a)da, i = 1,\ 2,$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i = 1.\ \ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s, P_1^*, P_2^*)ds\right], i = 1,\quad 2, a \geq 0,$

Which together with (3.3) lead to

$1=\int_o^\infty b_1(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_1(s, P_1^*, P_2^*)ds\right]da,$

(3.5) $1=\int_o^\infty b_2(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_2(s, P_1^*, P_2^*)ds\right]da$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1, P_2) = \int_o^\infty b_i(a, P_1, P_2)\exp\left[-\int_o^\infty f_i(s, P_1, P_2)ds\right]da,$

$i$ =1, 2, $P_1 \geq 0$, $P_2 \geq 0$,

**And suppose the following hold:**

(3.7) $A_{4.}$ $F_{i.}\ (0,\ 0) > 1,\ i = 1,2$:

$A_{5.}$ *there exists a positive constant c for which*

$F_1$(C,0)< $F_{2.}$(C,0),

(3.8) $F_1$(0.C)< $F_{2.}$(0,C),

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $F_1(P_1^*, P_2^*) = 1 = F_2(P_1^*, P_2^*)$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a) = [P_2^* \exp[-f_i(s, P_1^*, P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s, P_1^*, P_2^*)ds]da)$,

$i$ =1, 2, $a \geq 0$,

Proof Consider the elements of the surfaces defined by

Z= $F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0$,

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

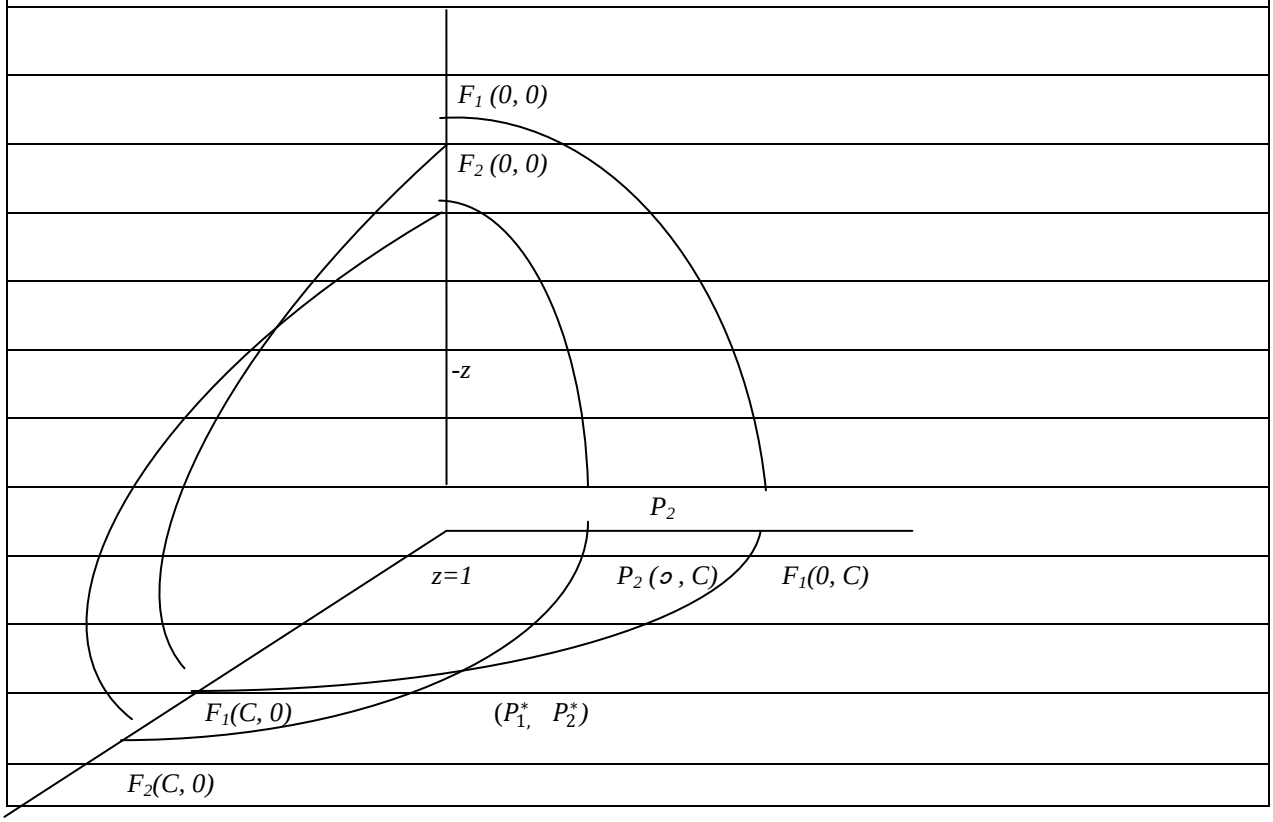
$P_1 \geq 0, P_2 \geq 0$, $i$ =1, 2,

$j$ =1, 2, Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, F_1(P_1, P_2 \geq 0$,

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.



(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11)
$$\begin{cases} p_i(a,t) = p^*_i(a) + u_i(a,t) \\ p_i(t) = p^*_i + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_i} p_i(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial b_i}{\partial p^*_i} p_i(t)\right\} da \end{cases}$$

Where $i = 1,2$ and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_i(a) \exp[\lambda t], \quad u_i(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty u_i(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{du_i}{da} + \lambda u_i = -f_i\left(a, p^*_1, p^*_2\right) u_i(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j} p^*_i\right\},$$

(3.15)

$$\mu_i(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\mu_i(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial b_i}{\partial p_j^*}\ p_i^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \ p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\mu_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1\left(a, p_1^*, p_2^*\right) - f_1\left(a, p_1^*, p_2^*\right)\right]\mu_i(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2\left(a, p_1^*, p_2^*\right) - f_2\left(a, p_1^*, p_2^*\right)\right]\mu_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\mu_i(a) = \left\{\mu_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\right\} - p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\mu_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\,e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)\,e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\varphi_1(a)$ and $\varphi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$\left(\lambda - A_{11}\right) p_1^* - A_{12} p_2^* \exp\left(-\lambda\tau\right) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21} p_1^* \exp\left(-\lambda\tau\right) + \left(\lambda - A_{22}\right) p_2^* = g_{21}(\lambda) p_1^* \exp\left(-\lambda\tau\right) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $$A_{i\varphi} = \int_0^\infty p_i^*(a) \left[\frac{\partial b_i}{\partial p_\varphi^*} - \frac{\partial f_i}{\partial p_\varphi^*}\right] da,$$

(3.23) $$g_{i\varphi}(\lambda) = \left[p_i^*(0) c_i\right]^{-1} \left\{\int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} da\right\} \varphi \left\{\int_0^\infty \left[p_i^*(a) \frac{\partial b_i}{\partial p_\varphi^*} - b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_\varphi^*} e^{\lambda s} ds\right)\right] da\right\}$$

$$- \int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_\varphi^*} e^{\lambda s} ds\right) da, \quad i, \varphi = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i\left(a, p_1^*, p_2^*\right) - f_i\left(a, p_1^*, p_2^*\right), \quad i, = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $\left(p_1^*, p_2^*\right)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et \begin{bmatrix} \lambda - \left(A_{11} + g_{11}(\lambda)\right) & -\left(A_{12} + g_{12}\right) e^{-\lambda\tau} \\ -\left(A_{21} + g_{21}\right) e^{-\lambda\tau} & \lambda - \left(A_{22} + g_{22}(\lambda)\right) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda\left(A_{11} + A_{22}\right) + A_{11}A_{22} - A_{12}A_{21} \exp\left[-2\lambda\tau\right] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = \left[A_{21} g_{12}(\lambda) + A_{12} g_{21}(\lambda) + g_{12}(\lambda) g_{21}(\lambda)\right] \exp\left[-2\lambda\tau\right]$

$$-A_{11} g_{22}(\lambda) - A_{22} g_{11}(\lambda) - g_{11}(\lambda) g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ =1,\ 2.)\ \ (see(A_1)\ and\ (A_2))$;

$A_7.\ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = |A_{11}\, g_{12}(0)| + |A_{12}\, g_{21}(0)| + |g_{12}(0)\, g_{21}(0)|$$

$$+ |A_{11}\, g_{22}(0)| + |A_{22}\, g_{11}(0)| + |g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$$F_1(\lambda, \tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau];$$

$$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau),$$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$, $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) | - F_2(\lambda, \tau) \right|$$

$$\geq |\lambda|^2 - |\lambda|\{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[(|\lambda| - P)^2 + (q)\right]^2$$

Where

$$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2,$$

$$q = \left\{4 | (A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| | - | A_{11} + A_{22} + \| S_1(0) \| |^2\right\}/2,$$

It will now follow, from (A) and (3.26), $F_1(\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions

of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends,

assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the “Grand design “is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner

wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the

controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## SECTION A(17)-2

## Twisted N = 2 5d Super Yang–Mills Theory

### INTRODUCTION—VARIABLES USED

**Letters in Mathematical Physics January 2016, Volume 106, Issue 1, pp 1-27 First online: 06 October 2015 On Twisted N = 2 5D Super Yang–Mills Theory Jian Qiu, Maxim Zabzine**

(1) The partition function of these theories localises on (eb) the contact instantons, however, the contact instanton equations are (=) not elliptic.
(2) It turns out that these equations can be embedded into (eb) the Haydys–Witten equations (which are (=) elliptic) in the same way the 4D anti-self-dual instanton equations are embedded in (eb) the

Vafa–Witten equations.

(3) Authors show that under (e&eb) some favourable circumstances, the latter equations will reduce to (e) the former by (e) proving some vanishing theorems.

(4) It was also known that the Haydys–Witten equations on (e&eb) product manifolds M5=M4×RM5=M4×R arise in (eb) the context of twisting (e&eb) the 5D maximally supersymmetric Yang–Mills theory.

**(5)** In this paper, authors present the construction of twisted N = 2 Yang–Mills theory on (e&eb) Sasaki–Einstein manifolds, and more generally on K-contact manifolds.

**(6)** The localisation locus of this new theory thus provides (eb) a covariant version of the Haydys–Witten equation. Keywords five-dimensional gauge theory contact geometry

**NOTATION**

**Module One**

The partition function of these theories localises on (eb) the contact instantons, however, the contact instanton equations are (=) not elliptic

$G_{13}$ : Category one of **partition function of these theories localises** contact instantons, however, the contact instanton equations are (=) not elliptic**;**

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of contact instantons, however, the contact instanton equations are (=) not elliptic **;partition function of these theories localises**

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

The partition function of these theories localises on the contact instantons, however, the contact instanton equations are (=) not elliptic

$G_{16}$ : Category one of elliptic

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

$T_{16}$ : Category one of partition function of these theories localises on the contact instantons, however, the contact instanton equations

$T_{17}$ : Category two of SAS

$T_{18}$ : Category three of SAS

**Module three**

It turns out that these equations can be embedded into (eb) the Haydys–Witten equations (which are (=) elliptic) in the same way the 4D anti-self-dual instanton equations are embedded in (eb) the Vafa–Witten

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| equations | |
| $G_{20}$ : Category one of these equations can be embedded<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of Haydys–Witten equations (which are (=) elliptic) in the same way the 4D anti-self-dual instanton equations are embedded in (eb) the Vafa–Witten equations<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>It turns out that these equations can be embedded into the Haydys–Witten equations (which are (=) elliptic) in the same way the 4D anti-self-dual instanton equations are embedded in (eb) the Vafa–Witten equations | |
| $G_{24}$ : Category one of these equations can be embedded into the Haydys–Witten equations (which are (=) elliptic) in the same way the 4D anti-self-dual instanton equations are embedded<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of Vafa–Witten equations<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>Authors show that under (e&eb) some favourable circumstances, the latter equations will reduce to (e) the former by proving some vanishing theorems | |
| $G_{28}$ : Category one of **latter equations;** former by proving some vanishing theorems<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of former by proving some vanishing theorems; **latter equations**<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>It was also known that the Haydys–Witten equations on (e&eb) product manifolds M5=M4×RM5=M4×R | |

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| arise in (eb) the context of twisting (e&eb) the 5D maximally supersymmetric Yang–Mills theory | |
| $G_{32}$ : Category one of **Haydys–Witten equations;** product manifolds M5=M4×RM5=M4×R arise in (eb) the context of twisting (e&eb) the 5D maximally supersymmetric Yang–Mills theory<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of product manifolds M5=M4×RM5=M4×R arise in (eb) the context of twisting (e&eb) the 5D maximally supersymmetric Yang–Mills theory **;Haydys–Witten equations**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>It was also known that the Haydys–Witten equations on product manifolds M5=M4×RM5=M4×R arise in (eb) the context of twisting (e&eb) the 5D maximally supersymmetric Yang–Mills theory | |
| $G_{36}$ : Category one of Haydys–Witten equations on product manifolds M5=M4×RM5=M4×R arise<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of context of twisting (e&eb) the 5D maximally supersymmetric Yang–Mills theory<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>It was also known that the Haydys–Witten equations on product manifolds M5=M4×RM5=M4×R arise in ) the context of twisting (e&eb) the 5D maximally supersymmetric Yang–Mills theory | |
| $G_{40}$ : Category one of **Haydys–Witten equations on product manifolds M5=M4×RM5=M4×R arise in ) the context of twisting;** 5D maximally supersymmetric Yang–Mills theory<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of5D maximally supersymmetric Yang–Mills theory; **Haydys–Witten equations on product manifolds M5=M4×RM5=M4×R arise in ) the context of twisting**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |

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| **Module Nine**<br><br>In this paper, authors present the construction of twisted N = 2 Yang–Mills theory on (e&eb) Sasaki–Einstein manifolds, and more generally on K-contact manifolds | |
| $G_{44}$ : Category one of **construction of twisted N = 2 Yang–Mills theory;** Sasaki–Einstein manifolds, and more generally on K-contact manifolds<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of Sasaki–Einstein manifolds, and more generally on K-contact manifolds **;construction of twisted N = 2 Yang–Mills theory**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION A(18)**<br><br>**Yang-Mills Theory And Magneto Statics** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Nuclear Physics B Volume 143, Issue 3, 23 October 1978, Pages 485-502Vacuum of the quantum Yang-Mills theory and magneto statics: Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. Heinz Pagels ⁎⁎, E. Tomboulis**<br><br>(1) It is argued that since in asymptotically free (e) Yang-Mills theories the quantum ground state is (=) not controlled by (e) perturbation theory, there is no a priori reason to believe (e) that individual orbits corresponding to(e&eb) minima of the classical action dominate (e) the Euclidean functional integral.<br>(2) To examine and classify (e&eb) the vacua of the quantum gauge theory, authors propose an effective action in (eb) which the gauge field coupling constant g is replaced by (e&eb) the effective coupling g(t), t = ln[Fμva)2μ4].<br>(3) The vacua of this model correspond to (e&eb) paramagnetism and (e&eb) perfect paramagnetism, for (e) which the gauge field is (=) Fμva = 0, and ferromagnetism, for (e) which (Fμva) 2 = λ2, i.e. spontaneous magnetization of (e) the vacuum occurs.<br>(4) Authors show that there are (=) no instanton solutions to (e) the quantum effective action.<br>(5) The equations for (e) a point classical source of (e) color spin are solved, and they show that the field infrared energy becomes (=) linearly divergent in (eb) the limit of spontaneous magnetization.<br>**(6)** This implies (eb) bag formation, and (e&eb) an electric Meissner effect confining (e) the bag contents. Research supported in part by NSF contract PHY 78-01221.Copyright © 1978 Published by Elsevier B.V. | |

| NOTATION | |
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| **Module One**<br>It is argued that since in asymptotically free Yang-Mills theories the quantum ground state is not controlled by (e) perturbation theory, there is no a priori reason to believe (e) that individual orbits corresponding to(e&eb) minima of the classical action dominate (e) the Euclidean functional integral | |
| $G_{13}$ : Category one of **asymptotically free Yang-Mills theories the quantum ground state is not controlled;** perturbation theory, there is no a priori reason to believe (e) that individual orbits corresponding to(e&eb) minima of the classical action dominate (e) the Euclidean functional integral<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of perturbation theory, there is no a priori reason to believe (e) that individual orbits corresponding to(e&eb) minima of the classical action dominate (e) the Euclidean functional integral; **asymptotically free Yang-Mills theories the quantum ground state is not controlled**<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>It is argued that since in asymptotically free Yang-Mills theories the quantum ground state is not controlled by perturbation theory, there is no a priori reason to believe (e) that individual orbits corresponding to(e&eb) minima of the classical action dominate (e) the Euclidean functional integral | |
| $G_{16}$ : Category one of individual orbits corresponding to(e&eb) minima of the classical action dominate (e) the Euclidean functional integral<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of asymptotically free Yang-Mills theories the quantum ground state is not controlled by perturbation theory, there is no a priori reason to believe<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>It is argued that since in asymptotically free Yang-Mills theories the quantum ground state is not controlled by perturbation theory, there is no a priori reason to believe that individual orbits corresponding to(e&eb) minima of the classical action dominate (e) the Euclidean functional integral | |
| $G_{20}$ : Category one of **asymptotically free Yang-Mills theories the quantum ground state is not controlled by perturbation theory, there is no a priori reason to believe that individual orbits;** minima of the classical action dominate (e) the Euclidean functional integral<br>$G_{21}$ : Category two of SAS | |

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| $G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of minima of the classical action dominate (e) the Euclidean functional integral **;asymptotically free Yang-Mills theories the quantum ground state is not controlled by perturbation theory, there is no a priori reason to believe that individual orbits**<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>It is argued that since in asymptotically free Yang-Mills theories the quantum ground state is not controlled by perturbation theory, there is no a priori reason to believe that individual orbits corresponding to minima of the classical action dominate (e) the Euclidean functional integral | |
| $G_{24}$ : Category one of Euclidean functional integral<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of asymptotically free Yang-Mills theories the quantum ground state is not controlled by perturbation theory, there is no a priori reason to believe that individual orbits corresponding to minima of the classical action dominate<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>To examine and classify (e&eb) the vacua of the quantum gauge theory, authors propose an effective action in (eb) which the gauge field coupling constant g is replaced by (e&eb) the effective coupling g(t), t = ln[Fμva)2μ4] | |
| $G_{28}$ : Category one of **vacua of the quantum gauge theory, authors propose an effective action in which the gauge field coupling constant g is replaced;** effective coupling g(t), t = ln[Fμva)2μ4]<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of effective coupling g(t), t = ln[Fμva)2μ4];**vacua of the quantum gauge theory, authors propose an effective action in which the gauge field coupling constant g is replaced**<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>effective coupling g(t), t = ln[Fμva)2μ4] | |

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| Relationship between t, mu and its concomitant predictions | |
| $G_{32}$ : Category one of LHS of the above equation<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of RHS of the above equation<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>The vacua of this model correspond to (e&eb) paramagnetism and (e&eb) perfect paramagnetism, for (e) which the gauge field is (=) Fμva = 0, and ferromagnetism, for (e) which (Fμva) 2 = λ2, i.e. spontaneous magnetization of (e) the vacuum occurs | |
| $G_{36}$ : Category one of **vacua of this model;** paramagnetism and (e&eb) perfect paramagnetism, for (e) which the gauge field is (=) Fμva = 0, and ferromagnetism, for (e) which (Fμva) 2 = λ2, i.e. spontaneous magnetization of (e) the vacuum occurs<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of paramagnetism and (e&eb) perfect paramagnetism, for (e) which the gauge field is (=) Fμva = 0, and ferromagnetism, for (e) which (Fμva) 2 = λ2, i.e. spontaneous magnetization of (e) the vacuum occurs **;vacua of this model**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The vacua of this model correspond to paramagnetism and (e&eb) perfect paramagnetism, for (e) which the gauge field is (=) Fμva = 0, and ferromagnetism, for (e) which (Fμva) 2 = λ2, i.e. spontaneous magnetization of (e) the vacuum occurs | |
| $G_{40}$ : Category one of **vacua of this model correspond to paramagnetism;** perfect paramagnetism, for (e) which the gauge field is (=) Fμva = 0, and ferromagnetism, for (e) which (Fμva) 2 = λ2, i.e. spontaneous magnetization of (e) the vacuum occurs<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |

$T_{40}$ : Category one of perfect paramagnetism, for (e) which the gauge field is (=) Fμva = 0, and ferromagnetism, for (e) which (Fμva) 2 = λ2, i.e. spontaneous magnetization of (e) the vacuum occurs; **vacua of this model correspond to paramagnetism**

$T_{41}$ : Category two of SAS

$T_{42}$ : Category three of SAS

**Module Nine**

The vacua of this model correspond to paramagnetism and perfect paramagnetism, for which the gauge field is (=) Fμva = 0, and ferromagnetism, for (e) which (Fμva) 2 = λ2, i.e. spontaneous magnetization of (e) the vacuum occurs

$G_{44}$ : Category one of vacua of this model correspond to paramagnetism and perfect paramagnetism, for which the gauge field

$G_{45}$ : Category two of SAS

$G_{46}$ : Category three of SAS

$T_{44}$ : Category one of Fμva = 0, and ferromagnetism, for (e) which (Fμva) 2 = λ2, i.e. spontaneous magnetization of (e) the vacuum occurs

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

**The Coefficients:**

$(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:
$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$
$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,
$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$
$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$
$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$
$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$

are Accentuation coefficients
$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$
$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$
$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}\ (a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,
$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$
$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,
$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,
$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,
are Dissipation coefficients

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|---|---|
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14},t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14},t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14},t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G,t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G,t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14},t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G,t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17},t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17},t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17},t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17},t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}),t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |

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| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a'_{24})^{(4)} + (a''_{24})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a'_{25})^{(4)} + (a''_{25})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a'_{26})^{(4)} + (a''_{26})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b'_{24})^{(4)} - (b''_{24})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b'_{25})^{(4)} - (b''_{25})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b'_{26})^{(4)} - (b''_{26})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a''_{24})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b''_{24})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a'_{28})^{(5)} + (a''_{28})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a'_{29})^{(5)} + (a''_{29})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a'_{30})^{(5)} + (a''_{30})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b'_{28})^{(5)} - (b''_{28})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b'_{29})^{(5)} - (b''_{29})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b'_{30})^{(5)} - (b''_{30})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a''_{28})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b''_{28})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a'_{32})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b'_{32})^{(6)} - (b''_{32})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b'_{33})^{(6)} - (b''_{33})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b'_{34})^{(6)} - (b''_{34})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a''_{32})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b'_{36})^{(7)} - (b''_{36})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |

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| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b'_{37})^{(7)} - (b''_{37})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b'_{38})^{(7)} - (b''_{38})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a''_{36})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a'_{41})^{(8)} + (a''_{41})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b'_{40})^{(8)} - (b''_{40})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b'_{41})^{(8)} - (b''_{41})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b'_{42})^{(8)} - (b''_{42})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b'_{44})^{(9)} - (b''_{44})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b'_{45})^{(9)} - (b''_{45})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b'_{46})^{(9)} - (b''_{46})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a''_{44})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b''_{44})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a'_{13})^{(1)}\boxed{+(a''_{13})^{(1)}(T_{14},t)} & \boxed{+(a''_{16})^{(2,2,)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a''_{36})^{(7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a'_{14})^{(1)}\boxed{+(a''_{14})^{(1)}(T_{14},t)} & \boxed{+(a''_{17})^{(2,2,)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a''_{37})^{(7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a'_{15})^{(1)}\boxed{+(a''_{15})^{(1)}(T_{14},t)} & \boxed{+(a''_{18})^{(2,2,)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *57* |
| Where $\boxed{(a''_{13})^{(1)}(T_{14},t)}$, $\boxed{(a''_{14})^{(1)}(T_{14},t)}$, $\boxed{(a''_{15})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a''_{16})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3 | |

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| $+(a_{20}^{''})^{(3,3,)}(T_{21},t)$ , $+(a_{21}^{''})^{(3,3,)}(T_{21},t)$ , $+(a_{22}^{''})^{(3,3,)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{38}^{''})^{(7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3<br>$+(a_{40}^{''})^{(8,8)}(T_{41},t)$ $+(a_{41}^{''})^{(8,8)}(T_{41},t)$ $+(a_{42}^{''})^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3<br>$+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ , $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ , $+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G,t) & -(b_{16}^{''})^{(2,2,)}(G_{19},t) & -(b_{20}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,)}(G_{39},t) & -(b_{40}^{''})^{(8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}-(b_{14}^{''})^{(1)}(G,t) & -(b_{17}^{''})^{(2,2,)}(G_{19},t) & -(b_{21}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,)}(G_{39},t) & -(b_{41}^{''})^{(8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G,t) & -(b_{18}^{''})^{(2,2,)}(G_{19},t) & -(b_{22}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,)}(G_{39},t) & -(b_{42}^{''})^{(8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$ | *60* |
| Where $-(b_{13}^{''})^{(1)}(G,t)$ , $-(b_{14}^{''})^{(1)}(G,t)$ , $-(b_{15}^{''})^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{16}^{''})^{(2,2,)}(G_{19},t)$ , $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$ , $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{20}^{''})^{(3,3,)}(G_{23},t)$ , $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$ , $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$ , $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$ , $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$ , $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$ , $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{37}^{''})^{(7,7,)}(G_{39},t)$ , $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$ , $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 | |

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| $-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt}=(a_{16})^{(2)}G_{17}-\begin{bmatrix}(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,)}(T_{14},t) & +(a_{20}^{''})^{(3,3,3)}(T_{21},t)\\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)\\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9)}(T_{45},t)\end{bmatrix}G_{16}$ | 61 |
| $\frac{dG_{17}}{dt}=(a_{17})^{(2)}G_{16}-\begin{bmatrix}(a_{17}^{'})^{(2)}+(a_{17}^{''})^{(2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,)}(T_{14},t) & +(a_{21}^{''})^{(3,3,3)}(T_{21},t)\\ +(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)\\ +(a_{37}^{''})^{(7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9)}(T_{45},t)\end{bmatrix}G_{17}$ | 62 |
| $\frac{dG_{18}}{dt}=(a_{18})^{(2)}G_{17}-\begin{bmatrix}(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17},t) & +(a_{15}^{''})^{(1,1,)}(T_{14},t) & +(a_{22}^{''})^{(3,3,3)}(T_{21},t)\\ +(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)\\ +(a_{38}^{''})^{(7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9)}(T_{45},t)\end{bmatrix}G_{18}$ | 63 |
| Where $+(a_{16}^{''})^{(2)}(T_{17},t)$, $+(a_{17}^{''})^{(2)}(T_{17},t)$, $+(a_{18}^{''})^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt}=(b_{16})^{(2)}T_{17}-\begin{bmatrix}(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t)\\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)\\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t)\end{bmatrix}T_{16}$ | 64 |

$$\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b'_{17})^{(2)}\boxed{-(b''_{17})^{(2)}(G_{19},t)} & \boxed{-(b''_{14})^{(1,1,)}(G,t)} & \boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{17} \quad (65)$$

$$\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b'_{18})^{(2)}\boxed{-(b''_{18})^{(2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,)}(G,t)} & \boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{18} \quad (66)$$

where $\boxed{-(b''_{16})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{17})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{18})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,)}(G,t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{44})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2 and 3

$$\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a'_{20})^{(3)}\boxed{+(a''_{20})^{(3)}(T_{21},t)} & \boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{20} \quad (67)$$

$$\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a'_{21})^{(3)}\boxed{+(a''_{21})^{(3)}(T_{21},t)} & \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{21} \quad (68)$$

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a'_{22})^{(3)}\boxed{+(a''_{22})^{(3)}(T_{21},t)} & \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{22} \quad (69)$$

$\boxed{+(a''_{20})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

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| $+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)}\,-(b_{20}^{''})^{(3)}(G_{23},t) & -(b_{16}^{''})^{(2,2,2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,1,)}(G,t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{20}$ | *70* |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)}\,-(b_{21}^{''})^{(3)}(G_{23},t) & -(b_{17}^{''})^{(2,2,2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,1,)}(G,t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21}$ | *71* |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b_{22}^{'})^{(3)}\,-(b_{22}^{''})^{(3)}(G_{23},t) & -(b_{18}^{''})^{(2,2,2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,1,)}(G,t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{22}$ | *72* |
| $-(b_{20}^{''})^{(3)}(G_{23},t)$, $-(b_{21}^{''})^{(3)}(G_{23},t)$, $-(b_{22}^{''})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*<br>$-(b_{16}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*<br>$-(b_{13}^{''})^{(1,1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition* | |

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| *coefficients for category 1, 2 and 3*<br>$\boxed{-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)}$ $\boxed{-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{46}^{''})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[\begin{matrix} (a_{24}^{'})^{(4)}\boxed{+(a_{24}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{24}$ | *73* |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[\begin{matrix} (a_{25}^{'})^{(4)}\boxed{+(a_{25}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{25}$ | *74* |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[\begin{matrix} (a_{26}^{'})^{(4)}\boxed{+(a_{26}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{26}$ | *75* |
| $\boxed{(a_{24}^{''})^{(4)}(T_{25},t)}$, $\boxed{(a_{25}^{''})^{(4)}(T_{25},t)}$, $\boxed{(a_{26}^{''})^{(4)}(T_{25},t)}$ *are first augmentation coefficients category* 1, 2 3<br>$\boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3<br>$\boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3<br>$\boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)}$ *are fourth augmentation coefficients for category* 1, 2 *and* 3<br>$\boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)}$ *are fifth augmentation coefficients for category* 1, 2 *and* 3<br>$\boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)}$ *are sixth augmentation coefficients for category* 1, 2 *and* 3<br>$\boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)}$ *are seventh augmentation coefficients for category* 1, 2 *and* 3<br>$\boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)}$ *are eighth augmentation coefficients for category* 1, 2 *and* 3<br>$\boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)}$ are ninth detrition coefficients for category 1 2 3 | |

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| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b'_{24})^{(4)}\boxed{-(b''_{24})^{(4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1)}(G,t)} & \boxed{-(b''_{16})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{24}$ | 76 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b'_{25})^{(4)}\boxed{-(b''_{25})^{(4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1)}(G,t)} & \boxed{-(b''_{17})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{25}$ | 77 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b'_{26})^{(4)}\boxed{-(b''_{26})^{(4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{26}$ | 78 |
| *Where* $\boxed{-(b''_{24})^{(4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4)}(G_{27},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b''_{28})^{(5,5,)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,)}(G_{31},t)}$ *are second detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b''_{32})^{(6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,)}(G_{35},t)}$ *are third detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b''_{13})^{(1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,1)}(G,t)}$<br>*are fourth detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b''_{16})^{(2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2)}(G_{19},t)}$<br>*are fifth detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b''_{20})^{(3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3)}(G_{23},t)}$<br>*are sixth detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b''_{36})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)}$<br>*are seventh detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b''_{40})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)}$<br>*are eighth detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b''_{46})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a'_{28})^{(5)}\boxed{+(a''_{28})^{(5)}(T_{29},t)} & \boxed{+(a''_{24})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{32})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{28}$ | 79 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a'_{29})^{(5)}\boxed{+(a''_{29})^{(5)}(T_{29},t)} & \boxed{+(a''_{25})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{33})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{29}$ | 80 |

$$\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[\begin{array}{ccc} (a'_{30})^{(5)} & +(a''_{30})^{(5)}(T_{29},t) & +(a''_{26})^{(4,4,)}(T_{25},t) \quad +(a''_{34})^{(6,6,6)}(T_{33},t) \\ +(a''_{15})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3)}(T_{21},t) \\ +(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9)}(T_{45},t) \end{array}\right] G_{30} \quad (81)$$

*Where* $+(a''_{28})^{(5)}(T_{29},t)$, $+(a''_{29})^{(5)}(T_{29},t)$, $+(a''_{30})^{(5)}(T_{29},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

*And* $+(a''_{24})^{(4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,)}(T_{25},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3

$+(a''_{32})^{(6,6,6)}(T_{33},t)$, $+(a''_{33})^{(6,6,6)}(T_{33},t)$, $+(a''_{34})^{(6,6,6)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3

$+(a''_{16})^{(2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3

$+(a''_{20})^{(3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3

$+(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t)$ , $+(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3

$+(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3

$+(a''_{46})^{(9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9)}(T_{45},t)$, $+(a''_{44})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[\begin{array}{ccc} (b'_{28})^{(5)} & -(b''_{28})^{(5)}(G_{31},t) & -(b''_{24})^{(4,4,)}(G_{27},t) \quad -(b''_{32})^{(6,6,6)}(G_{35},t) \\ -(b''_{13})^{(1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3)}(G_{23},t) \\ -(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9)}(G_{47},t) \end{array}\right] T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[\begin{array}{ccc} (b'_{29})^{(5)} & -(b''_{29})^{(5)}(G_{31},t) & -(b''_{25})^{(4,4,)}(G_{27},t) \quad -(b''_{33})^{(6,6,6)}(G_{35},t) \\ -(b''_{14})^{(1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3)}(G_{23},t) \\ -(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9)}(G_{47},t) \end{array}\right] T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[\begin{array}{ccc} (b'_{30})^{(5)} & -(b''_{30})^{(5)}(G_{31},t) & -(b''_{26})^{(4,4,)}(G_{27},t) \quad -(b''_{34})^{(6,6,6)}(G_{35},t) \\ -(b''_{15})^{(1,1,1,1,1,)}(G,t) & -(b''_{18})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3)}(G_{23},t) \\ -(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9)}(G_{47},t) \end{array}\right] T_{30} \quad (84)$$

*where* $-(b''_{28})^{(5)}(G_{31},t)$ , $-(b''_{29})^{(5)}(G_{31},t)$ , $-(b''_{30})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{24})^{(4,4,)}(G_{27},t)$, $-(b''_{25})^{(4,4,)}(G_{27},t)$, $-(b''_{26})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3

$-(b''_{32})^{(6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3

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| $-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3<br>$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3<br>$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3<br>$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3<br>$-(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3<br>$-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3 | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a_{32}^{'})^{(6)} & +(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,4,)}(T_{25},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{32}$ | *85* |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a_{33}^{'})^{(6)} & +(a_{33}^{''})^{(6)}(T_{33},t) & +(a_{29}^{''})^{(5,5,5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,4,)}(T_{25},t) \\ & +(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{33}$ | *86* |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \begin{bmatrix} (a_{34}^{'})^{(6)} & +(a_{34}^{''})^{(6)}(T_{33},t) & +(a_{30}^{''})^{(5,5,5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,4,)}(T_{25},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34}$ | *87* |
| $+(a_{32}^{''})^{(6)}(T_{33},t)$, $+(a_{33}^{''})^{(6)}(T_{33},t)$, $+(a_{34}^{''})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{28}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients<br>$+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients<br>$+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients<br>$+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients | |

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| $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients | |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b_{32}^{'})^{(6)} & -(b_{32}^{''})^{(6)}(G_{35},t) & -(b_{28}^{''})^{(5,5,5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,4,)}(G_{27},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{32}$ | *88* |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b_{33}^{'})^{(6)} & -(b_{33}^{''})^{(6)}(G_{35},t) & -(b_{29}^{''})^{(5,5,5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,4,)}(G_{27},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{33}$ | *89* |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b_{34}^{'})^{(6)} & -(b_{34}^{''})^{(6)}(G_{35},t) & -(b_{30}^{''})^{(5,5,5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,4,)}(G_{27},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{34}$ | *90* |
| | |
| $-(b_{32}^{''})^{(6)}(G_{35},t)$, $-(b_{33}^{''})^{(6)}(G_{35},t)$, $-(b_{34}^{''})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3 $-(b_{28}^{''})^{(5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3 $-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3 $-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3 $-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3 $-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3 $-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3 $-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3 $-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3 | |

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a'_{36})^{(7)}\boxed{+(a''_{36})^{(7)}(T_{37},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a'_{37})^{(7)}\boxed{+(a''_{37})^{(7)}(T_{37},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a'_{38})^{(7)}\boxed{+(a''_{38})^{(7)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad (93)$$

Where $\boxed{(a''_{36})^{(7)}(T_{37},t)}$, $\boxed{(a''_{37})^{(7)}(T_{37},t)}$, $\boxed{(a''_{38})^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = \quad (94)$$

$$(b_{36})^{(7)}T_{37} - \begin{bmatrix} (b'_{36})^{(7)}\boxed{-(b''_{36})^{(7)}(G_{39},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{37}}{dt} =$$

$$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b'_{37})^{(7)}\boxed{-(b''_{37})^{(7)}(G_{39},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} =$$

$$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b'_{38})^{(7)}\boxed{-(b''_{38})^{(7)}(G_{39},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a'_{40})^{(8)}\boxed{+(a''_{40})^{(8)}(T_{41},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a'_{41})^{(8)}\boxed{+(a''_{41})^{(8)}(T_{41},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a'_{42})^{(8)}\boxed{+(a''_{42})^{(8)}(T_{41},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{40})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b'_{40})^{(8)}\boxed{-(b''_{40})^{(8)}(G_{43},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} =$$

$$(b_{41})^{(8)}T_{40} - \begin{bmatrix} (b'_{41})^{(8)}\boxed{-(b''_{41})^{(8)}(G_{43},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b'_{42})^{(8)}\boxed{-(b''_{42})^{(8)}(G_{43},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{45}}{dt} = (a_{45})^{(9)} G_{44} - \left[ \begin{array}{l} (a'_{45})^{(9)} \boxed{+(a''_{45})^{(9)}(T_{45},t)} \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array} \right] G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)} G_{45} - \left[ \begin{array}{l} (a'_{46})^{(9)} \boxed{+(a''_{46})^{(9)}(T_{37},t)} \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array} \right] G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)} T_{45} - \left[ \begin{array}{l} (b'_{44})^{(9)} \boxed{-(b''_{44})^{(9)}(G_{47},t)} \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array} \right] T_{13}$$

$$\frac{dT_{45}}{dt} =$$

$$(b_{45})^{(9)}T_{44}-\begin{bmatrix}(b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}\\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}\\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}\end{bmatrix}T_{14}$$

$$\frac{dT_{46}}{dt}=$$

$$(b_{46})^{(9)}T_{45}-\begin{bmatrix}(b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}\\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}\\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}\end{bmatrix}T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

$(a_i^{''})^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$

$(b_i^{''})^{(1)}(G,t) \le (r_i)^{(1)} \le (b_i^{'})^{(1)} \le (\hat{B}_{13})^{(1)}$

$lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$ 98

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|---|---|
| $\lim_{G\to\infty}(b_i'')^{(1)}(G,t) = (r_i)^{(1)}$<br><br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br><br>Where $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}$ are positive constants and $i = 13,14,15$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(1)}(T_{14}',t) - (a_i'')^{(1)}(T_{14},t)\rvert \le (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}'\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\lvert(b_i'')^{(1)}(G',t) - (b_i'')^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G'\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(1)}(T_{14}',t)$ and $(a_i'')^{(1)}(T_{14},t)$ . $(T_{14}',t)$ and $(T_{14},t)$ are points belonging to the interval $[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}]$ . It is to be noted that $(a_i'')^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i'')^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i')^{(1)}, (b_i)^{(1)}, (b_i')^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i,j = 16,17,18$ | |
| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17},t) \le (p_i)^{(2)} \le (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19},t) \le (r_i)^{(2)} \le (b_i')^{(2)} \le (\hat{B}_{16})^{(2)}$ | 103 |

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| $\lim_{T_2\to\infty}(a_i^{''})^{(2)}(T_{17},t)=(p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty}(b_i^{''})^{(2)}\big((G_{19}),t\big)=\ (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)}$ :<br>Where $\boxed{(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)},(p_i)^{(2)},\ (r_i)^{(2)}}$ are positive constants and $\boxed{i=16{,}17{,}18}$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'},t)-(a_i^{''})^{(2)}(T_{17},t)\rvert\le(\hat{k}_{16})^{(2)}\lvert T_{17}-T_{17}^{'}\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'},t)-(b_i^{''})^{(2)}\big((G_{19}),t\big)\rvert<(\hat{k}_{16})^{(2)}\lvert\lvert(G_{19})-(G_{19})^{'}\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'},t)$ and$(a_i^{''})^{(2)}(T_{17},t)$ . $(T_{17}^{'},t)$ and $(T_{17},t)$ are points belonging to the interval $\big[(\hat{k}_{16})^{(2)},(\hat{M}_{16})^{(2)}\big]$. It is to be noted that $(a_i^{''})^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)}=1$ then the function $(a_i^{''})^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\ ,\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}<1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)},(\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)},(\hat{A}_{16})^{(2)}and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)},(a_i^{'})^{(2)},(b_i)^{(2)},(b_i^{'})^{(2)},(p_i)^{(2)},\ (r_i)^{(2)},i=16{,}17{,}18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)}+(a_i^{'})^{(2)}+\ (\hat{A}_{16})^{(2)}+\ (\hat{P}_{16})^{(2)}\ (\hat{k}_{16})^{(2)}]<1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[\ (b_i)^{(2)}+(b_i^{'})^{(2)}+\ (\hat{B}_{16})^{(2)}+\ (\hat{Q}_{16})^{(2)}\ \ (\hat{k}_{16})^{(2)}]<1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)},(a_i^{'})^{(3)},(a_i^{''})^{(3)},(b_i)^{(3)},(b_i^{'})^{(3)},(b_i^{''})^{(3)}>0,\quad i,j=20{,}21{,}22$<br>The functions $(a_i^{''})^{(3)},(b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(3)},\ (\mathrm{r_i})^{(3)}$: | 112 |

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| $(a_i^{''})^{(3)}(T_{21},t) \le (p_i)^{(3)} \le (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23},t) \le (r_i)^{(3)} \le (b_i^{'})^{(3)} \le (\hat{B}_{20})^{(3)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(3)}(T_{21},t) = (p_i)^{(3)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'},t) - (a_i^{''})^{(3)}(T_{21},t)\rvert \le (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'},t) - (b_i^{''})^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'},t)$ and $(a_i^{''})^{(3)}(T_{21},t)$ . $(T_{21}^{'},t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)},$ are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$: | 117 |

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|---|---|
| $(a_i^{''})^{(4)}(T_{25},t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}((G_{27}),t) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(4)}(T_{25},t) = (p_i)^{(4)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(4)}((G_{27}),t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'},t) - (a_i^{''})^{(4)}(T_{25},t)\rvert \leq (\hat{k}_{24})^{(4)}\lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'},t) - (b_i^{''})^{(4)}((G_{27}),t)\rvert < (\hat{k}_{24})^{(4)}\lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)}t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'},t)$ and$(a_i^{''})^{(4)}(T_{25},t)$ . $(T_{25}^{'},t)$ and $(T_{25},t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)}(\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)} + (b_i^{'})^{(4)} + (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}(\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded. | 122 |

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| **Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29},t) \le (p_i)^{(5)} \le (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}\big((G_{31}),t\big) \le (r_i)^{(5)} \le (b_i^{'})^{(5)} \le (\hat{B}_{28})^{(5)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(5)}(T_{29},t) = (p_i)^{(5)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(5)}(G_{31},t) = (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'},t) - (a_i^{''})^{(5)}(T_{29},t)\rvert \le (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'},t) - (b_i^{''})^{(5)}\big((G_{31}),t\big)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'},t)$ and$(a_i^{''})^{(5)}(T_{29},t)$ . $(T_{29}^{'},t)$ and $(T_{29},t)$ are points belonging to the interval $\big[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\big]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br><br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)}$, $(r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$ | 126 |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$ | 127 |

The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:

$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$

$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$

$lim_{T_2 \to \infty}(a_i^{''})^{(6)}(T_{33}, t) = (p_i)^{(6)}$

$\lim_{G \to \infty}(b_i^{''})^{(6)}\big((G_{35}), t\big) = (r_i)^{(6)}$

**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :

Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ 128

They satisfy Lipschitz condition:

$|(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)| \leq (\hat{k}_{32})^{(6)}|T_{33} - T_{33}^{'}|e^{-(\hat{M}_{32})^{(6)}t}$

$|(b_i^{''})^{(6)}((G_{35})^{'}, t) - (b_i^{''})^{(6)}\big((G_{35}), t\big)| < (\hat{k}_{32})^{(6)}||(G_{35}) - (G_{35})^{'}||e^{-(\hat{M}_{32})^{(6)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and $(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $\big[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\big]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ : 129

$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants

$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$

**Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ : 130

There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities

$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$

$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$

Where we suppose

(KKKKK) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i,j = 36,37,38$ 131

(LLLLL) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}$, $(r_i)^{(7)}$:

$$(a_i^{''})^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$$

$$(b_i^{''})^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$$

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*(MMMMM)* $\lim_{T_2 \to \infty} (a_i^{''})^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(NNNNN)*

$\lim_{G \to \infty} (b_i^{''})^{(7)}\big((G_{39}), t\big) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

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They satisfy Lipschitz condition:

$$|(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)} |T_{37} - T_{37}^{'}| e^{-(\hat{M}_{36})^{(7)} t}$$

$$|(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}\big((G_{39}), t\big)| < (\hat{k}_{36})^{(7)} ||(G_{39}) - (G_{39})^{'}|| e^{-(\hat{M}_{36})^{(7)} t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'}, t)$ and $(a_i^{''})^{(7)}(T_{37}, t)$ . $(T_{37}^{'}, t)$ and $(T_{37}, t)$ are points belonging to the interval $\big[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}\big]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ : 134

(OOOOO) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$, are positive constants

$$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$$

**Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ : 135

(PPPPP) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities

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| $\frac{1}{(\widehat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\widehat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty}(a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty}(b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)}\lvert T_{41} - T_{41}^{'}\rvert e^{-(\widehat{M}_{40})^{(8)}t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})^{'}, t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)}\lvert\lvert(G_{43}) - (G_{43})^{'}\rvert\rvert e^{-(\widehat{M}_{40})^{(8)}t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\widehat{M}_{40})^{(8)}]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\widehat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\widehat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants | |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ : | |

There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$

Satisfy the inequalities

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1 \quad (145)$$

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1 \quad (146)$$

Where we suppose

$$(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i,j = 44,45,46 \quad (146A)$$

The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:

$$(a_i'')^{(9)}(T_{45},t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$$

$$(b_i'')^{(9)}(G_{47},t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$$

$$\lim_{T_2\to\infty}(a_i'')^{(9)}(T_{45},t) = (p_i)^{(9)}$$

$$\lim_{G\to\infty}(b_i'')^{(9)}(G_{47},t) = (r_i)^{(9)}$$

**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :

Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$

They satisfy Lipschitz condition:

$$|(a_i'')^{(9)}(T_{45}',t) - (a_i'')^{(9)}(T_{45},t)| \le (\hat{k}_{44})^{(9)}|T_{45} - T_{45}'|e^{-(\hat{M}_{44})^{(9)}t}$$

$$|(b_i'')^{(9)}((G_{47})',t) - (b_i'')^{(9)}((G_{47}),t)| < (\hat{k}_{44})^{(9)}||(G_{47}) - (G_{47})'||e^{-(\hat{M}_{44})^{(9)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}',t)$ and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i'')^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants

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| $\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$ | |
| **Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :<br><br>There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}, i = 44,45,46,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$<br><br>$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$ | |
| **Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>Definition of $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 147 |
| **Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$<br><br>$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $G_i(0) = G_i^0 > 0$<br><br>$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | 148 |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br><br>$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions | 151 |

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| **Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le \left(\hat{P}_{13}\right)^{(1)}$ , $T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le \left(\hat{P}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{16})^{(2)}$, $T_i^0 \le (\hat{Q}_{16})^{(2)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[ (a_{18})^{(2)} G_{17}(s_{(16)}) - \left( (a_{18}')^{(2)} + (a_{18}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{18}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[ (b_{16})^{(2)} T_{17}(s_{(16)}) - \left( (b_{16}')^{(2)} - (b_{16}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{16}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[ (b_{17})^{(2)} T_{16}(s_{(16)}) - \left( (b_{17}')^{(2)} - (b_{17}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[ (b_{18})^{(2)} T_{17}(s_{(16)}) - \left( (b_{18}')^{(2)} - (b_{18}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{18}(s_{(16)}) \right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |

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| **Proof:**<br><br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{20})^{(3)}$ , $T_i^0 \leq (\hat{Q}_{20})^{(3)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br><br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a'_{20})^{(3)} + a''_{20})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a'_{21})^{(3)} + (a''_{21})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a'_{22})^{(3)} + (a''_{22})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b'_{20})^{(3)} - (b''_{20})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b'_{21})^{(3)} - (b''_{21})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b'_{22})^{(3)} - (b''_{22})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br><br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{24})^{(4)}$ , $T_i^0 \leq (\hat{Q}_{24})^{(4)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br><br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |

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| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[(b_{24})^{(4)} T_{25}(s_{(24)}) - \left((b_{24}')^{(4)} - (b_{24}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{24}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[(b_{25})^{(4)} T_{24}(s_{(24)}) - \left((b_{25}')^{(4)} - (b_{25}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b_{26}')^{(4)} - (b_{26}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{28})^{(5)}$ , $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a_{28}')^{(5)} + a_{28}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a_{29}')^{(5)} + (a_{29}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a_{30}')^{(5)} + (a_{30}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b_{28}')^{(5)} - (b_{28}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b_{29}')^{(5)} - (b_{29}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b_{30}')^{(5)} - (b_{30}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{32})^{(6)}$ , $T_i^0 \le (\hat{Q}_{32})^{(6)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
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| $\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a'_{32})^{(6)} + a''_{32})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b'_{32})^{(6)} - (b''_{32})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b'_{33})^{(6)} - (b''_{33})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b'_{34})^{(6)} - (b''_{34})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{36})^{(7)}$, $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a'_{36})^{(7)} + a''_{36})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b'_{36})^{(7)} - (b''_{36})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b'_{37})^{(7)} - (b''_{37})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b'_{38})^{(7)} - (b''_{38})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |

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| **Proof:**<br><br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{40})^{(8)}$ , $T_i^0 \leq (\hat{Q}_{40})^{(8)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| By<br><br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{40})^{(8)} + a''_{40})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{40}(s_{(40)}) \right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[ (a_{41})^{(8)} G_{40}(s_{(40)}) - \left( (a'_{41})^{(8)} + (a''_{41})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{41}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[ (a_{42})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{42})^{(8)} + (a''_{42})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{42}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[ (b_{40})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{40})^{(8)} - (b''_{40})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{40}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[ (b_{41})^{(8)} T_{40}(s_{(40)}) - \left( (b'_{41})^{(8)} - (b''_{41})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{41}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[ (b_{42})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{42})^{(8)} - (b''_{42})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{42}(s_{(40)}) \right] ds_{(40)}$<br><br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br><br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[ (a_{44})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{44}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[ (a_{45})^{(9)} G_{44}(s_{(44)}) - \left( (a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{45}(s_{(44)}) \right] ds_{(44)}$ | |

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| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{\mathrm{T}}_{46}(\mathrm{t}) = \mathrm{T}_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[((\hat{P}_{13})^{(1)} + G_{14}^0) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that<br>$(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left[((\hat{P}_{16})^{(2)} + G_{17}^0) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} \left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$<br>$\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}} \left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$ | 171 |

From which it follows that 172

$$(G_{20}(t)-G_{20}^{0})e^{-(\bar{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^{0})e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^{0}}{G_{21}^{0}}\right)}+(\hat{P}_{20})^{(3)}\right]$$

Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$

The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that 173

$$G_{24}(t) \le G_{24}^{0}+\int_{0}^{t}\left[(a_{24})^{(4)}\left(G_{25}^{0}+(\hat{P}_{24})^{(4)}e^{(\bar{M}_{24})^{(4)}s_{(24)}}\right)\right]ds_{(24)} =$$

$$(1+(a_{24})^{(4)}t)G_{25}^{0}+\frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left(e^{(\bar{M}_{24})^{(4)}t}-1\right)$$

From which it follows that 174

$$(G_{24}(t)-G_{24}^{0})e^{-(\bar{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^{0})e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^{0}}{G_{25}^{0}}\right)}+(\hat{P}_{24})^{(4)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 4

The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{28}(t) \le G_{28}^{0}+\int_{0}^{t}\left[(a_{28})^{(5)}\left(G_{29}^{0}+(\hat{P}_{28})^{(5)}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\right)\right]ds_{(28)} =$$

$$(1+(a_{28})^{(5)}t)G_{29}^{0}+\frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)}t}-1\right)$$

From which it follows that 175

$$(G_{28}(t)-G_{28}^{0})e^{-(\bar{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^{0})e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^{0}}{G_{29}^{0}}\right)}+(\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 176

$$G_{32}(t) \le G_{32}^{0}+\int_{0}^{t}\left[(a_{32})^{(6)}\left(G_{33}^{0}+(\hat{P}_{32})^{(6)}e^{(\bar{M}_{32})^{(6)}s_{(32)}}\right)\right]ds_{(32)} =$$

$$(1+(a_{32})^{(6)}t)G_{33}^{0}+\frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left(e^{(\bar{M}_{32})^{(6)}t}-1\right)$$

From which it follows that 177

$$(G_{32}(t)-G_{32}^{0})e^{-(\bar{M}_{32})^{(6)}t} \le \frac{(a_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^{0})e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^{0}}{G_{33}^{0}}\right)}+(\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}$ , $G_{26}, T_{24}, T_{25}, T_{26}$

(t) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 178

$$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} \left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$$
$$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$$

From which it follows that

$$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left[\left((\hat{P}_{36})^{(7)} + G_{37}^0\right) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} \left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$$
$$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$$ 180

From which it follows that 181

$$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left[\left((\hat{P}_{40})^{(8)} + G_{41}^0\right) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 8

Analogous inequalities hold also for $G_{41}$ , $G_{42}, T_{40}, T_{41}, T_{42}$

The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} \left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$$
$$\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left(e^{(\hat{M}_{44})^{(9)} t} - 1\right)$$

From which it follows that

$$(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left[\left((\hat{P}_{44})^{(9)} + G_{45}^0\right) e^{\left(-\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0}\right)} + (\hat{P}_{44})^{(9)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 9

Analogous inequalities hold also for $G_{45}$ , $G_{46}, T_{44}, T_{45}, T_{46}$

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| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\widehat{M}_{13})^{(1)}} < 1$ and to choose<br>$(\widehat{P}_{13})^{(1)}$ and $(\widehat{Q}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}\left[(\widehat{P}_{13})^{(1)} + \left((\hat{P}_{13})^{(1)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{13})^{(1)}\right] \leq (\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)}, T^{(1)}\right), \left(G^{(2)}, T^{(2)}\right)\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{13})^{(1)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G}, \tilde{T} : (\tilde{G}, \tilde{T}) = \mathcal{A}^{(1)}(G, T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{13})^{(1)}\left|G_{14}^{(1)} - G_{14}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\,ds_{(13)} +$<br>$\int_0^t\{(a'_{13})^{(1)}\left|G_{13}^{(1)} - G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} +$<br>$(a''_{13})^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right)\left|G_{13}^{(1)} - G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}} +$<br>$G_{13}^{(2)}\left|(a''_{13})^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right) - (a''_{13})^{(1)}\left(T_{14}^{(2)}, s_{(13)}\right)\right|\ e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$<br>Where $s_{(13)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses it follows | |
| $\left\|G^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}t} \leq$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)}, T^{(1)};\ G^{(2)}, T^{(2)}\right)\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | |

condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 2:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$

From 19 to 24 it results

$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(1)} - (a_i^{''})^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$

$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(1)}t)} > 0$ for t > 0

**Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$ : 187

**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if

$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a_{14}^{'})^{(1)}G_{14}$ and by integrating

$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$

In the same way , one can obtain

$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}^{'})^{(1)}$

If $G_{14}\ or\ G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively.

**Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}\ and\ G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. 188

**Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t), t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$. 189

**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :

Indeed let $t_1$ be so that for $t > t_1$

$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t), t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$

Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to

$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results

$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right),\quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.
The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t), t) = (b_{15}^{'})^{(1)}$

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| We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)}, (T_{19})^{(1)}), ((G_{19})^{(2)}, (T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)}\left\|G_{17}^{(1)} - G_{17}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\,ds_{(16)} +$<br>$\int_0^t\{(a_{16}')^{(2)}\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{-(\widehat{M}_{16})^{(2)}s_{(16)}} +$<br>$(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}} +$<br>$G_{16}^{(2)}\left\|(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}'')^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}')^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}\right)d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |

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| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ and $(\widehat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)}t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $((\widehat{M}_{16})^{(2)})_1, ((\widehat{M}_{16})^{(2)})_2$ and $((\widehat{M}_{16})^{(2)})_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq ((\widehat{M}_{16})^{(2)})_1 - (a_{17}^{'})^{(2)}G_{17}$ and by integrating<br><br>$G_{17} \leq ((\widehat{M}_{16})^{(2)})_2 = G_{17}^0 + 2(a_{17})^{(2)}((\widehat{M}_{16})^{(2)})_1/(a_{17}^{'})^{(2)}$<br><br>In the same way , one can obtain<br><br>$G_{18} \leq ((\widehat{M}_{16})^{(2)})_3 = G_{18}^0 + 2(a_{18})^{(2)}((\widehat{M}_{16})^{(2)})_2/(a_{18}^{'})^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | 201 |
| **Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(2)}((G_{19})(t), t)) = (b_{17}^{'})^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i^{''})^{(2)}((G_{19})(t), t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded. | 205 |

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| The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right)=(b_{18}')^{(2)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}<1$ and to choose<br>$(\widehat{P}_{20})^{(3)}$ and $(\widehat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[(\widehat{P}_{20})^{(3)}+\left((\widehat{P}_{20})^{(3)}+G_j^0\right)e^{-\left(\frac{(\widehat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right]\leq(\widehat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[\left((\widehat{Q}_{20})^{(3)}+T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)}+(\widehat{Q}_{20})^{(3)}\right]\leq(\widehat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | 211 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}},\widetilde{T_{23}}:\left(\widetilde{(G_{23})},\widetilde{(T_{23})}\right)=\mathcal{A}^{(3)}((G_{23}),(T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{20})^{(3)}\left\|G_{21}^{(1)}-G_{21}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)}+$<br>$\int_0^t\{(a_{20}')^{(3)}\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$(a_{20}'')^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$G_{20}^{(2)}\left\\|(a_{20}'')^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)-(a_{20}'')^{(3)}\left(T_{21}^{(2)},s_{(20)}\right)\right\|\ e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)}-G_{23}{}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}t}\leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)}+(a_{20}')^{(3)}+(\widehat{A}_{20})^{(3)}+(\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)},(T_{23})^{(1)};\ (G_{23})^{(2)},(T_{23})^{(2)})\right)$ | 214 |

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| And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ *and* $(\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(3)}-(a_i^{''})^{(3)}(T_{21}(s_{(20)}),s_{(20)})\}ds_{(20)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)}t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2\ and\ \left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}\ and\ G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)}G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br><br>If $G_{21}\ or\ G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}\ \ is$ bounded, from below, the same property holds for $G_{21}\ and\ G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}\,(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results | 220 |

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| $T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right), \quad t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded. The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $\widetilde{(G_{27})},\widetilde{(T_{27})}$ : $\left(\widetilde{(G_{27})},\widetilde{(T_{27})}\right) = \mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{24})^{(4)}\left\|G_{25}^{(1)} - G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)} +$<br>$\int_0^t\{(a_{24}^{'})^{(4)}\left\|G_{24}^{(1)} - G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}} +$<br>$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)} - G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}} +$<br>$G_{24}^{(2)}\left\|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right) - (a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\| \; e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)} - (G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t} \leq$ | 226 |

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| $\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)} + (a_{24}^{'})^{(4)} + (\widehat{A}_{24})^{(4)} + (\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i = 24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(4)}t\right)} > 0$ for t > 0 | 228 |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ :<br><br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if<br><br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}^{'})^{(4)}G_{25}$ and by integrating<br><br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a_{25}^{'})^{(4)}$<br><br>In the same way , one can obtain<br><br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a_{26}^{'})^{(4)}$<br><br>If $G_{25}\ or\ G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}\ and\ G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t),t)) = (b_{25}^{'})^{(4)}$ then $T_{25} \to \infty$.<br><br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br><br>Indeed let $t_4$ be so that for $t > t_4$<br><br>$(b_{25})^{(4)} - (b_i^{''})^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to | 232 |

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1-e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right), \quad t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42

Analogous inequalities hold also for $G_{29}, G_{30}, T_{28}, T_{29}, T_{30}$

It is now sufficient to take $\frac{(a_i)^{(5)}}{(\bar{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\bar{M}_{28})^{(5)}} < 1$ and to choose

$(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ large to have

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$$\frac{(a_i)^{(5)}}{(\bar{M}_{28})^{(5)}}\left[(\hat{P}_{28})^{(5)} + \left((\hat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{28})^{(5)}$$

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$$\frac{(b_i)^{(5)}}{(\bar{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{28})^{(5)}\right] \leq (\hat{Q}_{28})^{(5)}$$

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In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric

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$$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{28})^{(5)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{28})^{(5)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{31}}), (\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right) = \mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$

It results

$$\left|\tilde{G}_{28}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{28})^{(5)}\left|G_{29}^{(1)} - G_{29}^{(2)}\right|e^{-(\bar{M}_{28})^{(5)}s_{(28)}}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)} +$$

$$\int_0^t\{(a_{28}^{'})^{(5)}\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\bar{M}_{28})^{(5)}s_{(28)}}e^{-(\bar{M}_{28})^{(5)}s_{(28)}} +$$

$$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\bar{M}_{28})^{(5)}s_{(28)}}e^{(\bar{M}_{28})^{(5)}s_{(28)}} +$$

$$G_{28}^{(2)}\left|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right) - (a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right| \; e^{-(\bar{M}_{28})^{(5)}s_{(28)}}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$$

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| Where $s_{(28)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on it follows | |
| $\left\|(G_{31})^{(1)} - (G_{31})^{(2)}\right\| e^{-(\widehat{M}_{28})^{(5)}t} \leq$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)} + (a'_{28})^{(5)} + (\widehat{A}_{28})^{(5)} + (\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right) d\left(((G_{31})^{(1)}, (T_{31})^{(1)};\ (G_{31})^{(2)}, (T_{31})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a''_{28})^{(5)}$ and $(b''_{28})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(5)}$ and $(b''_i)^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(5)} - (a''_i)^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(5)}t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}\ and\ G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a'_{29})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a'_{29})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a'_{30})^{(5)}$<br><br>If $G_{29}\ or\ G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}\ and\ G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(5)}((G_{31})(t), t)) = (b'_{29})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ : | 242 |

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| Indeed let $t_5$ be so that for $t > t_5$<br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t), t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t), t\right) = (b_{30}^{'})^{(5)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br>Analogous inequalities hold also for $G_{33}, G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\bar{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\bar{M}_{32})^{(6)}} < 1$ and to choose<br>$(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\bar{M}_{32})^{(6)}}\left[(\hat{P}_{32})^{(6)} + \left((\hat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\bar{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{32})^{(6)}\right] \leq (\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)}, (T_{35})^{(1)}), ((G_{35})^{(2)}, (T_{35})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}), (\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}), (\widetilde{T_{35}})\right) = \mathcal{A}^{(6)}\left((G_{35}), (T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{32})^{(6)}\left|G_{33}^{(1)} - G_{33}^{(2)}\right|e^{-(\bar{M}_{32})^{(6)}s_{(32)}}e^{(\bar{M}_{32})^{(6)}s_{(32)}}\,ds_{(32)} +$<br>$\int_0^t\{(a_{32}^{'})^{(6)}\left|G_{32}^{(1)} - G_{32}^{(2)}\right|e^{-(\bar{M}_{32})^{(6)}s_{(32)}}e^{-(\bar{M}_{32})^{(6)}s_{(32)}} +$ | 247 |

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| $(a_{32}^{''})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)\left\|G_{32}^{(1)} - G_{32}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)} s_{(32)}} e^{(\widehat{M}_{32})^{(6)} s_{(32)}} +$ <br> $G_{32}^{(2)} \left\|(a_{32}^{''})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right) - (a_{32}^{''})^{(6)}\left(T_{33}^{(2)}, s_{(32)}\right)\right\| \; e^{-(\widehat{M}_{32})^{(6)} s_{(32)}} e^{(\widehat{M}_{32})^{(6)} s_{(32)}} \} ds_{(32)}$ <br> Where $s_{(32)}$ represents integrand that is integrated over the interval $[0, t]$ <br> From the hypotheses it follows | |
| $\left\|(G_{35})^{(1)} - (G_{35})^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)} t} \leq$ <br> $\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)} + (a_{32}^{'})^{(6)} + (\widehat{A}_{32})^{(6)} + (\widehat{P}_{32})^{(6)} (\widehat{k}_{32})^{(6)}\right) d\left(\left((G_{35})^{(1)}, (T_{35})^{(1)}; \; (G_{35})^{(2)}, (T_{35})^{(2)}\right)\right)$ <br> And analogous inequalities for $G_i \; and \; T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a_{32}^{''})^{(6)}$ and $(b_{32}^{''})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)} e^{(\widehat{M}_{32})^{(6)} t} \; and \; (\widehat{Q}_{32})^{(6)} e^{(\widehat{M}_{32})^{(6)} t}$ respectively of $\mathbb{R}_+$. <br> If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i = 32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})(and \; not \; on \; t)$ and hypothesis can replaced by a usual Lipschitz condition. | 249 |
| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0 \; and \; T_i(t) = 0$ <br> it results <br> $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(6)} - (a_i^{''})^{(6)} (T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$ <br> $T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(6)} t\right)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2 \; and \; \left((\widehat{M}_{32})^{(6)}\right)_3$ : <br> **Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33} \; and \; G_{34}$ . indeed if <br> $G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}^{'})^{(6)} G_{33}$ and by integrating <br> $G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}^{'})^{(6)}$ <br> In the same way , one can obtain <br> $G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}^{'})^{(6)}$ <br> If $G_{33} \; or \; G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33} \; and \; G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |

**Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$. 253

**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :

Indeed let $t_6$ be so that for $t > t_6$

$(b_{33})^{(6)} - (b_i^{''})^{(6)}\big((G_{35})(t),t\big) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$

Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to 254

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right), \quad t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.
The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

Analogous inequalities hold also for $G_{37}, G_{38}, T_{36}, T_{37}, T_{38}$ 255

It is now sufficient to take $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ and to choose
$(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ large to have

$$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[(\hat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)} \quad 256$$

$$\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)} \quad 257$$

In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric 258

$$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right) =$$
$$\sup_i\{\max_{t\in\mathbb{R}_+}|G_i^{(1)}(t) - G_i^{(2)}(t)|e^{-(\hat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}|T_i^{(1)}(t) - T_i^{(2)}(t)|e^{-(\hat{M}_{36})^{(7)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{39})}, \widetilde{(T_{39})}$ : $\left(\widetilde{(G_{39})}, \widetilde{(T_{39})}\right) = \mathcal{A}^{(7)}((G_{39}),(T_{39}))$
It results

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| $\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{36})^{(7)} \left\|G_{37}^{(1)}-G_{37}^{(2)}\right\| e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\, ds_{(36)} +$ $\int_0^t \{(a_{36}^{'})^{(7)} \left\|G_{36}^{(1)}-G_{36}^{(2)}\right\| e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} +$ $(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\| e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} e^{(\widehat{M}_{36})^{(7)}s_{(36)}} +$ $G_{36}^{(2)} \left\|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right)-(a_{36}^{''})^{(7)}\left(T_{37}^{(2)}, s_{(36)}\right)\right\| \; e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\} ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on it follows | |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\| e^{-(\widehat{M}_{36})^{(7)}t} \leq$ $\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}^{'})^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right) d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)}t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2$ *and* $\left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}$ *and* $G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)}G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1/(a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}^{'})^{(7)}$ | 262 |

If $G_{37}$ *or* $G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively.

**Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}$ *and* $G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. 263

**Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t),t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$. 264

**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :

Indeed let $t_7$ be so that for $t > t_7$

$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$

Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to 265

$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results

$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right)$, $t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.
The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t),t\right) = (b_{38}^{'})^{(7)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations

It is now sufficient to take $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ and to choose 266
$(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ large to have

$$\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\left[(\hat{P}_{40})^{(8)} + \left((\hat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{40})^{(8)} \quad 267$$

$$\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{40})^{(8)}\right] \leq (\hat{Q}_{40})^{(8)} \quad 268$$

In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric

$$d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right) = \quad 269$$
$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{40})^{(8)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{40})^{(8)}t}\}$$

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| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})}, \widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})}, \widetilde{(T_{43})}\right) = \mathcal{A}^{(8)}((G_{43}), (T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{40})^{(8)} \left\|G_{41}^{(1)} - G_{41}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}}\, ds_{(40)} +$<br>$\int_0^t \{(a'_{40})^{(8)} \left\|G_{40}^{(1)} - G_{40}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} +$<br>$(a''_{40})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}} +$<br>$G_{40}^{(2)} \left\|(a''_{40})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right) - (a''_{40})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)\right\| \; e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}}\} ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)} - (G_{43})^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} t} \leq$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)} + (a'_{40})^{(8)} + (\widehat{A}_{40})^{(8)} + (\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right) d\left(((G_{43})^{(1)}, (T_{43})^{(1)};\ (G_{43})^{(2)}, (T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a''_{40})^{(8)}$ and $(b''_{40})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)} t}$ *and* $(\widehat{Q}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(8)}$ and $(b_i'')^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(8)} - (a_i'')^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(8)} t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ *and* $G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a'_{41})^{(8)} G_{41}$ and by integrating | 276 |

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| $G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1/(a_{41}')^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2/(a_{42}')^{(8)}$<br><br>If $G_{41}$ *or* $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | |
| **Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}$ *and* $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(8)}((G_{43})(t),t)) = (b_{41}')^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i'')^{(8)}\left((G_{43})(t),t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$ | 278 |
| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right)$, $t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded. The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}'')^{(8)}\left((G_{43})(t),t(t),t\right) = (b_{42}')^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$ | |
| In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself | |
| The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric | |

$$d\left(((G_{47})^{(1)},(T_{47})^{(1)}),((G_{47})^{(2)},(T_{47})^{(2)})\right)=$$
$$sup_i\{\max_{t\in\mathbb{R}_+}|G_i^{(1)}(t)-G_i^{(2)}(t)|e^{-(\widehat{M}_{44})^{(9)}t},\max_{t\in\mathbb{R}_+}|T_i^{(1)}(t)-T_i^{(2)}(t)|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{47}}),(\widetilde{T_{47}}):\left((\widetilde{G_{47}}),(\widetilde{T_{47}})\right)=\mathcal{A}^{(9)}((G_{47}),(T_{47}))$
It results

$$|\tilde{G}_{44}^{(1)}-\tilde{G}_i^{(2)}|\le\int_0^t(a_{44})^{(9)}|G_{45}^{(1)}-G_{45}^{(2)}|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}ds_{(44)}+$$
$$\int_0^t\{(a_{44}^{'})^{(9)}|G_{44}^{(1)}-G_{44}^{(2)}|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$(a_{44}^{''})^{(9)}(T_{45}^{(1)},s_{(44)})|G_{44}^{(1)}-G_{44}^{(2)}|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$G_{44}^{(2)}|(a_{44}^{''})^{(9)}(T_{45}^{(1)},s_{(44)})-(a_{44}^{''})^{(9)}(T_{45}^{(2)},s_{(44)})|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$|(G_{47})^{(1)}-G^{(2)}|e^{-(\widehat{M}_{44})^{(9)}t}\le$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}((a_{44})^{(9)}+(a_{44}^{'})^{(9)}+(\widehat{A}_{44})^{(9)}+(\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)})d\left(((G_{47})^{(1)},(T_{47})^{(1)};\ (G_{47})^{(2)},(T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}\ and\ (\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i=44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

From 99 to 44 it results

$$G_i\ (t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(9)}-(a_i^{''})^{(9)}(T_{45}(s_{(44)}),s_{(44)})\}ds_{(44)}\right]}\ge 0$$

$$T_i\ (t)\ge T_i^0e^{(-(b_i^{'})^{(9)}t)}>0\quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1,\left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44}<(\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt}\le\left((\widehat{M}_{44})^{(9)}\right)_1-(a_{45}^{'})^{(9)}G_{45}$ and by integrating

$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}')^{(9)}$

In the same way , one can obtain

$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}')^{(9)}$

If $G_{45}$ *or* $G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}$ *and* $G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(9)}\left((G_{47})(t),t\right) = (b_{45}')^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i'')^{(9)}\left((G_{47})(t),t\right) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}'')^{(9)}\left((G_{47})(t),t\right) = (b_{46}')^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}')^{(1)} + (a_{14}')^{(1)} - (a_{13}'')^{(1)}(T_{14},t) + (a_{14}'')^{(1)}(T_{14},t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b_{13}')^{(1)} + (b_{14}')^{(1)} - (b_{13}'')^{(1)}(G,t) - (b_{14}'')^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$ , $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$ , $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)}, , (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$ , $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$ , $(\bar{u}_2)^{(1)} < 0$ the roots of the equations

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| $(a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$ | |
| **Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (v_0)^{(1)}$ :-<br><br>If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by<br>$(m_2)^{(1)} = (v_0)^{(1)}, (m_1)^{(1)} = (v_1)^{(1)},\ if\ (v_0)^{(1)} < (v_1)^{(1)}$<br><br>$(m_2)^{(1)} = (v_1)^{(1)}, (m_1)^{(1)} = (\bar{v}_1)^{(1)}, if\ (v_1)^{(1)} < (v_0)^{(1)} < (\bar{v}_1)^{(1)},$<br><br>and $\boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$<br><br>$(m_2)^{(1)} = (v_1)^{(1)}, (m_1)^{(1)} = (v_0)^{(1)},\ if\ (\bar{v}_1)^{(1)} < (v_0)^{(1)}$ | 283 |
| and analogously<br><br>$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)},\ if\ (u_0)^{(1)} < (u_1)^{(1)}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)},$<br><br>and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$<br><br>are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br><br>where $(p_i)^{(1)}$ is defined by equation<br><br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le\right.$<br><br>$\left.\frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a'_{15})^{(1)})}\left[e^{(S_1)^{(1)}t} - e^{-(a'_{15})^{(1)}t}\right] + G_{15}^0 e^{-(a'_{15})^{(1)}t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)} T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)} - (b'_{15})^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b'_{15})^{(1)}t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)}t} \le T_{15}(t) \le$ | 289 |

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| $\frac{(a_{15})^{(1)}T_{13}^{0}}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t}-e^{-(R_2)^{(1)}t}\right]+T_{15}^{0}e^{-(R_2)^{(1)}t}$ | |
| **Definition of** $(S_1)^{(1)},(S_2)^{(1)},(R_1)^{(1)},(R_2)^{(1)}$:- <br> Where $(S_1)^{(1)}=(a_{13})^{(1)}(m_2)^{(1)}-(a_{13}^{'})^{(1)}$ <br> $(S_2)^{(1)}=(a_{15})^{(1)}-(p_{15})^{(1)}$ <br> $(R_1)^{(1)}=(b_{13})^{(1)}(\mu_2)^{(1)}-(b_{13}^{'})^{(1)}$ <br> $(R_2)^{(1)}=(b_{15}^{'})^{(1)}-(r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** <br> **Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ : <br> $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)}\le-(a_{16}^{'})^{(2)}+(a_{17}^{'})^{(2)}-(a_{16}^{''})^{(2)}(T_{17},t)+(a_{17}^{''})^{(2)}(T_{17},t)\le-(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)}\le-(b_{16}^{'})^{(2)}+(b_{17}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19}),t)-(b_{17}^{''})^{(2)}((G_{19}),t)\le-(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)},(\nu_2)^{(2)},(u_1)^{(2)},(u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)}>0$ , $(\nu_2)^{(2)}<0$ and respectively $(u_1)^{(2)}>0$ , $(u_2)^{(2)}<0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}(\nu^{(2)})^{2}+(\sigma_1)^{(2)}\nu^{(2)}-(a_{16})^{(2)}=0$ | 297 |
| and $(b_{14})^{(2)}(u^{(2)})^{2}+(\tau_1)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)},,(\bar{\nu}_2)^{(2)},(\bar{u}_1)^{(2)},(\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)}>0$ , $(\bar{\nu}_2)^{(2)}<0$ and respectively $(\bar{u}_1)^{(2)}>0$ , $(\bar{u}_2)^{(2)}<0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}(\nu^{(2)})^{2}+(\sigma_2)^{(2)}\nu^{(2)}-(a_{16})^{(2)}=0$ | 301 |
| and $(b_{17})^{(2)}(u^{(2)})^{2}+(\tau_2)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ | 302 |
| **Definition of** $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)}=(\nu_0)^{(2)},(m_1)^{(2)}=(\nu_1)^{(2)}$, $\boldsymbol{if}$ $(\nu_0)^{(2)}<(\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)}=(\nu_1)^{(2)},(m_1)^{(2)}=(\bar{\nu}_1)^{(2)}$ , $\boldsymbol{if}$ $(\nu_1)^{(2)}<(\nu_0)^{(2)}<(\bar{\nu}_1)^{(2)}$, | 306 |

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| and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)},$ ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)},$ ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)},$<br>and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)},$ ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \leq G_{16}(t) \leq G_{16}^0 e^{(S_1)^{(2)}t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \leq G_{17}(t) \leq \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \leq G_{18}(t) \leq \right.$<br>$\left.\frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a'_{18})^{(2)})}\left[e^{(S_1)^{(2)}t} - e^{-(a'_{18})^{(2)}t}\right] + G_{18}^0 e^{-(a'_{18})^{(2)}t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)}t} \leq T_{16}(t) \leq T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)}t} \leq T_{16}(t) \leq \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b'_{18})^{(2)})}\left[e^{(R_1)^{(2)}t} - e^{-(b'_{18})^{(2)}t}\right] + T_{18}^0 e^{-(b'_{18})^{(2)}t} \leq T_{18}(t) \leq$<br>$\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)} + (r_{16})^{(2)} + (R_2)^{(2)})}\left[e^{((R_1)^{(2)} + (r_{16})^{(2)})t} - e^{-(R_2)^{(2)}t}\right] + T_{18}^0 e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a'_{16})^{(2)}$<br>$(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b'_{16})^{(2)}$<br>$(R_2)^{(2)} = (b'_{18})^{(2)} - (r_{18})^{(2)}$ | 318 |

**Behavior of the solutions** 319

**Theorem 3:** If we denote and define

**Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ :

$(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying

$$-(\sigma_2)^{(3)} \le -(a_{20}^{'})^{(3)} + (a_{21}^{'})^{(3)} - (a_{20}^{''})^{(3)}(T_{21}, t) + (a_{21}^{''})^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$$

$$-(\tau_2)^{(3)} \le -(b_{20}^{'})^{(3)} + (b_{21}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t) - (b_{21}^{''})^{(3)}\big((G_{23}), t\big) \le -(\tau_1)^{(3)}$$

**Definition of** $(\nu_1)^{(3)}, (\nu_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ : 320

By $(\nu_1)^{(3)} > 0$ , $(\nu_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$ , $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}\big(\nu^{(3)}\big)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\big(u^{(3)}\big)^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and

By $(\bar{\nu}_1)^{(3)} > 0$ , $(\bar{\nu}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$ , $(\bar{u}_2)^{(3)} < 0$ the

roots of the equations $(a_{21})^{(3)}\big(\nu^{(3)}\big)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\big(u^{(3)}\big)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$

**Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :- 321

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ , ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,

and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously 322

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$ , ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)}t}$$

| | |
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| $(p_i)^{(3)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)}-(p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)}G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)}-(p_{20})^{(3)}-(S_2)^{(3)})}\left[e^{((S_1)^{(3)}-(p_{20})^{(3)})t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \le G_{22}(t) \le \right.$ $\left.\frac{(a_{22})^{(3)}G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)}-(a'_{22})^{(3)})}[e^{(S_1)^{(3)}t} - e^{-(a'_{22})^{(3)}t}] + G_{22}^0 e^{-(a'_{22})^{(3)}t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)}+(r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)}+(r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)}T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)}-(b'_{22})^{(3)})}\left[e^{(R_1)^{(3)}t} - e^{-(b'_{22})^{(3)}t}\right] + T_{22}^0 e^{-(b'_{22})^{(3)}t} \le T_{22}(t) \le$ $\frac{(a_{22})^{(3)}T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)}+(r_{20})^{(3)}+(R_2)^{(3)})}\left[e^{((R_1)^{(3)}+(r_{20})^{(3)})t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$ | 327 |
| **Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:-<br>Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$<br>$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$<br>$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$<br>$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$ | 328 |
| **Behavior of the solutions of equation**<br>**Theorem:** If we denote and define<br>**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :<br>$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying<br>$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$<br>$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ :<br>By $(v_1)^{(4)} > 0$ , $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$ , $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$<br>and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and | 329 |
| **Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : | 330 |

By $(\bar{\nu}_1)^{(4)} > 0$ , $(\bar{\nu}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$ , $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)}$, $(\mu_2)^{(4)}$, $(\nu_0)^{(4)}$ :-

If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)}$, $(\mu_2)^{(4)}$ by

$(m_2)^{(4)} = (\nu_0)^{(4)}, (m_1)^{(4)} = (\nu_1)^{(4)}$, ***if*** $(\nu_0)^{(4)} < (\nu_1)^{(4)}$

$(m_2)^{(4)} = (\nu_1)^{(4)}, (m_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ , ***if*** $(\nu_4)^{(4)} < (\nu_0)^{(4)} < (\bar{\nu}_1)^{(4)}$,

and $\boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$

$(m_2)^{(4)} = (\nu_4)^{(4)}, (m_1)^{(4)} = (\nu_0)^{(4)}$, ***if*** $(\bar{\nu}_4)^{(4)} < (\nu_0)^{(4)}$

and analogously

$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$

$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,

and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$

$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$

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Then the solution of global equations satisfies the inequalities

$$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$$

where $(p_i)^{(4)}$ is defined by equation

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$$\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$$

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$$\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le\right.$$

*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t*

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$$\boxed{T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$$

$$\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$$

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$$\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b_{26}')^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}')^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)}t} \le T_{26}(t) \le$$

$$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)} + (r_{24})^{(4)} + (R_2)^{(4)})}\left[e^{((R_1)^{(4)} + (r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$$

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**Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-

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Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a'_{24})^{(4)}$

$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$

$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b'_{24})^{(4)}$

$(R_2)^{(4)} = (b'_{26})^{(4)} - (r_{26})^{(4)}$

**Behavior of the solutions of equation** 338

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :

$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying

$-(\sigma_2)^{(5)} \le -(a'_{28})^{(5)} + (a'_{29})^{(5)} - (a''_{28})^{(5)}(T_{29}, t) + (a''_{29})^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$

$-(\tau_2)^{(5)} \le -(b'_{28})^{(5)} + (b'_{29})^{(5)} - (b''_{28})^{(5)}((G_{31}), t) - (b''_{29})^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$

**Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ : 339

By $(v_1)^{(5)} > 0$ , $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$ , $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$
and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and

**Definition of** $(\bar{v}_1)^{(5)}, , (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ : 340

By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$
and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$

**Definition of** $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :-

If we define $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}$ by

$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$

$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,
and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$

$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$

and analogously 341

$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$

$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,

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| and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}, \boldsymbol{if}\ (\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)} t}$<br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \le G_{30}(t) \le\right.$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b'_{30})^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b'_{30})^{(5)} t}\right] + T_{30}^0 e^{-(b'_{30})^{(5)} t} \le T_{30}(t) \le$<br>$\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a'_{28})^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b'_{28})^{(5)}$<br>$(R_2)^{(5)} = (b'_{30})^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :<br>$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying<br>$-(\sigma_2)^{(6)} \le -(a'_{32})^{(6)} + (a'_{33})^{(6)} - (a''_{32})^{(6)}(T_{33}, t) + (a''_{33})^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$ | 349 |

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| $-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}((G_{35}),t) - (b''_{33})^{(6)}((G_{35}),t) \le -(\tau_1)^{(6)}$ | |
| **Definition of** $(v_1)^{(6)}, (v_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, v^{(6)}, u^{(6)}$ :<br><br>By $(v_1)^{(6)} > 0$ , $(v_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_1)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\left(u^{(6)}\right)^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{v}_1)^{(6)},, (\bar{v}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br><br>By $(\bar{v}_1)^{(6)} > 0$ , $(\bar{v}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_2)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\left(u^{(6)}\right)^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (v_0)^{(6)}$ :-<br><br>If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br><br>$(m_2)^{(6)} = (v_0)^{(6)}, (m_1)^{(6)} = (v_1)^{(6)}$, ***if*** $(v_0)^{(6)} < (v_1)^{(6)}$<br><br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (\bar{v}_6)^{(6)}$ , ***if*** $(v_1)^{(6)} < (v_0)^{(6)} < (\bar{v}_1)^{(6)}$,<br>and $\boxed{(v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$<br><br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (v_0)^{(6)}$, ***if*** $(\bar{v}_1)^{(6)} < (v_0)^{(6)}$ | 351 |
| and analogously<br><br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br><br>$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$<br><br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G^0_{32}e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)}G^0_{32}}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G^0_{34}e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |

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| $\boxed{T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)}T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)}-(b'_{34})^{(6)})}\left[e^{(R_1)^{(6)}t} - e^{-(b'_{34})^{(6)}t}\right] + T_{34}^0 e^{-(b'_{34})^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)}T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t} - e^{-(R_2)^{(6)}t}\right] + T_{34}^0 e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:- Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a'_{32})^{(6)}$ $(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$ $(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b'_{32})^{(6)}$ $(R_2)^{(6)} = (b'_{34})^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ : $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying $-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$ $-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}((G_{39}), t) - (b''_{37})^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)}, (v_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, v^{(7)}, u^{(7)}$ : By $(v_1)^{(7)} > 0, (v_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(v^{(7)})^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)}, , (\bar{v}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ : By $(\bar{v}_1)^{(7)} > 0, (\bar{v}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(v^{(7)})^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ **Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (v_0)^{(7)}$ :- If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by | 362 |

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| $(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)},$ ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ , ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,<br>and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)},$ ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)},$ ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$ , ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,<br>and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)},$ ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)} t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | 364 |
| $\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)} t}$ | 365 |
| $(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})} \left[ e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)} t} \right] + G_{38}^0 e^{-(S_2)^{(7)} t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a'_{38})^{(7)})} [e^{(S_1)^{(7)} t} - e^{-(a'_{38})^{(7)} t}] + G_{38}^0 e^{-(a'_{38})^{(7)} t})$ | 366 |
| $\boxed{T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b'_{38})^{(7)})} \left[ e^{(R_1)^{(7)} t} - e^{-(b'_{38})^{(7)} t} \right] + T_{38}^0 e^{-(b'_{38})^{(7)} t} \le T_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)} + (r_{36})^{(7)} + (R_2)^{(7)})} \left[ e^{((R_1)^{(7)} + (r_{36})^{(7)})t} - e^{-(R_2)^{(7)} t} \right] + T_{38}^0 e^{-(R_2)^{(7)} t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:- | 370 |

Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a'_{36})^{(7)}$

$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$

$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b'_{36})^{(7)}$

$(R_2)^{(7)} = (b'_{38})^{(7)} - (r_{38})^{(7)}$

**Behavior of the solutions of equation** 371

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :

$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying

$-(\sigma_2)^{(8)} \le -(a'_{40})^{(8)} + (a'_{41})^{(8)} - (a''_{40})^{(8)}(T_{41}, t) + (a''_{41})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$

$-(\tau_2)^{(8)} \le -(b'_{40})^{(8)} + (b'_{41})^{(8)} - (b''_{40})^{(8)}((G_{43}), t) - (b''_{41})^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$

**Definition of** $(\nu_1)^{(8)}, (\nu_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, \nu^{(8)}, u^{(8)}$ : 372

By $(\nu_1)^{(8)} > 0, (\nu_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}(\nu^{(8)})^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$
and $(b_{41})^{(8)}(u^{(8)})^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(8)}, , (\bar{\nu}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :

By $(\bar{\nu}_1)^{(8)} > 0, (\bar{\nu}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0, (\bar{u}_2)^{(8)} < 0$ the

roots of the equations $(a_{41})^{(8)}(\nu^{(8)})^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$

and $(b_{41})^{(8)}(u^{(8)})^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$

**Definition of** $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (\nu_0)^{(8)}$ :-

If we define $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by

$(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)}$, ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$

$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$, ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)}$,

and $\boxed{(\nu_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$

| | |
|---|---|
| $(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)},\ \boldsymbol{if}\ (\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},\ \boldsymbol{if}\ (u_0)^{(8)} < (u_1)^{(8)}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}, \boldsymbol{if}\ (u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}, \boldsymbol{if}\ (\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)} t}$<br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \le G_{42}(t) \le \right.$<br>$\left.\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)} t} - e^{-(a'_{42})^{(8)} t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)} t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b'_{42})^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b'_{42})^{(8)} t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)} t} \le T_{42}(t) \le$<br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$ | 381 |

$(R_2)^{(8)} = (b_{42}^{'})^{(8)} - (r_{42})^{(8)}$

**Behavior of the solutions of equation 37 to 92** 382

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$$-(\sigma_2)^{(9)} \le -(a_{44}^{'})^{(9)} + (a_{45}^{'})^{(9)} - (a_{44}^{''})^{(9)}(T_{45}, t) + (a_{45}^{''})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$$

$$-(\tau_2)^{(9)} \le -(b_{44}^{'})^{(9)} + (b_{45}^{'})^{(9)} - (b_{44}^{''})^{(9)}\big((G_{47}), t\big) - (b_{45}^{''})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$$

**Definition of** $(v_1)^{(9)}, (v_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, v^{(9)}, u^{(9)}$ :

By $(v_1)^{(9)} > 0$ , $(v_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(v^{(9)}\big)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{v}_1)^{(9)}, , (\bar{v}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{v}_1)^{(9)} > 0$ , $(\bar{v}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(v^{(9)}\big)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (v_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (v_0)^{(9)}, (m_1)^{(9)} = (v_1)^{(9)}$, ***if*** $(v_0)^{(9)} < (v_1)^{(9)}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (\bar{v}_1)^{(9)}$ , ***if*** $(v_1)^{(9)} < (v_0)^{(9)} < (\bar{v}_1)^{(9)}$,

and $\boxed{(v_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (v_0)^{(9)}$, ***if*** $(\bar{v}_1)^{(9)} < (v_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0 e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^0 e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G_{44}^0 e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}} G_{44}^0 e^{(S_1)^{(9)}t}$$

$$\left( \frac{(a_{46})^{(9)} G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})} \left[ e^{((S_1)^{(9)}-(p_{44})^{(9)})t} - e^{-(S_2)^{(9)}t} \right] + G_{46}^0 e^{-(S_2)^{(9)}t} \le G_{46}(t) \le \right.$$

$$\left. \frac{(a_{46})^{(9)} G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)}-(a'_{46})^{(9)})} [e^{(S_1)^{(9)}t} - e^{-(a'_{46})^{(9)}t}] + G_{46}^0 e^{-(a'_{46})^{(9)}t} \right)$$

$$\boxed{T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^0 e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}} T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}} T_{44}^0 e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)} T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)}-(b'_{46})^{(9)})} \left[ e^{(R_1)^{(9)}t} - e^{-(b'_{46})^{(9)}t} \right] + T_{46}^0 e^{-(b'_{46})^{(9)}t} \le T_{46}(t) \le$$

$$\frac{(a_{46})^{(9)} T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})} \left[ e^{((R_1)^{(9)}+(r_{44})^{(9)})t} - e^{-(R_2)^{(9)}t} \right] + T_{46}^0 e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain

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$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left( (a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14}, t) \right) - (a''_{14})^{(1)}(T_{14}, t) v^{(1)} - (a_{14})^{(1)} v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left( (a_{14})^{(1)} \left(v^{(1)}\right)^2 + (\sigma_2)^{(1)} v^{(1)} - (a_{13})^{(1)} \right) \le \frac{dv^{(1)}}{dt} \le -\left( (a_{14})^{(1)} \left(v^{(1)}\right)^2 + (\sigma_1)^{(1)} v^{(1)} - (a_{13})^{(1)} \right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(1)}, (\nu_0)^{(1)}$ :-

For $0 < \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (\nu_1)^{(1)} < (\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t) \geq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(\nu_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\nu_1)^{(1)}$

In the same manner , we get 384

$$\nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{\nu}_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\bar{\nu}_1)^{(1)}$

If $0 < (\nu_1)^{(1)} < (\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{\nu}_1)^{(1)}$ we find like in the previous case, 385

$$(\nu_1)^{(1)} \leq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}} \leq \nu^{(1)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (C)^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$ if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \le \frac{d\nu^{(2)}}{dt} \le -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \ge \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \le \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \le \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \le \nu^{(2)}(t) \le$$

| | |
|---|---|
| $\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$ | |
| If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br><br>$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :-<br><br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right) - (a_{21}^{''})^{(3)}(T_{21},t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| | 400 |

From which one obtains

For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

In the same manner , we get

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$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

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If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case,

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

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If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$ if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

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In the same manner , we get

$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

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$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

In the same manner , we get

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$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}'')^{(5)} = (b_{29}'')^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then

$(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(v_1)^{(5)}$ and $(\bar{v}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain 412

$$\frac{dv^{(6)}}{dt} = (a_{32})^{(6)} - \left((a'_{32})^{(6)} - (a'_{33})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right) - (a''_{33})^{(6)}(T_{33},t)v^{(6)} - (a_{33})^{(6)}v^{(6)}$$

**Definition of** $v^{(6)}$ :- $\boxed{v^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_2)^{(6)}v^{(6)} - (a_{32})^{(6)}\right) \le \frac{dv^{(6)}}{dt} \le -\left((a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_1)^{(6)}v^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(6)}, (v_0)^{(6)}$ :-

For $0 < \boxed{(v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}} < (v_1)^{(6)} < (\bar{v}_1)^{(6)}$

$$v^{(6)}(t) \ge \frac{(v_1)^{(6)} + (C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)} - (v_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)} - (v_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(v_1)^{(6)} - (v_0)^{(6)}}{(v_0)^{(6)} - (v_2)^{(6)}}}$$

it follows $(v_0)^{(6)} \le v^{(6)}(t) \le (v_1)^{(6)}$

In the same manner , we get 413

$$v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)} + (\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{v}_1)^{(6)} - (v_0)^{(6)}}{(v_0)^{(6)} - (\bar{v}_2)^{(6)}}}$$

From which we deduce $(v_0)^{(6)} \le v^{(6)}(t) \le (\bar{v}_1)^{(6)}$

If $0 < (v_1)^{(6)} < (v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}} < (\bar{v}_1)^{(6)}$ we find like in the previous case, 414

$$(v_1)^{(6)} \le \frac{(v_1)^{(6)} + (C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)} - (v_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)} - (v_2)^{(6)}\right)t\right]}} \le v^{(6)}(t) \le$$

$$\frac{(\bar{v}_1)^{(6)} + (\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}} \le (\bar{v}_1)^{(6)}$$

If $0 < (v_1)^{(6)} \le (\bar{v}_1)^{(6)} \le \boxed{(v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$ , we obtain 415

$$(v_1)^{(6)} \le v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)} + (\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}} \le (v_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}$, $\boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

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$$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$$

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If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain

$$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(C)^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

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**Definition of** $u^{(7)}(t)$ :-

$$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}, \quad \boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}'')^{(7)} = (a_{37}'')^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}'')^{(7)} = (b_{37}'')^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a'_{40})^{(8)} - (a'_{41})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right) - (a''_{41})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \le \frac{dv^{(8)}}{dt} \le -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(8)}, (v_0)^{(8)}$ :-

For $0 < \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}} < (v_1)^{(8)} < (\bar{v}_1)^{(8)}$

$$v^{(8)}(t) \ge \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(v_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (v_2)^{(8)}}}$$

it follows $(v_0)^{(8)} \le v^{(8)}(t) \le (v_1)^{(8)}$

In the same manner , we get

$$v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{v}_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (\bar{v}_2)^{(8)}}}$$

From which we deduce $(v_0)^{(8)} \le v^{(8)}(t) \le (\bar{v}_8)^{(8)}$

If $0 < (v_1)^{(8)} < (v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}} < (\bar{v}_1)^{(8)}$ we find like in the previous case,

$$(v_1)^{(8)} \le \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}} \le v^{(8)}(t) \le$$

$$\frac{(\bar{v}_1)^{(8)} + (C)^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (\bar{v}_1)^{(8)}$$

If $0 < (v_1)^{(8)} \le (\bar{v}_1)^{(8)} \le \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$ , we obtain

$$(v_1)^{(8)} \le v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (v_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

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**Definition of** $v^{(8)}(t)$ :-

$(m_2)^{(8)} \leq v^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{v^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $v^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$v^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)}-(\nu_0)^{(9)}}{(\nu_0)^{(9)}-(\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \le \frac{(\nu_1)^{(9)}+(C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}} \le \nu^{(9)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \le (\bar{\nu}_1)^{(9)} \le \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(\nu_1)^{(9)} \le \nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \le \nu^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \le u^{(9)}(t) \le (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)}G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

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| $(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,<br>$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$<br>$with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$ | 433 |

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| $(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$ | 436 A |

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| $(a'_{44})^{(9)}(a'_{45})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a'_{45})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b'_{44})^{(9)}(r_{45})^{(9)} - (b'_{45})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b'_{13})^{(1)} - (b''_{13})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b'_{14})^{(1)} - (b''_{14})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b'_{15})^{(1)} - (b''_{15})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b'_{16})^{(2)} - (b''_{16})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b'_{17})^{(2)} - (b''_{17})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b'_{18})^{(2)} - (b''_{18})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a'_{20})^{(3)} + (a''_{20})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a'_{21})^{(3)} + (a''_{21})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a'_{22})^{(3)} + (a''_{22})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b'_{20})^{(3)} - (b''_{20})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b'_{21})^{(3)} - (b''_{21})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b'_{22})^{(3)} - (b''_{22})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) +$ | 485 |

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| $(a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | |
| **Proof:**<br><br>(t) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |

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| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492<br>A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations | 498 |

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| $G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a'_{28})^{(5)}+(a''_{28})^{(5)}(T^*_{29})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a'_{30})^{(5)}+(a''_{30})^{(5)}(T^*_{29})]}$ | |
| **Definition and uniqueness of** $T^*_{33}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T^*_{33}$ for which $f(T^*_{33}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a'_{32})^{(6)}+(a''_{32})^{(6)}(T^*_{33})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a'_{34})^{(6)}+(a''_{34})^{(6)}(T^*_{33})]}$ | 499 |
| **Definition and uniqueness of** $T^*_{37}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T^*_{37}$ for which $f(T^*_{37}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a'_{36})^{(7)}+(a''_{36})^{(7)}(T^*_{37})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a'_{38})^{(7)}+(a''_{38})^{(7)}(T^*_{37})]}$ | 500 |
| **Definition and uniqueness of** $T^*_{41}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T^*_{41}$ for which $f(T^*_{41}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a'_{40})^{(8)}+(a''_{40})^{(8)}(T^*_{41})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a'_{42})^{(8)}+(a''_{42})^{(8)}(T^*_{41})]}$ | 501 |
| **Definition and uniqueness of** $T^*_{45}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T^*_{45}$ for which $f(T^*_{45}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a'_{44})^{(9)}+(a''_{44})^{(9)}(T^*_{45})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a'_{46})^{(9)}+(a''_{46})^{(9)}(T^*_{45})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br><br>$\left[(b'_{13})^{(1)}(b''_{14})^{(1)}(G) + (b'_{14})^{(1)}(b''_{13})^{(1)}(G)\right]+(b''_{13})^{(1)}(G)(b''_{14})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that $\varphi$ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that | 502 |

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| there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*})=0$ | |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br>$\varphi(G_{19})=(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)}-(b_{16})^{(2)}(b_{17})^{(2)}-$<br>$\left[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19})+(b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})\right]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19})=0$ | 503 |
| Where in $(G_{19})(G_{16},G_{17},G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*})=0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br>$\varphi(G_{23})=(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)}-(b_{20})^{(3)}(b_{21})^{(3)}-$<br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23})+(b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23})=0$<br>Where in $G_{23}(G_{20},G_{21},G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*})=0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br>$\varphi(G_{27})=(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)}-(b_{24})^{(4)}(b_{25})^{(4)}-$<br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27})+(b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27})=0$<br>Where in $(G_{27})(G_{24},G_{25},G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*})=0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31})=(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)}-(b_{28})^{(5)}(b_{29})^{(5)}-$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31})+(b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right]+(b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31})=0$<br>Where in $(G_{31})(G_{28},G_{29},G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*})=0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35})=(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)}-(b_{32})^{(6)}(b_{33})^{(6)}-$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35})+(b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right]+(b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35})=0$<br>Where in $(G_{35})(G_{32},G_{33},G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that | 508 |

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| there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0\,, T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0\,, T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |

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| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*}) = 0$ , $T_{21}^{*}$ given by $f(T_{21}^{*}) = 0$ and<br><br>$G_{20}^{*} = \frac{(a_{20})^{(3)} G_{21}^{*}}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*} = \frac{(a_{22})^{(3)} G_{21}^{*}}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br><br>$T_{20}^{*} = \frac{(b_{20})^{(3)} T_{21}^{*}}{[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*} = \frac{(b_{22})^{(3)} T_{21}^{*}}{[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br><br>$G_{25}^{*}$ given by $\varphi(G_{27}) = 0$ , $T_{25}^{*}$ given by $f(T_{25}^{*}) = 0$ and<br><br>$G_{24}^{*} = \frac{(a_{24})^{(4)} G_{25}^{*}}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*} = \frac{(a_{26})^{(4)} G_{25}^{*}}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*} = \frac{(b_{24})^{(4)} T_{25}^{*}}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*} = \frac{(b_{26})^{(4)} T_{25}^{*}}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br><br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*}) = 0$ , $T_{29}^{*}$ given by $f(T_{29}^{*}) = 0$ and<br><br>$G_{28}^{*} = \frac{(a_{28})^{(5)} G_{29}^{*}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*} = \frac{(a_{30})^{(5)} G_{29}^{*}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |
| $T_{28}^{*} = \frac{(b_{28})^{(5)} T_{29}^{*}}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)} T_{29}^{*}}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br><br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br><br>$G_{32}^{*} = \frac{(a_{32})^{(6)} G_{33}^{*}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)} G_{33}^{*}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)} T_{33}^{*}}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)} T_{33}^{*}}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br><br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and | 522 |

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| $G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)} (T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)} (T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)} ((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)} ((G_{39})^{*})]}$ | |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)} (T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)} (T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)} ((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)} ((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br><br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br><br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)} (T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)} (T_{45}^{*})]}$<br><br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)} ((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)} ((G_{47})^{*})]}$ | 523 A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:**_Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}} (T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j} (G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*} \mathbb{T}_{14}$ | 525 |

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| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}')^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}')^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}')^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^{*}\mathbb{G}_{j}\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}')^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^{*}\mathbb{G}_{j}\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}')^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^{*}\mathbb{G}_{j}\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i'')^{(2)}$ and $(b_i'')^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 532 |
| $\frac{\partial(a_{17}'')^{(2)}}{\partial T_{17}}(T_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial(b_i'')^{(2)}}{\partial G_j}((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}')^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^{*}\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}')^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}')^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}')^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}')^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}')^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i'')^{(3)}$ *and* $(b_i'')^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | 540 |

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| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)} \ , \ \frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^*\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^*\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^*\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^*\mathbb{G}_j\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^*\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^*\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ *and* $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)} \ , \ \frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^*) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right)$ | 554 |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^*) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^*\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^*\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^*\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |

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| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{37})^{(7)}}{\partial T_{37}}(T^*_{37}) = (q_{37})^{(7)}$ , $\frac{\partial(b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a'_{36})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G^*_{36}\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a'_{37})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G^*_{37}\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. | |

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| Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \ , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}((G_{43})^*) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^*\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^*\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b_{41}^{'})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T_{41}^*\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b_{42}^{'})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T_{42}^*\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS** **Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. **Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |

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| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$ $\left[\left(\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\right]$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\right)$ $+\left(\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\right)$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((b'_{13})^{(1)} + (b'_{14})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $+\left(\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$ $+\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G^*_{13}\right)$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T^*_{14} + (b_{14})^{(1)}s_{(13),(15)}T^*_{13}\right)\} = 0$ $+$ $\left((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)}\right)\{\left((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)}\right)$ $\left[\left(\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}^*_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^*_{16}\right)\right]$ $\left(\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^*_{17}\right)$ $+\left(\left((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}^*_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^*_{17}\right)$ $\left(\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^*_{16}\right)$ $\left(\left((\lambda)^{(2)}\right)^2 + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ $\left(\left((\lambda)^{(2)}\right)^2 + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ | |

$$+\left(\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\right)\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left(\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\right)$$

$$+\left(\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((b_{20}')^{(3)}+(b_{21}')^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}')^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}')^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}')^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}')^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}')^{(4)}+(a_{25}')^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((b_{24}')^{(4)}+(b_{25}')^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\}\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^2+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G^*_{33}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G^*_{32}\right)$$

$$\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T^*_{33}+(b_{33})^{(6)}s_{(32),(34)}T^*_{32}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b'_{38})^{(7)}-(r_{38})^{(7)}\}\{\left((\lambda)^{(7)}+(a'_{38})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(q_{36})^{(7)}G^*_{36}\right)\right]$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T^*_{37}+(b_{37})^{(7)}s_{(36),(37)}T^*_{37}\Big)$$

$$+\left(\left((\lambda)^{(7)}+(a'_{37})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G^*_{36}+(a_{36})^{(7)}(q_{37})^{(7)}G^*_{37}\right)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T^*_{37}+(b_{37})^{(7)}s_{(36),(36)}T^*_{36}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left(\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^*_{36}\right)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^*_{37}+(b_{37})^{(7)}s_{(36),(38)}T^*_{36}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^*_{40}\right)\right]$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^*_{41}+(b_{41})^{(8)}s_{(40),(41)}T^*_{41}\Big)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^*_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^*_{41}\right)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^*_{41}+(b_{41})^{(8)}s_{(40),(40)}T^*_{40}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^*_{40}\right)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^*_{41}+(b_{41})^{(8)}s_{(40),(42)}T^*_{40}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^*_{44}\right)\right]$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^*_{45}+(b_{45})^{(9)}s_{(44),(45)}T^*_{45}\Big)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^*_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45}+(b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left(\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45}+(b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

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# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i(x)\, xi$ where $x_i \in R^n_i$, $x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i(x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i(\mu)$ of the $A_i(x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k \tag{20}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to

replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 20. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,...x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i(x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,...,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t \geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t \leq 0}$

$\overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow *x.t restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set *K* is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** *x.t* is defined for all $t \geq 0$

**A4:** *x.t* is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\mathcal{A}(\tilde{A}), C \backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \longrightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) in\ ||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population *i* is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^{t}_{0} (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\to\infty}{limsup} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v = au_i(x) + w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |B_i(t,x)u_i(x)|(a+\alpha+\exp(-\beta t)|w|)$. Hence, $\limsup_{t\to\infty}\frac{1}{t} In\, B_i(t,x)\, v| \leq$

$$\limsup_{t\to\infty}\frac{1}{t} In\, B_i(t,x)\, u_i(x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty}\frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a-\alpha+\exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty}\frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty}\frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\, |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\, u_i(x)| = In\, |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\, u_i(x.t)||\, B_i(t,x)\, u_i(x)|$$

$$= \text{In } b_i(s,t.x)\, b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt}b_i(t,x) = \lim_{s\to 0}\frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0}\frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

$$=$$

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s), u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } | (x.t)_i |$$

$$\leq \limsup_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x) =$ Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of

irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x)u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$ where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ and $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as $n\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$. Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\le \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x)u_i^n(x), u_i^n(x)\rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop. 3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop. 3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)|d\mu(x) + |\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\, d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm. 2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\, ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i\, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak$^*$ limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu}\ \max_{p\epsilon\Delta}\ \sum_i\ Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \text{min max } \lambda i(\mu) = \min_{\mu}\ \max_{p\epsilon\Delta}\ \sum_i\ Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \{ \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i \epsilon I}{if\ i \notin I}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that \quad \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t$ −invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently *K* is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s =$

$s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_s$. If $r(x) = s(x), then\ x \in M_r$. Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume $x \mapsto A(x)$ is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists $j \in \{1,\ldots,k\}$ such that $M_i$ ----- $\{x \in C : x_l \gg 0 \text{ for } l \neq j\}$ Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_i M_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, *$\lambda_i(\mu) > 0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, ..., x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda(0)$, which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i(0) > 0$ for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the

system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j \ (x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1{}^1, \ldots, E_1{}^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i{}_2(E_1{}^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$$

$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > 1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria $E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$ and $\lambda_3(E_1)$ are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j, \qquad i = 1,\ldots,k;\ j = 1,\ldots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i\, a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i\, x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \qquad i = 1, \ldots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \ldots, k$$

It has been shown [12,ch.16.1,16.2] That for $k = 3$,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is

the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y \end{pmatrix}$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & O \\ \gamma & my\text{-}b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \begin{pmatrix} \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1-y) - b(N)\ y \end{pmatrix}$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*′ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\curlywedge_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\curlywedge_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\curlywedge_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + K p\, x_i x_i - 1}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions

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# Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a

qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and

orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.58) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.59) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.60) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{'} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$ (1.4)

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \epsilon [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1, \varphi_2, f_1, f_2, b_1, b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for

each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as* $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 20. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'}\ - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, \text{R}^-),$$

$A_3$. $\varphi_i\ (\cdot\ , s) \in L^2\ (R^+, R^+)\ \cap C(R^+, R^+)$ for each $s \in\ [-\tau, 0]$,

$i, = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[-\int_0^\infty f_i(a + s, P(t + S))ds\right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[-\int_o^x f_1(a + s, p(t + s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0)\exp\left[-\int_o^t f_1(a - t + s, p_1(s))\, ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

*(2.3)* $p_i(a,t) = B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i and\ L_i\ (i = 1,2)$ as follows:

$$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a,t,\ p_i(t)) = \exp[-\int_o^t f_1(a+s,\ p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

$$(2.6)\ p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da + \int_o^\infty \varphi_i(a, 0)\ L_i(a,t,p_i(t)da,$$

$$t > 0, i = 1,2,$$

$$(2.7)\ p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$$

$$(2.8)\ \beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.\left(a,t,P_i(t)\right)da$$

$$+\int_0^\infty b_i\left(a+t, P_i(t)\right)\varphi_i(a,0)L_i\left(a,t,P_i(t)\right)da$$

$$i = 1, 2, t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t) \phi(0) d_i^*] t \quad for \ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T], R^+), \quad B_i \in C([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in C([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau, T]$. Then for any

$(x, y) \in C^+[-\tau, T] \times C^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau, T], \quad C_T^2 = C^+[-\tau, T] \times C^+[-\tau, T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for $i = 1,2$.

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \pi C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \quad P_t = \ (P_{1t}, P_{2t}) \in C^2_{t*} \mid \|P_t - \Phi_t\|_{t*} \le r^*$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)\, da \ for\ s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)\, da \ for\ s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Phi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^* \right] \ t^*$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for t $\le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \le \left| \int_0^t B_1^{(1)}(a) \left\{ M_1(a, t, P_t^{(1)}) - M_1(a, t, P_t^{(2)}) \right\} da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a, t, P_t^{(2)}) da \right|$$

$$+ \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \; for \; x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \| ds\right) da$$

$$\leq d_1 \beta_1^* \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o)\left[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds\right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \| t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\text{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right]\right] d\sigma$$

$$+ \int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^{\infty} \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$= R_1 + R_2 + R_3 + R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a + \sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \, /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \text{ for } a \, \varepsilon [o, t^*]$$

And hence we have

$$|R_3(a) + R_4(a)| \leq \beta_1 \Phi_1(0) || P_t^{(1)} - P_t^{(2)} ||_{t^*} + \Phi_1(0) \beta_1 d_1 || P_t^{(1)} - P_t^{(2)} ||_{t^*}^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a - 0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) e^{\delta 1^a} \beta_1 \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) e^{\delta 1^a} \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| B_1^{*d}\sigma$$

$$+ \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \left\{ B_1 + B_2 d_1 a + B_1^* \left( B_2 a + d_1 a^2/2 \right) e^{\delta 1^a} \right\}$$

$$\leq \beta_1^* \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| do + \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \to \beta_1$ as $a \to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{g_1(a) + 0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))\text{-}b_1(a\text{-}\sigma, P_t^{(1)}(a))\, M_1(\sigma, a, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + /\!/ \Phi_t /\!/_{t*}\right) /\!/ \; P_t^{(1)}\text{-}P_t^{(2)} /\!/_{t*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right) /\!/ \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^{*})$$

As $t^{*} \to o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^{*}, P_t^{*}$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp\, P_2^{*}(a)}{da} + f_i(\mathrm{a}, P_1^{*}, P_2^{*})\, P_i^{*}(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^{*} = \int_o^{\infty} P_i^{*}(a)da, i = 1,\ 2,$$

$$(3,3)\ P_i^{*} = \int_o^{\infty} b_i(a,\ P_1^{*},\ P_2^{*}) P_i^{*}(a)da,\ i = 1.\ \ 2$$

All solutions of (3.1)are of the from

$$(3.4)\ P_i^{*}(a) = P_i^{*}(o) \exp\left[ - \int_o^{\infty} f_i(s, P_1^{*}, P_2^{*})ds \right], i = 1, \quad 2, a \geq 0,$$

Which together with (3.3) lead to

$$1=\int_o^{\infty} b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^{\infty} b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $$F_{i.}\ (P_1,P_2) = \int_o^{\infty} b_i(a,P_1,P_2)\exp\left[-\int_o^{\infty} f_i\,(s,P_1,P_2)ds\right]da,$$

$$i = 1,\ 2, \quad P_1 \geq 0, \quad P_2 \geq 0,$$

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_5$. *there exists a positive constant c for which*

$$F_1(\mathrm{C},0) < F_{2.}(\mathrm{C},0),$$

(3.8) $$F_1(0.\mathrm{C}) < F_{2.}(0,\mathrm{C}),$$

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $$F_1(P_1^*,P_2^*) = 1 = F_2(P_1^*,P_2^*)$$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $$P_2^*(a) = [P_2^*\exp[-f_i\,(s,P_1^*,P_2^*)ds])/(\int_o^{\infty}\exp[-\int_o^a f_i\,(s,P_1^*,P_2^*)ds]da),$$

$$i = 1,\ 2, \quad a \geq 0,$$

Proof Consider the elements of the surfaces defined by

$$\mathrm{Z}= F_1(P_1,P_2), z = F_2(P_1,P_2),\ P_1 \geq 0, P_2 \geq 0,$$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i/\partial P_j < 0\ and$ , by ($A_2$), $\partial b_i/\partial P_j \leq 0$ for all

$$P_1 \geq 0, P_2 \geq 0, \quad i = 1,\ 2,$$

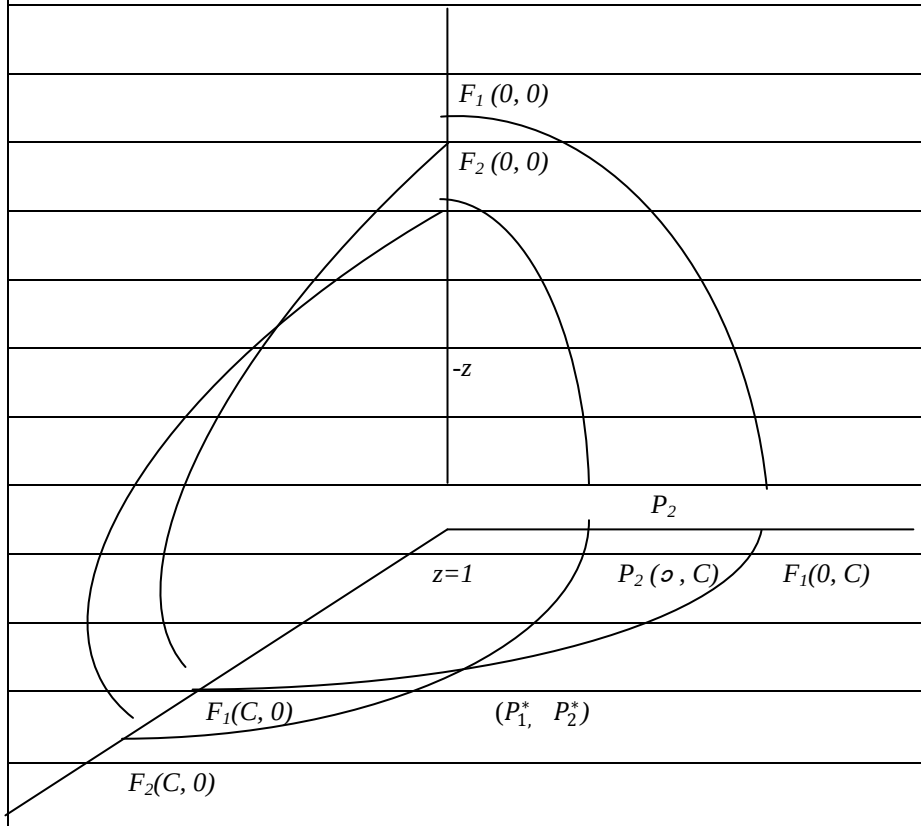
$j = 1,\ 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= 1, $P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$$F_1(P_1, P_2 = 1 \; and \; F_2(P_1, P_2) = 1, \; F_1(P_1, P_2 \geq 0,$$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

$$(3.11) \quad \begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \; i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*}\, p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square 1}^{2} \frac{\partial b_i}{\partial p_{\square}^*}\, p_{i\square}(t)\right\} da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_{\square}(a)\exp[\lambda t], \quad {}_{\square}(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty {}_{\square}(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d_{\square}}{da} + \lambda_{\square} = -f_i\left(a, p_1^*, p_2^*\right)_{\square}(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*}\, p_{i\square}^*\right\},$$

(3.15)

$$_{\square}(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)_{\square}(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*}\, p_{i\square}^*\right\} da,$$

Where $i = 1, 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^* \exp\{-\lambda\tau\}, \quad p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\Phi_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\Phi_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\Phi_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\Phi_i(a) = \{\Phi_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$

$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_j\right\} e^{\lambda s} ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\Phi_i(a)\left[p_i^*(a)/p_i^*(0)c_i \exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_j\right\} da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a}\left\{\int_0^a \left(\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_j\right) e^{\lambda s} ds\right\} da\right]$$

$$- p_i^*(a) e^{-\lambda a}\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Phi_j\right\} e^{\lambda s} ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\Phi_1(a)$ and $\Phi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^* \exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^* \exp(-\lambda\tau) + (\lambda - A_{22})p_2^* = g_{21}(\lambda)p_1^* \exp(-\lambda\tau) + g_{22}(\lambda)p_2^*$$

Where

(3.22) $A_{ij} = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_j^*} - \frac{\partial f_i}{\partial p_j^*}\right]da,$

(3.23) $g_{ij}(\lambda) = [p_i^*(0)c_i]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\,da\right\}$
$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_j^*} - b_i(a, p_1^*, p_2^*)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)\right]da\right\}$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)da,\quad i, j = 1, 2,$$

(3.24) $\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*),\quad i, = 1, 2.$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix}\lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda))\end{bmatrix} = 0$

Or equivalently

(3.25) $\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = [A_{21}g_{12}(\lambda) + A_{12}g_{21}(\lambda) + g_{12}(\lambda)g_{21}(\lambda)]\exp[-2\lambda\tau]$

$-A_{11}g_{22}(\lambda) - A_{22}g_{11}(\lambda) - g_{11}(\lambda)g_{22}(\lambda)$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i, = 1, 2.)\ (see(A_1)\ and\ (A_2))$;

$A_7.\ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\},$

$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$

$\| S_2(0) \| = |A_{11}g_{12}(0)| + |A_{12}g_{21}(0)| + |g_{12}(0)g_{21}(0)|$

$+ |A_{11}g_{22}(0)| + |A_{22}g_{11}(0)| + |g_{11}(0)g_{22}(0)|$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i(a, t) = \phi_i(a) \exp[\lambda t] \to 0$ as $t \to \infty$ for $i = 1, 2$ and $a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda$ $\mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$

$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \left|\right.$

$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$

$\geq [(|\lambda| - P)^2 + (q)]^2$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \{4|(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| | - |A_{11} + A_{22} + \| S_1(0) \| |^2\}/2,$

It will now follow, from (A) and (3.26), $F_1(\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder

concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power

obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy

withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation

from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## SECTION A(19)

## Chaoticity In Yang-Mills Field Theory

### INTRODUCTION—VARIABLES USED

**Entropy production from chaoticity in Yang-Mills field theory with use of the Husimi function Hidekazu Tsukiji, Hideaki Iida, Teiji Kunihiro, Akira Ohnishi, Toru T. Takahashi (Submitted on 15 Mar 2016)**

(1) **Hidekazu Tsukiji, Hideaki Iida, Teiji Kunihiro, Akira Ohnishi, Toru T. Takahashi** investigate possible entropy production in Yang-Mills (YM) field theory by using (e) a quantum distribution function called Husimi function fH(A,E,t) for (e) YM field, which is given by (e) a coarse graining of (e) Wigner function and non-negative.

(2) Authors calculate the Husimi-Wehrl (HW) entropy SHW (t)=−TrfHlogfH defined as (=) an integral over the phase-space, for which two adaptations of (e) the test-particle method are used combined with (e&eb) Monte-Carlo method.

(3) Authors utilize the semiclassical approximation to obtain (eb) the time evolution of (e) the distribution functions of (e) the YM field, which is known to show (eb) a chaotic behavior in the classical limit.

(4) Authors also make a simplification of (e) the multi-dimensional phase-space integrals by (e) making a product ansatz for (e) the Husimi function, which is found to give (eb) a 10-20 per cent over estimate of (e) the HW entropy for (e) a quantum system with (e&eb) a few degrees of freedom.

**(5)** Authors show that the quantum YM theory does exhibit (eb) the entropy production, and that the entropy production rate agrees with (=) the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting (eb) that the chaoticity of the classical YM field causes (eb)

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|---|---|
| the entropy production in the quantum YM theory. Subjects: High Energy Physics - Phenomenology (hep-ph); Statistical Mechanics (cond-mat.stat-mech); Nuclear Theory (nucl-th) Report number: KUNS-2616, YITP-16-40 Cite as: arXiv: 1603.04622 [hep-ph] (or arXiv: 1603.04622v1 [hep-ph] for this version) | |
| **NOTATION** | |
| **Module One**<br>Authors show that there are no instanton solutions to (e) the quantum effective action | |
| $G_{13}$ : Category one of quantum effective action<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of there are no instanton solutions<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>The equations for a point classical source of (e) color spin are solved, and they show that the field infrared energy becomes (=) linearly divergent in (eb) the limit of spontaneous magnetization | |
| $G_{16}$ : Category one of color spin are solved, and they show that the field infrared energy becomes (=) linearly divergent in (eb) the limit of spontaneous magnetization<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of equations for a point classical source<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>The equations for a point classical source of color spin are solved, and they show that the field infrared energy becomes (=) linearly divergent in (eb) the limit of spontaneous magnetization | |
| $G_{20}$ : Category one of equations for a point classical source of color spin are solved, and they show that the field infrared energy<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of linearly divergent in (eb) the limit of spontaneous magnetization<br>$T_{21}$ : Category two of SAS | |

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|---|---|
| $T_{22}$ : Category three of SAS | |
| **Module four**<br><br>The equations for a point classical source of color spin are solved, and they show that the field infrared energy becomes linearly divergent in the limit of spontaneous magnetization | |
| $G_{24}$ : Category one of equations for a point classical source of color spin are solved, and they show that the field infrared energy becomes linearly divergent<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of limit of spontaneous magnetization<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>This implies (eb) bag formation, and an electric Meissner effect confining (e) the bag contents. | |
| $G_{28}$ : Category one of equations for a point classical source of color spin are solved, and they show that the field infrared energy becomes linearly divergent in the limit of spontaneous magnetization<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of bag formation, and an electric Meissner effect confining the bag contents.<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>**Hidekazu Tsukiji, Hideaki Iida, Teiji Kunihiro, Akira Ohnishi, Toru T. Takahashi** investigate possible entropy production in Yang-Mills (YM) field theory by using (e) a quantum distribution function called Husimi function fH(A,E,t) for (e) YM field, which is given by (e) a coarse graining of (e) Wigner function and non-negative | |
| $G_{32}$ : Category one of quantum distribution function called Husimi function fH(A,E,t) for (e) YM field, which is given by (e) a coarse graining of (e) Wigner function and non-negative<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of entropy production in Yang-Mills (YM) field theory | |

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|---|---|
| $T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>**Hidekazu Tsukiji, Hideaki Iida, Teiji Kunihiro, Akira Ohnishi, Toru T. Takahashi** investigate possible entropy production in Yang-Mills (YM) field theory by using a quantum distribution function called Husimi function fH(A,E,t) for (e) YM field, which is given by (e) a coarse graining of (e) Wigner function and non-negative | |
| $G_{36}$ : Category one of YM field, which is given by (e) a coarse graining of (e) Wigner function and non-negative<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of entropy production in Yang-Mills (YM) field theory by using a quantum distribution function called Husimi function fH(A,E,t)<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>**Hidekazu Tsukiji, Hideaki Iida, Teiji Kunihiro, Akira Ohnishi, Toru T. Takahashi** investigate possible entropy production in Yang-Mills (YM) field theory by using a quantum distribution function called Husimi function fH(A,E,t) for YM field, which is given by (e) a coarse graining of (e) Wigner function and non-negative | |
| $G_{40}$ : Category one of entropy production in Yang-Mills (YM) field theory by using a quantum distribution function called Husimi function fH(A,E,t) for YM field,<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of coarse graining of (e) Wigner function and non-negative<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>coarse graining of (e) Wigner function and non-negative | |
| $G_{44}$ : Category one of Wigner function and non-negative | |

| | |
|---|---|
| $G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of coarse graining<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION A(19)-2

## Husimi Function

**INTRODUCTION—VARIABLES USED**

**Entropy production from chaoticity in Yang-Mills field theory with use of the Husimi function Hidekazu Tsukiji, Hideaki Iida, Teiji Kunihiro, Akira Ohnishi, Toru T. Takahashi (Submitted on 15 Mar 2016)**

(1) **Hidekazu Tsukiji, Hideaki Iida, Teiji Kunihiro, Akira Ohnishi, Toru T. Takahashi** calculate the Husimi-Wehrl (HW) entropy SHW (t)=−TrfHlogfH defined as (=) an integral over the phase-space, for which two adaptations of (e) the test-particle method are used combined with (e&eb) Monte-Carlo method.

(2) Authors utilize the semiclassical approximation to obtain (eb) the time evolution of (e) the distribution functions of (e) the YM field, which is known to show (eb) a chaotic behavior in the classical limit.

**(3)** Authors also make a simplification of (e) the multi-dimensional phase-space integrals by (e) making a product ansatz for (e) the Husimi function, which is found to give (eb) a 10-20 per cent over estimate of (e) the HW entropy for (e) a quantum system with (e&eb) a few degrees of freedom.

**(4)** Authors show that the quantum YM theory does exhibit (eb) the entropy production, and that the entropy production rate agrees with (=) the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting (eb) that the chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory. Subjects: High Energy Physics - Phenomenology (hep-ph); Statistical Mechanics (cond-mat.stat-mech); Nuclear Theory (nucl-th) Report number: KUNS-2616, YITP-16-40 Cite as: arXiv: 1603.04622 [hep-ph] (or arXiv: 1603.04622v1 [hep-ph] for this version)

**NOTATION**

**Module One**

**Hidekazu Tsukiji, Hideaki Iida, Teiji Kunihiro, Akira Ohnishi, Toru T. Takahashi** calculate the Husimi-Wehrl (HW) entropy SHW (t)=−TrfHlogfH defined as (=) an integral over the phase-space, for which two adaptations of (e) the test-particle method are used combined with (e&eb) Monte-Carlo method

$G_{13}$ : Category one of Husimi-Wehrl (HW) entropy SHW (t)=−TrfHlogfH

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|---|---|
| $G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of integral over the phase-space, for which two adaptations of (e) the test-particle method are used combined with (e&eb) Monte-Carlo method<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>**Hidekazu Tsukiji, Hideaki Iida, Teiji Kunihiro, Akira Ohnishi, Toru T. Takahashi** calculate the Husimi-Wehrl (HW) entropy SHW (t)=−TrfHlogfH defined as an integral over the phase-space, for which two adaptations of (e) the test-particle method are used combined with (e&eb) Monte-Carlo method | |
| $G_{16}$ : Category one of test-particle method are used combined with Monte-Carlo method<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of Husimi-Wehrl (HW) entropy SHW (t)=−TrfHlogfH defined as an integral over the phase-space, for which two adaptations<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>**Hidekazu Tsukiji, Hideaki Iida, Teiji Kunihiro, Akira Ohnishi, Toru T. Takahashi** calculate the Husimi-Wehrl (HW) entropy SHW (t)=−TrfHlogfH defined as an integral over the phase-space, for which two adaptations of the test-particle method are used combined with (e&eb) Monte-Carlo method | |
| $G_{20}$ : Category one of **Husimi-Wehrl (HW) entropy SHW (t)=−TrfHlogfH defined as an integral over the phase-space, for which two adaptations of the test-particle method are used combined;** Monte-Carlo method<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of Monte-Carlo method **;Husimi-Wehrl (HW) entropy SHW (t)=−TrfHlogfH defined as an integral over the phase-space, for which two adaptations of the test-particle method are used combined**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Authors utilize the semiclassical approximation to obtain (eb) the time evolution of (e) the distribution | |

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|---|---|
| functions of (e) the YM field, which is known to show (eb) a chaotic behavior in the classical limit | |
| $G_{24}$ : Category one of semiclassical approximation<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of time evolution of (e) the distribution functions of (e) the YM field, which is known to show (eb) a chaotic behavior in the classical limit<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>Authors utilize the semiclassical approximation to obtain the time evolution of (e) the distribution functions of (e) the YM field, which is known to show (eb) a chaotic behavior in the classical limit | |
| $G_{28}$ : Category one of distribution functions of (e) the YM field, which is known to show (eb) a chaotic behavior in the classical limit<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of semiclassical approximation to obtain the time evolution<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>Authors utilize the semiclassical approximation to obtain the time evolution of the distribution functions of (e) the YM field, which is known to show (eb) a chaotic behavior in the classical limit | |
| $G_{32}$ : Category one of YM field, which is known to show (eb) a chaotic behavior in the classical limit<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of semiclassical approximation to obtain the time evolution of the distribution functions<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br>Authors utilize the semiclassical approximation to obtain the time evolution of the distribution functions of | |

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| the YM field, which is known to show (eb) a chaotic behavior in the classical limit | |
| $G_{36}$ : Category one of semiclassical approximation to obtain the time evolution of the distribution functions of the YM field, which is known to show<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of chaotic behavior in the classical limit<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Authors also make a simplification of the multi-dimensional phase-space integrals by (e) making a product ansatz for (e) the Husimi function, which is found to give (eb) a 10-20 per cent over estimate of (e) the HW entropy for (e) a quantum system with (e&eb) a few degrees of freedom | |
| $G_{40}$ : Category one of a **simplification of the multi-dimensional phase-space integrals;** making a product ansatz for (e) the Husimi function, which is found to give (eb) a 10-20 per cent over estimate of (e) the HW entropy for (e) a quantum system with (e&eb) a few degrees of freedom<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of a making a product ansatz for (e) the Husimi function, which is found to give (eb) a 10-20 per cent over estimate of (e) the HW entropy for (e) a quantum system with (e&eb) a few degrees of freedom **;simplification of the multi-dimensional phase-space integrals**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Authors also make a simplification of the multi-dimensional phase-space integrals by making a product ansatz for (e) the Husimi function, which is found to give (eb) a 10-20 per cent over estimate of (e) the HW entropy for (e) a quantum system with (e&eb) a few degrees of freedom | |
| $G_{44}$ : Category one of Husimi function, which is found to give (eb) a 10-20 per cent over estimate of (e) the HW entropy for (e) a quantum system with (e&eb) a few degrees of freedom<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |

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| $T_{44}$ : Category one of simplification of the multi-dimensional phase-space integrals by making a product ansatz<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION A(19)-3**<br><br>**Simplification Of The Multi-Dimensional Phase-Space Integrals** | |
| **INTRODUCTION—VARIABLES USED** | |
| **NOTATION** | |
| **Module One**<br>Authors also make a simplification of the multi-dimensional phase-space integrals by making a product ansatz for the Husimi function, which is found to give (eb) a 10-20 per cent over estimate of (e) the HW entropy for (e) a quantum system with (e&eb) a few degrees of freedom | |
| $G_{13}$ : Category one of simplification of the multi-dimensional phase-space integrals by making a product ansatz for the Husimi function<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of10-20 per cent over estimate of (e) the HW entropy for (e) a quantum system with (e&eb) a few degrees of freedom<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>Authors also make a simplification of the multi-dimensional phase-space integrals by making a product ansatz for the Husimi function, which is found to give a 10-20 per cent over estimate of (e) the HW entropy for (e) a quantum system with (e&eb) a few degrees of freedom | |
| $G_{16}$ : Category one of HW entropy for (e) a quantum system with (e&eb) a few degrees of freedom<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of simplification of the multi-dimensional phase-space integrals by making a product ansatz for the Husimi function, which is found to give a 10-20 per cent over estimate<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |

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| **Module three**<br><br>Authors also make a simplification of the multi-dimensional phase-space integrals by making a product ansatz for the Husimi function, which is found to give a 10-20 per cent over estimate of (e) the HW entropy for a quantum system with (e&eb) a few degrees of freedom | |
| $G_{20}$ : Category one of **simplification of the multi-dimensional phase-space integrals by making a product ansatz for the Husimi function, which is found to give a 10-20 per cent over estimate of (e) the HW entropy for a quantum system;** few degrees of freedom<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of few degrees of freedom **;simplification of the multi-dimensional phase-space integrals by making a product ansatz for the Husimi function, which is found to give a 10-20 per cent over estimate of (e) the HW entropy for a quantum system**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Authors show that the quantum YM theory does exhibit (eb) the entropy production, and that the entropy production rate agrees with (=) the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting (eb) that the chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory.<br><br>Subjects: High Energy Physics - Phenomenology (hep-ph); Statistical Mechanics (cond-mat.stat-mech); Nuclear Theory (nucl-th) Report number: KUNS-2616, YITP-16-40 Cite as: arXiv: 1603.04622 [hep-ph] (or arXiv: 1603.04622v1 [hep-ph] for this version) | |
| $G_{24}$ : Category one of quantum YM theory<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of entropy production, and that the entropy production rate agrees with (=) the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting (eb) that the chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory.<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Authors show that the quantum YM theory does exhibit the entropy production, and that the entropy production rate agrees with (=) the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting (eb) that the chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory | |
| $G_{28}$ : Category one of quantum YM theory does exhibit the entropy production, and that the entropy | |

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| production rate<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting (eb) that the chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Authors show that the quantum YM theory does exhibit the entropy production, and that the entropy production rate agrees with the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting (eb) that the chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory | |
| $G_{32}$ : Category one of quantum YM theory does exhibit the entropy production, and that the entropy production rate agrees with the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Authors show that the quantum YM theory does exhibit the entropy production, and that the entropy production rate agrees with the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting that the chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory | |
| $G_{36}$ : Category one of classical YM field causes (eb) the entropy production in the quantum YM theory<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |

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|---|---|
| $T_{36}$ : Category one of quantum YM theory does exhibit the entropy production, and that the entropy production rate agrees with the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting that the chaoticity<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Authors show that the quantum YM theory does exhibit the entropy production, and that the entropy production rate agrees with the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting that the chaoticity of the classical YM field causes (eb) the entropy production in the quantum YM theory | |
| $G_{40}$ : Category one of quantum YM theory does exhibit the entropy production, and that the entropy production rate agrees with the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting that the chaoticity of the classical YM field<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of entropy production in the quantum YM theory<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Authors show that the quantum YM theory does exhibit the entropy production, and that the entropy production rate agrees with the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting that the chaoticity of the classical YM field causes the entropy production in the quantum YM theory | |
| $G_{44}$ : Category one of quantum YM theory does exhibit the entropy production, and that the entropy production rate agrees with the sum of positive Lyapunov exponents or the Kolmogorov-Sinai entropy, suggesting that the chaoticity of the classical YM field causes the entropy production<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of quantum YM theory<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION A(21)

## Thermalization Of Isolated Quantum Systems

### INTRODUCTION—VARIABLES USED

**Entropy production in quantum Yang-Mills mechanics in semi-classical approximation Hidekazu Tsukiji, Hideaki Iida, Teiji Kunihiro, Akira Ohnishi, Toru T. Takahashi(Submitted on 18 May 2015 (v1), last revised 12 Nov 2015 (this version, v2))**

(1) Authors discuss thermalization of (e) isolated quantum systems by using (e) the Husimi-Wehrl entropy evaluated in (eb) the semiclassical treatment.
(2) The Husimi-Wehrl entropy is (=) the Wehrl entropy obtained by using (e) the Husimi function for (e) the phase space distribution.
(3) The time evolution of the Husimi function is given by (e) smearing the Wigner function, whose time evolution is obtained in (eb) the semiclassical approximation.
(4) Authors show the efficiency and usefulness of (e) this semiclassical treatment in describing entropy production of (e) a couple of quantum mechanical systems, whose classical counter systems are(=) known to be chaotic.
(5) Authors propose two methods to evaluate the time evolution of (e) the Husimi-Wehrl entropy, the test-particle method and (e&eb) the two-step Monte-Carlo method.
**(6)** They demonstrate the characteristics of (e) the two methods by numerical calculations, and show that (eb) the simultaneous application of the two methods ensures (eb) the reliability of the results of (e) the Husimi-Wehrl entropy at a given time. Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th); Quantum Physics (quant-ph) Journal reference: PTEP 2015 (2015) 083A01 DOI: 10.1093/ptep/ptv107 Report number: KUNS-2561, YITP-15-42 Cite as: arXiv:1505.04698 [hep-ph] (or arXiv:1505.04698v2 [hep-ph] for this version)

### NOTATION

### Module One

Authors discuss thermalization of (e) isolated quantum systems by using (e) the Husimi-Wehrl entropy evaluated in (eb) the semiclassical treatment

$G_{13}$ : Category one of isolated quantum systems by using (e) the Husimi-Wehrl entropy evaluated in (eb) the semiclassical treatment

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of thermalization

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

### Module Two

| | |
|---|---|
| Authors discuss thermalization of isolated quantum systems by using (e) the Husimi-Wehrl entropy evaluated in (eb) the semiclassical treatment | |
| $G_{16}$ : Category one of Husimi-Wehrl entropy evaluated in (eb) the semiclassical treatment<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of thermalization of isolated quantum systems<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>Authors discuss thermalization of isolated quantum systems by using the Husimi-Wehrl entropy evaluated in (eb) the semiclassical treatment | |
| $G_{20}$ : Category one of thermalization of isolated quantum systems by using the Husimi-Wehrl entropy evaluated<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of semiclassical treatment<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>The Husimi-Wehrl entropy is (=) the Wehrl entropy obtained by using (e) the Husimi function for (e) the phase space distribution | |
| $G_{24}$ : Category one of Husimi-Wehrl entropy<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of Wehrl entropy obtained by using (e) the Husimi function for (e) the phase space distribution<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>The Husimi-Wehrl entropy is the Wehrl entropy obtained by using (e) the Husimi function for (e) the phase space distribution | |

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|---|---|
| $G_{28}$ : Category one of Husimi function for (e) the phase space distribution<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of Husimi-Wehrl entropy is the Wehrl entropy<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>Husimi function for (e) the phase space distribution | |
| $G_{32}$ : Category one of phase space distribution<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of Husimi function<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br>The time evolution of the Husimi function is given by (e) smearing the Wigner function, whose time evolution is obtained in (eb) the semiclassical approximation | |
| $G_{36}$ : Category one of time evolution of the Husimi function<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of smearing the Wigner function, whose time evolution is obtained in (eb) the semiclassical approximation<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>The time evolution of the Husimi function is given by smearing the Wigner function, whose time evolution is obtained in (eb) the semiclassical approximation | |

$G_{40}$ : Category one of time evolution of the Husimi function is given by smearing the Wigner function, whose time evolution

$G_{41}$ : Category two of SAS

$G_{42}$ : Category three of SAS

$T_{40}$ : Category one of semiclassical approximation

$T_{41}$ : Category two of SAS

$T_{42}$ : Category three of SAS

**Module Nine**

Authors show the efficiency and usefulness of (e) this semiclassical treatment in describing entropy production of (e) a couple of quantum mechanical systems, whose classical counter systems are(=) known to be chaotic

$G_{44}$ : Category one of couple of quantum mechanical systems, whose classical counter systems are(=) known to be chaotic

$G_{45}$ : Category two of SAS

$G_{46}$ : Category three of SAS

$T_{44}$ : Category one of entropy production

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

**The Coefficients:**

$(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:
$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$
$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,
$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$
$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$
$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$
$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$

are Accentuation coefficients
$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$
$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$
$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}\ (a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,
$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$
$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,
$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,

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| $(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)},$ are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}, t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}, t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21}, t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23}, t) =$ First detritions factor | |

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| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25}, t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}, t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}), t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}), t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}), t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25}, t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}), t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29}, t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}, t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}), t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}), t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}), t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29}, t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}), t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33}, t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}, t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}), t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}), t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}), t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33}, t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}, t)\right]G_{37}$ | *38* |

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| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{array}{ccc} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for | |

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| category 1, 2 and 3<br>$+(a_{16}'')^{(2,2,)}(T_{17},t)$, $+(a_{17}'')^{(2,2,)}(T_{17},t)$, $+(a_{18}'')^{(2,2,)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}'')^{(3,3,)}(T_{21},t)$, $+(a_{21}'')^{(3,3,)}(T_{21},t)$, $+(a_{22}'')^{(3,3,)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}'')^{(4,4,4,4,)}(T_{25},t)$, $+(a_{25}'')^{(4,4,4,4,)}(T_{25},t)$, $+(a_{26}'')^{(4,4,4,4,)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}'')^{(5,5,5,5,)}(T_{29},t)$, $+(a_{29}'')^{(5,5,5,5,)}(T_{29},t)$, $+(a_{30}'')^{(5,5,5,5,)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}'')^{(6,6,6,6,)}(T_{33},t)$, $+(a_{33}'')^{(6,6,6,6,)}(T_{33},t)$, $+(a_{34}'')^{(6,6,6,6,)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{38}'')^{(7,7)}(T_{37},t)$ $+(a_{37}'')^{(7,7)}(T_{37},t)$ $+(a_{36}'')^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3<br>$+(a_{40}'')^{(8,8)}(T_{41},t)$ $+(a_{41}'')^{(8,8)}(T_{41},t)$ $+(a_{42}'')^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3<br>$+(a_{44}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}')^{(1)}\left[-(b_{13}'')^{(1)}(G,t)\right] & \left[-(b_{16}'')^{(2,2,)}(G_{19},t)\right] & \left[-(b_{20}'')^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{24}'')^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{28}'')^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{32}'')^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{36}'')^{(7,7,)}(G_{39},t)\right] & \left[-(b_{40}'')^{(8,8)}(G_{43},t)\right] & \left[-(b_{44}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}')^{(1)}\left[-(b_{14}'')^{(1)}(G,t)\right] & \left[-(b_{17}'')^{(2,2,)}(G_{19},t)\right] & \left[-(b_{21}'')^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{25}'')^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{29}'')^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{33}'')^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{37}'')^{(7,7,)}(G_{39},t)\right] & \left[-(b_{41}'')^{(8,8)}(G_{43},t)\right] & \left[-(b_{45}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}')^{(1)}\left[-(b_{15}'')^{(1)}(G,t)\right] & \left[-(b_{18}'')^{(2,2,)}(G_{19},t)\right] & \left[-(b_{22}'')^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{26}'')^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{30}'')^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{34}'')^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{38}'')^{(7,7,)}(G_{39},t)\right] & \left[-(b_{42}'')^{(8,8)}(G_{43},t)\right] & \left[-(b_{46}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{15}$ | *60* |
| Where $-(b_{13}'')^{(1)}(G,t)$, $-(b_{14}'')^{(1)}(G,t)$, $-(b_{15}'')^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{16}'')^{(2,2,)}(G_{19},t)$, $-(b_{17}'')^{(2,2,)}(G_{19},t)$, $-(b_{18}'')^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{20}'')^{(3,3,)}(G_{23},t)$, $-(b_{21}'')^{(3,3,)}(G_{23},t)$, $-(b_{22}'')^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}'')^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}'')^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}'')^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}'')^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}'')^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}'')^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}'')^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}'')^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}'')^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for | |

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| category 1, 2 and 3 $\boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[\begin{array}{ccc} (a_{16}^{'})^{(2)} & \boxed{+(a_{16}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)} \quad \boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)} \end{array}\right] G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[\begin{array}{ccc} (a_{17}^{'})^{(2)} & \boxed{+(a_{17}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)} \quad \boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)} \end{array}\right] G_{17}$ | *62* |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[\begin{array}{ccc} (a_{18}^{'})^{(2)} & \boxed{+(a_{18}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)} \quad \boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)} \end{array}\right] G_{18}$ | *63* |
| Where $\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)}$ are first augmentation coefficients for category 1, 2 and 3 $\boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)}$ are second augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)}$ are seventh augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)}$ are eight augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)}$ are ninth augmentation coefficient for category 1, 2 and 3 | |

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|---|---|
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)} & -(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{16}$ | 64 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)} & -(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{17}$ | 65 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)} & -(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{18}$ | 66 |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$ , $-(b_{17}^{''})^{(2)}(G_{19},t)$ , $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3<br>$-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a_{20}^{'})^{(3)} & +(a_{20}^{''})^{(3)}(T_{21},t) & +(a_{16}^{''})^{(2,2,2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t) & \\ +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9)}(T_{45},t) & \end{bmatrix} G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a_{21}^{'})^{(3)} & +(a_{21}^{''})^{(3)}(T_{21},t) & +(a_{17}^{''})^{(2,2,2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t) & \\ +(a_{37}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9)}(T_{45},t) & \end{bmatrix} G_{21}$ | 68 |

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a'_{22})^{(3)}\boxed{+(a''_{22})^{(3)}(T_{21},t)} & \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{22} \quad (69)$$

$\boxed{+(a''_{20})^{(3)}(T_{21},t)}, \boxed{+(a''_{21})^{(3)}(T_{21},t)}, \boxed{+(a''_{22})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)}, \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)}, \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)}, \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)}, \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)}$ are third augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)}, \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)}, \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)}, \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)}, \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)}, \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)}, \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)}, \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)}, \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)}, \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)}, \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)}$ are eight augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)}, \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)}, \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1, 2 and 3

$$\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b'_{20})^{(3)}\boxed{-(b''_{20})^{(3)}(G_{23},t)} & \boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{13})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{20} \quad (70)$$

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b'_{21})^{(3)}\boxed{-(b''_{21})^{(3)}(G_{23},t)} & \boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{14})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)}\boxed{-(b''_{22})^{(3)}(G_{23},t)} & \boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{22} \quad (72)$$

$\boxed{-(b''_{20})^{(3)}(G_{23},t)}, \boxed{-(b''_{21})^{(3)}(G_{23},t)}, \boxed{-(b''_{22})^{(3)}(G_{23},t)}$ *are first detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)}, \boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)}, \boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)}$ *are second detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{13})^{(1,1,1,)}(G,t)}, \boxed{-(b''_{14})^{(1,1,1,)}(G,t)}, \boxed{-(b''_{15})^{(1,1,1,)}(G,t)}$ *are third detrition coefficients for category 1,2 and 3*

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|---|---|
| $-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*<br>$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$ $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a_{24}^{'})^{(4)} & +(a_{24}^{''})^{(4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,)}(T_{29},t) & +(a_{32}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24}$ | *73* |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a_{25}^{'})^{(4)} & +(a_{25}^{''})^{(4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,)}(T_{29},t) & +(a_{33}^{''})^{(6,6)}(T_{33},t) \\ & +(a_{14}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25}$ | *74* |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a_{26}^{'})^{(4)} & +(a_{26}^{''})^{(4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,)}(T_{29},t) & +(a_{34}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26}$ | *75* |

$(a_{24}^{''})^{(4)}(T_{25},t)$, $(a_{25}^{''})^{(4)}(T_{25},t)$, $(a_{26}^{''})^{(4)}(T_{25},t)$ are first augmentation coefficients category 1, 2 3

$+(a_{28}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,)}(T_{29},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,)}(T_{33},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1, 2 and 3

$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3

$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1, 2 and 3

$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b_{24}^{'})^{(4)} & -(b_{24}^{''})^{(4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,)}(G_{35},t) \\ & -(b_{13}^{''})^{(1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3)}(G_{23},t) \\ & -(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b_{25}^{'})^{(4)} & -(b_{25}^{''})^{(4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,)}(G_{35},t) \\ & -(b_{14}^{''})^{(1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3)}(G_{23},t) \\ & -(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b_{26}^{'})^{(4)} & -(b_{26}^{''})^{(4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,)}(G_{35},t) \\ & -(b_{15}^{''})^{(1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3)}(G_{23},t) \\ & -(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{26} \quad (78)$$

Where $-(b_{24}^{''})^{(4)}(G_{27},t)$, $-(b_{25}^{''})^{(4)}(G_{27},t)$, $-(b_{26}^{''})^{(4)}(G_{27},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,)}(G_{31},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,)}(G_{35},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2 and 3

| | |
|---|---|
| $-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 $-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3 $-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a_{28}^{'})^{(5)} & +(a_{28}^{''})^{(5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,)}(T_{25},t) & +(a_{32}^{''})^{(6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{28}$ | 79 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a_{29}^{'})^{(5)} & +(a_{29}^{''})^{(5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,)}(T_{25},t) & +(a_{33}^{''})^{(6,6,6)}(T_{33},t) \\ & +(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{29}$ | 80 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a_{30}^{'})^{(5)} & +(a_{30}^{''})^{(5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,)}(T_{25},t) & +(a_{34}^{''})^{(6,6,6)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{30}$ | 81 |
| Where $+(a_{28}^{''})^{(5)}(T_{29},t)$, $+(a_{29}^{''})^{(5)}(T_{29},t)$, $+(a_{30}^{''})^{(5)}(T_{29},t)$ are first augmentation coefficients for category 1, 2 and 3 And $+(a_{24}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,)}(T_{25},t)$ are second augmentation coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ are third augmentation coefficient for category 1, 2 and 3 $+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3 $+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3 $+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3 $+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3 $+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3 $+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b'_{28})^{(5)} & -(b''_{28})^{(5)}(G_{31},t) & -(b''_{24})^{(4,4,)}(G_{27},t) & -(b''_{32})^{(6,6,6)}(G_{35},t) \\ -(b''_{13})^{(1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b'_{29})^{(5)} & -(b''_{29})^{(5)}(G_{31},t) & -(b''_{25})^{(4,4,)}(G_{27},t) & -(b''_{33})^{(6,6,6)}(G_{35},t) \\ -(b''_{14})^{(1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b'_{30})^{(5)} & -(b''_{30})^{(5)}(G_{31},t) & -(b''_{26})^{(4,4,)}(G_{27},t) & -(b''_{34})^{(6,6,6)}(G_{35},t) \\ -(b''_{15})^{(1,1,1,1,1,)}(G,t) & -(b''_{18})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{30} \quad (84)$$

*where* $-(b''_{28})^{(5)}(G_{31},t)$ , $-(b''_{29})^{(5)}(G_{31},t)$ , $-(b''_{30})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{24})^{(4,4,)}(G_{27},t)$, $-(b''_{25})^{(4,4,)}(G_{27},t)$, $-(b''_{26})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3

$-(b''_{32})^{(6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b''_{13})^{(1,1,1,1,1)}(G,t)$, $-(b''_{14})^{(1,1,1,1,1)}(G,t)$ , $-(b''_{15})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3

$-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3

$-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)$, $-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)$, $-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3

$-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3

$-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3

$-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a'_{32})^{(6)} & +(a''_{32})^{(6)}(T_{33},t) & +(a''_{28})^{(5,5,5)}(T_{29},t) & +(a''_{24})^{(4,4,4,)}(T_{25},t) \\ +(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a'_{33})^{(6)} & +(a''_{33})^{(6)}(T_{33},t) & +(a''_{29})^{(5,5,5)}(T_{29},t) & +(a''_{25})^{(4,4,4,)}(T_{25},t) \\ +(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)} G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)} T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)} T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)} T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{28})^{(5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)} & +(a_{36}^{''})^{(7)}(T_{37},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)} & +(a_{37}^{''})^{(7)}(T_{37},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)} & +(a_{38}^{''})^{(7)}(T_{37},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15} \quad (93)$$

Where $(a_{36}^{''})^{(7)}(T_{37},t)$, $(a_{37}^{''})^{(7)}(T_{37},t)$, $(a_{38}^{''})^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation

coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3 $+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}(G_{39},t) & -(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

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$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}-(b_{37}^{''})^{(7)}(G_{39},t) & -(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}(G_{39},t) & -(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3 $-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3 $-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3 $-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{46}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}')^{(8)} & +(a_{40}'')^{(8)}(T_{41},t) & +(a_{16}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{24}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}'')^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13} \quad (95)$$

$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}')^{(8)} & +(a_{41}'')^{(8)}(T_{41},t) & +(a_{17}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{25}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}'')^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}')^{(8)} & +(a_{42}'')^{(8)}(T_{41},t) & +(a_{18}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{26}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{15}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}'')^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$$

Where $+(a_{40}'')^{(8)}(T_{41},t)$, $+(a_{41}'')^{(8)}(T_{41},t)$, $+(a_{42}'')^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & -(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & -(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & -(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

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$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{13}$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[\begin{array}{ccc} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$\frac{dT_{44}}{dt} =$

$$(b_{44})^{(9)}T_{45} - \left[\begin{array}{ccc} (b_{44}^{'})^{(9)}\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[\begin{array}{ccc} (b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[\begin{array}{ccc} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

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| **Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br>$(a_i'')^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$<br>$(b_i'')^{(1)}(G,t) \le (r_i)^{(1)} \le (b_i')^{(1)} \le (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2\to\infty}(a_i'')^{(1)}(T_{14},t) = (p_i)^{(1)}$<br>$\lim_{G\to\infty}(b_i'')^{(1)}(G,t) = (r_i)^{(1)}$<br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(1)}(T_{14}',t) - (a_i'')^{(1)}(T_{14},t)\rvert \le (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}'\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br>$\lvert(b_i'')^{(1)}(G',t) - (b_i'')^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G'\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(1)}(T_{14}',t)$ and $(a_i'')^{(1)}(T_{14},t)$ . $(T_{14}',t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i'')^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i'')^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i')^{(1)}, (b_i)^{(1)}, (b_i')^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i,j = 16,17,18$ | |

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|---|---|
| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17},t) \le (p_i)^{(2)} \le (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19},t) \le (r_i)^{(2)} \le (b_i')^{(2)} \le (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2\to\infty}(a_i'')^{(2)}(T_{17},t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty}(b_i'')^{(2)}((G_{19}),t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}$ are positive constants and $i = 16,17,18$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(2)}(T_{17}',t) - (a_i'')^{(2)}(T_{17},t)\rvert \le (\hat{k}_{16})^{(2)}\lvert T_{17} - T_{17}'\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i'')^{(2)}((G_{19})',t) - (b_i'')^{(2)}((G_{19}),t)\rvert < (\hat{k}_{16})^{(2)}\lvert\lvert(G_{19}) - (G_{19})'\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}',t)$ and $(a_i'')^{(2)}(T_{17},t)$ . $(T_{17}',t)$ and $(T_{17},t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}]$ . It is to be noted that $(a_i'')^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i'')^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i')^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ | 111 |

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|---|---|
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(3)} (T_{21}, t) = (p_i)^{(3)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(3)} (G_{23}, t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'}, t) - (a_i^{''})^{(3)}(T_{21}, t)\rvert \leq (\hat{k}_{20})^{(3)} \lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)} t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'}, t) - (b_i^{''})^{(3)}(G_{23}, t)\rvert < (\hat{k}_{20})^{(3)} \lvert\lvert G_{23} - G_{23}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)} t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'}, t)$ and$(a_i^{''})^{(3)}(T_{21}, t)$ . $(T_{21}^{'}, t)$ And $(T_{21}, t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$<br>satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$ | 116 |

| | |
|---|---|
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}\big((G_{27}), t\big) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(4)}\big((G_{27}), t\big) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\rvert \leq (\hat{k}_{24})^{(4)} \lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)} t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'}, t) - (b_i^{''})^{(4)}\big((G_{27}), t\big)\rvert < (\hat{k}_{24})^{(4)} \lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)} t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $\big[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\big]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |

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| $\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i^{'})^{(4)} +\ (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}\big((G_{31}), t\big) \leq\ (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(5)}\,(T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(5)}\,(G_{31}, t) =\ (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'}, t) - (b_i^{''})^{(5)}\big((G_{31}), t\big)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $\big[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\big]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br><br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\ , \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities | 126 |

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| $\frac{1}{(\widehat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\widehat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(6)}(T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(6)}((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)}\lvert T_{33} - T_{33}^{'}\rvert e^{-(\widehat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})^{'}, t) - (b_i^{''})^{(6)}((G_{35}), t)\rvert < (\hat{k}_{32})^{(6)}\lvert\lvert(G_{35}) - (G_{35})^{'}\rvert\rvert e^{-(\widehat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and$(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\widehat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\widehat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\widehat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\widehat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants | 130 |

$(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$
satisfy the inequalities

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

Where we suppose

(QQQQQ) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i, j = 36,37,38$ 131

(RRRRR) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$:

$$(a_i^{''})^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$$

$$(b_i^{''})^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$$

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*(SSSSS)* $\lim_{T_2 \to \infty} (a_i^{''})^{(7)}(T_{37}, t) = (p_i)^{(7)}$
*(TTTTT)*
$\lim_{G \to \infty} (b_i^{''})^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, \ (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)} |T_{37} - T_{37}^{'}| e^{-(\hat{M}_{36})^{(7)} t}$$

$$|(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)} ||(G_{39}) - (G_{39})^{'}|| e^{-(\hat{M}_{36})^{(7)} t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'}, t)$ and $(a_i^{''})^{(7)}(T_{37}, t)$ . $(T_{37}^{'}, t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ : 134

(UUUUU) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants

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| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ : <br><br> (VVVVV) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities <br><br> $\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ <br><br> $\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}, (r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ : <br><br> Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})', t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})'\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would | |

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| be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i, j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45}, t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47}, t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2 \to \infty} (a_i'')^{(9)}(T_{45}, t) = (p_i)^{(9)}$<br>$\lim_{G \to \infty} (b_i'')^{(9)}(G_{47}, t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}', t) - (a_i'')^{(9)}(T_{45}, t)\rvert \le (\hat{k}_{44})^{(9)} \lvert T_{45} - T_{45}' \rvert e^{-(\hat{M}_{44})^{(9)} t}$<br>$\lvert(b_i'')^{(9)}((G_{47})', t) - (b_i'')^{(9)}((G_{47}), t)\rvert < (\hat{k}_{44})^{(9)} \lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}', t)$ | |

and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)},(\widehat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{44})^{(9)}=1$ then the function $(a_i'')^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$ :

$(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},$ are positive constants

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\ ,\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}<1$$

**Definition of** $(\hat{P}_{44})^{(9)},(\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},(\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)},(a_i')^{(9)},(b_i)^{(9)},(b_i')^{(9)},(p_i)^{(9)},\ (r_i)^{(9)},i=44,45,46,$ satisfy the inequalities

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[(a_i)^{(9)}+(a_i')^{(9)}+\ (\hat{A}_{44})^{(9)}+\ (\hat{P}_{44})^{(9)}\ (\hat{k}_{44})^{(9)}]<1$$

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[\ (b_i)^{(9)}+(b_i')^{(9)}+\ (\hat{B}_{44})^{(9)}+\ (\hat{Q}_{44})^{(9)}\ \ (\hat{k}_{44})^{(9)}]<1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0)\ ,T_i(0)$ :

$$G_i(t)\le\ (\hat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{G_i(0)=G_i^0>0}$$

$$T_i(t)\le\ (\hat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{T_i(0)=T_i^0>0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0)\ ,T_i(0)$

$$G_i(t)\le\ (\hat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t)\le\ (\hat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

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| **Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$ | 153 B |

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| $T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |

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| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} G_{21}(s_{(20)}) - \left((a_{20}')^{(3)} + a_{20}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{20}(s_{(20)})\right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[(a_{21})^{(3)} G_{20}(s_{(20)}) - \left((a_{21}')^{(3)} + (a_{21}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}')^{(3)} - (b_{21}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{28})^{(5)}$, $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[ (b_{29})^{(5)} T_{28}(s_{(28)}) - \left( (b'_{29})^{(5)} - (b''_{29})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[ (b_{30})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{30})^{(5)} - (b''_{30})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{30}(s_{(28)}) \right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** | |

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| Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[ (a_{32})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{32}(s_{(32)}) \right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[ (a_{33})^{(6)} G_{32}(s_{(32)}) - \left( (a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[ (a_{34})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{34}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[ (b_{32})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{32}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[ (b_{33})^{(6)} T_{32}(s_{(32)}) - \left( (b'_{33})^{(6)} - (b''_{33})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[ (b_{34})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{34})^{(6)} - (b''_{34})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{34}(s_{(32)}) \right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |

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| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}^{'})^{(8)} + a_{40}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^{0} + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^{0} + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^{0} + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{13}(t) \leq G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \leq \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}$ , $G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \leq G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that | 170 |

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| $(G_{16}(t)-G_{16}^0)e^{-(\bar{M}_{16})^{(2)}t} \le \frac{(a_{16})^{(2)}}{(\bar{M}_{16})^{(2)}}\left[((\hat{P}_{16})^{(2)}+G_{17}^0)e^{\left(-\frac{(\hat{P}_{16})^{(2)}+G_{17}^0}{G_{17}^0}\right)}+(\hat{P}_{16})^{(2)}\right]$ | |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0+(\hat{P}_{20})^{(3)}e^{(\bar{M}_{20})^{(3)}s_{(20)}}\right)\right] ds_{(20)} =$<br>$(1+(a_{20})^{(3)}t)G_{21}^0 + \frac{(a_{20})^{(3)}(\hat{P}_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left(e^{(\bar{M}_{20})^{(3)}t}-1\right)$ | 171 |
| From which it follows that<br>$(G_{20}(t)-G_{20}^0)e^{-(\bar{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that<br>$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\bar{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$<br>$(1+(a_{24})^{(4)}t)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left(e^{(\bar{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that<br>$(G_{24}(t)-G_{24}^0)e^{-(\bar{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$<br>$(1+(a_{28})^{(5)}t)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)}t}-1\right)$ | |
| From which it follows that<br>$(G_{28}(t)-G_{28}^0)e^{-(\bar{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 5 | 175 |
| The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious | 176 |

that

$$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$$

$$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$$

From which it follows that 177

$$(G_{32}(t) - G_{32}^0) e^{-(\hat{M}_{32})^{(6)} t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)} + G_{33}^0) e^{\left(-\frac{(\hat{P}_{32})^{(6)} + G_{33}^0}{G_{33}^0}\right)} + (\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

(u) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 178

$$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)}\left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$$

$$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$$

From which it follows that

$$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)} + G_{37}^0) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)}\left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} = 180$$

$$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$$

From which it follows that 181

$$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)} + G_{41}^0) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 8

Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$

The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)}\left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$$

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| $\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$ | |
| From which it follows that<br>$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t}\leq\frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{P}_{44})^{(9)}+G_{45}^{0}\right)e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ and to choose<br>$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{13})^{(1)}\left\|G_{14}^{(1)}-G_{14}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$<br>$\int_0^t\{(a_{13}')^{(1)}\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{-(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$G_{13}^{(2)}\left\\|(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}'')^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right\\|\ e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$ | |

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| Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | |
| $\left\|G^{(1)} - G^{(2)}\right\| e^{-(\widehat{M}_{13})^{(1)}t} \leq$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right) d\left(\left(G^{(1)}, T^{(1)};\ G^{(2)}, T^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(1)}$ and $(b''_i)^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(1)} - (a''_i)^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1 / (a'_{14})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2 / (a'_{15})^{(1)}$<br><br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(1)}(G(t), t)) = (b'_{14})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ : | 189 |

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| Indeed let $t_1$ be so that for $t > t_1$<br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ $and$ $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |

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| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$(a_{16}^{''})^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)} \left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}} \left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)} (\widehat{k}_{16})^{(2)}\right) d\left(\left((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)}\right)\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)} (T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0 \quad \text{for } t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)} \left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$<br>In the same way , one can obtain | 201 |

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| $G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$, $G_{18}$ and $G_{16}$, $G_{17}$ respectively. | |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$. The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}$, $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}} < 1$ and to choose<br><br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$, $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\bar{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\bar{M}_{20})^{(3)}t}\}$ | 211 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :\left( \widetilde{(G_{23})}, \widetilde{(T_{23})} \right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)} \left\|G_{21}^{(1)} - G_{21}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\, ds_{(20)} +$<br><br>$\int_0^t \{(a_{20}^{'})^{(3)} \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$G_{20}^{(2)} \left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}^{''})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\} ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} t} \leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a_{20}^{'})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)} (\widehat{k}_{20})^{(3)}\right) d\left(\left((G_{23})^{(1)}, (T_{23})^{(1)}; (G_{23})^{(2)}, (T_{23})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ *and* $(\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)} t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating | 217 |

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| $G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}')^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}')^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(3)}\left((G_{23})(t),t\right) = (b_{21}')^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i'')^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}'')^{(3)}\left((G_{23})(t),t\right) = (b_{22}')^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}\left[(\hat{P}_{24})^{(4)} + \left((\hat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}}\left[\left((\hat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{24})^{(4)}\right] \leq (\hat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$ | 225 |

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{24})^{(4)}\left|G_{25}^{(1)}-G_{25}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$$

$$\int_0^t\{(a_{24}^{'})^{(4)}\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$G_{24}^{(2)}\left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)}-(G_{27})^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}t}\le$$
$$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows

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**Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}\ and\ (\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

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**Remark 17:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

it results

$$G_i\ (t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]}\ge 0$$

$$T_i\ (t)\ge T_i^0e^{\left(-(b_i^{'})^{(4)}t\right)}>0\quad \text{for t} > 0$$

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**Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1,\left((\widehat{M}_{24})^{(4)}\right)_2\ and\ \left((\widehat{M}_{24})^{(4)}\right)_3$ :

**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if

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| $G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a'_{25})^{(4)} G_{25}$ and by integrating<br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a'_{25})^{(4)}$<br>In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a'_{26})^{(4)}$<br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t),t)) = (b'_{25})^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded.<br>The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}\left((G_{27})(t),t\right) = (b'_{26})^{(4)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + \left((\widehat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\widehat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{28})^{(5)}\right] \leq (\widehat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying | |

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| Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}((G_{31}),(T_{31}))$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)}+$<br>$\int_0^t\{(a_{28}^{'})^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\le$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}^{'})^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}^{''})^{(5)}$ and $(b_{28}^{''})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(5)}-(a_i^{''})^{(5)}(T_{29}(s_{(28)}),s_{(28)})\}ds_{(28)}\right]}\ge 0$ | 239 |

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| $T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(5)}t)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}^{'})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1/(a_{29}^{'})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2/(a_{30}^{'})^{(5)}$<br><br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t),t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right)$, $t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |

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| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{32})^{(6)}\right]\leq(\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}ds_{(32)}+$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a''_{32})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a'_{32})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(6)}$ and $(b_i'')^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 249 |

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| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(6)} - (a_i'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(6)} t)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}')^{(6)} G_{33}$ and by integrating<br><br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}')^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}')^{(6)}$<br><br>If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(6)}((G_{35})(t), t)) = (b_{33}')^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i'')^{(6)}((G_{35})(t), t) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty} (b_{34}'')^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}')^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have | 255 |

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| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)}+\left((\hat{P}_{36})^{(7)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{36})^{(7)}\right]\leq(\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}')^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}'')^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}')^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}'')^{(7)}$ and $(b_{36}'')^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$. | 260 |

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| If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2 / (a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t), t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t), t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right),\quad t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded. | 265 |

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| The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t),t\right)=(b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}<1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)}+\left((\hat{P}_{40})^{(8)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{40})^{(8)}\right]\leq(\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}ds_{(40)}+$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\leq$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ $and$ $T_i$. Taking into account the hypothesis the result follows | 273 |

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| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(8)}-(a_i^{''})^{(8)}(T_{41}(s_{(40)}),s_{(40)})\}ds_{(40)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2\ and\ \left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)}G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1/(a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2/(a_{42}^{'})^{(8)}$<br><br>If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}\,((G_{43})(t),t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t),t\right) < \varepsilon_8, T_{40}\,(t) > (m)^{(8)}$ | 278 |

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| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.<br>The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}^{'})^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$ | |
| In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying 39,35,36 into itself | |
| The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{47})^{(1)}, (T_{47})^{(1)}), ((G_{47})^{(2)}, (T_{47})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \left(\widetilde{(G_{47})}, \widetilde{(T_{47})}\right) = \mathcal{A}^{(9)}\left((G_{47}), (T_{47})\right)$<br>It results<br><br>$\left\|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$<br>$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right) - (a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right| \; e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$<br><br>Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on 45,46,47,28 and 29 it follows | |
| $\left|(G_{47})^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t} \leq$<br>$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)}, (T_{47})^{(1)}; (G_{47})^{(2)}, (T_{47})^{(2)})\right)$ | |

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44{,}45{,}46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$

From 99 to 44 it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(9)}t\right)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)}G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t), t\right) < \varepsilon_9, T_{44}\,(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded. The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b''_{46})^{(9)}\left((G_{47})(t),t\right) = (b'_{46})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a'_{13})^{(1)} + (a'_{14})^{(1)} - (a''_{13})^{(1)}(T_{14}, t) + (a''_{14})^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b'_{13})^{(1)} + (b'_{14})^{(1)} - (b''_{13})^{(1)}(G,t) - (b''_{14})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,

and $\boxed{(u_0)^{(1)} = \frac{T^0_{13}}{T^0_{14}}}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$

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| are defined | |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le \right.$<br>$\left.\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a'_{15})^{(1)})}[e^{(S_1)^{(1)}t} - e^{-(a'_{15})^{(1)}t}] + G_{15}^0 e^{-(a'_{15})^{(1)}t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b'_{15})^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b'_{15})^{(1)}t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)}t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$<br>$(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ :<br>$(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a'_{16})^{(2)} + (a'_{17})^{(2)} - (a''_{16})^{(2)}(T_{17}, t) + (a''_{17})^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b'_{16})^{(2)} + (b'_{17})^{(2)} - (b''_{16})^{(2)}((G_{19}), t) - (b''_{17})^{(2)}((G_{19}), t) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(v_1)^{(2)}, (v_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |

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| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\left(u^{(2)}\right)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}$,, $(\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\left(u^{(2)}\right)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously $(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$ $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities $G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)}t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le\right.$ | 312 |

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| $\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_2)^{(2)}((S_1)^{(2)}-(a_{18}')^{(2)})}[e^{(S_1)^{(2)}t}-e^{-(a_{18}')^{(2)}t}]+G_{18}^{0}e^{-(a_{18}')^{(2)}t})$ | |
| $\boxed{T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}}T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le\frac{1}{(\mu_2)^{(2)}}T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)}T_{16}^{0}}{(\mu_1)^{(2)}((R_1)^{(2)}-(b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t}-e^{-(b_{18}')^{(2)}t}\right]+T_{18}^{0}e^{-(b_{18}')^{(2)}t}\le T_{18}(t)\le$ $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)},(S_2)^{(2)},(R_1)^{(2)},(R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)}=(a_{16})^{(2)}(m_2)^{(2)}-(a_{16}')^{(2)}$ $(S_2)^{(2)}=(a_{18})^{(2)}-(p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)}=(b_{16})^{(2)}(\mu_2)^{(1)}-(b_{16}')^{(2)}$ $(R_2)^{(2)}=(b_{18}')^{(2)}-(r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$: $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)}\le-(a_{20}')^{(3)}+(a_{21}')^{(3)}-(a_{20}'')^{(3)}(T_{21},t)+(a_{21}'')^{(3)}(T_{21},t)\le-(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)}\le-(b_{20}')^{(3)}+(b_{21}')^{(3)}-(b_{20}'')^{(3)}(G_{23},t)-(b_{21}'')^{(3)}((G_{23}),t)\le-(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)},(v_2)^{(3)},(u_1)^{(3)},(u_2)^{(3)}$: By $(v_1)^{(3)}>0,(v_2)^{(3)}<0$ and respectively $(u_1)^{(3)}>0,(u_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_1)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_1)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ and By $(\bar{v}_1)^{(3)}>0,(\bar{v}_2)^{(3)}<0$ and respectively $(\bar{u}_1)^{(3)}>0,(\bar{u}_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_2)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_2)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ | 320 |
| **Definition of** $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$:- | 321 |

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$, ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,

and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously

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$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$$

$(p_i)^{(3)}$ is defined by equation

$$\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t} \quad (323)$$

$$\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$$

$$\left. \frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right) \quad (324)$$

$$\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}} \quad (325)$$

$$\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t} \quad (326)$$

$$\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$$

$$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t} \quad (327)$$

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- (328)

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$$

$$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$$

$$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}\big((G_{27}), t\big) - (b''_{25})^{(4)}\big((G_{27}), t\big) \le -(\tau_1)^{(4)}$$

**Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : 329

By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{v}_1)^{(4)} > 0$, $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (v_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}, \ \textbf{\textit{if}}\ (v_0)^{(4)} < (v_1)^{(4)}$$

$$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}, \textbf{\textit{if}}\ (v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)},$$

and $\boxed{(v_0)^{(4)} = \frac{G^0_{24}}{G^0_{25}}}$

$$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}, \ \textbf{\textit{if}}\ (\bar{v}_4)^{(4)} < (v_0)^{(4)}$$

and analogously 331

$$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}, \ \textbf{\textit{if}}\ (u_0)^{(4)} < (u_1)^{(4)}$$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}, \textbf{\textit{if}}\ (u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)},$$

and $\boxed{(u_0)^{(4)} = \frac{T^0_{24}}{T^0_{25}}}$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \textbf{\textit{if}}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)} \text{ where } (u_1)^{(4)}, (\bar{u}_1)^{(4)}$$

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| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}')^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}')^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)}t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(\nu_1)^{(5)}, (\nu_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, \nu^{(5)}, u^{(5)}$ :<br>By $(\nu_1)^{(5)} > 0$, $(\nu_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}(\nu^{(5)})^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)} = 0$ | 339 |

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| and $(b_{29})^{(5)}\left(u^{(5)}\right)^2+(\tau_1)^{(5)}u^{(5)}-(b_{28})^{(5)}=0$ and | |
| **Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ :<br><br>By $(\bar{v}_1)^{(5)}>0$ , $(\bar{v}_2)^{(5)}<0$ and respectively $(\bar{u}_1)^{(5)}>0$ , $(\bar{u}_2)^{(5)}<0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_2)^{(5)}v^{(5)}-(a_{28})^{(5)}=0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2+(\tau_2)^{(5)}u^{(5)}-(b_{28})^{(5)}=0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)},(v_0)^{(5)}$ :-<br><br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)}$ by<br><br>$(m_2)^{(5)}=(v_0)^{(5)},(m_1)^{(5)}=(v_1)^{(5)}$, ***if*** $(v_0)^{(5)}<(v_1)^{(5)}$<br><br>$(m_2)^{(5)}=(v_1)^{(5)},(m_1)^{(5)}=(\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)}<(v_0)^{(5)}<(\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)}=\frac{G_{28}^0}{G_{29}^0}}$<br><br>$(m_2)^{(5)}=(v_1)^{(5)},(m_1)^{(5)}=(v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)}<(v_0)^{(5)}$ | 340 |
| and analogously<br><br>$(\mu_2)^{(5)}=(u_0)^{(5)},(\mu_1)^{(5)}=(u_1)^{(5)}$, ***if*** $(u_0)^{(5)}<(u_1)^{(5)}$<br><br>$(\mu_2)^{(5)}=(u_1)^{(5)},(\mu_1)^{(5)}=(\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)}<(u_0)^{(5)}<(\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)}=\frac{T_{28}^0}{T_{29}^0}}$<br><br>$(\mu_2)^{(5)}=(u_1)^{(5)},(\mu_1)^{(5)}=(u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)}<(u_0)^{(5)}$ where $(u_1)^{(5)},(\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{28}^0e^{((S_1)^{(5)}-(p_{28})^{(5)})t}\le G_{28}(t)\le G_{28}^0e^{(S_1)^{(5)}t}$<br><br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}}G_{28}^0e^{((S_1)^{(5)}-(p_{28})^{(5)})t}\le G_{29}(t)\le\frac{1}{(m_2)^{(5)}}G_{28}^0e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)}G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)}-(p_{28})^{(5)}-(S_2)^{(5)})}\left[e^{((S_1)^{(5)}-(p_{28})^{(5)})t}-e^{-(S_2)^{(5)}t}\right]+G_{30}^0e^{-(S_2)^{(5)}t}\le G_{30}(t)\le\right.$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0e^{(R_1)^{(5)}t}\le T_{28}(t)\le T_{28}^0e^{((R_1)^{(5)}+(r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}}T_{28}^0e^{(R_1)^{(5)}t}\le T_{28}(t)\le\frac{1}{(\mu_2)^{(5)}}T_{28}^0e^{((R_1)^{(5)}+(r_{28})^{(5)})t}$ | 346 |

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| $\frac{(b_{30})^{(5)}T_{28}^{0}}{(\mu_1)^{(5)}((R_1)^{(5)}-(b_{30}^{'})^{(5)})}\left[e^{(R_1)^{(5)}t}-e^{-(b_{30}^{'})^{(5)}t}\right]+T_{30}^{0}e^{-(b_{30}^{'})^{(5)}t}\le T_{30}(t)\le$ $\frac{(a_{30})^{(5)}T_{28}^{0}}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t}-e^{-(R_2)^{(5)}t}\right]+T_{30}^{0}e^{-(R_2)^{(5)}t}$ | 347 |
| **Definition of** $(S_1)^{(5)},(S_2)^{(5)},(R_1)^{(5)},(R_2)^{(5)}$:- Where $(S_1)^{(5)}=(a_{28})^{(5)}(m_2)^{(5)}-(a_{28}^{'})^{(5)}$ $(S_2)^{(5)}=(a_{30})^{(5)}-(p_{30})^{(5)}$ $(R_1)^{(5)}=(b_{28})^{(5)}(\mu_2)^{(5)}-(b_{28}^{'})^{(5)}$ $(R_2)^{(5)}=(b_{30}^{'})^{(5)}-(r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define | 349 |
| **Definition of** $(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ : $(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ four constants satisfying $-(\sigma_2)^{(6)}\le-(a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}-(a_{32}^{''})^{(6)}(T_{33},t)+(a_{33}^{''})^{(6)}(T_{33},t)\le-(\sigma_1)^{(6)}$ $-(\tau_2)^{(6)}\le-(b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(b_{32}^{''})^{(6)}\big((G_{35}),t\big)-(b_{33}^{''})^{(6)}\big((G_{35}),t\big)\le-(\tau_1)^{(6)}$ | 350 |
| **Definition of** $(\nu_1)^{(6)},(\nu_2)^{(6)},(u_1)^{(6)},(u_2)^{(6)},\nu^{(6)},u^{(6)}$ : By $(\nu_1)^{(6)}>0$, $(\nu_2)^{(6)}<0$ and respectively $(u_1)^{(6)}>0$, $(u_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2+(\sigma_1)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\big(u^{(6)}\big)^2+(\tau_1)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$ and | 350 |
| **Definition of** $(\bar{\nu}_1)^{(6)},,(\bar{\nu}_2)^{(6)},(\bar{u}_1)^{(6)},(\bar{u}_2)^{(6)}$ : By $(\bar{\nu}_1)^{(6)}>0$, $(\bar{\nu}_2)^{(6)}<0$ and respectively $(\bar{u}_1)^{(6)}>0$, $(\bar{u}_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2+(\sigma_2)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\big(u^{(6)}\big)^2+(\tau_2)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$ **Definition of** $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)},(\nu_0)^{(6)}$ :- If we define $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)}$ by $(m_2)^{(6)}=(\nu_0)^{(6)},(m_1)^{(6)}=(\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)}<(\nu_1)^{(6)}$ $(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\bar{\nu}_6)^{(6)}$, ***if*** $(\nu_1)^{(6)}<(\nu_0)^{(6)}<(\bar{\nu}_1)^{(6)}$, and $\boxed{(\nu_0)^{(6)}=\frac{G_{32}^{0}}{G_{33}^{0}}}$ $(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)}<(\nu_0)^{(6)}$ | 351 |

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| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},\ \boldsymbol{if}\ (u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}, \boldsymbol{if}\ (u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)},$<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}, \boldsymbol{if}\ (\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le\right.$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b_{34}')^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b_{34}')^{(6)} t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$<br>$(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ : | |

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| $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying<br><br>$-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$<br><br>$-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}((G_{39}), t) - (b''_{37})^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ :<br><br>By $(\nu_1)^{(7)} > 0, (\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ :<br><br>By $(\bar{\nu}_1)^{(7)} > 0, (\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$<br>**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-<br><br>If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by<br><br>$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,<br>and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | 362 |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,<br>and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities | 364 |

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| $G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^{0}e^{(S_1)^{(7)}t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | |
| $\frac{1}{(m_7)^{(7)}} G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^{0}e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^{0}e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}^{'})^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}^{'})^{(7)}t}] + G_{38}^{0}e^{-(a_{38}^{'})^{(7)}t})$ | 366 |
| $\boxed{T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)}T_{36}^{0}}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}^{'})^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}^{'})^{(7)}t}\right] + T_{38}^{0}e^{-(b_{38}^{'})^{(7)}t} \le T_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)}T_{36}^{0}}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^{0}e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}^{'})^{(7)}$<br><br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br><br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}^{'})^{(7)}$<br><br>$(R_2)^{(7)} = (b_{38}^{'})^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br><br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br><br>$-(\sigma_2)^{(8)} \le -(a_{40}^{'})^{(8)} + (a_{41}^{'})^{(8)} - (a_{40}^{''})^{(8)}(T_{41}, t) + (a_{41}^{''})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br><br>$-(\tau_2)^{(8)} \le -(b_{40}^{'})^{(8)} + (b_{41}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}), t) - (b_{41}^{''})^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |
| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : | 372 |

By $(v_1)^{(8)} > 0\,, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0\,, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$
and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and

**Definition of** $(\bar{v}_1)^{(8)},, (\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :

By $(\bar{v}_1)^{(8)} > 0\,, (\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0\,, (\bar{u}_2)^{(8)} < 0$ the

roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$

and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$

**Definition of** $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :-

If we define $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by

$(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)},$ ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}$ , ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)},$
and $\boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (v_0)^{(8)},$ ***if*** $(\bar{v}_1)^{(8)} < (v_0)^{(8)}$

and analogously

$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},$ ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$

$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$
and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$

$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)},$ ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$

Then the solution of global equations satisfies the inequalities

$$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)}t} \quad (375)$$

where $(p_i)^{(8)}$ is defined by equation

$$\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t} \quad (376)$$

| | |
|---|---|
| $\left(\frac{(a_{42})^{(8)}G_{40}^{0}}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t}-e^{-(S_2)^{(8)}t}\right]+G_{42}^{0}e^{-(S_2)^{(8)}t}\leq G_{42}(t)\leq \frac{(a_{42})^{(8)}G_{40}^{0}}{(m_2)^{(8)}((S_1)^{(8)}-(a_{42}^{'})^{(8)})}[e^{(S_1)^{(8)}t}-e^{-(a_{42}^{'})^{(8)}t}]+G_{42}^{0}e^{-(a_{42}^{'})^{(8)}t}\right)$ | 377 |
| $\boxed{T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}}T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq \frac{1}{(\mu_2)^{(8)}}T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^{0}}{(\mu_1)^{(8)}((R_1)^{(8)}-(b_{42}^{'})^{(8)})}\left[e^{(R_1)^{(8)}t}-e^{-(b_{42}^{'})^{(8)}t}\right]+T_{42}^{0}e^{-(b_{42}^{'})^{(8)}t}\leq T_{42}(t)\leq$ $\frac{(a_{42})^{(8)}T_{40}^{0}}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t}-e^{-(R_2)^{(8)}t}\right]+T_{42}^{0}e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)},(S_2)^{(8)},(R_1)^{(8)},(R_2)^{(8)}$:-<br>Where $(S_1)^{(8)}=(a_{40})^{(8)}(m_2)^{(8)}-(a_{40}^{'})^{(8)}$<br>$(S_2)^{(8)}=(a_{42})^{(8)}-(p_{42})^{(8)}$<br>$(R_1)^{(8)}=(b_{40})^{(8)}(\mu_2)^{(8)}-(b_{40}^{'})^{(8)}$<br>$(R_2)^{(8)}=(b_{42}^{'})^{(8)}-(r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)}\leq -(a_{44}^{'})^{(9)}+(a_{45}^{'})^{(9)}-(a_{44}^{''})^{(9)}(T_{45},t)+(a_{45}^{''})^{(9)}(T_{45},t)\leq -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)}\leq -(b_{44}^{'})^{(9)}+(b_{45}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47}),t)-(b_{45}^{''})^{(9)}((G_{47}),t)\leq -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(\nu_1)^{(9)},(\nu_2)^{(9)},(u_1)^{(9)},(u_2)^{(9)},\nu^{(9)},u^{(9)}$ :<br>By $(\nu_1)^{(9)}>0$ , $(\nu_2)^{(9)}<0$ and respectively $(u_1)^{(9)}>0$ , $(u_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_1)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$<br>and $(b_{45})^{(9)}(u^{(9)})^2+(\tau_1)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(9)},,(\bar{\nu}_2)^{(9)},(\bar{u}_1)^{(9)},(\bar{u}_2)^{(9)}$ :<br>By $(\bar{\nu}_1)^{(9)}>0$ , $(\bar{\nu}_2)^{(9)}<0$ and respectively $(\bar{u}_1)^{(9)}>0$ , $(\bar{u}_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_2)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$ | |

and $(b_{45})^{(9)}\left(u^{(9)}\right)^2+(\tau_2)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$

**Definition of** $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)},(\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)}$ by

$(m_2)^{(9)}=(\nu_0)^{(9)},(m_1)^{(9)}=(\nu_1)^{(9)},$ ***if*** $(\nu_0)^{(9)}<(\nu_1)^{(9)}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)}<(\nu_0)^{(9)}<(\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)}=\frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\nu_0)^{(9)},$ ***if*** $(\bar{\nu}_1)^{(9)}<(\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)}=(u_0)^{(9)},(\mu_1)^{(9)}=(u_1)^{(9)},$ ***if*** $(u_0)^{(9)}<(u_1)^{(9)}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)}<(u_0)^{(9)}<(\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)}=\frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(u_0)^{(9)},$ ***if*** $(\bar{u}_1)^{(9)}<(u_0)^{(9)}$ where $(u_1)^{(9)},(\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{44}(t)\le G_{44}^0e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{45}(t)\le\frac{1}{(m_2)^{(9)}}G_{44}^0e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)}G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^0e^{-(S_2)^{(9)}t}\le G_{46}(t)\le\right.$$

$$\left.\frac{(a_{46})^{(9)}G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}')^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}')^{(9)}t}]+\ G_{46}^0e^{-(a_{46}')^{(9)}t}\right)$$

$$\boxed{T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le\frac{1}{(\mu_2)^{(9)}}T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}')^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}')^{(9)}t}\right]+T_{46}^0e^{-(b_{46}')^{(9)}t}\le T_{46}(t)\le$$

$$\frac{(a_{46})^{(9)}T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^0e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain (383)

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right) - (a''_{14})^{(1)}(T_{14}, t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get (384)

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{v}_1)^{(1)}$ we find like in the previous case, (385)

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}'')^{(1)} = (a_{14}'')^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)}G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}'')^{(1)} = (b_{14}'')^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)}T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}')^{(2)} - (a_{17}')^{(2)} + (a_{16}'')^{(2)}(T_{17}, t)\right) - (a_{17}'')^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$$

If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain 394

$$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(2)}(t)$ :- 395

$$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}, \quad \boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$$

In a completely analogous way, we obtain 396

**Definition of** $u^{(2)}(t)$ :-

$$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}, \quad \boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$$

| | |
|---|---|
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$ if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}')^{(3)} - (a_{21}')^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| From which one obtains<br><br>For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$<br><br>$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)} + (C)^{(3)}(\nu_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}$ , $\boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\nu_2)^{(3)}}}$<br><br>it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$ | 400 |
| In the same manner , we get<br><br>$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)} + (\bar{C})^{(3)}(\bar{\nu}_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\bar{\nu}_2)^{(3)}}}$<br><br>**Definition of** $(\bar{\nu}_1)^{(3)}$ :-<br><br>From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$ | 401 |
| If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, | 402 |

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

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$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right) - (a_{25}^{''})^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\quad \boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

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$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} =$

$(\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b''_{24})^{(4)} = (b''_{25})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a'_{28})^{(5)} - (a'_{29})^{(5)} + (a''_{28})^{(5)}(T_{29}, t)\right) - (a''_{29})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\quad \boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get 413

$$v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{v}_1)^{(6)}-(v_0)^{(6)}}{(v_0)^{(6)}-(\bar{v}_2)^{(6)}}}$$

From which we deduce $(v_0)^{(6)} \le v^{(6)}(t) \le (\bar{v}_1)^{(6)}$

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If $0 < (v_1)^{(6)} < (v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{v}_1)^{(6)}$ we find like in the previous case,

$$(v_1)^{(6)} \le \frac{(v_1)^{(6)}+(C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}} \le v^{(6)}(t) \le$$

$$\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (\bar{v}_1)^{(6)}$$

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If $0 < (v_1)^{(6)} \le (\bar{v}_1)^{(6)} \le \boxed{(v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(v_1)^{(6)} \le v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (v_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$(m_2)^{(6)} \le v^{(6)}(t) \le (m_1)^{(6)}$, $\boxed{v^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \le u^{(6)}(t) \le (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(v_1)^{(6)} = (\bar{v}_1)^{(6)}$ if in addition $(v_0)^{(6)} = (v_1)^{(6)}$ then $v^{(6)}(t) = (v_0)^{(6)}$ and as a consequence $G_{32}(t) = (v_0)^{(6)}G_{33}(t)$ **this also defines** $(v_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(v_1)^{(6)}$ and $(\bar{v}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right) - (a_{37}^{''})^{(7)}(T_{37},t)v^{(7)} - (a_{37})^{(7)}v^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

$$\nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \leq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}} \leq \nu^{(7)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\bar{\nu}_1)^{(7)}$$

418

If $0 < (\nu_1)^{(7)} \leq (\bar{\nu}_1)^{(7)} \leq \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain

$$(\nu_1)^{(7)} \leq \nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \leq \nu^{(7)}(t) \leq (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

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In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :- 420

$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)} G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain 421

$$\frac{d\nu^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right) - (a_{41}^{''})^{(8)}(T_{41}, t)\nu^{(8)} - (a_{41})^{(8)}\nu^{(8)}$$

**Definition of** $\nu^{(8)}$ :- $\boxed{\nu^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right) \leq \frac{d\nu^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)}-(\nu_0)^{(8)}}{(\nu_0)^{(8)}-(\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \le \nu^{(8)}(t) \le (\bar{\nu}_8)^{(8)}$

423

If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case,

$$(\nu_1)^{(8)} \le \frac{(\nu_1)^{(8)}+(C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}} \le \nu^{(8)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\bar{\nu}_1)^{(8)}$$

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If $0 < (\nu_1)^{(8)} \le (\bar{\nu}_1)^{(8)} \le \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$, we obtain

$$(\nu_1)^{(8)} \le \nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \le \nu^{(8)}(t) \le (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \le u^{(8)}(t) \le (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)}G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)}T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right) - (a''_{45})^{(9)}(T_{45},t)v^{(9)} - (a_{45})^{(9)}v^{(9)}$$

**Definition of** $v^{(9)}$ :- $\boxed{v^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)}\right) \le \frac{dv^{(9)}}{dt} \le -\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(9)}, (v_0)^{(9)}$ :-

For $0 < \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (v_1)^{(9)} < (\bar{v}_1)^{(9)}$

$$v^{(9)}(t) \ge \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(v_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (v_2)^{(9)}}}$$

it follows $(v_0)^{(9)} \le v^{(9)}(t) \le (v_9)^{(9)}$

In the same manner , we get

$$v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{v}_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (\bar{v}_2)^{(9)}}}$$

From which we deduce $(v_0)^{(9)} \le v^{(9)}(t) \le (\bar{v}_1)^{(9)}$

If $0 < (v_1)^{(9)} < (v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}} < (\bar{v}_1)^{(9)}$ we find like in the previous case,

$$(v_1)^{(9)} \le \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}} \le v^{(9)}(t) \le$$

$$\frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (\bar{v}_1)^{(9)}$$

If $0 < (v_1)^{(9)} \le (\bar{v}_1)^{(9)} \le \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$ , we obtain

$$(v_1)^{(9)} \le v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (v_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(9)}(t)$ :-

$$(m_2)^{(9)} \le v^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{v^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}$, $\boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ 430

$with\ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations 431

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$

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| $with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | |
| We can prove the following<br><br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br><br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br><br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br><br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$ | 435 |

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| $with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |

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| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27})\big)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\big((G_{27})\big)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\big((G_{27})\big)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |

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| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |

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| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(u) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) +$ | 489 |

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| $(a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that | 496 |

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| there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_{i}^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_{i}^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_{i}^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{32}=\frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}=\frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_{i}^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{36}=\frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}=\frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |
| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_{i}^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*})=0$. With this value , we obtain from the three first equations | 501 |

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| $G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br>$\left[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G) + (b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)\right]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*}) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br><br>$\left[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19}) + (b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})\right]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br><br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ | 506 |

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| is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if $\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$ $\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$ Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if $\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$ $\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$ Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if $\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$ $\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$ Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if $\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$ $\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$ Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if $\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$ $\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$ Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution | 511 |

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| $G_{14}^{*}$ given by $\varphi(G^{*})=0$ , $T_{14}^{*}$ given by $f(T_{14}^{*})=0$ and<br>$G_{13}^{*}=\frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*}=\frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br>$T_{13}^{*}=\frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*}=\frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br>Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*})=0$ , $T_{17}^{*}$ given by $f(T_{17}^{*})=0$ and | 512 |
| $G_{16}^{*}=\frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*}=\frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*}=\frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*}=\frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*})=0$ , $T_{21}^{*}$ given by $f(T_{21}^{*})=0$ and<br>$G_{20}^{*}=\frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*}=\frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br>$T_{20}^{*}=\frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*}=\frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^{*}$ given by $\varphi(G_{27})=0$ , $T_{25}^{*}$ given by $f(T_{25}^{*})=0$ and<br>$G_{24}^{*}=\frac{(a_{24})^{(4)}G_{25}^{*}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*}=\frac{(a_{26})^{(4)}G_{25}^{*}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*}=\frac{(b_{24})^{(4)}T_{25}^{*}}{[(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*}=\frac{(b_{26})^{(4)}T_{25}^{*}}{[(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}((G_{27})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*})=0$ , $T_{29}^{*}$ given by $f(T_{29}^{*})=0$ and<br>$G_{28}^{*}=\frac{(a_{28})^{(5)}G_{29}^{*}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*}=\frac{(a_{30})^{(5)}G_{29}^{*}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |

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| $T_{28}^{*} = \frac{(b_{28})^{(5)}T_{29}^{*}}{[(b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)}T_{29}^{*}}{[(b_{30}^{'})^{(5)}-(b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br>$G_{32}^{*} = \frac{(a_{32})^{(6)}G_{33}^{*}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)}G_{33}^{*}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)}T_{33}^{*}}{[(b_{32}^{'})^{(6)}-(b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)}T_{33}^{*}}{[(b_{34}^{'})^{(6)}-(b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br>$G_{36}^{*} = \frac{(a_{36})^{(7)}G_{37}^{*}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)}G_{37}^{*}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br>$T_{36}^{*} = \frac{(b_{36})^{(7)}T_{37}^{*}}{[(b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)}T_{37}^{*}}{[(b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br>$G_{40}^{*} = \frac{(a_{40})^{(8)}G_{41}^{*}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)}G_{41}^{*}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br>$T_{40}^{*} = \frac{(b_{40})^{(8)}T_{41}^{*}}{[(b_{40}^{'})^{(8)}-(b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)}T_{41}^{*}}{[(b_{42}^{'})^{(8)}-(b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)}G_{45}^{*}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)}G_{45}^{*}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)}T_{45}^{*}}{[(b_{44}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)}T_{45}^{*}}{[(b_{46}^{'})^{(9)}-(b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | 523 A |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^*) = (q_{14})^{(1)} \quad , \frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^*) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)}G_{13}^*\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^*\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^*\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^*\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^*\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^*\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)} \quad , \frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |

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| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)} \;\; , \frac{\partial(b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_{j}\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_{j}\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_{j}\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ *and* $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 548 |

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| $\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^{*}) = (q_{25})^{(4)}$ , $\frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^{*}) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^{*}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^{*}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^{*}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^{*}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^{*}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^{*}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^{*}) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^{*}) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^{*}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^{*}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^{*}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^{*}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^{*}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^{*}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 563 |

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| **Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^*\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^*\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^*\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^*\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}((G_{39})^{**}) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |

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|---|---|
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{41})^{(8)}}{\partial T_{41}}(T^*_{41}) = (q_{41})^{(8)}$ , $\frac{\partial(b_i^{''})^{(8)}}{\partial G_j}((G_{43})^*) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a'_{40})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G^*_{40}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a'_{41})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G^*_{41}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a'_{42})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G^*_{42}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |

| | |
|---|---|
| | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \ \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b_{44}^{'})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T_{44}^*\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b_{45}^{'})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T_{45}^*\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b_{46}^{'})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T_{46}^*\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)$<br>$\left[\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(q_{13})^{(1)}G_{13}^*\right)\Big]$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T_{14}^* + (b_{14})^{(1)}s_{(13),(14)}T_{14}^*\Big)$<br>$+\left(\left((\lambda)^{(1)} + (a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^* + (a_{13})^{(1)}(q_{14})^{(1)}G_{14}^*\right)$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^* + (b_{14})^{(1)}s_{(13),(13)}T_{13}^*\Big)$<br>$\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$<br>$\left(\left((\lambda)^{(1)}\right)^2 + \left((b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$<br>$+\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$<br>$+\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^*\right)$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^* + (b_{14})^{(1)}s_{(13),(15)}T_{13}^*\Big)\} = 0$ | |

$$+$$

$$((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)})\{((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)})$$

$$\left[\left(((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)})(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)\right]$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(17)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^{*}_{17}\right)$$

$$+\left(((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)})(q_{16})^{(2)}\mathrm{G}^{*}_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(16)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^{*}_{16}\right)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left(\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(18)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(18)}\mathrm{T}^{*}_{16}\right)\} = 0$$

$$+$$

$$((\lambda)^{(3)} + (b'_{22})^{(3)} - (r_{22})^{(3)})\{((\lambda)^{(3)} + (a'_{22})^{(3)} + (p_{22})^{(3)})$$

$$\left[\left(((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)})(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)\right]$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(21)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(21)}T^{*}_{21}\right)$$

$$+\left(((\lambda)^{(3)} + (a'_{21})^{(3)} + (p_{21})^{(3)})(q_{20})^{(3)}G^{*}_{20} + (a_{20})^{(3)}(q_{21})^{(1)}G^{*}_{21}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(20)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(20)}T^{*}_{20}\right)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((b'_{20})^{(3)} + (b'_{21})^{(3)} - (r_{20})^{(3)} + (r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left(\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(22)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(22)}T^{*}_{20}\right)\} = 0$$

$$+$$

$$\left((\lambda)^{(4)}+(b'_{26})^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a'_{26})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a'_{24})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G^{*}_{25}+(a_{25})^{(4)}(q_{24})^{(4)}G^{*}_{24}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(25)}T^{*}_{25}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a'_{25})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G^{*}_{24}+(a_{24})^{(4)}(q_{25})^{(4)}G^{*}_{25}\right)$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(24)}T^{*}_{24}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((a'_{24})^{(4)}+(a'_{25})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((b'_{24})^{(4)}+(b'_{25})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^{2}+\left((a'_{24})^{(4)}+(a'_{25})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a'_{24})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G^{*}_{25}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G^{*}_{24}\right)$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(26)}T^{*}_{24}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(5)}+(b'_{30})^{(5)}-(r_{30})^{(5)}\right)\{\left((\lambda)^{(5)}+(a'_{30})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a'_{28})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G^{*}_{29}+(a_{29})^{(5)}(q_{28})^{(5)}G^{*}_{28}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(29)}T^{*}_{29}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a'_{29})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G^{*}_{28}+(a_{28})^{(5)}(q_{29})^{(5)}G^{*}_{29}\right)$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(28)}T^{*}_{28}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((a'_{28})^{(5)}+(a'_{29})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((b'_{28})^{(5)}+(b'_{29})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^{2}+\left((a'_{28})^{(5)}+(a'_{29})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a'_{28})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G^{*}_{29}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G^{*}_{28}\right)$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(30)}T^{*}_{28}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\right)\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\right)\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\right)\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)\right]$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(41)}T^{*}_{41}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^{*}_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^{*}_{41}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(40)}T^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$+\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(42)}T^{*}_{40}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\right)\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)\right]$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(45)}T^{*}_{45}\right)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^{*}_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^{*}_{45}\right)$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(44)}T^{*}_{44}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^{2}+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$+\left(\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)$$

$\left(\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\right)\} = 0$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i \in R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i\ X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor

bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k \tag{21}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 21. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,...x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone $\{ x_i \epsilon C_i: \prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i(x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \in I, x \in K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow *x.t restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\,\delta\,(K)= \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\mathcal{A}(\tilde{A}), C\backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^{+}$ with $u_i(x)|=|v_i(x)|=1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population *i* is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int_0^t (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \epsilon R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v = au_i(x) + w$ with $a \epsilon R$ and $w \epsilon F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |B_i(t,x)u_i(x)|(a + \alpha + exp(-\beta t)|w|)$. Hence, $\limsup_{t\to\infty} \frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\limsup_{t\to\infty} \frac{1}{t} In\ B_i(t,x)\ u_i(x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In } b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x - bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

=

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \le 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \le \gamma$ for all $t \ge 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \text{In } |\, (x.t)_i\, |$$

$$\le \operatorname{limsup}_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0. If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\ge 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\operatorname{limsup}_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x) =$ Integral $h(x.t)d\,\mu(x)$for all $t \ge 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure*. $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$ , $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x) \rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1 \le j \le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |\mathrm{h}(x.t) - h(x^n.t)|d\mu_n(x)| + |\int_c h(x^n.t)|d\mu_n(x) - \int_c \mathrm{h}(x)d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x)d\mu_n(x) - \int_c h(x)d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $\mathrm{h}_i^n(x) = \langle \mathrm{A}_i^n(x)\mathrm{u}_i^n(x), \mathrm{u}_i^n(x)\rangle$ where $\mathrm{u}_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $\mathrm{A}_i^n(x) \to \mathrm{A}_i(x)$ uniformly for $x \in N_1((\mathrm{A}))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $\mathrm{u}_i^n(x) \to \mathrm{u}_i(x)$ converges uniformly for $x \in N_1(\Lambda(\mathrm{A}))$. Given $\in > 0$, choose $n$ sufficiently large so that $|\mathrm{h}_i^n(x) - \mathrm{h}_i(x)| \leq \in$ for all $x \in N_1(\Lambda(\mathrm{A}))$ and $|\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$\lambda_i(\mu) \leq \int_c |\mathrm{h}_i(x) - \mathrm{h}_i^n(x)|d\mu(x) + |\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)|$

$+ \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)$

$\leq \quad 2\in$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\, \mathrm{ds}$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument

implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

(6) $\min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$ $\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_{1,}\ldots.,\tilde{A}_k)$ by

$$\tilde{A}_i(x) = \begin{cases} \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} & if\ i \epsilon I \\ if\ i \notin I \end{cases}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t >$ $0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in Kand\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set $K$ if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$?then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$?and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume x* $\mapsto$ *A(x) is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists j* $\epsilon$ { 1,….,k} *such that $M_i$ ----- { $x \in C : x_l \gg 0$ for $l \neq j$} Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_iM_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----{1,…,k} such that $\mu(---_i \in_I C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures μ with support in ∂C, $\lambda_i(\mu)>0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $$\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j \qquad i = 1,..,k \qquad j=1,\ldots,m$$

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge

to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j$ ($x^j_i$) = 0 i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product ($E_1^1$ ,…,$E_1^m$), and the matrix $A^2$ ($E_1$) is close to the diagonal matrix diag($f^i_2(E_1{}^j)$)$^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f_2^j (E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j (E^j_1)>0$ and $\max_j f_2^j (E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial$C is given by two equilibria {0,$E_1$}, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j (E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j\ x_2^j\ - \alpha^j\ x_3^j\ ) + \sum d_1^{jk}\ \ x_1^k\ - e_1^j\ x_1^j$

$\frac{dx_2^j}{dt} = x_2^j(1 - \alpha^j\ x_1^j\ - x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_2^{jk}\ \ x_2^k\ - e_2^j\ x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1 - \beta^j\ x_1^j\ - \alpha^j x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_3^{jk}\ \ x_3^k\ - e_3^j\ x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d_i^{l,j}$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2\ (E_1)$and $\lambda_3\ (E_1)$are close to $\max_j\ 1 - a^j$ and $\max_j\ 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j\ 1 - a^j$ and $\max_j\ 1 - \beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1 - a^j) > -\max_j\ (1 - \beta^j)$$

Equivalently,

$$2 > \min_j\ a^j + \min_j\ \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \ \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i\left(r_i - \sum_s a_{is}x_s\right), \qquad i = 1,\dots,k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$$

It has been shown [12,ch.16.1,16.2] That for *k* =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \ \ with\ \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1 \to E_2 \to E_3 \to E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing

holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & O \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left[ \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1 - y) - b(N)\ y \right]$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*′ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon+x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords

Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation

of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1$ *(t)* and $P_2$ *(t)* denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial

alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.61) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.62) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.63) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_{1,}$ $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_{i,}$ $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic ($as$ $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 21. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2)) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, \text{R}^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1, 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) \; P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^\infty f_i(a + s, P(t + S)) ds \right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) \, exp\left[ - \int_o^x f_1(a + s, p(t + s)) ds \right]$

For $a \geq t$ *we choose* $(a,t) = (a-t,0)$ in (1.3) and $x = t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0) \exp\left[ - \int_o^t f_1\left(a - t + s, p_1(s)\right) ds \right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(\text{a+s}, p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da+\int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$$t> 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \epsilon[-\tau, 0], i = 1,2,$

(2.8) $\beta_i(\text{t}) = \int_0^t \beta_i(\text{a}) b_i(\text{t-a}, P_i(t)) M_i.\left(a, t, P_i(t)\right)da$

$$+\int_0^\infty b_i\left(a + t, P_i(t)\right)\varphi_i(a, 0)L_i\left(a, t, P_i(t)\right)da$$

$$i = 1, 2, \text{t} > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t)\, \phi(0) d_i^*]t \quad for\ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T],R^+), \quad B_i \in C([0,T],R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau,T] = \{f \in C([-\tau,T],R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau,T]$. Then for any

$(x, y) \in C^+[-\tau,T] \times C^+[-\tau,T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau,T], \quad C_T^2 = C^+[-\tau,T] \times C^+[-\tau,T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau,T]$ *and* $t \in [0,T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$B: C_T^2 \to C_T^2$ and $\Pi C_T^2 \to C_T^2$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = P_t \left\{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right)$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Pi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Pi$ is a contraction.

Let $P_t^{(1)}$ and $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Pi(P_t^{(1)}) - \Pi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Pi(P^{(1)}(t)) - \Pi(P^{(2)}(t)| \le \int_0^t B_1^{(1)}(a) \left\{ M_1(a, t, P_t^{(1)}) - M_1(a, t, P_t^{(2)}) \right\} da$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left( \int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_* ds \right) da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o) \left[ \int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds \right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \|_t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$=\int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\mathrm{X}[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \, /\!/ P_t^{(1)} - P_t^{(2)} /\!/ \text{ for } a \, \varepsilon [o, t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \leq \beta_1 \Phi_1(0) || P_t^{(1)} - P_t^{(2)} ||_t^* + \Phi_1(0) \beta_1 d_1 || P_t^{(1)} - P_t^{(2)} ||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) \, e^{\delta 1^a} \beta_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) \, e^{\delta 1^a} /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| B_1^{*d}\sigma$$

$$+ \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \left\{ B_1 + B_2 d_1 a + B_1^* \left(B_2 a + d_1 a^2/2\right) e^{\delta 1^a} \right\}$$

$$\leq \beta_1^* \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| do + \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \rightarrow \beta_1$ as $a \rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{g_1(a) + 0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a)) - b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a)) do \right|$$

$$\left| B_1^* \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} 0(a) as\ a \rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi (P_t^{(1)} - \pi \left(P_t^{(2)}\right) /\!/ \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^*)$$

As $t^* \rightarrow o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp P_2^*(a)}{da} + f_i(a, P_1^*, P_2^*) P_i^*(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^* = \int_o^\infty P_i^*(a) da, i = 1,\ 2,$$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i=1.\ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s,P_1^*,P_2^*)ds\right], i=1,\quad 2, a\geq 0,$

Which together with (3.3) lead to

$$1=\int_o^\infty b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^\infty b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1,P_2) = \int_o^\infty b_i(a,P_1,P_2)\exp\left[-\int_o^\infty f_i(s,P_1,P_2)ds\right]da,$

$i=1,\ 2,\quad P_1\geq 0,\quad P_2\geq 0,$

**And suppose the following hold:**

(3.7) $A_4.\quad F_{i.}\ (0,\ 0)>1,\ i=1,2$:

$A_5.$ *there exists a positive constant c for which*

$$F_1(C,0)< F_{2.}(C,0),$$

(3.8) $$F_1(0.C)< F_{2.}(0,C),$$

Then there exists a unique pair $(P_1^*,P_2^*)$ of real numbers $P_1^*>o,$

$P_2^*>0$ such that

(3,9) $F_1(P_1^*,P_2^*)=1=F_2(P_1^*,P_2^*)$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a)=[P_2^*\exp[-f_i(s,P_1^*,P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s,P_1^*,P_2^*)ds]da),$

$i=1,2,\quad a\geq 0,$

Proof Consider the elements of the surfaces defined by

Z= $F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0,$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

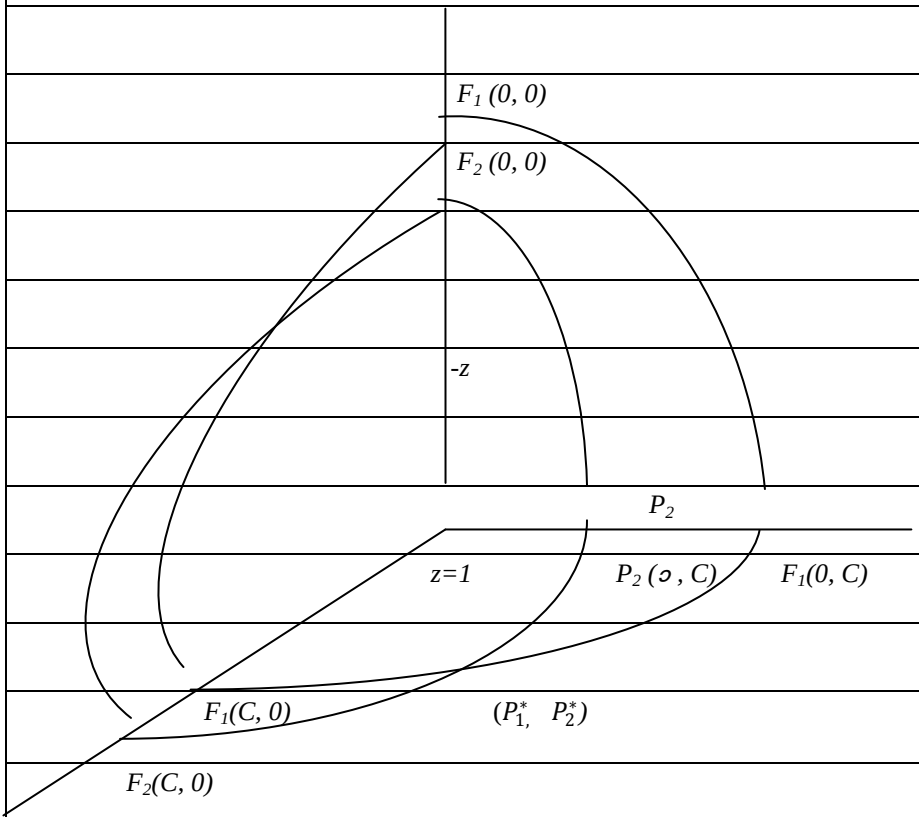
$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0,$ These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, F_1(P_1, P_2 \geq 0,$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11) $$\begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12) $$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square=1}^{2} \frac{\partial b_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14) $$\begin{cases} u_i(a,t) = u\square(a)\exp[\lambda t], \quad \square(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty \square(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d\square}{da} + \lambda\square = -f_i\left(a, p_1^*, p_2^*\right)\square(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_{i\square}^*\right\},$$

(3.15)

$$\square(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\square(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*} p_{i\square}^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^{*} = p_{1}^{*}, \quad p_{12}^{*} = p_{2}^{*} \exp\{-\lambda\tau\},$$

$$p_{21}^{*} = p_{1}^{*}, \exp\{-\lambda\tau\}, \ p_{22}^{*} = p_{2}^{*}.$$

If we integrate the first of (3.15) and use the fact that $\mu_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\mu_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\mu_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\mu_i(a) = \{\mu_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\mu_i\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\mu_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\mu_i\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\mu_i\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\mu_i\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)}\int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}da, \quad i = 1,\ 2,$$

Now supplying $\mu_1(a)$ and $\mu_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^*\exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^*\exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^* \exp(-\lambda\tau) + (\lambda - A_{22})\, p_2^* = g_{21}(\lambda)\, p_1^* \exp(-\lambda\tau) + g_{22}(\lambda)\, p_2^*$$

Where

(3.22) $$A_{i} = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_{\square}^*} - \frac{\partial f_i}{\partial p_{\square}^*}\right]da,$$

(3.23) $$g_{i}(\lambda) = \left[p_i^*(0)c_i\right]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\, da\right\}$$
$$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_{\square}^*} - b_i(a, p_1^*, p_2^*)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{\square}^*}\, e^{\lambda s}\, ds\right)\right]da\right\}$$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{\square}^*}\, e^{\lambda s}\, ds\right)da, \quad i, = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*), \quad i, = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})\, e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$$S_2(\lambda, \tau) = \left[A_{21}\, g_{12}(\lambda) + A_{12}\, g_{21}(\lambda) + g_{12}(\lambda)\, g_{21}(\lambda)\right]\exp[-2\lambda\tau]$$

$$-A_{11}\, g_{22}(\lambda) - A_{22}\, g_{11}(\lambda) - g_{11}(\lambda)\, g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6$. $0 < \beta_i^* < d_i^*$ $(i, = 1, 2.)$ $(see(A_1)\ and\ (A_2))$;

$A_7$. $\{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu\ \geq 0$ and $w\ \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - \left| F_2(\lambda, \tau) \right|$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \,| - |A_{11} + A_{22} + \| S_1(0) \| \,|^2 \right\}/2,$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational

ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the

Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have

obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is:

individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## SECTION A(22)

## Dyons And Dyonic Black Holes

### INTRODUCTION—VARIABLES USED

**Dyons and dyonic black holes in su(N) Einstein-Yang-Mills theory in anti–de Sitter spacetime Ben L. Shepherd and Elizabeth Winstanley Phys. Rev. D 93, 064064 – Published 25 March 2016**

(1) Authors present new spherically symmetric, dyonic soliton and black hole solutions of (e) the su(N) Einstein-Yang-Mills equations in (eb) four-dimensional asymptotically anti–de Sitter spacetime.
(2) The gauge field has (e) nontrivial electric and magnetic components and is described by (e) N−1 magnetic gauge field functions and (e&eb) N−1 electric gauge field functions.
(3) Authors explore the phase space of solutions in detail for (e) su (2) and su (3) gauge groups.
(4) Combinations of the electric gauge field functions are monotonic and have (e) no zeros; in general the magnetic gauge field functions may have (e) zeros.
(5) The phase space of solutions is extremely rich, and authors find solutions in which the magnetic gauge field functions have (e) more than fifty zeros.
(6) Of particular interest are (=) solutions for which the magnetic gauge field functions have (e) no zeros, which exist when (e) the negative cosmological constant has sufficiently large magnitude.
**(7)** Authors conjecture that at least some of these nodeless solutions may be (=) stable under linear,

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| spherically symmetric, perturbations. | |
| **NOTATION** | |
| **Module One**<br>Authors show the efficiency and usefulness of this semiclassical treatment in describing entropy production of a couple of quantum mechanical systems, whose classical counter systems are(=) known to be chaotic | |
| $G_{13}$ : Category one of entropy production of a couple of quantum mechanical systems, whose classical counter systems<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of chaotic<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>Authors propose two methods to evaluate the time evolution of (e) the Husimi-Wehrl entropy, the test-particle method and (e&eb) the two-step Monte-Carlo method | |
| $G_{16}$ : Category one of **test-particle method;** two-step Monte-Carlo method<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of two-step Monte-Carlo method **;test-particle method**<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>They demonstrate the characteristics of the two methods by numerical calculations, and show that (eb) the simultaneous application of the two methods ensures (eb) the reliability of the results of (e) the Husimi-Wehrl entropy at a given time.<br>Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th); Quantum Physics (quant-ph) Journal reference: PTEP 2015 (2015) 083A01 DOI: 10.1093/ptep/ptv107 Report number: KUNS-2561, YITP-15-42 Cite as: arXiv:1505.04698 [hep-ph] (or arXiv:1505.04698v2 [hep-ph] for this version) | |
| $G_{20}$ : Category one of characteristics of the two methods by numerical calculations,<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of simultaneous application of the two methods ensures (eb) the reliability of the results | |

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| of (e) the Husimi-Wehrl entropy at a given time.<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>They demonstrate the characteristics of the two methods by numerical calculations, and show that the simultaneous application of the two methods ensures (eb) the reliability of the results of (e) the Husimi-Wehrl entropy at a given time. | |
| $G_{24}$ : Category one of characteristics of the two methods by numerical calculations, and show that the simultaneous application of the two methods<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of reliability of the results of the Husimi-Wehrl entropy at a given time.<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Authors present new spherically symmetric, dyonic soliton and black hole solutions of (e) the su(N) Einstein-Yang-Mills equations in (eb) four-dimensional asymptotically anti–de Sitter spacetime | |
| $G_{28}$ : Category one of su(N) Einstein-Yang-Mills equations in (eb) four-dimensional asymptotically anti–de Sitter spacetime<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of spherically symmetric, dyonic soliton and black hole solutions<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Authors present new spherically symmetric, dyonic soliton and black hole solutions of the su(N) Einstein-Yang-Mills equations in (eb) four-dimensional asymptotically anti–de Sitter spacetime | |
| $G_{32}$ : Category one of spherically symmetric, dyonic soliton and black hole solutions of the su(N) Einstein-Yang-Mills equations<br><br>$G_{33}$ : Category two of SAS | |

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| $G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of four-dimensional asymptotically anti–de Sitter spacetime<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>The gauge field has (e) nontrivial electric and magnetic components and is described by (e) N−1 magnetic gauge field functions and (e&eb) N−1 electric gauge field functions | |
| $G_{36}$ : Category one of nontrivial electric and magnetic components and is described by (e) N−1 magnetic gauge field functions and (e&eb) N−1 electric gauge field functions<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of gauge field<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The gauge field has nontrivial electric and magnetic components and is described by N−1 magnetic gauge field functions and (e&eb) N−1 electric gauge field functions | |
| $G_{40}$ : Category one of **gauge field has nontrivial electric and magnetic components and is described by N−1 magnetic gauge field functions;** N−1 electric gauge field functions<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of N−1 electric gauge field functions **;gauge field has nontrivial electric and magnetic components and is described by N−1 magnetic gauge field functions**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Authors explore the phase space of solutions in detail for (e) su (2) and su (3) gauge groups | |
| $G_{44}$ : Category one of **phase space of solutions;** su (2) and su (3) gauge groups | |

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| $G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of su (2) and su (3) gauge groups **;phase space of solutions**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION A(23)

## 2 + 1 Dimensional Gravity

**INTRODUCTION—VARIABLES USED**

**Nuclear Physics B Volume 311, Issue 1, 19 December 1988, Pages 46-78 2 + 1 dimensional gravity as an exactly soluble system Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. Edward Witten ∗**

(1) By disentangling the Hamiltonian constraint equations, 2 + 1 dimensional gravity (with or without a cosmological constant) is shown to be (=) exactly soluble at (eb) the classical and quantum levels.

**(2)** Indeed, it is closely related to (e&eb) Yang-Mills theory with purely the Chern-Simons action, which recently has turned out to define (eb) a soluble quantum field theory. 2 + 1 dimensional gravity has (e) a straightforward renormalized perturbation expansion, with (e&eb) vanishing beta function. 2 + 1 dimensional quantum gravity may provide (eb) a testing ground for (e) understanding the role of classical singularities in quantum mechanics, may be related to (e&eb) the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry. Copyright © 1988 Published by Elsevier B.V.

**NOTATION**

**Module One**

**Dyons and dyonic black holes in su(N) Einstein-Yang-Mills theory in anti–de Sitter spacetime Ben L. Shepherd and Elizabeth Winstanley Phys. Rev. D 93, 064064 – Published 25 March 2016**

(1) Combinations of the electric gauge field functions are monotonic and have (e) no zeros; in general the magnetic gauge field functions may have (e) zeros.
(2) The phase space of solutions is extremely rich, and authors find solutions in which the magnetic gauge field functions have (e) more than fifty zeros.
(3) Of particular interest are (=) solutions for which the magnetic gauge field functions have (e) no zeros, which exist when (e) the negative cosmological constant has sufficiently large magnitude.
(4) Authors conjecture that at least some of these nodeless solutions may be (=) stable under linear, spherically symmetric, perturbations.

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| $G_{13}$ : Category one of no zeros; in general the magnetic gauge field functions may have (e) zeros<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of Combinations of the electric gauge field functions are monotonic<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>the magnetic gauge field functions may have (e) zeros | |
| $G_{16}$ : Category one of zeros<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of magnetic gauge field functions<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>The phase space of solutions is extremely rich, and authors find solutions in which the magnetic gauge field functions have (e) more than fifty zeros | |
| $G_{20}$ : Category one of more than fifty zeros<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one ofphase space of solutions is extremely rich, and authors find solutions in which the magnetic gauge field functions h<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>Of particular interest are (=) solutions for which the magnetic gauge field functions have (e) no zeros, which exist when (e) the negative cosmological constant has sufficiently large magnitude | |
| $G_{24}$ : Category one of negative cosmological constant has sufficiently large magnitude<br>$G_{25}$ : Category two of SAS | |

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| $G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of solutions for which the magnetic gauge field functions have no zeros exist<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>Authors conjecture that at least some of these nodeless solutions may be (=) stable under linear, spherically symmetric, perturbations | |
| $G_{28}$ : Category one of some of these nodeless solutions<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of stable under linear, spherically symmetric, perturbations<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>By disentangling the Hamiltonian constraint equations, 2 + 1 dimensional gravity (with or without a cosmological constant) is shown to be (=) exactly soluble at (eb) the classical and quantum levels | |
| $G_{32}$ : Category one of Hamiltonian constraint equations, 2 + 1 dimensional gravity (with or without a cosmological constant) disentangling<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of exactly soluble at (eb) the classical and quantum levels<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br>Hamiltonian constraint equations, disentangled 2 + 1 dimensional gravity (with or without a cosmological constant) is shown to be exactly soluble at (eb) the classical and quantum levels | |
| $G_{36}$ : Category one of Hamiltonian constraint equations, disentangled 2 + 1 dimensional gravity (with or without a cosmological constant) is shown to be exactly soluble<br>$G_{37}$ : Category two of SAS | |

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| $G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of classical and quantum levels<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Indeed, Hamiltonian constraint equations is closely related to (e&eb) Yang-Mills theory with purely the Chern-Simons action, which recently has turned out to define (eb) a soluble quantum field theory. 2 + 1 dimensional gravity has (e) a straightforward renormalized perturbation expansion, with (e&eb) vanishing beta function. 2 + 1 dimensional quantum gravity may provide (eb) a testing ground for (e) understanding the role of classical singularities in quantum mechanics, may be related to (e&eb) the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry.<br><br>Copyright © 1988 Published by Elsevier B.V. | |
| $G_{40}$ : Category one of **Hamiltonian constraint equations;** Yang-Mills theory with purely the Chern-Simons action<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of Yang-Mills theory with purely the Chern-Simons action; **Hamiltonian constraint equations**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Indeed, Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define (eb) a soluble quantum field theory. 2 + 1 dimensional gravity has (e) a straightforward renormalized perturbation expansion, with (e&eb) vanishing beta function. 2 + 1 dimensional quantum gravity may provide (eb) a testing ground for (e) understanding the role of classical singularities in quantum mechanics, may be related to (e&eb) the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry.<br><br>Copyright © 1988 Published by Elsevier B.V. | |
| $G_{44}$ : Category one of Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define<br><br>$G_{45}$ : Category two of SAS | |

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| $G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of soluble quantum field theory. 2 + 1 dimensional gravity has (e) a straightforward renormalized perturbation expansion, with (e&eb) vanishing beta function. 2 + 1 dimensional quantum gravity may provide (eb) a testing ground for (e) understanding the role of classical singularities in quantum mechanics, may be related to (e&eb) the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry.<br>$T_{45}$ : Category two of SAS<br>$T_{46}$ : Category three of SAS | |

## SECTION A(23)-2

## Hamiltonian Constraint Equations Is Closely Related To Yang-Mills Theory

**INTRODUCTION—VARIABLES USED**

**NOTATION**

**Module One**

Indeed, Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define (eb) a soluble quantum field theory. 2 + 1 dimensional gravity has (e) a straightforward renormalized perturbation expansion, with (e&eb) vanishing beta function. 2 + 1 dimensional quantum gravity may provide (eb) a testing ground for (e) understanding the role of classical singularities in quantum mechanics, may be related to (e&eb) the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry.

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| $G_{13}$ : Category one of Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of soluble quantum field theory. 2 + 1 dimensional gravity has (e) a straightforward renormalized perturbation expansion, with (e&eb) vanishing beta function. 2 + 1 dimensional quantum gravity may provide (eb) a testing ground for (e) understanding the role of classical singularities in quantum mechanics, may be related to (e&eb) the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry.<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two** | |

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| Indeed, Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define a soluble quantum field theory. 2 + 1 dimensional gravity has (e) a straightforward renormalized perturbation expansion, with (e&eb) vanishing beta function. 2 + 1 dimensional quantum gravity may provide (eb) a testing ground for (e) understanding the role of classical singularities in quantum mechanics, may be related to (e&eb) the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry | |
| $G_{16}$ : Category one ofstraightforward renormalized perturbation expansion, with (e&eb) vanishing beta function. 2 + 1 dimensional quantum gravity may provide (eb) a testing ground for (e) understanding the role of classical singularities in quantum mechanics, may be related to (e&eb) the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define a soluble quantum field theory. 2 + 1 dimensional gravity<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>Indeed, Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define a soluble quantum field theory. 2 + 1 dimensional gravity has a straightforward renormalized perturbation expansion, with (e&eb) vanishing beta function. 2 + 1 dimensional quantum gravity may provide (eb) a testing ground for (e) understanding the role of classical singularities in quantum mechanics, may be related to (e&eb) the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry | |
| $G_{20}$ : Category one of**Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define a soluble quantum field theory. 2 + 1 dimensional gravity has a straightforward renormalized perturbation expansion;** vanishing beta function. 2 + 1 dimensional quantum gravity may provide (eb) a testing ground for (e) understanding the role of classical singularities in quantum mechanics, may be related to (e&eb) the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of vanishing beta function. 2 + 1 dimensional quantum gravity may provide (eb) a testing ground for (e) understanding the role of classical singularities in quantum mechanics, may be related to (e&eb) the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry; **Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define a soluble quantum field theory. 2 + 1 dimensional gravity has a straightforward renormalized perturbation expansion**<br><br>$T_{21}$ : Category two of SAS | |

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| $T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Indeed, Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define a soluble quantum field theory. 2 + 1 dimensional gravity has a straightforward renormalized perturbation expansion, with vanishing beta function. 2 + 1 dimensional quantum gravity may provide (eb) a testing ground for (e) understanding the role of classical singularities in quantum mechanics, may be related to (e&eb) the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry | |
| $G_{24}$ : Category one of Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define a soluble quantum field theory. 2 + 1 dimensional gravity has a straightforward renormalized perturbation expansion, with vanishing beta function. 2 + 1 dimensional quantum gravity<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one oftesting ground for (e) understanding the role of classical singularities in quantum mechanics, may be related to (e&eb) the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Indeed, Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define a soluble quantum field theory. 2 + 1 dimensional gravity has a straightforward renormalized perturbation expansion, with vanishing beta function. 2 + 1 dimensional quantum gravity may provide a testing ground for (e) understanding the role of classical singularities in quantum mechanics, may be related to (e&eb) the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry | |
| $G_{28}$ : Category one ofunderstanding the role of classical singularities in quantum mechanics, may be related to (e&eb) the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define a soluble quantum field theory. 2 + 1 dimensional gravity has a straightforward renormalized perturbation expansion, with vanishing beta function. 2 + 1 dimensional quantum gravity may provide a testing ground<br><br>$T_{29}$ : Category two of SAS | |

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| $T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Indeed, Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define a soluble quantum field theory. 2 + 1 dimensional gravity has a straightforward renormalized perturbation expansion, with vanishing beta function. 2 + 1 dimensional quantum gravity may provide a testing ground for understanding the role of classical singularities in quantum mechanics, may be related to (e&eb) the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry | |
| $G_{32}$ : Category one of**Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define a soluble quantum field theory. 2 + 1 dimensional gravity has a straightforward renormalized perturbation expansion, with vanishing beta function. 2 + 1 dimensional quantum gravity may provide a testing ground for understanding the role of classical singularities in quantum mechanics;** discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry **;Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define a soluble quantum field theory. 2 + 1 dimensional gravity has a straightforward renormalized perturbation expansion, with vanishing beta function. 2 + 1 dimensional quantum gravity may provide a testing ground for understanding the role of classical singularities in quantum mechanics**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Indeed, Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define a soluble quantum field theory. 2 + 1 dimensional gravity has a straightforward renormalized perturbation expansion, with vanishing beta function. 2 + 1 dimensional quantum gravity may provide a testing ground for understanding the role of classical singularities in quantum mechanics, may be related to the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in (eb) studying three-dimensional geometry | |
| $G_{36}$ : Category one of Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define a soluble quantum field theory. 2 + 1 dimensional gravity has a straightforward renormalized perturbation expansion, with vanishing beta function. 2 + 1 dimensional quantum gravity may provide a testing ground for understanding the role of classical singularities in quantum mechanics, may be related to the discrete series of Virasoro representations in 1 + 1 dimensions; **three-dimensional geometry** | |

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| $G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of **three-dimensional geometry;** Hamiltonian constraint equations is closely related toYang-Mills theory with purely the Chern-Simons action, which recently has turned out to define a soluble quantum field theory. 2 + 1 dimensional gravity has a straightforward renormalized perturbation expansion, with vanishing beta function. 2 + 1 dimensional quantum gravity may provide a testing ground for understanding the role of classical singularities in quantum mechanics, may be related to the discrete series of Virasoro representations in 1 + 1 dimensions;<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>**Abelian Tensor Hierarchy in 4D, N=1 Superspace Katrin Becker, Melanie Becker, William D Linch III, Daniel Robbins (Submitted on 12 Jan 2016)**<br><br>(1) With the goal of constructing the supersymmetric action for all fields, massless and massive, obtained by Kaluza-Klein compactification from (e) type II theory or M-theory in a closed form, authors embed the (Abelian) tensor hierarchy of p-forms in (eb) four-dimensional, N=1 superspace and construct its Chern-Simons-like invariants.<br>(2) When specialized to the case in which the tensors arise from (e) a higher-dimensional theory, the invariants may be interpreted as (=) higher-dimensional Chern-Simons forms reduced to (e&eb) four dimensions.<br>(3) As an application of the formalism, authors construct the eleven-dimensional Chern-Simons form in terms of (e&eb) four-dimensional, N=1 superfields. Subjects: High Energy Physics - Theory (hep-th) DOI: 10.1007/JHEP03 (2016)052 Report number: MI-TH-1605 Cite as: arXiv:1601.03066 [hep-th] (or arXiv:1601.03066v1 [hep-th] for this version) | |
| $G_{40}$ : Category one of type II theory or M-theory in a closed form, authors embed the (Abelian) tensor hierarchy of p-forms in (eb) four-dimensional, N=1 superspace and construct its Chern-Simons-like invariants.<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of construction of the supersymmetric action for all fields, massless and massive, obtained by Kaluza-Klein compactification<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>With the goal of constructing the supersymmetric action for all fields, massless and massive, obtained by | |

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| Kaluza-Klein compactification from type II theory or M-theory in a closed form, authors embed the (Abelian) tensor hierarchy of p-forms in (eb) four-dimensional, N=1 superspace and construct its Chern-Simons-like invariants | |
| $G_{44}$ : Category one of construction of the supersymmetric action for all fields, massless and massive, obtained by Kaluza-Klein compactification from type II theory or M-theory in a closed form, authors embed the (Abelian) tensor hierarchy of p-forms<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of four-dimensional, N=1 superspace and construct its Chern-Simons-like invariants<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$<br>$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$<br>$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}$ $(a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,<br>$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$<br>$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,<br>$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,<br>$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b'_{13})^{(1)} - (b''_{13})^{(1)}(G, t)\right]T_{13}$ | 4 |

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| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G,t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G,t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14},t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G,t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17},t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17},t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17},t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19}),t)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}((G_{19}),t)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19}),t)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17},t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}((G_{19}),t) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |

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| $-(b_{24}^{''})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |

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| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{array}{ccc} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3 | |

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| $+(a_{38}^{''})^{(7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3<br>$+(a_{40}^{''})^{(8,8)}(T_{41},t)$ $+(a_{41}^{''})^{(8,8)}(T_{41},t)$ $+(a_{42}^{''})^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3<br>$+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)} & -(b_{13}^{''})^{(1)}(G,t) & -(b_{16}^{''})^{(2,2,)}(G_{19},t) & -(b_{20}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,)}(G_{39},t) & -(b_{40}^{''})^{(8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)} & -(b_{14}^{''})^{(1)}(G,t) & -(b_{17}^{''})^{(2,2,)}(G_{19},t) & -(b_{21}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,)}(G_{39},t) & -(b_{41}^{''})^{(8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)} & -(b_{15}^{''})^{(1)}(G,t) & -(b_{18}^{''})^{(2,2,)}(G_{19},t) & -(b_{22}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,)}(G_{39},t) & -(b_{42}^{''})^{(8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$ | *60* |
| Where $-(b_{13}^{''})^{(1)}(G,t)$, $-(b_{14}^{''})^{(1)}(G,t)$, $-(b_{15}^{''})^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{16}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{20}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{37}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |

$$\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}')^{(2)} & +(a_{16}'')^{(2)}(T_{17},t) & +(a_{13}'')^{(1,1,)}(T_{14},t) & +(a_{20}'')^{(3,3,3)}(T_{21},t) \\ +(a_{24}'')^{(4,4,4,4,4)}(T_{25},t) & +(a_{28}'')^{(5,5,5,5,5)}(T_{29},t) & +(a_{32}'')^{(6,6,6,6,6)}(T_{33},t) & \\ +(a_{36}'')^{(7,7,7)}(T_{37},t) & +(a_{40}'')^{(8,8,8)}(T_{41},t) & +(a_{44}'')^{(9,9)}(T_{45},t) & \end{bmatrix} G_{16} \quad (61)$$

$$\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}')^{(2)} & +(a_{17}'')^{(2)}(T_{17},t) & +(a_{14}'')^{(1,1,)}(T_{14},t) & +(a_{21}'')^{(3,3,3)}(T_{21},t) \\ +(a_{25}'')^{(4,4,4,4,4)}(T_{25},t) & +(a_{29}'')^{(5,5,5,5,5)}(T_{29},t) & +(a_{33}'')^{(6,6,6,6,6)}(T_{33},t) & \\ +(a_{37}'')^{(7,7,7)}(T_{37},t) & +(a_{41}'')^{(8,8,8)}(T_{41},t) & +(a_{45}'')^{(9,9)}(T_{45},t) & \end{bmatrix} G_{17} \quad (62)$$

$$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}')^{(2)} & +(a_{18}'')^{(2)}(T_{17},t) & +(a_{15}'')^{(1,1,)}(T_{14},t) & +(a_{22}'')^{(3,3,3)}(T_{21},t) \\ +(a_{26}'')^{(4,4,4,4,4)}(T_{25},t) & +(a_{30}'')^{(5,5,5,5,5)}(T_{29},t) & +(a_{34}'')^{(6,6,6,6,6)}(T_{33},t) & \\ +(a_{38}'')^{(7,7,7)}(T_{37},t) & +(a_{42}'')^{(8,8,8)}(T_{41},t) & +(a_{46}'')^{(9,9)}(T_{45},t) & \end{bmatrix} G_{18} \quad (63)$$

Where $+(a_{16}'')^{(2)}(T_{17},t)$, $+(a_{17}'')^{(2)}(T_{17},t)$, $+(a_{18}'')^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{13}'')^{(1,1,)}(T_{14},t)$, $+(a_{14}'')^{(1,1,)}(T_{14},t)$, $+(a_{15}'')^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}'')^{(3,3,3)}(T_{21},t)$, $+(a_{21}'')^{(3,3,3)}(T_{21},t)$, $+(a_{22}'')^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}'')^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}'')^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}'')^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}'')^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}'')^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{30}'')^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}'')^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{33}'')^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{34}'')^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{36}'')^{(7,7,7)}(T_{37},t)$, $+(a_{37}'')^{(7,7,7)}(T_{37},t)$, $+(a_{38}'')^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3

$+(a_{40}'')^{(8,8,8)}(T_{41},t)$, $+(a_{41}'')^{(8,8,8)}(T_{41},t)$, $+(a_{42}'')^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3

$+(a_{44}'')^{(9,9)}(T_{45},t)$, $+(a_{45}'')^{(9,9)}(T_{45},t)$, $+(a_{46}'')^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3

$$\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}')^{(2)} & -(b_{16}'')^{(2)}(G_{19},t) & -(b_{13}'')^{(1,1,)}(G,t) & -(b_{20}'')^{(3,3,3,)}(G_{23},t) \\ -(b_{24}'')^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}'')^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}'')^{(6,6,6,6,6)}(G_{35},t) & \\ -(b_{36}'')^{(7,7,7)}(G_{39},t) & -(b_{40}'')^{(8,8,8)}(G_{43},t) & -(b_{44}'')^{(9,9)}(G_{47},t) & \end{bmatrix} T_{16} \quad (64)$$

$$\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}')^{(2)} & -(b_{17}'')^{(2)}(G_{19},t) & -(b_{14}'')^{(1,1,)}(G,t) & -(b_{21}'')^{(3,3,3,)}(G_{23},t) \\ -(b_{25}'')^{(4,4,4,4,4)}(G_{27},t) & -(b_{29}'')^{(5,5,5,5,5)}(G_{31},t) & -(b_{33}'')^{(6,6,6,6,6)}(G_{35},t) & \\ -(b_{37}'')^{(7,7,7)}(G_{39},t) & -(b_{41}'')^{(8,8,8)}(G_{43},t) & -(b_{45}'')^{(9,9)}(G_{47},t) & \end{bmatrix} T_{17} \quad (65)$$

$$\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b'_{18})^{(2)} & \boxed{-(b''_{18})^{(2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,)}(G,t)} & \boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6)}(G_{35},t)} & \\ \boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9)}(G_{47},t)} & \end{bmatrix} T_{18} \quad (66)$$

where $\boxed{-(b''_{16})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{17})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{18})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,)}(G,t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{44})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2 and 3

$$\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a'_{20})^{(3)} & \boxed{+(a''_{20})^{(3)}(T_{21},t)} & \boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)} & \\ \boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)} & \end{bmatrix} G_{20} \quad (67)$$

$$\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a'_{21})^{(3)} & \boxed{+(a''_{21})^{(3)}(T_{21},t)} & \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)} & \\ \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)} & \end{bmatrix} G_{21} \quad (68)$$

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a'_{22})^{(3)} & \boxed{+(a''_{22})^{(3)}(T_{21},t)} & \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)} & \\ \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)} & \end{bmatrix} G_{22} \quad (69)$$

$\boxed{+(a''_{20})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)}$ are third augmentation coefficients

| | |
|---|---|
| for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)} & -(b_{20}^{''})^{(3)}(G_{23},t) & -(b_{16}^{''})^{(2,2,2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,1,)}(G,t) \\ & -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ & -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{20}$ | *70* |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)} & -(b_{21}^{''})^{(3)}(G_{23},t) & -(b_{17}^{''})^{(2,2,2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,1,)}(G,t) \\ & -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ & -(b_{37}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21}$ | *71* |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b_{22}^{'})^{(3)} & -(b_{22}^{''})^{(3)}(G_{23},t) & -(b_{18}^{''})^{(2,2,2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,1,)}(G,t) \\ & -(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ & -(b_{38}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{22}$ | *72* |
| $-(b_{20}^{''})^{(3)}(G_{23},t)$, $-(b_{21}^{''})^{(3)}(G_{23},t)$, $-(b_{22}^{''})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*<br>$-(b_{16}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*<br>$-(b_{13}^{''})^{(1,1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*<br>$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 | |

$\boxed{-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{46}^{''})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt}=(a_{24})^{(4)}G_{25}-\begin{bmatrix}(a_{24}^{'})^{(4)}\boxed{+(a_{24}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)}\\ \boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt}=(a_{25})^{(4)}G_{24}-\begin{bmatrix}(a_{25}^{'})^{(4)}\boxed{+(a_{25}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6)}(T_{33},t)}\\ \boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt}=(a_{26})^{(4)}G_{25}-\begin{bmatrix}(a_{26}^{'})^{(4)}\boxed{+(a_{26}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)}\\ \boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{26} \quad (75)$$

$\boxed{(a_{24}^{''})^{(4)}(T_{25},t)}$, $\boxed{(a_{25}^{''})^{(4)}(T_{25},t)}$, $\boxed{(a_{26}^{''})^{(4)}(T_{25},t)}$ *are first augmentation coefficients category* 1, 2 3

$\boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)}$
*are fourth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)}$
*are fifth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)}$
*are sixth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)}$
*are seventh augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)}$
*are eighth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)}$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)}\boxed{-(b_{24}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,)}(G_{35},t)}\\ \boxed{-(b_{13}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)}\\ \boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}\end{bmatrix}T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b'_{25})^{(4)} & -(b''_{25})^{(4)}(G_{27},t) & -(b''_{29})^{(5,5,)}(G_{31},t) & -(b''_{33})^{(6,6,)}(G_{35},t) \\ & -(b''_{14})^{(1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b'_{26})^{(4)} & -(b''_{26})^{(4)}(G_{27},t) & -(b''_{30})^{(5,5,)}(G_{31},t) & -(b''_{34})^{(6,6,)}(G_{35},t) \\ & -(b''_{15})^{(1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{26} \quad (78)$$

Where $-(b''_{24})^{(4)}(G_{27},t)$, $-(b''_{25})^{(4)}(G_{27},t)$, $-(b''_{26})^{(4)}(G_{27},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b''_{28})^{(5,5,)}(G_{31},t)$, $-(b''_{29})^{(5,5,)}(G_{31},t)$, $-(b''_{30})^{(5,5,)}(G_{31},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b''_{32})^{(6,6,)}(G_{35},t)$, $-(b''_{33})^{(6,6,)}(G_{35},t)$, $-(b''_{34})^{(6,6,)}(G_{35},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b''_{13})^{(1,1,1,1)}(G,t)$, $-(b''_{14})^{(1,1,1,1)}(G,t)$, $-(b''_{15})^{(1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b''_{16})^{(2,2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b''_{20})^{(3,3,3,3)}(G_{23},t)$, $-(b''_{21})^{(3,3,3,3)}(G_{23},t)$, $-(b''_{22})^{(3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b''_{36})^{(7,7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)$, $-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b''_{40})^{(8,8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b''_{46})^{(9,9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a'_{28})^{(5)} & +(a''_{28})^{(5)}(T_{29},t) & +(a''_{24})^{(4,4,)}(T_{25},t) & +(a''_{32})^{(6,6,6)}(T_{33},t) \\ & +(a''_{13})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{28} \quad (79)$$

$$\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a'_{29})^{(5)} & +(a''_{29})^{(5)}(T_{29},t) & +(a''_{25})^{(4,4,)}(T_{25},t) & +(a''_{33})^{(6,6,6)}(T_{33},t) \\ & +(a''_{14})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{29} \quad (80)$$

$$\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a'_{30})^{(5)} & +(a''_{30})^{(5)}(T_{29},t) & +(a''_{26})^{(4,4,)}(T_{25},t) & +(a''_{34})^{(6,6,6)}(T_{33},t) \\ & +(a''_{15})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{30} \quad (81)$$

Where $+(a''_{28})^{(5)}(T_{29},t)$, $+(a''_{29})^{(5)}(T_{29},t)$, $+(a''_{30})^{(5)}(T_{29},t)$ are first augmentation coefficients for category 1, 2 and 3

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| $And\ \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)}, \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)}, \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)}\ are\ second\ augmentation\ coefficient\ for\ category\ 1,2\ and\ 3$ $\boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)}, \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)}, \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)}\ are\ third\ augmentation\ coefficient\ for\ category\ 1,2\ and\ 3$ $\boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)}, \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)}, \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3 $\boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation coefficients for category 1,2,and 3 $\boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)}$ are sixth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$ , $\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}, \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1,2, 3 $\boxed{+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}, \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}, \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$ are eighth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)}, \boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)}, \boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1,2, 3 | |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b_{28}^{'})^{(5)}\boxed{-(b_{28}^{''})^{(5)}(G_{31},t)} & \boxed{-(b_{24}^{''})^{(4,4,)}(G_{27},t)} & \boxed{-(b_{32}^{''})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{28}$ | *82* |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b_{29}^{'})^{(5)}\boxed{-(b_{29}^{''})^{(5)}(G_{31},t)} & \boxed{-(b_{25}^{''})^{(4,4,)}(G_{27},t)} & \boxed{-(b_{33}^{''})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{29}$ | *83* |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b_{30}^{'})^{(5)}\boxed{-(b_{30}^{''})^{(5)}(G_{31},t)} & \boxed{-(b_{26}^{''})^{(4,4,)}(G_{27},t)} & \boxed{-(b_{34}^{''})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{30}$ | *84* |
| $where\ \boxed{-(b_{28}^{''})^{(5)}(G_{31},t)}, \boxed{-(b_{29}^{''})^{(5)}(G_{31},t)}, \boxed{-(b_{30}^{''})^{(5)}(G_{31},t)}\ are\ first\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{24}^{''})^{(4,4,)}(G_{27},t)}, \boxed{-(b_{25}^{''})^{(4,4,)}(G_{27},t)}, \boxed{-(b_{26}^{''})^{(4,4,)}(G_{27},t)}\ are\ second\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{32}^{''})^{(6,6,6)}(G_{35},t)}, \boxed{-(b_{33}^{''})^{(6,6,6)}(G_{35},t)}, \boxed{-(b_{34}^{''})^{(6,6,6)}(G_{35},t)}\ are\ third\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)}, \boxed{-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)}, \boxed{-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)}$ are fourth detrition coefficients for category 1,2, and 3 $\boxed{-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)}, \boxed{-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)}, \boxed{-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1,2, and 3 $\boxed{-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)}, \boxed{-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)}, \boxed{-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients | |

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| for category 1,2, and 3<br>$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3<br>$-(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3<br>$-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3 | |
| $\frac{dG_{32}}{dt}=(a_{32})^{(6)}G_{33}$<br>$-\begin{bmatrix}(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{32}$ | *85* |
| $\frac{dG_{33}}{dt}=(a_{33})^{(6)}G_{32}-\begin{bmatrix}(a_{33}^{'})^{(6)}+(a_{33}^{''})^{(6)}(T_{33},t) & +(a_{29}^{''})^{(5,5,5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{33}$ | *86* |
| $\frac{dG_{34}}{dt}=(a_{34})^{(6)}G_{33}-\begin{bmatrix}(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33},t) & +(a_{30}^{''})^{(5,5,5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{34}$ | *87* |
| $+(a_{32}^{''})^{(6)}(T_{33},t)$, $+(a_{33}^{''})^{(6)}(T_{33},t)$, $+(a_{34}^{''})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{28}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients<br>$+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients<br>$+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients<br>$+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients<br>$+(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients<br>$+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients | |

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{28})^{(5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b''_{24})^{(4,4,4,)}(G_{27},t)$, $-(b''_{25})^{(4,4,4,)}(G_{27},t)$, $-(b''_{26})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b''_{13})^{(1,1,1,1,1,1)}(G,t)$, $-(b''_{14})^{(1,1,1,1,1,1)}(G,t)$, $-(b''_{15})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3

$-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3

$-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3

$-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3

$-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a'_{36})^{(7)} & +(a''_{36})^{(7)}(T_{37},t) & +(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t) & \\ +(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a''_{40})^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}')^{(7)}\boxed{+(a_{37}'')^{(7)}(T_{37},t)} & \boxed{+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}')^{(7)}\boxed{+(a_{38}'')^{(7)}(T_{37},t)} & \boxed{+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad (93)$$

Where $\boxed{(a_{36}'')^{(7)}(T_{37},t)}$, $\boxed{(a_{37}'')^{(7)}(T_{37},t)}$, $\boxed{(a_{38}'')^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{40}'')^{(8,8,8,8,8,8,8,8,)}(T_{41},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}')^{(7)}\boxed{-(b_{36}'')^{(7)}(G_{39},t)} & \boxed{-(b_{16}'')^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}'')^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}'')^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}'')^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}'')^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}'')^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}'')^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13} \quad (94)$$

$$\frac{dT_{37}}{dt} =$$

$$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b'_{37})^{(7)}\boxed{-(b''_{37})^{(7)}(G_{39},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} =$$

$$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b'_{38})^{(7)}\boxed{-(b''_{38})^{(7)}(G_{39},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a'_{40})^{(8)}\boxed{+(a''_{40})^{(8)}(T_{41},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a'_{41})^{(8)}\boxed{+(a''_{41})^{(8)}(T_{41},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a'_{42})^{(8)}\boxed{+(a''_{42})^{(8)}(T_{41},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{40})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b'_{40})^{(8)}\boxed{-(b''_{40})^{(8)}(G_{43},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b'_{41})^{(8)}\boxed{-(b''_{41})^{(8)}(G_{43},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b'_{42})^{(8)}\boxed{-(b''_{42})^{(8)}(G_{43},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13} \quad (96)$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} =$$

$$(b_{46})^{(9)}T_{45} - \begin{bmatrix} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

$(a_i^{''})^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$

$(b_i^{''})^{(1)}(G,t) \le (r_i)^{(1)} \le (b_i^{'})^{(1)} \le (\hat{B}_{13})^{(1)}$

$lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$ 98

$\lim_{G\to\infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$

**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :

Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$

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| They satisfy Lipschitz condition:<br>$\|(a_i^{''})^{(1)}(T_{14}^{'},t)-(a_i^{''})^{(1)}(T_{14},t)\| \leq (\hat{k}_{13})^{(1)}\|T_{14}-T_{14}^{'}\|e^{-(\hat{M}_{13})^{(1)}t}$<br>$\|(b_i^{''})^{(1)}(G^{'},t)-(b_i^{''})^{(1)}(G,t)\| < (\hat{k}_{13})^{(1)}\|G-G^{'}\|e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $[(\hat{k}_{13})^{(1)},(\hat{M}_{13})^{(1)}]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)}=1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)}$, are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)},(\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)},(\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)},(a_i^{'})^{(1)},(b_i)^{(1)},(b_i^{'})^{(1)},(p_i)^{(1)}$, $(r_i)^{(1)}, i=13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)}+(a_i^{'})^{(1)}+(\hat{A}_{13})^{(1)}+(\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}]<1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)}+(b_i^{'})^{(1)}+(\hat{B}_{13})^{(1)}+(\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}]<1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)},(a_i^{'})^{(2)},(a_i^{''})^{(2)},(b_i)^{(2)},(b_i^{'})^{(2)},(b_i^{''})^{(2)}>0, \quad i,j=16,17,18$ | |
| The functions $(a_i^{''})^{(2)},(b_i^{''})^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i^{''})^{(2)}(T_{17},t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i^{''})^{(2)}(G_{19},t) \leq (r_i)^{(2)} \leq (b_i^{'})^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(2)}(T_{17},t)=(p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty}(b_i^{''})^{(2)}((G_{19}),t)=(r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)}$ : | 106 |

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| Where $\boxed{(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}}$ are positive constants and $\boxed{i = 16,17,18}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'}, t) - (a_i^{''})^{(2)}(T_{17}, t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}^{'}\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'}, t) - (b_i^{''})^{(2)}((G_{19}), t)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})^{'}\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'}, t)$ and $(a_i^{''})^{(2)}(T_{17}, t)$ . $(T_{17}^{'}, t)$ and $(T_{17}, t)$ are points belonging to the interval $\left[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}\right]$. It is to be noted that $(a_i^{''})^{(2)}(T_{17}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i^{''})^{(2)}(T_{17}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br><br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br><br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br><br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i^{'})^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i, j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(r_i)^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(3)}(T_{21}, t) = (p_i)^{(3)}$ | 113 |

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| $\lim_{G\to\infty}(b_i'')^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(3)}(T_{21}',t) - (a_i'')^{(3)}(T_{21},t)\rvert \le (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}'\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i'')^{(3)}(G_{23}',t) - (b_i'')^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}'\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(3)}(T_{21}',t)$ and$(a_i'')^{(3)}(T_{21},t)$ . $(T_{21}',t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i'')^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i'')^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i')^{(3)}, (b_i)^{(3)}, (b_i')^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i')^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i')^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i')^{(4)}, (a_i'')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (b_i'')^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i'')^{(4)}, (b_i'')^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i'')^{(4)}(T_{25},t) \le (p_i)^{(4)} \le (\hat{A}_{24})^{(4)}$<br><br>$(b_i'')^{(4)}\big((G_{27}),t\big) \le (r_i)^{(4)} \le (b_i')^{(4)} \le (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2\to\infty}(a_i'')^{(4)}(T_{25},t) = (p_i)^{(4)}$ | 118 |

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| $\lim_{G\to\infty}(b_i'')^{(4)}\left((G_{27}),t\right) = \ (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, \ (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24{,}25{,}26}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(4)}(T_{25}',t) - (a_i'')^{(4)}(T_{25},t)\rvert \le (\hat{k}_{24})^{(4)}\lvert T_{25} - T_{25}'\rvert e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\lvert(b_i'')^{(4)}((G_{27})',t) - (b_i'')^{(4)}\left((G_{27}),t\right)\rvert < (\hat{k}_{24})^{(4)}\lvert\lvert(G_{27}) - (G_{27})'\rvert\rvert e^{-(\hat{M}_{24})^{(4)}t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(4)}(T_{25}',t)$ and $(a_i'')^{(4)}(T_{25},t)$ . $(T_{25}',t)$ and $(T_{25},t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i'')^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i'')^{(4)}(T_{25},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (p_i)^{(4)}, \ (r_i)^{(4)}, i = 24{,}25{,}26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[\,(a_i)^{(4)} + (a_i')^{(4)} + \ (\hat{A}_{24})^{(4)} + \ (\hat{P}_{24})^{(4)}\,(\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i')^{(4)} + \ (\hat{B}_{24})^{(4)} + \ (\hat{Q}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i')^{(5)}, (a_i'')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (b_i'')^{(5)} > 0, \quad i,j = 28{,}29{,}30$<br><br>The functions $(a_i'')^{(5)}, (b_i'')^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}, \ (r_i)^{(5)}$:<br><br>$(a_i'')^{(5)}(T_{29},t) \le (p_i)^{(5)} \le (\hat{A}_{28})^{(5)}$<br><br>$(b_i'')^{(5)}\left((G_{31}),t\right) \le \ (r_i)^{(5)} \le (b_i')^{(5)} \le (\hat{B}_{28})^{(5)}$ | 122 |

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| $lim_{T_2\to\infty}(a_i'')^{(5)}(T_{29},t)=(p_i)^{(5)}$<br><br>$\lim_{G\to\infty}(b_i'')^{(5)}(G_{31},t)=(r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)},(\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)},(\hat{B}_{28})^{(5)},(p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i=28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(5)}(T_{29}',t)-(a_i'')^{(5)}(T_{29},t)\rvert\le(\hat{k}_{28})^{(5)}\lvert T_{29}-T_{29}'\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br><br>$\lvert(b_i'')^{(5)}((G_{31})',t)-(b_i'')^{(5)}\big((G_{31}),t\big)\rvert<(\hat{k}_{28})^{(5)}\lvert\lvert(G_{31})-(G_{31})'\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(5)}(T_{29}',t)$ and $(a_i'')^{(5)}(T_{29},t)$ . $(T_{29}',t)$ and $(T_{29},t)$ are points belonging to the interval $\left[(\hat{k}_{28})^{(5)},(\hat{M}_{28})^{(5)}\right]$. It is to be noted that $(a_i'')^{(5)}(T_{29},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)}=1$ then the function $(a_i'')^{(5)}(T_{29},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)},(\hat{k}_{28})^{(5)}$ :<br><br>$(\hat{M}_{28})^{(5)},(\hat{k}_{28})^{(5)},$ are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\ ,\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}}<1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)},(\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)},(\hat{k}_{28})^{(5)},(\hat{A}_{28})^{(5)}and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)},(a_i')^{(5)},(b_i)^{(5)},(b_i')^{(5)},(p_i)^{(5)},\ (r_i)^{(5)},i=28,29,30,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)}+(a_i')^{(5)}+(\hat{A}_{28})^{(5)}+(\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}]<1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)}+(b_i')^{(5)}+(\hat{B}_{28})^{(5)}+(\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}]<1$ | 126 |
| Where we suppose | |
| $(a_i)^{(6)},(a_i')^{(6)},(a_i'')^{(6)},(b_i)^{(6)},(b_i')^{(6)},(b_i'')^{(6)}>0,\quad i,j=32,33,34$<br><br>The functions $(a_i'')^{(6)},(b_i'')^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)},\ (r_i)^{(6)}$:<br><br>$(a_i'')^{(6)}(T_{33},t)\le(p_i)^{(6)}\le(\hat{A}_{32})^{(6)}$ | 127 |

| | |
|---|---|
| $(b_i^{''})^{(6)}((G_{35}),t) \leq \ (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(6)}(T_{33},t) = (p_i)^{(6)}$<br>$\lim_{G\to\infty}(b_i^{''})^{(6)}\left((G_{35}),t\right) = \ (r_i)^{(6)}$<br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'},t) - (a_i^{''})^{(6)}(T_{33},t)\rvert \leq (\hat{k}_{32})^{(6)}\lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br>$\lvert(b_i^{''})^{(6)}((G_{35})^{'},t) - (b_i^{''})^{(6)}\left((G_{35}),t\right)\rvert < (\hat{k}_{32})^{(6)}\lvert\lvert(G_{35}) - (G_{35})^{'}\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'},t)$ and$(a_i^{''})^{(6)}(T_{33},t)$ . $(T_{33}^{'},t)$ and $(T_{33},t)$ are points belonging to the interval $\left[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\right]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and \ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + \ (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$<br>$\frac{1}{(\hat{M}_{32})^{(6)}}[\ (b_i)^{(6)} + (b_i^{'})^{(6)} + \ (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)}\ (\hat{k}_{32})^{(6)}] < 1$ | 130 |
| Where we suppose | |
| (WWWWW) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i,j = 36,37,38$<br>(XXXXX) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$: | 131 |

$(a_i^{''})^{(7)}(T_{37},t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$

$(b_i^{''})^{(7)}(G_{39},t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$

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*(YYYYY)* $\lim_{T_2\to\infty} (a_i^{''})^{(7)}(T_{37},t) = (p_i)^{(7)}$

*(ZZZZZ)*

$\lim_{G\to\infty}(b_i^{''})^{(7)}\big((G_{39}),t\big) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

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They satisfy Lipschitz condition:

$|(a_i^{''})^{(7)}(T_{37}^{'},t) - (a_i^{''})^{(7)}(T_{37},t)| \leq (\hat{k}_{36})^{(7)}|T_{37} - T_{37}^{'}|e^{-(\hat{M}_{36})^{(7)}t}$

$|(b_i^{''})^{(7)}((G_{39})^{'},t) - (b_i^{''})^{(7)}\big((G_{39}),t\big)| < (\hat{k}_{36})^{(7)}||(G_{39}) - (G_{39})^{'}||e^{-(\hat{M}_{36})^{(7)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'},t)$ and$(a_i^{''})^{(7)}(T_{37},t)$ . $(T_{37}^{'},t)$ and $(T_{37},t)$ are points belonging to the interval $\big[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}\big]$ . It is to be noted that $(a_i^{''})^{(7)}(T_{37},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

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**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :

(AAAAAA) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants

$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$

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**Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :

(BBBBBB) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38,$ satisfy the inequalities

$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$

$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$

| | |
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| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})^{'}, t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})^{'}\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $\left[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}\right]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i^{'})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (p_i)^{(8)}$, $(r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |

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| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i,j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45},t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47},t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2\to\infty}(a_i'')^{(9)}(T_{45},t) = (p_i)^{(9)}$<br>$\lim_{G\to\infty}(b_i'')^{(9)}(G_{47},t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}',t) - (a_i'')^{(9)}(T_{45},t)\rvert \le (\hat{k}_{44})^{(9)}\lvert T_{45} - T_{45}'\rvert e^{-(\hat{M}_{44})^{(9)}t}$<br>$\lvert(b_i'')^{(9)}((G_{47})',t) - (b_i'')^{(9)}((G_{47}),t)\rvert < (\hat{k}_{44})^{(9)}\lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}',t)$ and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i'')^{(9)}(T_{45},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :<br>$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants<br>$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$ | |
| **Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :<br>There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and (\hat{B}_{44})^{(9)}$ and the constants | |

$(a_i)^{(9)}, (a_i')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (p_i)^{(9)}, \ (r_i)^{(9)}, i = 44,45,46,$
satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i')^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i')^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0), T_i(0)$ :

$$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}, \quad \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}, \quad \boxed{T_i(0) = T_i^0 > 0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0), T_i(0)$

$$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}, \quad G_i(0) = G_i^0 > 0$$

$$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}, \quad T_i(0) = T_i^0 > 0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}, \quad G_i(0) = G_i^0 > 0$$

$$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}, \quad T_i(0) = T_i^0 > 0$$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

**Definition of** $G_i(0), T_i(0)$ :

$$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}, \quad \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}, \quad \boxed{T_i(0) = T_i^0 > 0}$$

**Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 151

**Definition of** $G_i(0), T_i(0)$ :

$$G_i(t) \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}, \quad \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}, \quad \boxed{T_i(0) = T_i^0 > 0}$$

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| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0)\,, T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le \left(\hat{Q}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le \left(\hat{P}_{13}\right)^{(1)}$ , $T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le \left(\hat{P}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |
| $0 \le T_i(t) - T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 157 |
| By<br><br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}\left(s_{(13)}\right) - \left((a_{13}')^{(1)} + a_{13}'')^{(1)}\left(T_{14}\left(s_{(13)}\right), s_{(13)}\right)\right) G_{13}\left(s_{(13)}\right)\right] ds_{(13)}$ | 158 |

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| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{16})^{(2)}$, $T_i^0 \le (\hat{Q}_{16})^{(2)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} G_{17}(s_{(16)}) - \left((a_{16}')^{(2)} + a_{16}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{16}(s_{(16)})\right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[(a_{17})^{(2)} G_{16}(s_{(16)}) - \left((a_{17}')^{(2)} + (a_{17}'')^{(2)}(T_{17}(s_{(16)}), s_{(17)})\right) G_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{20})^{(3)}$ , $T_i^0 \leq (\hat{Q}_{20})^{(3)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a'_{20})^{(3)} + a''_{20})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a'_{21})^{(3)} + (a''_{21})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a'_{22})^{(3)} + (a''_{22})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b'_{20})^{(3)} - (b''_{20})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b'_{21})^{(3)} - (b''_{21})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b'_{22})^{(3)} - (b''_{22})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{24})^{(4)}$ , $T_i^0 \leq (\hat{Q}_{24})^{(4)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |

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| $\overline{\mathrm{T}}_{26}(\mathrm{t}) = \mathrm{T}_{26}^{0} + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{28})^{(5)}$ , $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a_{28}^{'})^{(5)} + a_{28}^{''})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\overline{\mathrm{T}}_{30}(\mathrm{t}) = \mathrm{T}_{30}^{0} + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{32})^{(6)}$ , $T_i^0 \le (\hat{Q}_{32})^{(6)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a_{32}^{'})^{(6)} + a_{32}^{''})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |

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| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{36})^{(7)}$, $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a_{36}^{'})^{(7)} + a_{36}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ | |

which satisfy

$G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{40})^{(8)}$ , $T_i^0 \leq (\hat{Q}_{40})^{(8)}$,

$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$

$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$

By 166

$$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{40})^{(8)} + a''_{40})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{40}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[ (a_{41})^{(8)} G_{40}(s_{(40)}) - \left( (a'_{41})^{(8)} + (a''_{41})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{41}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[ (a_{42})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{42})^{(8)} + (a''_{42})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{42}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[ (b_{40})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{40})^{(8)} - (b''_{40})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{40}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[ (b_{41})^{(8)} T_{40}(s_{(40)}) - \left( (b'_{41})^{(8)} - (b''_{41})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{41}(s_{(40)}) \right] ds_{(40)}$$

$$\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[ (b_{42})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{42})^{(8)} - (b''_{42})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{42}(s_{(40)}) \right] ds_{(40)}$$

Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$

**Proof:** 166 A

Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy

$G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$,

$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$

$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$

By

$$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[ (a_{44})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{44}(s_{(44)}) \right] ds_{(44)}$$

$$\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[ (a_{45})^{(9)} G_{44}(s_{(44)}) - \left( (a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{45}(s_{(44)}) \right] ds_{(44)}$$

$$\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[ (a_{46})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{46}(s_{(44)}) \right] ds_{(44)}$$

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| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that<br>$(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} \left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$<br>$\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}} \left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$ | 171 |
| From which it follows that | 172 |

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| $(G_{20}(t)-G_{20}^0)e^{-(\hat{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that $G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\hat{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$ $\left(1+(a_{24})^{(4)}t\right)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that $(G_{24}(t)-G_{24}^0)e^{-(\hat{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$ $\left(1+(a_{28})^{(5)}t\right)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left(e^{(\hat{M}_{28})^{(5)}t}-1\right)$ | |
| From which it follows that $(G_{28}(t)-G_{28}^0)e^{-(\hat{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 5 | 175 |
| The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0+(\hat{P}_{32})^{(6)}e^{(\hat{M}_{32})^{(6)}s_{(32)}}\right)\right] ds_{(32)} =$ $\left(1+(a_{32})^{(6)}t\right)G_{33}^0 + \frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)}t}-1\right)$ | 176 |
| From which it follows that $(G_{32}(t)-G_{32}^0)e^{-(\hat{M}_{32})^{(6)}t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^0)e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^0}{G_{33}^0}\right)}+(\hat{P}_{32})^{(6)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 6 | 177 |

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| Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$ | |
| (v) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} \left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$<br>$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$ | 178 |
| From which it follows that<br>$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left[\left((\hat{P}_{36})^{(7)} + G_{37}^0\right) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 7 | |
| The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} \left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$<br>$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$ | 180 |
| From which it follows that<br>$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left[\left((\hat{P}_{40})^{(8)} + G_{41}^0\right) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} \left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$<br>$\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left(e^{(\hat{M}_{44})^{(9)} t} - 1\right)$ | |
| From which it follows that<br>$(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left[\left((\hat{P}_{44})^{(9)} + G_{45}^0\right) e^{\left(-\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0}\right)} + (\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ and to choose | 182 |

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| $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{13})^{(1)}\left\|G_{14}^{(1)}-G_{14}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$<br>$\int_0^t\{(a_{13}')^{(1)}\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{-(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$G_{13}^{(2)}\left\\|(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}'')^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right\\|\ e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$<br>Where $s_{(13)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | |
| $\left\|G^{(1)}-G^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}t}\leq$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}\left((a_{13})^{(1)}+(a_{13}')^{(1)}+(\hat{A}_{13})^{(1)}+(\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)},T^{(1)};\ G^{(2)},T^{(2)}\right)\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a_{13}'')^{(1)}$ and $(b_{13}'')^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by | |

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| $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(1)} - (a_i^{''})^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a_{14}^{'})^{(1)}G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}^{'})^{(1)}$<br><br>If $G_{14}\ or\ G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}\ and\ G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t), t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t), t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right),\quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t), t) = (b_{15}^{'})^{(1)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |

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| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)}, (T_{19})^{(1)}), ((G_{19})^{(2)}, (T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)}\left\|G_{17}^{(1)} - G_{17}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\,ds_{(16)} +$<br>$\int_0^t\{(a_{16}')^{(2)}\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{-(\hat{M}_{16})^{(2)}s_{(16)}} +$<br>$(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}} +$<br>$G_{16}^{(2)}\left\|(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}'')^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}t} \leq$<br>$\frac{1}{(\hat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}\right)d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}'')^{(2)}$ and $(b_{16}'')^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | 199 |

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| condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ and $(\widehat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)}t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1, \left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)}G_{17}$ and by integrating<br><br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1/(a_{17}^{'})^{(2)}$<br><br>In the same way , one can obtain<br><br>$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}^{'})^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | 201 |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(2)}((G_{19})(t), t)) = (b_{17}^{'})^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i^{''})^{(2)}((G_{19})(t), t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}^{''})^{(2)}\left((G_{19})(t), t\right) = (b_{18}^{'})^{(2)}$ | 205 |

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| We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)}, (T_{23})^{(1)}), ((G_{23})^{(2)}, (T_{23})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{20})^{(3)}t}\}$ | 211 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :(\widetilde{(G_{23})}, \widetilde{(T_{23})}) = \mathcal{A}^{(3)}((G_{23}), (T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)}\left\|G_{21}^{(1)} - G_{21}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)} +$<br>$\int_0^t\{(a'_{20})^{(3)}\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{-(\hat{M}_{20})^{(3)}s_{(20)}} +$<br>$(a''_{20})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right)\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}} +$<br>$G_{20}^{(2)}\left\|(a''_{20})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a''_{20})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}t} \leq$<br>$\frac{1}{(\hat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a'_{20})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)}, (T_{23})^{(1)};\ (G_{23})^{(2)}, (T_{23})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |

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| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ $and$ $(\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(3)}t)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2\ and\ \left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}\ and\ G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)}G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br><br>If $G_{21}\ or\ G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ $is$ bounded, from below, the same property holds for $G_{21}\ and\ G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t), t\right)) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t), t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded. | 220 |

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| The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}'')^{(3)}\left((G_{23})(t),t\right)=(b_{22}')^{(3)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}<1$ and to choose<br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)}+\left((\hat{P}_{24})^{(4)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right]\le(\hat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\hat{Q}_{24})^{(4)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{24})^{(4)}\right]\le(\hat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{24})^{(4)}\left\|G_{25}^{(1)}-G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}ds_{(24)}+$<br>$\int_0^t\{(a_{24}')^{(4)}\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$(a_{24}'')^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$G_{24}^{(2)}\left\|(a_{24}'')^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}'')^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)}-(G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t}\le$<br>$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}')^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$ | 226 |

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| And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i = 24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(4)} - (a_i^{''})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(4)}t\right)} > 0$ for t > 0 | 228 |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ :<br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if<br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}^{'})^{(4)}G_{25}$ and by integrating<br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a_{25}^{'})^{(4)}$<br>In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a_{26}^{'})^{(4)}$<br>If $G_{25}\ or\ G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}\ and\ G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t), t)) = (b_{25}^{'})^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i^{''})^{(4)}((G_{27})(t), t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results | 232 |

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right), \quad t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42

Analogous inequalities hold also for $G_{29}, G_{30}, T_{28}, T_{29}, T_{30}$

It is now sufficient to take $\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ and to choose

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$(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ large to have

$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[(\hat{P}_{28})^{(5)} + \left((\hat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{28})^{(5)} \qquad 234$$

$$\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{28})^{(5)}\right] \leq (\hat{Q}_{28})^{(5)} \qquad 235$$

In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric 236

$$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{31}}), (\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}), (\widetilde{T_{31}})\right) = \mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$

It results

$$\left|\tilde{G}_{28}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{28})^{(5)}\left|G_{29}^{(1)} - G_{29}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)} +$$

$$\int_0^t\{(a_{28}^{'})^{(5)}\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{-(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)}, s_{(28)}\right)\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$G_{28}^{(2)}\left|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)}, s_{(28)}\right) - (a_{28}^{''})^{(5)}\left(T_{29}^{(2)}, s_{(28)}\right)\right|\ e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$$

Where $s_{(28)}$ represents integrand that is integrated over the interval [0, t]

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| From the hypotheses on it follows | |
| $\left\|(G_{31})^{(1)} - (G_{31})^{(2)}\right\| e^{-(\widehat{M}_{28})^{(5)}t} \leq$ $\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)} + (a'_{28})^{(5)} + (\widehat{A}_{28})^{(5)} + (\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right) d\left(((G_{31})^{(1)}, (T_{31})^{(1)};\ (G_{31})^{(2)}, (T_{31})^{(2)})\right)$ And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a''_{28})^{(5)}$ and $(b''_{28})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ and $(\widehat{Q}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(5)}$ and $(b''_i)^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t) = 0$ and $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(5)} - (a''_i)^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(5)}t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ and $\left((\widehat{M}_{28})^{(5)}\right)_3$ : **Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ and $G_{30}$ . indeed if $G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a'_{29})^{(5)} G_{29}$ and by integrating $G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a'_{29})^{(5)}$ In the same way , one can obtain $G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a'_{30})^{(5)}$ If $G_{29}$ or $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ is bounded, from below, the same property holds for $G_{29}$ and $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |

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| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t),t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1-e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}, G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}\left[(\hat{P}_{32})^{(6)} + \left((\hat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{32})^{(6)}\right] \leq (\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{32})^{(6)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{35})}, \widetilde{(T_{35})}$ : $\left(\widetilde{(G_{35})}, \widetilde{(T_{35})}\right) = \mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br><br>It results | 247 |

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| $\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{32})^{(6)} \left\|G_{33}^{(1)}-G_{33}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\, ds_{(32)} +$<br>$\int_0^t \{(a_{32}^{'})^{(6)} \left\|G_{32}^{(1)}-G_{32}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$(a_{32}^{''})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$G_{32}^{(2)} \left\|(a_{32}^{''})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)-(a_{32}^{''})^{(6)}\left(T_{33}^{(2)}, s_{(32)}\right)\right\| \; e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\} ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}t} \leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a_{32}^{'})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right) d\left(((G_{35})^{(1)},(T_{35})^{(1)};\,(G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a_{32}^{''})^{(6)}$ and $(b_{32}^{''})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i = 32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 249 |
| **Remark 27:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$<br>it results<br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(6)}-(a_i^{''})^{(6)}(T_{33}(s_{(32)}),s_{(32)})\}ds_{(32)}\right]} \geq 0$<br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(6)}t\right)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ :<br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if<br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}^{'})^{(6)}G_{33}$ and by integrating<br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1/(a_{33}^{'})^{(6)}$<br>In the same way , one can obtain<br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2/(a_{34}^{'})^{(6)}$ | 251 |

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| If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i^{''})^{(6)}\big((G_{35})(t),t\big) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ large to have | 255 |
| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[(\hat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote | 258 |

**Definition of** $\widetilde{(G_{39})}, \widetilde{(T_{39})} : \left( \widetilde{(G_{39})}, \widetilde{(T_{39})} \right) = \mathcal{A}^{(7)}((G_{39}), (T_{39}))$
It results

$\left|\tilde{G}_{36}^{(1)} - \tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{36})^{(7)} \left|G_{37}^{(1)} - G_{37}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}}\, ds_{(36)} +$
$\int_0^t \{(a_{36}^{'})^{(7)} \left|G_{36}^{(1)} - G_{36}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} +$
$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) \left|G_{36}^{(1)} - G_{36}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}} +$
$G_{36}^{(2)} \left|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) - (a_{36}^{''})^{(7)}\left(T_{37}^{(2)}, s_{(36)}\right)\right| \; e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}} \} ds_{(36)}$

Where $s_{(36)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on it follows

$\left|(G_{39})^{(1)} - (G_{39})^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} t} \leq$
$\frac{1}{(\widehat{M}_{36})^{(7)}} \left((a_{36})^{(7)} + (a_{36}^{'})^{(7)} + (\widehat{A}_{36})^{(7)} + (\widehat{P}_{36})^{(7)} (\widehat{k}_{36})^{(7)}\right) d\left(((G_{39})^{(1)}, (T_{39})^{(1)};\ (G_{39})^{(2)}, (T_{39})^{(2)})\right)$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

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**Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)} e^{(\widehat{M}_{36})^{(7)} t}$ *and* $(\widehat{Q}_{36})^{(7)} e^{(\widehat{M}_{36})^{(7)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition.

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**Remark 32:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$

it results

$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)} (T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$

$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0

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**Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2$ *and* $\left((\widehat{M}_{36})^{(7)}\right)_3$ :

**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}$ *and* $G_{38}$ . indeed if

$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating

$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)} \left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$

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In the same way , one can obtain

$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}')^{(7)}$

If $G_{37}$ *or* $G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively.

**Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}$ *and* $G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. 263

**Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(7)}((G_{39})(t),t)) = (b_{37}')^{(7)}$ then $T_{37} \to \infty$. 264

**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :

Indeed let $t_7$ be so that for $t > t_7$

$(b_{37})^{(7)} - (b_i'')^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$

Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to 265

$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results

$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right)$, $t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.
The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}'')^{(7)}\left((G_{39})(t),t\right) = (b_{38}')^{(7)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations

It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose 266
$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have

$$\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)} + \left((\widehat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{40})^{(8)} \quad 267$$

$$\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\widehat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{40})^{(8)}\right] \leq (\widehat{Q}_{40})^{(8)} \quad 268$$

In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric

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| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$ $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\,ds_{(40)}+$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\le$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ *and* $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i=40,41,42$ depend only on $\mathrm{T}_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$<br><br>it results<br><br>$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(8)}-(a_i^{''})^{(8)}(T_{41}(s_{(40)}),s_{(40)})\}ds_{(40)}\right]}\ge 0$<br><br>$T_i\,(t)\ge T_i^0e^{\left(-(b_i^{'})^{(8)}t\right)}>0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1,\left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ : | 276 |

| | |
|---|---|
| **Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a'_{41})^{(8)}G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1/(a'_{41})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2/(a'_{42})^{(8)}$<br><br>If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | |
| **Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t),t)) = (b'_{41})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t),t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$ | 278 |
| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right),\ t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.<br>The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t),t(t),t\right) = (b'_{42})^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$ | |

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\left(((G_{47})^{(1)},(T_{47})^{(1)}),((G_{47})^{(2)},(T_{47})^{(2)})\right) =$$
$$sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})},\widetilde{(T_{47})} : (\widetilde{(G_{47})},\widetilde{(T_{47})}) = \mathcal{A}^{(9)}((G_{47}),(T_{47}))$
It results

$$\left|\tilde{G}_{44}^{(1)}-\tilde{G}_i^{(2)}\right| \le \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)}-G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)-(a_{44}^{''})^{(9)}\left(T_{45}^{(2)},s_{(44)}\right)\right|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t} \le$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)}+(a_{44}^{'})^{(9)}+(\widehat{A}_{44})^{(9)}+(\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)},(T_{47})^{(1)};\ (G_{47})^{(2)},(T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44{,}45{,}46$ depend only on $\mathrm{T}_{45}$ and respectively on $(G_{47})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$

From 99 to 44 it results

$$G_i(t) \ge G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(9)}-(a_i^{''})^{(9)}(T_{45}(s_{(44)}),s_{(44)})\}ds_{(44)}\right]} \ge 0$$

$$T_i(t) \ge T_i^0 e^{(-(b_i^{'})^{(9)}t)} > 0 \quad \text{for t} > 0$$

**Definition of** $((\widehat{M}_{44})^{(9)})_1, ((\widehat{M}_{44})^{(9)})_2$ *and* $((\widehat{M}_{44})^{(9)})_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)} G_{45}$ and by integrating

$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$

In the same way , one can obtain

$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t),t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t),t\right) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right),\quad t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\left((G_{47})(t),t\right) = (b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} - (a_{13}^{''})^{(1)}(T_{14},t) + (a_{14}^{''})^{(1)}(T_{14},t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(v_1)^{(1)}, (v_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, v^{(1)}, u^{(1)}$ : 281

By $(v_1)^{(1)} > 0$ , $(v_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$ , $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

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| **Definition of** $(\bar{v}_1)^{(1)},,(\bar{v}_2)^{(1)},(\bar{u}_1)^{(1)},(\bar{u}_2)^{(1)}$ :<br><br>By $(\bar{v}_1)^{(1)}>0$ , $(\bar{v}_2)^{(1)}<0$ and respectively $(\bar{u}_1)^{(1)}>0$ , $(\bar{u}_2)^{(1)}<0$ the roots of the equations $(a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_2)^{(1)}v^{(1)}-(a_{13})^{(1)}=0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2+(\tau_2)^{(1)}u^{(1)}-(b_{13})^{(1)}=0$ | 282 |
| **Definition of** $(m_1)^{(1)}$ , $(m_2)^{(1)}$ , $(\mu_1)^{(1)},(\mu_2)^{(1)},(\nu_0)^{(1)}$ :-<br><br>If we define $(m_1)^{(1)}$ , $(m_2)^{(1)}$ , $(\mu_1)^{(1)},(\mu_2)^{(1)}$ by<br>$(m_2)^{(1)}=(\nu_0)^{(1)},(m_1)^{(1)}=(\nu_1)^{(1)},\ if\ (\nu_0)^{(1)}<(\nu_1)^{(1)}$<br><br>$(m_2)^{(1)}=(\nu_1)^{(1)},(m_1)^{(1)}=(\bar{\nu}_1)^{(1)}\ ,if\ (\nu_1)^{(1)}<(\nu_0)^{(1)}<(\bar{\nu}_1)^{(1)},$<br><br>and $\boxed{(\nu_0)^{(1)}=\frac{G_{13}^0}{G_{14}^0}}$<br><br>$(\ m_2)^{(1)}=(\nu_1)^{(1)},(m_1)^{(1)}=(\nu_0)^{(1)},\ if\ (\bar{\nu}_1)^{(1)}<(\nu_0)^{(1)}$ | 283 |
| and analogously<br><br>$(\mu_2)^{(1)}=(u_0)^{(1)},(\mu_1)^{(1)}=(u_1)^{(1)},\ if\ (u_0)^{(1)}<(u_1)^{(1)}$<br><br>$(\mu_2)^{(1)}=(u_1)^{(1)},(\mu_1)^{(1)}=(\bar{u}_1)^{(1)}\ ,if\ (u_1)^{(1)}<(u_0)^{(1)}<(\bar{u}_1)^{(1)},$<br><br>and $\boxed{(u_0)^{(1)}=\frac{T_{13}^0}{T_{14}^0}}$<br><br>$(\ \mu_2)^{(1)}=(u_1)^{(1)},(\mu_1)^{(1)}=(u_0)^{(1)},if\ (\bar{u}_1)^{(1)}<(u_0)^{(1)}$ where $(u_1)^{(1)},(\bar{u}_1)^{(1)}$<br><br>are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{13}^0e^{((S_1)^{(1)}-(p_{13})^{(1)})t}\le G_{13}(t)\le G_{13}^0e^{(S_1)^{(1)}t}$<br><br>where $(p_i)^{(1)}$ is defined by equation<br><br>$\frac{1}{(m_1)^{(1)}}G_{13}^0e^{((S_1)^{(1)}-(p_{13})^{(1)})t}\le G_{14}(t)\le\frac{1}{(m_2)^{(1)}}G_{13}^0e^{(S_1)^{(1)}t}$ | 285 |
| $(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t}-e^{-(S_2)^{(1)}t}\right]+G_{15}^0e^{-(S_2)^{(1)}t}\le G_{15}(t)\le$<br><br>$\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a_{15}^{'})^{(1)})}[e^{(S_1)^{(1)}t}-e^{-(a_{15}^{'})^{(1)}t}]+\ G_{15}^0e^{-(a_{15}^{'})^{(1)}t})$ | 286 |
| $\boxed{T_{13}^0e^{(R_1)^{(1)}t}\le T_{13}(t)\le T_{13}^0e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}}T_{13}^0e^{(R_1)^{(1)}t}\le T_{13}(t)\le\frac{1}{(\mu_2)^{(1)}}T_{13}^0e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |

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| $\frac{(b_{15})^{(1)}T_{13}^{0}}{(\mu_1)^{(1)}((R_1)^{(1)}-(b_{15}^{'})^{(1)})}\left[e^{(R_1)^{(1)}t}-e^{-(b_{15}^{'})^{(1)}t}\right]+T_{15}^{0}e^{-(b_{15}^{'})^{(1)}t}\leq T_{15}(t)\leq$ $\frac{(a_{15})^{(1)}T_{13}^{0}}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t}-e^{-(R_2)^{(1)}t}\right]+T_{15}^{0}e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)},(S_2)^{(1)},(R_1)^{(1)},(R_2)^{(1)}$:- Where $(S_1)^{(1)}=(a_{13})^{(1)}(m_2)^{(1)}-(a_{13}^{'})^{(1)}$ $(S_2)^{(1)}=(a_{15})^{(1)}-(p_{15})^{(1)}$ $(R_1)^{(1)}=(b_{13})^{(1)}(\mu_2)^{(1)}-(b_{13}^{'})^{(1)}$ $(R_2)^{(1)}=(b_{15}^{'})^{(1)}-(r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ : $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)}\leq-(a_{16}^{'})^{(2)}+(a_{17}^{'})^{(2)}-(a_{16}^{''})^{(2)}(\mathrm{T}_{17},t)+(a_{17}^{''})^{(2)}(\mathrm{T}_{17},t)\leq-(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)}\leq-(b_{16}^{'})^{(2)}+(b_{17}^{'})^{(2)}-(b_{16}^{''})^{(2)}\big((G_{19}),t\big)-(b_{17}^{''})^{(2)}\big((G_{19}),t\big)\leq-(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(v_1)^{(2)},(v_2)^{(2)},(u_1)^{(2)},(u_2)^{(2)}$ : | 295 |
| By $(v_1)^{(2)}>0$ , $(v_2)^{(2)}<0$ and respectively $(u_1)^{(2)}>0$ , $(u_2)^{(2)}<0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(v^{(2)}\big)^2+(\sigma_1)^{(2)}v^{(2)}-(a_{16})^{(2)}=0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2+(\tau_1)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ and | 298 |
| **Definition of** $(\bar{v}_1)^{(2)},,(\bar{v}_2)^{(2)},(\bar{u}_1)^{(2)},(\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{v}_1)^{(2)}>0$ , $(\bar{v}_2)^{(2)}<0$ and respectively $(\bar{u}_1)^{(2)}>0$ , $(\bar{u}_2)^{(2)}<0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(v^{(2)}\big)^2+(\sigma_2)^{(2)}v^{(2)}-(a_{16})^{(2)}=0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2+(\tau_2)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ | 302 |
| **Definition of** $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)}=(v_0)^{(2)},(m_1)^{(2)}=(v_1)^{(2)}$, ***if*** $(v_0)^{(2)}<(v_1)^{(2)}$ | 305 |

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| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , $\boldsymbol{if}$ $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, $\boldsymbol{if}$ $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, $\boldsymbol{if}$ $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , $\boldsymbol{if}$ $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$,<br>and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, $\boldsymbol{if}$ $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)} t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)} t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)} t}\right] + G_{18}^0 e^{-(S_2)^{(2)} t} \le G_{18}(t) \le \right.$<br>$\left. \frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a'_{18})^{(2)})}\left[e^{(S_1)^{(2)} t} - e^{-(a'_{18})^{(2)} t}\right] + G_{18}^0 e^{-(a'_{18})^{(2)} t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b'_{18})^{(2)})}\left[e^{(R_1)^{(2)} t} - e^{-(b'_{18})^{(2)} t}\right] + T_{18}^0 e^{-(b'_{18})^{(2)} t} \le T_{18}(t) \le$<br>$\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)} + (r_{16})^{(2)} + (R_2)^{(2)})}\left[e^{((R_1)^{(2)} + (r_{16})^{(2)})t} - e^{-(R_2)^{(2)} t}\right] + T_{18}^0 e^{-(R_2)^{(2)} t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a'_{16})^{(2)}$<br>$(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b'_{16})^{(2)}$ | 318 |

$(\mathrm{R}_2)^{(2)} = (b_{18}^{'})^{(2)} - (r_{18})^{(2)}$

**Behavior of the solutions** 319

**Theorem 3:** If we denote and define

**Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ :

$(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying

$$-(\sigma_2)^{(3)} \le -(a_{20}^{'})^{(3)} + (a_{21}^{'})^{(3)} - (a_{20}^{''})^{(3)}(T_{21}, t) + (a_{21}^{''})^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$$

$$-(\tau_2)^{(3)} \le -(b_{20}^{'})^{(3)} + (b_{21}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t) - (b_{21}^{''})^{(3)}\big((G_{23}), t\big) \le -(\tau_1)^{(3)}$$

**Definition of** $(\nu_1)^{(3)}, (\nu_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ : 320

By $(\nu_1)^{(3)} > 0$ , $(\nu_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$ , $(u_2)^{(3)} < 0$ the roots of the equations

$(a_{21})^{(3)}\big(\nu^{(3)}\big)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\big(u^{(3)}\big)^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and

By $(\bar{\nu}_1)^{(3)} > 0$ , $(\bar{\nu}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$ , $(\bar{u}_2)^{(3)} < 0$ the

roots of the equations $(a_{21})^{(3)}\big(\nu^{(3)}\big)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\big(u^{(3)}\big)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$

**Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :- 321

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ , ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,

and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously 322

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$ , ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

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| $G_{20}^{0}e^{((S_1)^{(3)}-(p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^{0}e^{(S_1)^{(3)}t}$<br>$(p_i)^{(3)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(3)}}G_{20}^{0}e^{((S_1)^{(3)}-(p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}}G_{20}^{0}e^{(S_1)^{(3)}t}$ | 323 |
| $(\frac{(a_{22})^{(3)}G_{20}^{0}}{(m_1)^{(3)}((S_1)^{(3)}-(p_{20})^{(3)}-(S_2)^{(3)})}\left[e^{((S_1)^{(3)}-(p_{20})^{(3)})t}-e^{-(S_2)^{(3)}t}\right]+G_{22}^{0}e^{-(S_2)^{(3)}t} \le G_{22}(t) \le$<br>$\frac{(a_{22})^{(3)}G_{20}^{0}}{(m_2)^{(3)}((S_1)^{(3)}-(a_{22}^{'})^{(3)})}[e^{(S_1)^{(3)}t}-e^{-(a_{22}^{'})^{(3)}t}]+G_{22}^{0}e^{-(a_{22}^{'})^{(3)}t})$ | 324 |
| $\boxed{T_{20}^{0}e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^{0}e^{((R_1)^{(3)}+(r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}}T_{20}^{0}e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}}T_{20}^{0}e^{((R_1)^{(3)}+(r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)}T_{20}^{0}}{(\mu_1)^{(3)}((R_1)^{(3)}-(b_{22}^{'})^{(3)})}\left[e^{(R_1)^{(3)}t}-e^{-(b_{22}^{'})^{(3)}t}\right]+T_{22}^{0}e^{-(b_{22}^{'})^{(3)}t} \le T_{22}(t) \le$<br>$\frac{(a_{22})^{(3)}T_{20}^{0}}{(\mu_2)^{(3)}((R_1)^{(3)}+(r_{20})^{(3)}+(R_2)^{(3)})}\left[e^{((R_1)^{(3)}+(r_{20})^{(3)})t}-e^{-(R_2)^{(3)}t}\right]+T_{22}^{0}e^{-(R_2)^{(3)}t}$ | 327 |
| **Definition of** $(S_1)^{(3)},(S_2)^{(3)},(R_1)^{(3)},(R_2)^{(3)}$:-<br>Where $(S_1)^{(3)}=(a_{20})^{(3)}(m_2)^{(3)}-(a_{20}^{'})^{(3)}$<br>$(S_2)^{(3)}=(a_{22})^{(3)}-(p_{22})^{(3)}$<br>$(R_1)^{(3)}=(b_{20})^{(3)}(\mu_2)^{(3)}-(b_{20}^{'})^{(3)}$<br>$(R_2)^{(3)}=(b_{22}^{'})^{(3)}-(r_{22})^{(3)}$ | 328 |
| **Behavior of the solutions of equation**<br>**Theorem:** If we denote and define<br>**Definition of** $(\sigma_1)^{(4)},(\sigma_2)^{(4)},(\tau_1)^{(4)},(\tau_2)^{(4)}$ :<br>$(\sigma_1)^{(4)},(\sigma_2)^{(4)},(\tau_1)^{(4)},(\tau_2)^{(4)}$ four constants satisfying<br>$-(\sigma_2)^{(4)} \le -(a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}-(a_{24}^{''})^{(4)}(T_{25},t)+(a_{25}^{''})^{(4)}(T_{25},t) \le -(\sigma_1)^{(4)}$<br>$-(\tau_2)^{(4)} \le -(b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(b_{24}^{''})^{(4)}\big((G_{27}),t\big)-(b_{25}^{''})^{(4)}\big((G_{27}),t\big) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)},(v_2)^{(4)},(u_1)^{(4)},(u_2)^{(4)},v^{(4)},u^{(4)}$ :<br>By $(v_1)^{(4)}>0$ , $(v_2)^{(4)}<0$ and respectively $(u_1)^{(4)}>0$ , $(u_2)^{(4)}<0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2+(\sigma_1)^{(4)}v^{(4)}-(a_{24})^{(4)}=0$<br>and $(b_{25})^{(4)}\big(u^{(4)}\big)^2+(\tau_1)^{(4)}u^{(4)}-(b_{24})^{(4)}=0$ and | 329 |

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| **Definition of** $(\bar{v}_1)^{(4)},,(\bar{v}_2)^{(4)},(\bar{u}_1)^{(4)},(\bar{u}_2)^{(4)}$ :<br><br>By $(\bar{v}_1)^{(4)}>0$ , $(\bar{v}_2)^{(4)}<0$ and respectively $(\bar{u}_1)^{(4)}>0$ , $(\bar{u}_2)^{(4)}<0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2+(\sigma_2)^{(4)}v^{(4)}-(a_{24})^{(4)}=0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2+(\tau_2)^{(4)}u^{(4)}-(b_{24})^{(4)}=0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)},(v_0)^{(4)}$ :-<br><br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)}$ by<br><br>$(m_2)^{(4)}=(v_0)^{(4)},(m_1)^{(4)}=(v_1)^{(4)}$, ***if*** $(v_0)^{(4)}<(v_1)^{(4)}$<br><br>$(m_2)^{(4)}=(v_1)^{(4)},(m_1)^{(4)}=(\bar{v}_1)^{(4)}$ , ***if*** $(v_4)^{(4)}<(v_0)^{(4)}<(\bar{v}_1)^{(4)}$,<br>and $\boxed{(v_0)^{(4)}=\frac{G_{24}^0}{G_{25}^0}}$<br><br>$(m_2)^{(4)}=(v_4)^{(4)},(m_1)^{(4)}=(v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)}<(v_0)^{(4)}$ | 330 |
| and analogously<br><br>$(\mu_2)^{(4)}=(u_0)^{(4)},(\mu_1)^{(4)}=(u_1)^{(4)}$, ***if*** $(u_0)^{(4)}<(u_1)^{(4)}$<br><br>$(\mu_2)^{(4)}=(u_1)^{(4)},(\mu_1)^{(4)}=(\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)}<(u_0)^{(4)}<(\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)}=\frac{T_{24}^0}{T_{25}^0}}$<br><br>$(\mu_2)^{(4)}=(u_1)^{(4)},(\mu_1)^{(4)}=(u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)}<(u_0)^{(4)}$ where $(u_1)^{(4)},(\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{24}^0e^{((S_1)^{(4)}-(p_{24})^{(4)})t}\le G_{24}(t)\le G_{24}^0e^{(S_1)^{(4)}t}$<br><br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}}G_{24}^0e^{((S_1)^{(4)}-(p_{24})^{(4)})t}\le G_{25}(t)\le\frac{1}{(m_2)^{(4)}}G_{24}^0e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)}G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t}-e^{-(S_2)^{(4)}t}\right]+G_{26}^0e^{-(S_2)^{(4)}t}\le G_{26}(t)\le\right.$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0e^{(R_1)^{(4)}t}\le T_{24}(t)\le T_{24}^0e^{((R_1)^{(4)}+(r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}}T_{24}^0e^{(R_1)^{(4)}t}\le T_{24}(t)\le\frac{1}{(\mu_2)^{(4)}}T_{24}^0e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)}T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}^{'})^{(4)})}\left[e^{(R_1)^{(4)}t}-e^{-(b_{26}^{'})^{(4)}t}\right]+T_{26}^0e^{-(b_{26}^{'})^{(4)}t}\le T_{26}(t)\le$<br><br>$\frac{(a_{26})^{(4)}T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t}-e^{-(R_2)^{(4)}t}\right]+T_{26}^0e^{-(R_2)^{(4)}t}$ | 336 |

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| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a'_{24})^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b'_{24})^{(4)}$<br>$(R_2)^{(4)} = (b'_{26})^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a'_{28})^{(5)} + (a'_{29})^{(5)} - (a''_{28})^{(5)}(T_{29}, t) + (a''_{29})^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b'_{28})^{(5)} + (b'_{29})^{(5)} - (b''_{28})^{(5)}((G_{31}), t) - (b''_{29})^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ :<br>By $(v_1)^{(5)} > 0$, $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | 339 |
| **Definition of** $(\bar{v}_1)^{(5)},, (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ :<br>By $(\bar{v}_1)^{(5)} > 0$, $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$, $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$<br>and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :-<br>If we define $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}$ by<br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$, ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}}}$<br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$ | 341 |

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| $(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , $\boldsymbol{if}$ $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$, and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$ $(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}, \boldsymbol{if}$ $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | |
| Then the solution of global equations satisfies the inequalities $G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)} t}$ where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \le G_{30}(t) \le$ *(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b_{30}')^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b_{30}')^{(5)} t}\right] + T_{30}^0 e^{-(b_{30}')^{(5)} t} \le T_{30}(t) \le$ $\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:- Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a_{28}')^{(5)}$ $(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$ $(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b_{28}')^{(5)}$ $(R_2)^{(5)} = (b_{30}')^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ : $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying $-(\sigma_2)^{(6)} \le -(a_{32}')^{(6)} + (a_{33}')^{(6)} - (a_{32}'')^{(6)}(T_{33}, t) + (a_{33}'')^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$ | 349 |

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| $-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}\big((G_{35}),t\big) - (b''_{33})^{(6)}\big((G_{35}),t\big) \le -(\tau_1)^{(6)}$ | |
| **Definition of** $(v_1)^{(6)},(v_2)^{(6)},(u_1)^{(6)},(u_2)^{(6)},v^{(6)},u^{(6)}$ :<br><br>By $(v_1)^{(6)} > 0$ , $(v_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(v^{(6)}\big)^2 + (\sigma_1)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{v}_1)^{(6)},,(\bar{v}_2)^{(6)},(\bar{u}_1)^{(6)},(\bar{u}_2)^{(6)}$ :<br><br>By $(\bar{v}_1)^{(6)} > 0$ , $(\bar{v}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(v^{(6)}\big)^2 + (\sigma_2)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}$ , $(m_2)^{(6)}$ , $(\mu_1)^{(6)},(\mu_2)^{(6)},(v_0)^{(6)}$ :-<br><br>If we define $(m_1)^{(6)}$ , $(m_2)^{(6)}$ , $(\mu_1)^{(6)},(\mu_2)^{(6)}$ by<br><br>$(m_2)^{(6)} = (v_0)^{(6)}, (m_1)^{(6)} = (v_1)^{(6)}$, ***if*** $(v_0)^{(6)} < (v_1)^{(6)}$<br><br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (\bar{v}_6)^{(6)}$ , ***if*** $(v_1)^{(6)} < (v_0)^{(6)} < (\bar{v}_1)^{(6)}$,<br>and $\boxed{(v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$<br><br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (v_0)^{(6)}$, ***if*** $(\bar{v}_1)^{(6)} < (v_0)^{(6)}$ | 351 |
| and analogously<br><br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br><br>$G^0_{32}e^{((S_1)^{(6)}-(p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$<br><br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}}G^0_{32}e^{((S_1)^{(6)}-(p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}}G^0_{32}e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)}G^0_{32}}{(m_1)^{(6)}((S_1)^{(6)}-(p_{32})^{(6)}-(S_2)^{(6)})}\left[e^{((S_1)^{(6)}-(p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G^0_{34}e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |

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| $\boxed{T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)}T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)}-(b_{34}')^{(6)})}\left[e^{(R_1)^{(6)}t} - e^{-(b_{34}')^{(6)}t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)}T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t} - e^{-(R_2)^{(6)}t}\right] + T_{34}^0 e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:- Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$ $(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$ $(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$ $(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ :<br><br>$(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying<br><br>$-(\sigma_2)^{(7)} \le -(a_{36}')^{(7)} + (a_{37}')^{(7)} - (a_{36}'')^{(7)}(T_{37}, t) + (a_{37}'')^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$<br><br>$-(\tau_2)^{(7)} \le -(b_{36}')^{(7)} + (b_{37}')^{(7)} - (b_{36}'')^{(7)}\big((G_{39}), t\big) - (b_{37}'')^{(7)}\big((G_{39}), t\big) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)}, (v_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, v^{(7)}, u^{(7)}$ :<br><br>By $(v_1)^{(7)} > 0, (v_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(v^{(7)}\big)^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)},, (\bar{v}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ :<br><br>By $(\bar{v}_1)^{(7)} > 0, (\bar{v}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(v^{(7)}\big)^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$<br>**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (v_0)^{(7)}$ :- | 362 |

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, $\boldsymbol{if}\ (\nu_0)^{(7)} < (\nu_1)^{(7)}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, $\boldsymbol{if}\ (\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,

and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, $\boldsymbol{if}\ (\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$

and analogously

$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, $\boldsymbol{if}\ (u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, $\boldsymbol{if}\ (u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,

and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}, \boldsymbol{if}\ (\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

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Then the solution of global equations satisfies the inequalities

$$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)} t}$$

where $(p_i)^{(7)}$ is defined by equation

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$$\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)} t}$$

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$$\left(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})}\left[e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)} t}\right] + G_{38}^0 e^{-(S_2)^{(7)} t} \le G_{38}(t) \le \right.$$

$$\left. \frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a_{38}')^{(7)})}\left[e^{(S_1)^{(7)} t} - e^{-(a_{38}')^{(7)} t}\right] + G_{38}^0 e^{-(a_{38}')^{(7)} t}\right)$$

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$$\boxed{T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}}$$

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$$\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}$$

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$$\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b_{38}')^{(7)})}\left[e^{(R_1)^{(7)} t} - e^{-(b_{38}')^{(7)} t}\right] + T_{38}^0 e^{-(b_{38}')^{(7)} t} \le T_{38}(t) \le$$

$$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)} + (r_{36})^{(7)} + (R_2)^{(7)})}\left[e^{((R_1)^{(7)} + (r_{36})^{(7)})t} - e^{-(R_2)^{(7)} t}\right] + T_{38}^0 e^{-(R_2)^{(7)} t}$$

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| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a'_{36})^{(7)}$<br><br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br><br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b'_{36})^{(7)}$<br><br>$(R_2)^{(7)} = (b'_{38})^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br><br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br><br>$-(\sigma_2)^{(8)} \le -(a'_{40})^{(8)} + (a'_{41})^{(8)} - (a''_{40})^{(8)}(T_{41}, t) + (a''_{41})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br><br>$-(\tau_2)^{(8)} \le -(b'_{40})^{(8)} + (b'_{41})^{(8)} - (b''_{40})^{(8)}\big((G_{43}), t\big) - (b''_{41})^{(8)}\big((G_{43}), t\big) \le -(\tau_1)^{(8)}$ | 371 |
| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ :<br><br>By $(v_1)^{(8)} > 0, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$<br>and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | 372 |
| **Definition of** $(\bar{v}_1)^{(8)},, (\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :<br><br>By $(\bar{v}_1)^{(8)} > 0, (\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0, (\bar{u}_2)^{(8)} < 0$ the<br><br>roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$<br><br>and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$<br><br>**Definition of** $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :-<br><br>If we define $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by<br><br>$(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)}$, ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$<br><br>$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}$, ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)}$, | |

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| and $\boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br><br>$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (v_0)^{(8)},\ \boldsymbol{if}\ (\bar{v}_1)^{(8)} < (v_0)^{(8)}$ | |
| and analogously<br><br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},\ \boldsymbol{if}\ (u_0)^{(8)} < (u_1)^{(8)}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}, \boldsymbol{if}\ (u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}, \boldsymbol{if}\ (\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)} t}$<br><br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \le G_{42}(t) \le \right.$<br>$\left.\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)} t} - e^{-(a'_{42})^{(8)} t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)} t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b'_{42})^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b'_{42})^{(8)} t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)} t} \le T_{42}(t) \le$<br><br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br><br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br><br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$ | 381 |

$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$

$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$

**Behavior of the solutions of equation 37 to 92** 382

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$

$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}\big((G_{47}), t\big) - (b''_{45})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$

**Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :

By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(9)}, , (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{\nu}_1)^{(9)} > 0$ , $(\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}$, ***if*** $(\nu_0)^{(9)} < (\nu_1)^{(9)}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}$, ***if*** $(\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T^0_{44}}{T^0_{45}}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^{0}e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^{0}e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^{0}e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}}G_{44}^{0}e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^{0}e^{-(S_2)^{(9)}t} \le G_{46}(t) \le \right.$$

$$\left.\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}^{'})^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}^{'})^{(9)}t}]+\ G_{46}^{0}e^{-(a_{46}^{'})^{(9)}t}\right)$$

$$\boxed{T_{44}^{0}e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^{0}e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}}T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^{0}}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}^{'})^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}^{'})^{(9)}t}\right]+T_{46}^{0}e^{-(b_{46}^{'})^{(9)}t} \le T_{46}(t) \le$$

$$\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a_{44}^{'})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b_{44}^{'})^{(9)}$

$(R_2)^{(9)} = (b_{46}^{'})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a_{13}^{'})^{(1)} - (a_{14}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14},t)\right) - (a_{14}^{''})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}(v^{(1)})^{2} + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}(v^{(1)})^{2} + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

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For $0 < \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (\nu_1)^{(1)} < (\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t) \geq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(\nu_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\nu_1)^{(1)}$

In the same manner , we get 384

$$\nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{\nu}_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\bar{\nu}_1)^{(1)}$

If $0 < (\nu_1)^{(1)} < (\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{\nu}_1)^{(1)}$ we find like in the previous case, 385

$$(\nu_1)^{(1)} \leq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}} \leq \nu^{(1)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (C)^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$ if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \le \frac{d\nu^{(2)}}{dt} \le -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \ge \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \le \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \le \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \le \nu^{(2)}(t) \le$$

| | |
|---|---|
| $\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$ | |
| If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br><br>$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :-<br><br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right) - (a_{21}^{''})^{(3)}(T_{21},t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| | 400 |

From which one obtains

For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

In the same manner , we get

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$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

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If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case,

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

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If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$ if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

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In the same manner , we get

$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

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$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

In the same manner , we get

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$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}'')^{(5)} = (b_{29}'')^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then

$(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a'_{32})^{(6)} - (a'_{33})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right) - (a''_{33})^{(6)}(T_{33},t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \le \frac{d\nu^{(6)}}{dt} \le -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \ge \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \le \nu^{(6)}(t) \le (\nu_1)^{(6)}$

In the same manner , we get

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$$\nu^{(6)}(t) \le \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \le \nu^{(6)}(t) \le (\bar{\nu}_1)^{(6)}$

If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

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$$(\nu_1)^{(6)} \le \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}} \le \nu^{(6)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \le (\bar{\nu}_1)^{(6)}$$

If $0 < (\nu_1)^{(6)} \le (\bar{\nu}_1)^{(6)} \le \boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$ , we obtain

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$$(\nu_1)^{(6)} \le \nu^{(6)}(t) \le \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \le (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}$, $\boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

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$$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$

If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case, 418

$$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$$

If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain 419

$$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(C)^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :- 420

$$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}, \quad \boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}'')^{(7)} = (a_{37}'')^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}'')^{(7)} = (b_{37}'')^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a'_{40})^{(8)} - (a'_{41})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right) - (a''_{41})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \le \frac{dv^{(8)}}{dt} \le -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(8)}, (v_0)^{(8)}$ :-

For $0 < \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}} < (v_1)^{(8)} < (\bar{v}_1)^{(8)}$

$$v^{(8)}(t) \ge \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(v_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (v_2)^{(8)}}}$$

it follows $(v_0)^{(8)} \le v^{(8)}(t) \le (v_1)^{(8)}$

In the same manner , we get

$$v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{v}_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (\bar{v}_2)^{(8)}}}$$

From which we deduce $(v_0)^{(8)} \le v^{(8)}(t) \le (\bar{v}_8)^{(8)}$

If $0 < (v_1)^{(8)} < (v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}} < (\bar{v}_1)^{(8)}$ we find like in the previous case,

$$(v_1)^{(8)} \le \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}} \le v^{(8)}(t) \le$$

$$\frac{(\bar{v}_1)^{(8)} + (C)^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (\bar{v}_1)^{(8)}$$

If $0 < (v_1)^{(8)} \le (\bar{v}_1)^{(8)} \le \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$ , we obtain

$$(v_1)^{(8)} \le v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (v_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

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**Definition of** $v^{(8)}(t)$ :-

$(m_2)^{(8)} \leq v^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{v^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $v^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)}-(\nu_0)^{(9)}}{(\nu_0)^{(9)}-(\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \le \frac{(\nu_1)^{(9)}+(C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}} \le \nu^{(9)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \le (\bar{\nu}_1)^{(9)} \le \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(\nu_1)^{(9)} \le \nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \le \nu^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \le u^{(9)}(t) \le (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)}G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

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| $(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,<br>$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$<br>$with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$ | 433 |

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| $(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$ | 436 A |

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| $(a'_{44})^{(9)}(a'_{45})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a'_{45})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b'_{44})^{(9)}(r_{45})^{(9)} - (b'_{45})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b'_{13})^{(1)} - (b''_{13})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b'_{14})^{(1)} - (b''_{14})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b'_{15})^{(1)} - (b''_{15})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b'_{16})^{(2)} - (b''_{16})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b'_{17})^{(2)} - (b''_{17})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b'_{18})^{(2)} - (b''_{18})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a'_{20})^{(3)} + (a''_{20})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a'_{21})^{(3)} + (a''_{21})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a'_{22})^{(3)} + (a''_{22})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b'_{20})^{(3)} - (b''_{20})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b'_{21})^{(3)} - (b''_{21})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b'_{22})^{(3)} - (b''_{22})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) +$ | 485 |

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| $(a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | |
| **Proof:**<br><br>(v) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |

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| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492<br>A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations | 498 |

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| $G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |
| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 501 |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br><br>$\left[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G) + (b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)\right]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that $\varphi$ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that | 502 |

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| there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*})=0$ | |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br>$\varphi(G_{19})=(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)}-(b_{16})^{(2)}(b_{17})^{(2)}-$<br>$\left[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19})+(b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})\right]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19})=0$ | 503 |
| Where in $(G_{19})(G_{16},G_{17},G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*})=0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br>$\varphi(G_{23})=(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)}-(b_{20})^{(3)}(b_{21})^{(3)}-$<br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23})+(b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23})=0$<br>Where in $G_{23}(G_{20},G_{21},G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*})=0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br>$\varphi(G_{27})=(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)}-(b_{24})^{(4)}(b_{25})^{(4)}-$<br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27})+(b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27})=0$<br>Where in $(G_{27})(G_{24},G_{25},G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*})=0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31})=(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)}-(b_{28})^{(5)}(b_{29})^{(5)}-$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31})+(b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right]+(b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31})=0$<br>Where in $(G_{31})(G_{28},G_{29},G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*})=0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35})=(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)}-(b_{32})^{(6)}(b_{33})^{(6)}-$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35})+(b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right]+(b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35})=0$<br>Where in $(G_{35})(G_{32},G_{33},G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that | 508 |

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| there exists a unique $G_{33}^{*}$ such that $\varphi(G_{35}{}^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi(G_{39}{}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi(G_{43}{}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0\,, T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0\,, T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |

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| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^*$ given by $\varphi((G_{23})^*) = 0$ , $T_{21}^*$ given by $f(T_{21}^*) = 0$ and<br>$G_{20}^* = \frac{(a_{20})^{(3)} G_{21}^*}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^*)]}$ , $G_{22}^* = \frac{(a_{22})^{(3)} G_{21}^*}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^*)]}$<br>$T_{20}^* = \frac{(b_{20})^{(3)} T_{21}^*}{[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}{}^*)]}$ , $T_{22}^* = \frac{(b_{22})^{(3)} T_{21}^*}{[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}{}^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^*$ given by $\varphi(G_{27}) = 0$ , $T_{25}^*$ given by $f(T_{25}^*) = 0$ and<br>$G_{24}^* = \frac{(a_{24})^{(4)} G_{25}^*}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}^*)]}$ , $G_{26}^* = \frac{(a_{26})^{(4)} G_{25}^*}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}^*)]}$ | 516 |
| $T_{24}^* = \frac{(b_{24})^{(4)} T_{25}^*}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27})^*)]}$ , $T_{26}^* = \frac{(b_{26})^{(4)} T_{25}^*}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^*$ given by $\varphi((G_{31})^*) = 0$ , $T_{29}^*$ given by $f(T_{29}^*) = 0$ and<br>$G_{28}^* = \frac{(a_{28})^{(5)} G_{29}^*}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^*)]}$ , $G_{30}^* = \frac{(a_{30})^{(5)} G_{29}^*}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^*)]}$ | 518 |
| $T_{28}^* = \frac{(b_{28})^{(5)} T_{29}^*}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^*)]}$ , $T_{30}^* = \frac{(b_{30})^{(5)} T_{29}^*}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^*$ given by $\varphi((G_{35})^*) = 0$ , $T_{33}^*$ given by $f(T_{33}^*) = 0$ and<br>$G_{32}^* = \frac{(a_{32})^{(6)} G_{33}^*}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^*)]}$ , $G_{34}^* = \frac{(a_{34})^{(6)} G_{33}^*}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^*)]}$ | 520 |
| $T_{32}^* = \frac{(b_{32})^{(6)} T_{33}^*}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^*)]}$ , $T_{34}^* = \frac{(b_{34})^{(6)} T_{33}^*}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^*$ given by $\varphi((G_{39})^*) = 0$ , $T_{37}^*$ given by $f(T_{37}^*) = 0$ and | 522 |

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| $G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)} (T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)} (T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)} ((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)} ((G_{39})^{*})]}$ | |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)} (T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)} (T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)} ((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)} ((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br><br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br><br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)} (T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)} (T_{45}^{*})]}$<br><br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)} ((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)} ((G_{47})^{*})]}$ | 523 A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:**_Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}} (T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j} (G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)} \mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*} \mathbb{T}_{14}$ | 525 |

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| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^{*}\mathbb{G}_{j}\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^{*}\mathbb{G}_{j}\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^{*}\mathbb{G}_{j}\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 532 |
| $\frac{\partial(a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial(b_i^{''})^{(2)}}{\partial G_j}((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^{*}\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | 540 |

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| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)} \ , \ \frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^*\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^*\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^*\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^*\mathbb{G}_j\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^*\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^*\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ *and* $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)} \ , \ \frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^*) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right)$ | 554 |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^*) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^*\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^*\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^*\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |

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| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}')^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^{*}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}')^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^{*}\mathbb{G}_{j}\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}')^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^{*}\mathbb{G}_{j}\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}')^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^{*}\mathbb{G}_{j}\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i'')^{(7)}$ *and* $(b_i'')^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{37}'')^{(7)}}{\partial T_{37}}(T_{37}^{*}) = (q_{37})^{(7)}$ , $\frac{\partial(b_i'')^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}')^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^{*}\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}')^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^{*}\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}')^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^{*}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}')^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^{*}\mathbb{G}_{j}\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}')^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^{*}\mathbb{G}_{j}\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}')^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^{*}\mathbb{G}_{j}\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i'')^{(8)}$ *and* $(b_i'')^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. | |

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| Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \ , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}((G_{43})^*) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^*\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^*\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b_{41}^{'})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T_{41}^*\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b_{42}^{'})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T_{42}^*\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS** **Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. **Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |

| | |
|---|---|
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$ $\left[\left(\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\right]$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\right)$ $+\left(\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\right)$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((b'_{13})^{(1)} + (b'_{14})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $+\left(\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$ $+\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G^*_{13}\right)$ $\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T^*_{14} + (b_{14})^{(1)}s_{(13),(15)}T^*_{13}\right)\} = 0$ $+$ $\left((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)}\right)\{\left((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)}\right)$ $\left[\left(\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}^*_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^*_{16}\right)\right]$ $\left(\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^*_{17}\right)$ $+\left(\left((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}^*_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^*_{17}\right)$ $\left(\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^*_{16}\right)$ $\left(\left((\lambda)^{(2)}\right)^2 + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ $\left(\left((\lambda)^{(2)}\right)^2 + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ | |

$$+\left(\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\right)\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left(\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\right)$$

$$+\left(\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((b_{20}')^{(3)}+(b_{21}')^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}')^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}')^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}')^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}')^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}')^{(4)}+(a_{25}')^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((b_{24}')^{(4)}+(b_{25}')^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\}\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^2+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G^*_{33}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G^*_{32}\right)$$

$$\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T^*_{33}+(b_{33})^{(6)}s_{(32),(34)}T^*_{32}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b'_{38})^{(7)}-(r_{38})^{(7)}\}\{\left((\lambda)^{(7)}+(a'_{38})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(q_{36})^{(7)}G^*_{36}\right)\right]$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T^*_{37}+(b_{37})^{(7)}s_{(36),(37)}T^*_{37}\Big)$$

$$+\left(\left((\lambda)^{(7)}+(a'_{37})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G^*_{36}+(a_{36})^{(7)}(q_{37})^{(7)}G^*_{37}\right)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T^*_{37}+(b_{37})^{(7)}s_{(36),(36)}T^*_{36}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left(\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^*_{36}\right)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^*_{37}+(b_{37})^{(7)}s_{(36),(38)}T^*_{36}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^*_{40}\right)\right]$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^*_{41}+(b_{41})^{(8)}s_{(40),(41)}T^*_{41}\Big)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^*_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^*_{41}\right)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^*_{41}+(b_{41})^{(8)}s_{(40),(40)}T^*_{40}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^*_{40}\right)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^*_{41}+(b_{41})^{(8)}s_{(40),(42)}T^*_{40}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^*_{44}\right)\right]$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^*_{45}+(b_{45})^{(9)}s_{(44),(45)}T^*_{45}\Big)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^*_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45}+(b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left(\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45}+(b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

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# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i(x)\, xi$ where $x_i \in R^n_i$, $x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i(x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i(\mu)$ of the $A_i(x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k \tag{22}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to

replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 22. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon\ C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \ i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0}$

$\overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow *x.t restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set *K* is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** *x.t* is defined for all $t \geq 0$

**A4:** *x.t* is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\mathcal{A}(\tilde{A}), C \backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) In\ ||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population *i* is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^{t}_{0} (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\to\infty}{limsup} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v = au_i(x) + w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |B_i(t,x)u_i(x)|(a + \alpha + exp(-\beta t)|w|)$. Hence, $\underset{t\to\infty}{limsup}\frac{1}{t} In\, B_i(t,x)\, v| \leq$

$$\underset{t\to\infty}{limsup}\frac{1}{t} In\, B_i(t,x)\, u_i(x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \underset{t\to\infty}{\text{limsup}}\frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \underset{t\to\infty}{\text{limsup}}\frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \underset{t\to\infty}{\text{limsup}}\frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\, |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\, u_i(x)| = In\, |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\, u_i(x.t)||\, B_i(t,x)\, u_i(x)|$$

$$= \text{In } b_i(s,t.x)\, b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt}b_i(t,x) = \lim_{s\to 0}\frac{bi(t+s,x - bi(t,x)}{S-}$$

$$= \lim_{s\to 0}\frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

$$=$$

$$=(A_i(x.t)u_i(x.s), u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s), u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } | (x.t)_i |$$

$$\leq \limsup_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x) =$ Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ *- almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of

irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x)u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$ where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ and $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as $n\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n$ − perturbations of (2), then the weak* limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak* compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak* limit points of $\mu_n$. Let $\mu$ be such a weak* limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\le \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x)u_i^n(x), u_i^n(x)\rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)|d\mu(x) + |\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\,d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\,ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i\,dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\,d\,v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak$^*$ limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^\infty$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \{ \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i \epsilon I}{if\ i \notin I}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that \quad \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t$ −invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently *K* is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s =$

$s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_s$. If $r(x) = s(x), then\ x \in M_r$. Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume $x \mapsto A(x)$ is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists $j \in \{1,\ldots,k\}$ such that $M_i$ ----- $\{x \in C : x_l \gg 0 \text{ for } l \neq j\}$ Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_i M_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, *$\lambda_i(\mu) > 0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, ..., x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda(0)$, which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i(0) > 0$ for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the

system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j \ (x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1{}^1, \ldots, E_1{}^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i{}_2(E_1{}^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j \ x_2^j \ - \alpha^j \ x_3^j) + \sum d_1^{jk} \ x_1^k \ - e_1^j \ x_1^j$$

$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j\, x_1^j - x_2^j - \beta^j\, x_3^j) + \sum d_2^{jk}\, x_2^k - e_2^j\, x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j\, x_1^j - \alpha^j x_2^j - \beta^j\, x_3^j) + \sum d_3^{jk}\, x_3^k - e_3^j\, x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2$ $(E_1)$and $\lambda_3$ $(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i\,(0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\Big(r_i - \sum_s a_{is} x_s^j\Big) + \sum_l d_i^{jl}\, x_i^l - e_i^j x_i^j, \qquad i = 1,\ldots,k;\ j = 1,\ldots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i\, a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i\, x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \qquad i = 1, \ldots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \ldots, k$$

It has been shown [12,ch.16.1,16.2] That for $k = 3$,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is

the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1-y) - b(N)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*′ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\curlywedge_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\curlywedge_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\curlywedge_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + K p\, x_i x_i - 1}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i)-1 & \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions

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# Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a

qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and

orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.64) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.65) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.66) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{'} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$ (1.4)

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1, \varphi_2, f_1, f_2, b_1, b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for

each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as* $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 22. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'}\ - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1 / \partial P_1, \partial f_1 / \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2 / \partial P_1, \partial f_2 / \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, R^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, R^-),$$

$A_3$. $\varphi_i\ (\cdot\ , s) \in L^2\ (R^+, R^+)\ \cap C(R^+, R^+)$ for each $s \in\ [-\tau, 0]$,

$i, = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[-\int_0^\infty f_i(a + s, P(t + S))ds\right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[-\int_o^x f_1(a + s, p(t + s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0)\exp\left[-\int_o^t f_1(a - t + s, p_1(s))\, ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

*(2.3)* $p_i(a,t) = B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i and\ L_i\ (i = 1,2)$ as follows:

$$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a,t,\ p_i(t)) = \exp[-\int_o^t f_1(a+s,\ p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

$$(2.6)\ p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da + \int_o^\infty \varphi_i(a, 0)\ L_i(a,t,p_i(t)da,$$

$$t > 0, i = 1,2,$$

$$(2.7)\ p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$$

$$(2.8)\ \beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.\left(a,t,P_i(t)\right)da$$

$$+\int_0^\infty b_i\left(a+t, P_i(t)\right)\varphi_i(a,0)L_i\left(a,t,P_i(t)\right)da$$

$$i = 1, 2, t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t) \phi(0) d_i^*] t \quad for \ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T], R^+), \quad B_i \in C([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in C([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau, T]$. Then for any

$(x, y) \in C^+[-\tau, T] \times C^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau, T], \quad C_T^2 = C^+[-\tau, T] \times C^+[-\tau, T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for $i = 1,2$.

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \pi C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \quad P_t = \ (P_{1t}, P_{2t}) \in C^2_{t*} \mid \|P_t - \Phi_t\|_{t*} \le r^*$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)\, da \ for\ s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)\, da \ for\ s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Phi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^* \right] \quad t^*$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \le \left| \int_0^t B_1^{(1)}(a) \left\{ M_1(a, t, P_t^{(1)}) - M_1(a, t, P_t^{(2)}) \right\} da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a, t, P_t^{(2)}) da \right|$$

$$+ \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \; for \; x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \| ds\right) da$$

$$\leq d_1 \beta_1^* \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o)\left[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds\right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \| t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\text{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right]\right] d\sigma$$

$$+ \int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^{\infty} \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$= R_1 + R_2 + R_3 + R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a + \sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \, /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \text{ for } a \, \varepsilon [o, t^*]$$

And hence we have

$$|R_3(a) + R_4(a)| \leq \beta_1 \Phi_1(0) || P_t^{(1)} - P_t^{(2)} ||_{t^*} + \Phi_1(0) \beta_1 d_1 || P_t^{(1)} - P_t^{(2)} ||_{t^*}^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a - 0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) e^{\delta 1^a} \beta_1 \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) e^{\delta 1^a} \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| B_1^{*d}\sigma$$

$$+ \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \left\{ B_1 + B_2 d_1 a + B_1^* \left( B_2 a + d_1 a^2/2 \right) e^{\delta 1^a} \right\}$$

$$\leq \beta_1^* \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| do + \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \to \beta_1$ as $a \to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{g_1(a) + 0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))\text{-}b_1(a\text{-}\sigma, P_t^{(1)}(a))\, M_1(\sigma, a, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + /\!/ \Phi_t /\!/_{t*}\right) /\!/ \; P_t^{(1)}\text{-}P_t^{(2)} /\!/_{t*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right) /\!/ \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^{*})$$

As $t^{*} \to o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^{*}, P_t^{*}$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp\, P_2^{*}(a)}{da} + f_i(\mathrm{a}, P_1^{*}, P_2^{*})\, P_i^{*}(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^{*} = \int_o^{\infty} P_i^{*}(a)da, i = 1,\ 2,$$

$$(3,3)\ P_i^{*} = \int_o^{\infty} b_i(a,\ P_1^{*},\ P_2^{*}) P_i^{*}(a)da,\ i = 1.\ \ 2$$

All solutions of (3.1)are of the from

$$(3.4)\ P_i^{*}(a) = P_i^{*}(o) \exp\left[ - \int_o^{\infty} f_i(s, P_1^{*}, P_2^{*})ds \right], i = 1, \quad 2, a \geq 0,$$

Which together with (3.3) lead to

$$1=\int_o^{\infty} b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^{\infty} b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $$F_{i.}\ (P_1,P_2) = \int_o^{\infty} b_i(a,P_1,P_2)\exp\left[-\int_o^{\infty} f_i\,(s,P_1,P_2)ds\right]da,$$

$$i = 1,\ 2, \quad P_1 \geq 0, \quad P_2 \geq 0,$$

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_5$. *there exists a positive constant c for which*

$$F_1(\mathrm{C},0) < F_{2.}(\mathrm{C},0),$$

(3.8) $$F_1(0.\mathrm{C}) < F_{2.}(0,\mathrm{C}),$$

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $$F_1(P_1^*,P_2^*) = 1 = F_2(P_1^*,P_2^*)$$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $$P_2^*(a) = [P_2^*\exp[-f_i\,(s,P_1^*,P_2^*)ds])/(\int_o^{\infty}\exp[-\int_o^a f_i\,(s,P_1^*,P_2^*)ds]da),$$

$$i = 1,\ 2, \quad a \geq 0,$$

Proof Consider the elements of the surfaces defined by

$$\mathrm{Z}= F_1(P_1,P_2), z = F_2(P_1,P_2),\ P_1 \geq 0, P_2 \geq 0,$$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i/\partial P_j < 0\ and$ , by ($A_2$), $\partial b_i/\partial P_j \leq 0$ for all

$$P_1 \geq 0, P_2 \geq 0, \quad i = 1,\ 2,$$

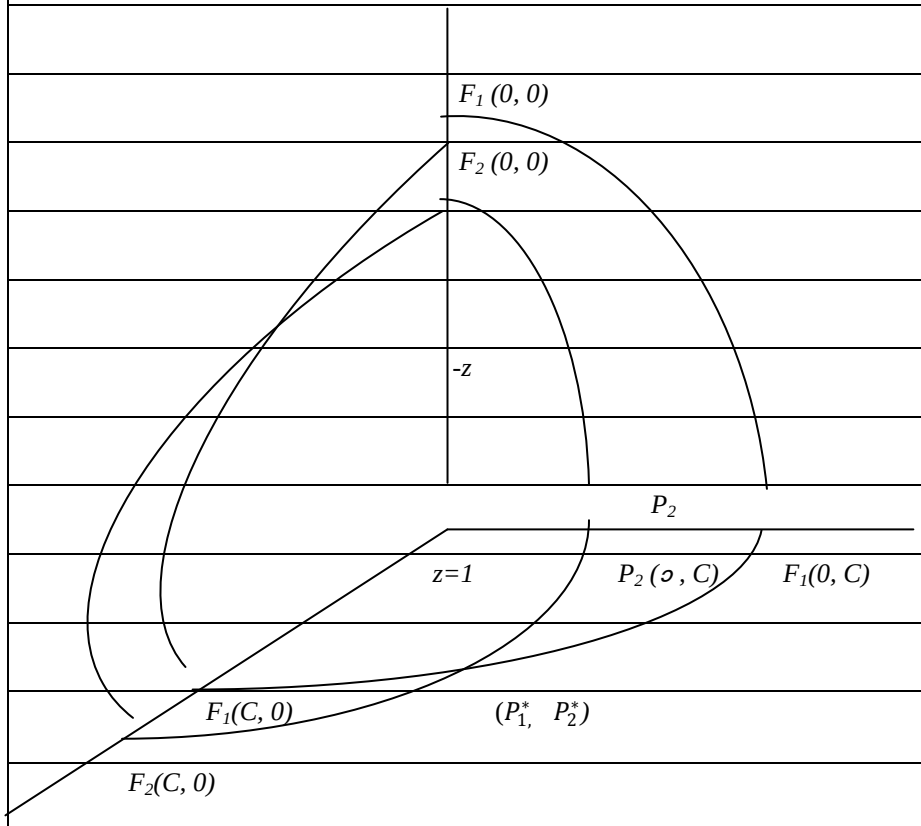
$j = 1,\ 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= 1, $P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$$F_1(P_1, P_2 = 1 \; and \; F_2(P_1, P_2) = 1, \; F_1(P_1, P_2 \geq 0,$$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility 1< $F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p_1^*, p_2^*)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p_1^*, p_2^*$, we let

$$(3.11) \quad \begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \; i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*}\, p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square 1}^{2} \frac{\partial b_i}{\partial p_{\square}^*}\, p_{i\square}(t)\right\} da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_{\square}(a)\exp[\lambda t], \quad \square(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty \square(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d\square}{da} + \lambda\square = -f_i\left(a, p_1^*, p_2^*\right)\square(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*}\, p_{i\square}^*\right\},$$

(3.15)

$$\square(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\square(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*}\, p_{i\square}^*\right\} da,$$

Where $i = 1, 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \quad p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\Phi_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\Phi_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\Phi_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\Phi_i(a) = \{\Phi_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_{ij}^*\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\Phi_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_{ij}^*\right\}da\right.$$
$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\, e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_{ij}^*\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)\, e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_{ij}^*\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)}\int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}da, \quad i = 1, 2,$$

Now supplying $\Phi_1(a)$ and $\Phi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^*\exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^*\exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^*\exp(-\lambda\tau) + (\lambda - A_{22})p_2^* = g_{21}(\lambda)p_1^*\exp(-\lambda\tau) + g_{22}(\lambda)p_2^*$$

Where

(3.22) $$A_{ij} = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_j^*} - \frac{\partial f_i}{\partial p_j^*}\right]da,$$

(3.23) $$g_{ij}(\lambda) = \left[p_i^*(0)c_i\right]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\,da\right\}\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_j^*} - b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)\right]da\right\}$$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)da,\quad i, j = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i\left(a, p_1^*, p_2^*\right) - f_i\left(a, p_1^*, p_2^*\right),\quad i, = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et$ $$\begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})\,e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda,\tau)$$

Where $$S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda),$$

$$S_2(\lambda,\tau) = \left[A_{21}g_{12}(\lambda) + A_{12}g_{21}(\lambda) + g_{12}(\lambda)g_{21}(\lambda)\right]\exp[-2\lambda\tau]$$

$$-A_{11}g_{22}(\lambda) - A_{22}g_{11}(\lambda) - g_{11}(\lambda)g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6$. $0 < \beta_i^* < d_i^*$ $(i, = 1, 2.)$ $(see(A_1)\ and\ (A_2))$;

$A_7$. $$\{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\},$$

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = |A_{11}g_{12}(0)| + |A_{12}g_{21}(0)| + |g_{12}(0)g_{21}(0)|$$

$$+ |A_{11}g_{22}(0)| + |A_{22}g_{11}(0)| + |g_{11}(0)g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i(a, t) = \phi_i(a) \exp[\lambda t] \to 0$ as $t \to \infty$ for $i = 1, 2$ and $a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda$ $\mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$

$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \left|\right.$

$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$

$\geq [(|\lambda| - P)^2 + (q)]^2$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \{4 | (A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| | - | A_{11} + A_{22} + \| S_1(0) \| |^2\}/2,$

It will now follow, from (A) and (3.26), $F_1(\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder

concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power

obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy

withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation

from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## SECTION A(24)

## Yang-Mills Theory And Gravity

### INTRODUCTION—VARIABLES USED

**Nuclear Physics B Volume 530, Issues 1–2, 12 October 1998, Pages 401–456 On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences Z. Berna, 1, L. Dixonb, 2, D.C. Dunbarc, 3, M. Perelsteinb, 2, J.S. Rozowskya, 1**

**(1)** String theory implies (eb) that field theories containing (e) gravity are in a certain sense ‘products’ of (e) gauge theories.
**(2)** Authors make this product structure explicit up to (e) two loops for the relatively simple case of (e) N = 8 supergravity four-point amplitudes, demonstrating that they are (=) ‘squares’ of N = 4 super-Yang-Mills amplitudes.
**(3)** This is accomplished by obtaining an explicit expression for (e) the D-dimensional two-loop contribution to (e) the four-particle S-matrix for (e) N = 8 supergravity, which authors compare to (e&eb) the corresponding N = 4 Yang-Mills result.
**(4)** From these expressions authors also obtain (eb) the two-loop ultraviolet divergences in dimensions D = 7 through D = 11.
**(5)** The analysis relies on (e) the unitarity cuts of the two theories, many of which can be (=) recycled from (e) a one-loop computation.
**(6)** The two-particle cuts, which may be (=) iterated to all loop orders, suggest (eb) that squaring relations between the two theories exist at (eb) any loop order.
**(7)** The loop-momentum power-counting implied by (e) proposed two-particle cut analysis indicates (eb) that in four dimensions the first four-point divergence in (eb) N = 8 supergravity should appear at (eb) five loops, contrary to (e&eb) the earlier expectation, based on (e) superspace arguments, of

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| (e) a three-loop counterterm. PACS 04.50.+h; 04.65.+e; 04.60.-m Keywords Unitarity; Cutting; Ultraviolet divergences; Supersymmetry; Supergravity; Two-loops; Multi-loops | |
| **NOTATION** | |
| **Module One**<br>**Abelian Tensor Hierarchy in 4D, N=1 Superspace Katrin Becker, Melanie Becker, William D Linch III, Daniel Robbins (Submitted on 12 Jan 2016)**<br>**(1)** When specialized to the case in which the tensors arise from (e) a higher-dimensional theory, the invariants may be interpreted as (=) higher-dimensional Chern-Simons forms reduced to (e&eb) four dimensions.<br>**(2)** As an application of the formalism, authors construct the eleven-dimensional Chern-Simons form in terms of (e&eb) four-dimensional, N=1 superfields. Subjects: High Energy Physics - Theory (hep-th) DOI: 10.1007/JHEP03 (2016)052 Report number: MI-TH-1605 Cite as: arXiv:1601.03066 [hep-th] (or arXiv:1601.03066v1 [hep-th] for this version)<br>When specialized to the case in which the tensors arise from a higher-dimensional theory, the invariants may be interpreted as (=) higher-dimensional Chern-Simons forms reduced to (e&eb) four dimensions | |
| $G_{13}$ : Category one of specialized to the case in which the tensors arise from a higher-dimensional theory<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of invariants may be interpreted as (=) higher-dimensional Chern-Simons forms reduced to (e&eb) four dimensions<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>When specialized to the case in which the tensors arise from a higher-dimensional theory, the invariants may be interpreted as (=) higher-dimensional Chern-Simons forms reduced to (e&eb) four dimensions | |
| $G_{16}$ : Category one of specialized to the case in which the tensors arise from a higher-dimensional theory, the invariants may be interpreted<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of higher-dimensional Chern-Simons forms reduced to (e&eb) four dimensions<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>When specialized to the case in which the tensors arise from a higher-dimensional theory, the invariants may | |

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| be interpreted as higher-dimensional Chern-Simons forms reduced to (e&eb) four dimensions | |
| $G_{20}$ : Category one of **specialized to the case in which the tensors arise from a higher-dimensional theory, the invariants may be interpreted as higher-dimensional Chern-Simons forms reduced;** four dimensions<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of four dimensions; **specialized to the case in which the tensors arise from a higher-dimensional theory, the invariants may be interpreted as higher-dimensional Chern-Simons forms reduced**<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>As an application of the formalism, authors construct the eleven-dimensional Chern-Simons form in terms of (e&eb) four-dimensional, N=1 superfields.<br>Subjects: High Energy Physics - Theory (hep-th) DOI: 10.1007/JHEP03 (2016)052 Report number: MI-TH-1605 Cite as: arXiv:1601.03066 [hep-th] (or arXiv:1601.03066v1 [hep-th] for this version) | |
| $G_{24}$ : Category one of **eleven-dimensional Chern-Simons form;** four-dimensional, N=1 superfields<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of four-dimensional, N=1 superfields **;eleven-dimensional Chern-Simons form**<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>String theory implies (eb) that field theories containing (e) gravity are in a certain sense 'products' of (e) gauge theories | |
| $G_{28}$ : Category one of String theory<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of field theories containing (e) gravity are in a certain sense 'products' of (e) gauge theories<br>$T_{29}$ : Category two of SAS | |

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| $T_{30}$ : Category three of SAS | |
| **Module six**<br><br>String theory implies that field theories containing (e) gravity are in a certain sense 'products' of (e) gauge theories | |
| $G_{32}$ : Category one of gravity are in a certain sense 'products' of (e) gauge theories<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of String theory implies that field theories containing<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>String theory implies that field theories containing gravity are in a certain sense 'products' of (e) gauge theories | |
| $G_{36}$ : Category one of gauge theories<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of String theory implies that field theories containing gravity are in a certain sense 'products'<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Authors make this product structure explicit up to two loops for the relatively simple case of (e) N = 8 supergravity four-point amplitudes, demonstrating that they are (=) 'squares' of N = 4 super-Yang-Mills amplitudes. | |
| $G_{40}$ : Category one of N = 8 supergravity four-point amplitudes, demonstrating that they are (=) 'squares' of N = 4 super-Yang-Mills amplitudes.<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |

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| $T_{40}$ : Category one of product structure explicit up to two loops for the relatively simple case<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Authors make this product structure explicit up to two loops for the relatively simple case of N = 8 supergravity four-point amplitudes, demonstrating that they are 'squares' of N = 4 super-Yang-Mills amplitudes. | |
| $G_{44}$ : Category one of product structure explicit up to two loops for the relatively simple case of N = 8 supergravity four-point amplitudes<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of squares' of N = 4 super-Yang-Mills amplitudes.<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION A(24)-2**<br><br>**D-Dimensional Two-Loop Contribution To The Four-Particle S-Matrix For N = 8 Supergravity** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Nuclear Physics B Volume 530, Issues 1–2, 12 October 1998, Pages 401–456 On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences Z. Berna, 1, L. Dixonb, 2, D.C. Dunbarc, 3, M. Perelsteinb, 2, J.S. Rozowskya, 1**<br><br>**(1)** product structure explicit up to two loops for the relatively simple case of N = 8 supergravity four-point amplitudes, demonstrating that they are 'squares' of N = 4 super-Yang-Mills amplitudes is accomplished by obtaining an explicit expression for (e) the D-dimensional two-loop contribution to (e) the four-particle S-matrix for (e) N = 8 supergravity, which authors compare to (e&eb) the corresponding N = 4 Yang-Mills result.<br>**(2)** From these expressions authors also obtain (eb) the two-loop ultraviolet divergences in dimensions D = 7 through D = 11.<br>**(3)** The analysis relies on (e) the unitarity cuts of the two theories, many of which can be (=) recycled from (e) a one-loop computation.<br>**(4)** The two-particle cuts, which may be (=) iterated to all loop orders, suggest (eb) that squaring relations between the two theories exist at (eb) any loop order.<br>**(5)** The loop-momentum power-counting implied by (e) proposed two-particle cut analysis indicates | |

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| (eb) that in four dimensions the first four-point divergence in (eb) N = 8 supergravity should appear at (eb) five loops, contrary to (e&eb) the earlier expectation, based on (e) superspace arguments, of (e) a three-loop counterterm. PACS 04.50.+h; 04.65.+e; 04.60.-m Keywords Unitarity; Cutting; Ultraviolet divergences; Supersymmetry; Supergravity; Two-loops; Multi-loops | |
| **NOTATION** | |
| **Module One**<br>product structure explicit up to two loops for the relatively simple case of N = 8 supergravity four-point amplitudes, demonstrating that they are 'squares' of N = 4 super-Yang-Mills amplitudes is accomplished by obtaining an explicit expression for (e) the D-dimensional two-loop contribution to (e) the four-particle S-matrix for (e) N = 8 supergravity, which authors compare to (e&eb) the corresponding N = 4 Yang-Mills result | |
| $G_{13}$ : Category one of D-dimensional two-loop contribution to (e) the four-particle S-matrix for (e) N = 8 supergravity, which authors compare to (e&eb) the corresponding N = 4 Yang-Mills result<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of product structure explicit up to two loops for the relatively simple case of N = 8 supergravity four-point amplitudes, demonstrating that they are 'squares' of N = 4 super-Yang-Mills amplitudes is accomplished by obtaining an explicit expression<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>product structure explicit up to two loops for the relatively simple case of N = 8 supergravity four-point amplitudes, demonstrating that they are 'squares' of N = 4 super-Yang-Mills amplitudes is accomplished by obtaining an explicit expression for the D-dimensional two-loop contribution to (e) the four-particle S-matrix for (e) N = 8 supergravity, which authors compare to (e&eb) the corresponding N = 4 Yang-Mills result | |
| $G_{16}$ : Category one of four-particle S-matrix for (e) N = 8 supergravity, which authors compare to (e&eb) the corresponding N = 4 Yang-Mills result<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of product structure explicit up to two loops for the relatively simple case of N = 8 supergravity four-point amplitudes, demonstrating that they are 'squares' of N = 4 super-Yang-Mills amplitudes is accomplished by obtaining an explicit expression for the D-dimensional two-loop contribution<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three** | |

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| product structure explicit up to two loops for the relatively simple case of N = 8 supergravity four-point amplitudes, demonstrating that they are 'squares' of N = 4 super-Yang-Mills amplitudes is accomplished by obtaining an explicit expression for the D-dimensional two-loop contribution to (e) the four-particle S-matrix for (e) N = 8 supergravity, which authors compare to (e&eb) the corresponding N = 4 Yang-Mills result | |
| $G_{20}$ : Category one of four-particle S-matrix for (e) N = 8 supergravity, which authors compare to (e&eb) the corresponding N = 4 Yang-Mills result<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of product structure explicit up to two loops for the relatively simple case of N = 8 supergravity four-point amplitudes, demonstrating that they are 'squares' of N = 4 super-Yang-Mills amplitudes is accomplished by obtaining an explicit expression for the D-dimensional two-loop contribution<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>product structure explicit up to two loops for the relatively simple case of N = 8 supergravity four-point amplitudes, demonstrating that they are 'squares' of N = 4 super-Yang-Mills amplitudes is accomplished by obtaining an explicit expression for the D-dimensional two-loop contribution to the four-particle S-matrix for (e) N = 8 supergravity, which authors compare to (e&eb) the corresponding N = 4 Yang-Mills result | |
| $G_{24}$ : Category one of N = 8 supergravity, which authors compare to (e&eb) the corresponding N = 4 Yang-Mills result<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of product structure explicit up to two loops for the relatively simple case of N = 8 supergravity four-point amplitudes, demonstrating that they are 'squares' of N = 4 super-Yang-Mills amplitudes is accomplished by obtaining an explicit expression for the D-dimensional two-loop contribution to the four-particle S-matrix<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>product structure explicit up to two loops for the relatively simple case of N = 8 supergravity four-point amplitudes, demonstrating that they are 'squares' of N = 4 super-Yang-Mills amplitudes is accomplished by obtaining an explicit expression for the D-dimensional two-loop contribution to the four-particle S-matrix for N = 8 supergravity, which authors compare to (e&eb) the corresponding N = 4 Yang-Mills result | |
| $G_{28}$ : Category one of **product structure explicit up to two loops for the relatively simple case of N = 8 supergravity four-point amplitudes, demonstrating that they are 'squares' of N = 4 super-Yang-Mills amplitudes is accomplished by obtaining an explicit expression for the D-dimensional two-loop** | |

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| **contribution to the four-particle S-matrix for N = 8 supergravity;** corresponding N = 4 Yang-Mills result<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of corresponding N = 4 Yang-Mills result **;product structure explicit up to two loops for the relatively simple case of N = 8 supergravity four-point amplitudes, demonstrating that they are 'squares' of N = 4 super-Yang-Mills amplitudes is accomplished by obtaining an explicit expression for the D-dimensional two-loop contribution to the four-particle S-matrix for N = 8 supergravity**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>From these expressions authors also obtain (eb) the two-loop ultraviolet divergences in dimensions D = 7 through D = 11 | |
| $G_{32}$ : Category one of these expressions<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of two-loop ultraviolet divergences in dimensions D = 7 through D = 11<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>The analysis relies on (e) the unitarity cuts of the two theories, many of which can be (=) recycled from (e) a one-loop computation | |
| $G_{36}$ : Category one of unitarity cuts of the two theories, many of which can be (=) recycled from (e) a one-loop computation<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of analysis<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |

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| **Module eight**<br><br>The analysis relies on the unitarity cuts of the two theories, many of which can be (=) recycled from (e) a one-loop computation | |
| $G_{40}$ : Category one of analysis relies on the unitarity cuts of the two theories<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of recycled from (e) a one-loop computation<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The analysis relies on the unitarity cuts of the two theories, many of which can be recycled from (e) a one-loop computation | |
| $G_{44}$ : Category one of one-loop computation<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of analysis relies on the unitarity cuts of the two theories, many of which can be recycled<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION A(25)**<br><br>**Yang–Mills theory** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Yang–Mills theory from Wikipedia, the free encyclopedia Question dropshade.png unsolved problem in physics:**<br><br>**(1) Yang–Mills theory in the non-perturbative regime**: The equations of Yang–Mills remain unsolved at (eb) energy scales relevant for (e) describing atomic nuclei. How does Yang–Mills theory give rise to (eb) the physics of nuclei and nuclear constituents? Yang–Mills theory is (=) a gauge theory based on (e) the SU (N) group, or more generally any compact, semi-simple Lie group. Yang–Mills theory seeks to describe (eb) the behavior of elementary particles using (e) these | |

non-Abelian Lie groups and is (=) at the core of the unification of (e) the electromagnetic and weak forces (i.e. U (1) × SU (2)) as well as quantum chromodynamics, the theory of the strong force (based on SU(3)). Thus it forms the basis of our understanding of (e) particle physics, the Standard Model. **Yang–Mills theory from Wikipedia, the free encyclopedia Question dropshade.png Unsolved problem in physics**

**History and theoretical description**

(2) In a private correspondence, Wolfgang Pauli formulated in 1953 a six-dimensional theory of Einstein's field equations of general relativity, extending the five-dimensional theory of Kaluza, Klein, Fock and others to (e&eb) a higher-dimensional internal space.[1]
(3) However, there is no evidence that Pauli developed (eb) the Lagrangian of a gauge field or the quantization of it.
(4) Because Pauli found that his theory "leads to (eb) some rather unphysical shadow particles", he refrained from (e) publishing his results formally.[1]
(5) Although Pauli did not publish his six-dimensional theory, he gave two talks about it in Zürich.[2] Recent research shows that an extended Kaluza–Klein theory is in general not (e) equivalent to Yang–Mills theory, as (e) the former contains additional terms.[3]
(6) In early 1954, Chen Ning Yang and Robert Mills [4] extended the concept of gauge theory for (e) abelian groups, e.g. quantum electrodynamics, to (e&eb) nonabelian groups to provide (eb) an explanation for (e) strong interactions.
**(7)** The idea by Yang–Mills was criticized by Pauli,[5] as the quanta of the Yang–Mills field must be (=) massless in order to (e) maintain gauge invariance

**NOTATION**

**Module One**

**Nuclear Physics B Volume 530, Issues 1–2, 12 October 1998, Pages 401–456 On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences Z. Berna, 1, L. Dixonb, 2, D.C. Dunbarc, 3, M. Perelsteinb, 2, J.S. Rozowskya, 1**

(1) The two-particle cuts, which may be iterated to all loop orders, suggest (eb) that squaring relations between the two theories exist at (eb) any loop order.
(2) The loop-momentum power-counting implied by (e) proposed two-particle cut analysis indicates (eb) that in four dimensions the first four-point divergence in (eb) N = 8 supergravity should appear at (eb) five loops, contrary to (e&eb) the earlier expectation, based on (e) superspace arguments, of (e) a three-loop counterterm. PACS 04.50.+h; 04.65.+e; 04.60.-m Keywords Unitarity; Cutting; Ultraviolet divergences; Supersymmetry; Supergravity; Two-loops; Multi-loops

$G_{13}$ : Category one of two-particle cuts, which may be iterated to all loop orders

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of squaring relations between the two theories exist at (eb) any loop order.

$T_{14}$ : Category two of SAS

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| $T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>The two-particle cuts, which may be iterated to all loop orders, suggest that squaring relations between the two theories exist at (eb) any loop order | |
| $G_{16}$ : Category one of two-particle cuts, which may be iterated to all loop orders, suggest that squaring relations between the two theories exist<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of any loop order<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>The loop-momentum power-counting implied by (e) proposed two-particle cut analysis indicates (eb) that in four dimensions the first four-point divergence in (eb) N = 8 supergravity should appear at (eb) five loops, contrary to (e&eb) the earlier expectation, based on (e) superspace arguments, of (e) a three-loop counterterm. PACS 04.50.+h; 04.65.+e; 04.60.-m<br><br>Keywords Unitarity; Cutting; Ultraviolet divergences; Supersymmetry; Supergravity; Two-loops; Multi-loops | |
| $G_{20}$ : Category one of proposed two-particle cut analysis indicates (eb) that in four dimensions the first four-point divergence in (eb) N = 8 supergravity should appear at (eb) five loops, contrary to (e&eb) the earlier expectation, based on (e) superspace arguments, of (e) a three-loop counterterm. PACS 04.50.+h; 04.65.+e; 04.60.-m<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of loop-momentum power-counting<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>The loop-momentum power-counting implied by proposed two-particle cut analysis indicates (eb) that in four dimensions the first four-point divergence in (eb) N = 8 supergravity should appear at (eb) five loops, contrary to (e&eb) the earlier expectation, based on (e) superspace arguments, of (e) a three-loop counterterm. PACS 04.50.+h; 04.65.+e; 04.60.-m | |
| $G_{24}$ : Category one of loop-momentum power-counting implied by proposed two-particle cut analysis<br><br>$G_{25}$ : Category two of SAS | |

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| $G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of four dimensions the first four-point divergence in (eb) N = 8 supergravity should appear at (eb) five loops, contrary to (e&eb) the earlier expectation, based on (e) superspace arguments, of (e) a three-loop counterterm. PACS 04.50.+h; 04.65.+e; 04.60.-m<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>The loop-momentum power-counting implied by proposed two-particle cut analysis indicates that in four dimensions the first four-point divergence in (eb) N = 8 supergravity should appear at (eb) five loops, contrary to (e&eb) the earlier expectation, based on (e) superspace arguments, of (e) a three-loop counterterm. PACS 04.50.+h; 04.65.+e; 04.60.-m | |
| $G_{28}$ : Category one of loop-momentum power-counting implied by proposed two-particle cut analysis indicates that in four dimensions the first four-point divergence<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of N = 8 supergravity should appear at (eb) five loops, contrary to (e&eb) the earlier expectation, based on (e) superspace arguments, of (e) a three-loop counterterm. PACS 04.50.+h; 04.65.+e; 04.60.-m<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>The loop-momentum power-counting implied by proposed two-particle cut analysis indicates that in four dimensions the first four-point divergence in N = 8 supergravity should appear at five loops, contrary to (e&eb) the earlier expectation, based on (e) superspace arguments, of (e) a three-loop counterterm. PACS 04.50.+h; 04.65.+e; 04.60.-m | |
| $G_{32}$ : Category one of **loop-momentum power-counting implied by proposed two-particle cut analysis indicates that in four dimensions the first four-point divergence in N = 8 supergravity should appear at five loops;** earlier expectation, based on (e) superspace arguments, of (e) a three-loop counterterm. PACS 04.50.+h; 04.65.+e; 04.60.-m<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of earlier expectation, based on (e) superspace arguments, of (e) a three-loop counterterm. PACS 04.50.+h; 04.65.+e; 04.60.-m; **loop-momentum power-counting implied by proposed two-particle cut analysis indicates that in four dimensions the first four-point divergence in N = 8** | |

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| **supergravity should appear at five loops**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>The loop-momentum power-counting implied by proposed two-particle cut analysis indicates that in four dimensions the first four-point divergence in N = 8 supergravity should appear at five loops, contrary to the earlier expectation, based on (e) superspace arguments, of (e) a three-loop counterterm. PACS 04.50.+h; 04.65.+e; 04.60.-m | |
| $G_{36}$ : Category one of superspace arguments, of (e) a three-loop counterterm. PACS 04.50.+h; 04.65.+e; 04.60.-m<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one ofloop-momentum power-counting implied by proposed two-particle cut analysis indicates that in four dimensions the first four-point divergence in N = 8 supergravity should appear at five loops, contrary to the earlier expectation<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The loop-momentum power-counting implied by proposed two-particle cut analysis indicates that in four dimensions the first four-point divergence in N = 8 supergravity should appear at five loops, contrary to the earlier expectation, based on superspace arguments, of (e) a three-loop counterterm. PACS 04.50.+h; 04.65.+e; 04.60.-m | |
| $G_{40}$ : Category one of three-loop counterterm<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one ofloop-momentum power-counting implied by proposed two-particle cut analysis indicates that in four dimensions the first four-point divergence in N = 8 supergravity should appear at five loops, contrary to the earlier expectation, based on superspace arguments<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine** | |

Keywords Unitarity; Cutting; Ultraviolet divergences; Supersymmetry; **Supergravity;** Two-loops; Multi-loops

$G_{44}$ : Category one of Ultraviolet divergences; **Supergravity;**

$G_{45}$ : Category two of SAS

$G_{46}$ : Category three of SAS

$T_{44}$ : Category one of **Supergravity;;** Ultraviolet divergences

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

## SECTION A(25)-1

## Yang–Mills Theory In The Non-Perturbative Regime

### INTRODUCTION—VARIABLES USED

**(1) Yang–Mills theory in the non-perturbative regime**: The equations of Yang–Mills remain unsolved at (eb) energy scales relevant for (e) describing atomic nuclei. How does Yang–Mills theory give rise to (eb) the physics of nuclei and nuclear constituents? Yang–Mills theory is (=) a gauge theory based on (e) the SU (N) group, or more generally any compact, semi-simple Lie group. Yang–Mills theory seeks to describe (eb) the behavior of elementary particles using (e) these non-Abelian Lie groups and is (=) at the core of the unification of (e) the electromagnetic and weak forces (i.e. U (1) × SU (2)) as well as quantum chromodynamics, the theory of the strong force (based on SU(3)). Thus it forms the basis of our understanding of (e) particle physics, the Standard Model. **Yang–Mills theory from Wikipedia, the free encyclopedia Question dropshade.png Unsolved problem in physics**

(2) In a private correspondence, Wolfgang Pauli formulated in 1953 a six-dimensional theory of Einstein's field equations of general relativity, extending the five-dimensional theory of Kaluza, Klein, Fock and others to (e&eb) a higher-dimensional internal space.[1]

(3) However, there is no evidence that Pauli developed (eb) the Lagrangian of a gauge field or the quantization of it.

(4) Because Pauli found that his theory "leads to (eb) some rather unphysical shadow particles", he refrained from (e) publishing his results formally.[1]

(5) Although Pauli did not publish his six-dimensional theory, he gave two talks about it in Zürich.[2] Recent research shows that an extended Kaluza–Klein theory is in general not (e) equivalent to Yang–Mills theory, as (e) the former contains additional terms.[3]

**(6)** In early 1954, Chen Ning Yang and Robert Mills [4] extended the concept of gauge theory for (e) abelian groups, e.g. quantum electrodynamics, to (e&eb) nonabelian groups to provide (eb) an explanation for (e) strong interactions.

**(7)** The idea by Yang–Mills was criticized by Pauli,[5] as the quanta of the Yang–Mills field must be (=) massless in order to (e) maintain gauge invariance

| NOTATION | |
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| **Module One**<br>**Yang–Mills theory in the non-perturbative regime**:<br><br>The equations of Yang–Mills remain unsolved at (eb) energy scales relevant for (e) describing atomic nuclei. How does Yang–Mills theory give rise to (eb) the physics of nuclei and nuclear constituents? Yang–Mills theory is (=) a gauge theory based on (e) the SU (N) group, or more generally any compact, semi-simple Lie group. Yang–Mills theory seeks to describe (eb) the behavior of elementary particles using (e) these non-Abelian Lie groups and is (=) at the core of the unification of (e) the electromagnetic and weak forces (i.e. U (1) × SU (2)) as well as quantum chromodynamics, the theory of the strong force (based on SU(3)). Thus it forms the basis of our understanding of (e) particle physics, the Standard Model<br><br>. **Yang–Mills theory from Wikipedia, the free encyclopedia Question dropshade.png Unsolved problem in physics** | |
| $G_{13}$ : Category one of equations of Yang–Mills remain unsolved<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of energy scales relevant for (e) describing atomic nuclei. How does Yang–Mills theory give rise to (eb) the physics of nuclei and nuclear constituents? Yang–Mills theory is (=) a gauge theory based on (e) the SU (N) group, or more generally any compact, semi-simple Lie group. Yang–Mills theory seeks to describe (eb) the behavior of elementary particles using (e) these non-Abelian Lie groups and is (=) at the core of the unification of (e) the electromagnetic and weak forces (i.e. U (1) × SU (2)) as well as quantum chromodynamics, the theory of the strong force (based on SU(3)). Thus it forms the basis of our understanding of (e) particle physics, the Standard Model<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>**Yang–Mills theory in the non-perturbative regime**:<br><br>The equations of Yang–Mills remain unsolved at energy scales relevant for (e) describing atomic nuclei.<br><br>How does Yang–Mills theory give rise to (eb) the physics of nuclei and nuclear constituents? Yang–Mills theory is (=) a gauge theory based on (e) the SU (N) group, or more generally any compact, semi-simple Lie group. Yang–Mills theory seeks to describe (eb) the behavior of elementary particles using (e) these non-Abelian Lie groups and is (=) at the core of the unification of (e) the electromagnetic and weak forces (i.e. U (1) × SU (2)) as well as quantum chromodynamics, the theory of the strong force (based on SU(3)). Thus it forms the basis of our understanding of (e) particle physics, the Standard Model<br><br>. **Yang–Mills theory from Wikipedia, the free encyclopedia Question dropshade.png Unsolved problem in physics** | |
| $G_{16}$ : Category one of describing atomic nuclei | |

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| $G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of equations of Yang–Mills remain unsolved at energy scales relevant<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>How does Yang–Mills theory give rise to (eb) the physics of nuclei and nuclear constituents?<br>Yang–Mills theory is (=) a gauge theory based on (e) the SU (N) group, or more generally any compact, semi-simple Lie group. Yang–Mills theory seeks to describe (eb) the behavior of elementary particles using (e) these non-Abelian Lie groups and is (=) at the core of the unification of (e) the electromagnetic and weak forces (i.e. U (1) × SU (2)) as well as quantum chromodynamics, the theory of the strong force (based on SU(3)). Thus it forms the basis of our understanding of (e) particle physics, the Standard Model<br>. **Yang–Mills theory from Wikipedia, the free encyclopedia Question dropshade.png Unsolved problem in physics**<br>**Model gives the mechanism by means of which Yang–Mills theory give rise to the physics of nuclei and nuclear constituents** | |
| $G_{20}$ : Category one of Yang–Mills theory<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of physics of nuclei and nuclear constituents<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>Yang–Mills theory is (=) a gauge theory based on (e) the SU (N) group, or more generally any compact, semi-simple Lie group.<br>Yang–Mills theory seeks to describe (eb) the behavior of elementary particles using (e) these non-Abelian Lie groups and is (=) at the core of the unification of (e) the electromagnetic and weak forces (i.e. U (1) × SU (2)) as well as quantum chromodynamics, the theory of the strong force (based on SU(3)). Thus it forms the basis of our understanding of (e) particle physics, the Standard Model<br>. **Yang–Mills theory from Wikipedia, the free encyclopedia Question dropshade.png Unsolved problem in physics** | |
| $G_{24}$ : Category one of Yang–Mills theory | |

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|---|---|
| $G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of gauge theory based on (e) the SU (N) group, or more generally any compact, semi-simple Lie group.<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>Yang–Mills theory is a gauge theory based on (e) the SU (N) group, or more generally any compact, semi-simple Lie group. | |
| $G_{28}$ : Category one of SU (N) group, or more generally any compact, semi-simple Lie group.<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of Yang–Mills theory is a gauge theory<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>Yang–Mills theory seeks to describe (eb) the behavior of elementary particles using (e) these non-Abelian Lie groups and is (=) at the core of the unification of (e) the electromagnetic and weak forces (i.e. U (1) × SU (2)) as well as quantum chromodynamics, the theory of the strong force (based on SU(3)).<br>Thus it forms the basis of our understanding of (e) particle physics, the Standard Model<br>. **Yang–Mills theory from Wikipedia, the free encyclopedia Question dropshade.png Unsolved problem in physics** | |
| $G_{32}$ : Category one of Yang–Mills theory seeks to describe<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of behavior of elementary particles using (e) these non-Abelian Lie groups and is (=) at the core of the unification of (e) the electromagnetic and weak forces (i.e. U (1) × SU (2)) as well as quantum chromodynamics, the theory of the strong force (based on SU(3))<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |

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| **Module seven**<br><br>Yang–Mills theory seeks to describe the behavior of elementary particles using (e) these non-Abelian Lie groups and is (=) at the core of the unification of (e) the electromagnetic and weak forces (i.e. U (1) × SU (2)) as well as quantum chromodynamics, the theory of the strong force (based on SU(3)).<br><br>Thus it forms the basis of our understanding of (e) particle physics, the Standard Model<br><br>. **Yang–Mills theory from Wikipedia, the free encyclopedia Question dropshade.png Unsolved problem in physics** | |
| $G_{36}$ : Category one of these non-Abelian Lie groups and is (=) at the core of the unification of (e) the electromagnetic and weak forces (i.e. U (1) × SU (2)) as well as quantum chromodynamics, the theory of the strong force (based on SU(3)).<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of Yang–Mills theory seeks to describe the behavior of elementary particles<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Yang–Mills theory seeks to describe the behavior of elementary particles using these non-Abelian Lie groups and is (=) at the core of the unification of (e) the electromagnetic and weak forces (i.e. U (1) × SU (2)) as well as quantum chromodynamics, the theory of the strong force (based on SU(3)) | |
| $G_{40}$ : Category one of Yang–Mills theory seeks to describe the behavior of elementary particles using these non-Abelian Lie groups<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of core of the unification of (e) the electromagnetic and weak forces (i.e. U (1) × SU (2)) as well as quantum chromodynamics, the theory of the strong force (based on SU(3))<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Yang–Mills theory seeks to describe the behavior of elementary particles using these non-Abelian Lie groups and is at the core of the unification of (e) the electromagnetic and weak forces (i.e. U (1) × SU (2)) as well as quantum chromodynamics, the theory of the strong force (based on SU(3)) | |

| | |
|---|---|
| $G_{44}$ : Category one of electromagnetic and weak forces (i.e. U (1) × SU (2)) as well as quantum chromodynamics, the theory of the strong force (based on SU(3))<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of Yang–Mills theory seeks to describe the behavior of elementary particles using these non-Abelian Lie groups and is at the core of the unification<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION A(25)-2

## Gauge Theory For Abelian Groups

**INTRODUCTION—VARIABLES USED**

**Source; Wikipedia**

(1) In a private correspondence, Wolfgang Pauli formulated in 1953 a six-dimensional theory of Einstein's field equations of general relativity, extending the five-dimensional theory of Kaluza, Klein, Fock and others to (e&eb) a higher-dimensional internal space.[1]
(2) However, there is no evidence that Pauli developed (eb) the Lagrangian of a gauge field or the quantization of it.
(3) Because Pauli found that his theory "leads to (eb) some rather unphysical shadow particles", he refrained from (e) publishing his results formally.[1]
(4) Although Pauli did not publish his six-dimensional theory, he gave two talks about it in Zürich.[2] Recent research shows that an extended Kaluza–Klein theory is in general not (e) equivalent to Yang–Mills theory, as (e) the former contains additional terms.[3]
**(5)** In early 1954, Chen Ning Yang and Robert Mills [4] extended the concept of gauge theory for (e) abelian groups, e.g. quantum electrodynamics, to (e&eb) nonabelian groups to provide (eb) an explanation for (e) strong interactions.
**(6)** The idea by Yang–Mills was criticized by Pauli,[5] as the quanta of the Yang–Mills field must be (=) massless in order to (e) maintain gauge invariance

**NOTATION**

**Module One**

Thus it forms the basis of our understanding of particle physics, the Standard Model

$G_{13}$ : Category one of Yang Mills Theory (classification can be based on the numerous applications or all the three categories can be taken as equivalent)

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| $G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of basis of our understanding of particle physics, the Standard Model<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>In a private correspondence, Wolfgang Pauli formulated in 1953 a six-dimensional theory of Einstein's field equations of general relativity, extending the five-dimensional theory of Kaluza, Klein, Fock and others to (e&eb) a higher-dimensional internal space.[1] | |
| $G_{16}$ : Category one of **six-dimensional theory of Einstein's field equations of general relativity, extending the five-dimensional theory of Kaluza, Klein, Fock and others;** higher-dimensional internal space.[1]<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of higher-dimensional internal space.[1];**six-dimensional theory of Einstein's field equations of general relativity, extending the five-dimensional theory of Kaluza, Klein, Fock and others**<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>However, there is no evidence that Pauli developed (eb) the Lagrangian of a gauge field or the quantization of it | |
| $G_{20}$ : Category one of gauge field or the quantization of it<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of Lagrangian<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>Because Pauli found that his theory "leads to (eb) some rather unphysical shadow particles", he refrained from (e) publishing his results formally.[1] | |

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| $G_{24}$ : Category one of Pauli's Theory<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of some rather unphysical shadow particles<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>Although Pauli did not publish his six-dimensional theory, he gave two talks about it in Zürich.[2]<br>Recent research shows that an extended Kaluza–Klein theory is in general not (e) equivalent to Yang–Mills theory, as (e) the former contains additional terms.[3] | |
| $G_{28}$ : Category one of equivalent to Yang–Mills theory, as (e) the former contains additional terms.[3]<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of extended Kaluza–Klein theory is in general<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>Recent research shows that an extended Kaluza–Klein theory is in general not equivalent to Yang–Mills theory, as (e) the former contains additional terms.[3] | |
| $G_{32}$ : Category one of extended Kaluza–Klein theory is in general not equivalent to Yang–Mills theory; **former contains additional terms.[3]**<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of **former contains additional terms.[3];**extended Kaluza–Klein theory is in general not equivalent to Yang–Mills theory<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br>In early 1954, Chen Ning Yang and Robert Mills [4] extended the concept of gauge theory for (e) abelian | |

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| groups, e.g. quantum electrodynamics, to (e&eb) nonabelian groups to provide (eb) an explanation for (e) strong interactions | |
| $G_{36}$ : Category one of **concept of gauge theory for abelian groups, e.g. quantum electrodynamics;** nonabelian groups to provide (eb) an explanation for (e) strong interactions<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of nonabelian groups to provide (eb) an explanation for (e) strong interactions **;concept of gauge theory for abelian groups, e.g. quantum electrodynamics**<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>In early 1954, Chen Ning Yang and Robert Mills [4] extended the concept of gauge theory for abelian groups, e.g. quantum electrodynamics, to nonabelian groups to provide (eb) an explanation for (e) strong interactions | |
| $G_{40}$ : Category one of concept of gauge theory for abelian groups, e.g. quantum electrodynamics, to nonabelian groups<br>$G_{41}$ : Category two of SAS<br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of explanation for strong interactions<br>$T_{41}$ : Category two of SAS<br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br>The idea by Yang–Mills was criticized by Pauli,[5] as the quanta of the Yang–Mills field must be (=) massless in order to (e) maintain gauge invariance | |
| $G_{44}$ : Category one of quanta of the Yang–Mills field<br>$G_{45}$ : Category two of SAS<br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of massless in order to maintain gauge invariance<br>$T_{45}$ : Category two of SAS | |

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| $T_{46}$ : Category three of SAS | |

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| **SECTION A(26)**<br><br>**Yang–Mills Theory-2** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Source: Wikipedia**<br><br>(1) The idea was set aside until 1960, when the concept of particles acquiring mass through (e&eb) symmetry breaking in massless theories was put forward, initially by Jeffrey Goldstone, Yoichiro Nambu, and Giovanni Jona-Lasinio.<br>(2) This prompted a significant restart of Yang–Mills theory studies that proved successful in the formulation of both electroweak unification and quantum chromodynamics (QCD).<br>(3) The electroweak interaction is described by (e) SU (2) × U (1) group while QCD is (=) an SU (3) Yang–Mills theory.<br>(4) The electroweak theory is obtained by (e) combining SU (2) with (e&eb) U (1), where quantum electrodynamics (QED) is described by (e) a U (1) group, and is replaced in the unified electroweak theory by (e&eb) a U(1) group representing a weak hypercharge rather than electric charge.<br>(5) The massless bosons from the SU (2) × U (1) theory mix after spontaneous symmetry breaking to produce (eb) the 3 massive weak bosons, and the photon field.<br>(6) The Standard Model combines the strong interaction with (e&eb) the unified electroweak interaction (unifying the weak and (e&eb) electromagnetic interaction) through (e&eb) the symmetry group SU (2) × U (1) × SU (3).<br>(7) In the current epoch the strong interaction is not (e) unified with the electroweak interaction, but from the observed running of the coupling constants it is believed they all converge to (e&eb) a single value at very high energies.<br>(8) Phenomenology at lower energies in (eb) quantum chromodynamics is not completely understood due to (e) the difficulties of managing such a theory with (e&eb) a strong coupling.<br>(9) This may be the reason why confinement has not (e) been theoretically proven, though (e) it is a consistent experimental observation.<br>**(10)** Proof that QCD confines at (eb) low energy is a mathematical problem of great relevance, and an award has been proposed by the Clay Mathematics Institute for whoever is also able to show that the Yang–Mills theory has (e) a mass gap and its existence. | |
| **NOTATION** | |
| **Module One**<br>The idea was set aside until 1960, when the concept of particles acquiring mass through (e&eb) symmetry breaking in massless theories was put forward, initially by Jeffrey Goldstone, Yoichiro Nambu, and Giovanni Jona-Lasinio | |
| $G_{13}$ : Category one of **concept of particles acquiring mass;** symmetry breaking in massless theories | |

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| $G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of symmetry breaking in massless theories ;**concept of particles acquiring mass**<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>The electroweak interaction is described by (e) SU (2) × U (1) group while QCD is (=) an SU (3) Yang–Mills theory | |
| $G_{16}$ : Category one of SU (2) × U (1) group<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of electroweak interaction is described<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>QCD is (=) an SU (3) Yang–Mills theory | |
| $G_{20}$ : Category one of QCD<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of SU (3) Yang–Mills theory<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>The electroweak theory is obtained by (e) combining SU (2) with (e&eb) U (1), where quantum electrodynamics (QED) is described by (e) a U (1) group, and is replaced in the unified electroweak theory by (e&eb) a U(1) group representing a weak hypercharge rather than electric charge | |
| $G_{24}$ : Category one ofcombining SU (2) with (e&eb) U (1), where quantum electrodynamics (QED) is described by (e) a U (1) group, and is replaced in the unified electroweak theory by (e&eb) a U(1) group representing a weak hypercharge rather than electric charge<br>$G_{25}$ : Category two of SAS | |

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| $G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of electroweak theory<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>The electroweak theory is obtained by combining SU (2) with (e&eb) U (1), where quantum electrodynamics (QED) is described by (e) a U (1) group, and is replaced in the unified electroweak theory by (e&eb) a U(1) group representing a weak hypercharge rather than electric charge | |
| $G_{28}$ : Category one of **electroweak theory is obtained by combining SU (2)** ; U (1), where quantum electrodynamics (QED) is described by (e) a U (1) group, and is replaced in the unified electroweak theory by (e&eb) a U(1) group representing a weak hypercharge rather than electric charge<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of U (1), where quantum electrodynamics (QED) is described by (e) a U (1) group, and is replaced in the unified electroweak theory by (e&eb) a U(1) group representing a weak hypercharge rather than electric charge;**electroweak theory is obtained by combining SU (2)**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>The electroweak theory is obtained by combining SU (2) with U (1), where quantum electrodynamics (QED) is described by (e) a U (1) group, and is replaced in the unified electroweak theory by (e&eb) a U(1) group representing a weak hypercharge rather than electric charge | |
| $G_{32}$ : Category one of U (1) group, and is replaced in the unified electroweak theory by (e&eb) a U(1) group representing a weak hypercharge rather than electric charge<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of electroweak theory is obtained by combining SU (2) with U (1), where quantum electrodynamics (QED) is described<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven** | |

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| The electroweak theory is obtained by combining SU (2) with U (1), where quantum electrodynamics (QED) is described by a U (1) group, and is replaced in the unified electroweak theory by (e&eb) a U(1) group representing a weak hypercharge rather than electric charge | |
| $G_{36}$ : Category one of **electroweak theory is obtained by combining SU (2) with U (1), where quantum electrodynamics (QED) is described by a U (1) group, and is replaced in the unified electroweak theory;** U(1) group representing a weak hypercharge rather than electric charge<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of U(1) group representing a weak hypercharge rather than electric charge **;electroweak theory is obtained by combining SU (2) with U (1), where quantum electrodynamics (QED) is described by a U (1) group, and is replaced in the unified electroweak theory**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The massless bosons from the SU (2) × U (1) theory mix after spontaneous symmetry breaking to produce (eb) the 3 massive weak bosons, and the photon field | |
| $G_{40}$ : Category one of massless bosons from the SU (2) × U (1) theory mix after spontaneous symmetry breaking<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of 3 massive weak bosons, and the photon field<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The Standard Model combines the strong interaction with (e&eb) the unified electroweak interaction (unifying the weak and (e&eb) electromagnetic interaction) through (e&eb) the symmetry group SU (2) × U (1) × SU (3). | |
| $G_{44}$ : Category one of **Standard Model combines the strong interaction**; unified electroweak interaction (unifying the weak and (e&eb) electromagnetic interaction) through (e&eb) the symmetry group SU (2) × U (1) × SU (3).<br><br>$G_{45}$ : Category two of SAS | |

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| $G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of unified electroweak interaction (unifying the weak and (e&eb) electromagnetic interaction) through (e&eb) the symmetry group SU (2) × U (1) × SU (3).;**Standard Model combines the strong interaction**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION A(26)-1**<br><br>**QCD Confines At Low Energy** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Source: Wikipedia**<br><br>(1) In the current epoch the strong interaction is not (e) unified with the electroweak interaction, but from the observed running of the coupling constants it is believed they all converge to (e&eb) a single value at very high energies.<br>(2) Phenomenology at lower energies in (eb) quantum chromodynamics is not completely understood due to (e) the difficulties of managing such a theory with (e&eb) a strong coupling.<br>**(3)** This may be the reason why confinement has not (e) been theoretically proven, though (e) it is a consistent experimental observation.<br>**(4)** Proof that QCD confines at (eb) low energy is a mathematical problem of great relevance, and an award has been proposed by the Clay Mathematics Institute for whoever is also able to show that the Yang–Mills theory has (e) a mass gap and its existence. | |
| **NOTATION** | |
| **Module One**<br>The Standard Model combines the strong interaction with the unified electroweak interaction (unifying the weak and (e&eb) electromagnetic interaction) through (e&eb) the symmetry group SU (2) × U (1) × SU (3). | |
| $G_{13}$ : Category one of **Standard Model combines the strong interaction with the unified electroweak interaction (unifying the weak;** electromagnetic interaction) through (e&eb) the symmetry group SU (2) × U (1) × SU (3).<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of electromagnetic interaction) through (e&eb) the symmetry group SU (2) × U (1) × SU (3).; **Standard Model combines the strong interaction with the unified electroweak interaction (unifying the weak**<br><br>$T_{14}$ : Category two of SAS | |

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| $T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>The Standard Model combines the strong interaction with the unified electroweak interaction (unifying the weak and electromagnetic interaction) through (e&eb) the symmetry group SU (2) × U (1) × SU (3). | |
| $G_{16}$ : Category one of **Standard Model combines the strong interaction with the unified electroweak interaction (unifying the weak and electromagnetic interaction);** symmetry group SU (2) × U (1) × SU (3).<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of symmetry group SU (2) × U (1) × SU (3).;**Standard Model combines the strong interaction with the unified electroweak interaction (unifying the weak and electromagnetic interaction)**<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>In the current epoch the strong interaction is not (e) unified with the electroweak interaction, but from the observed running of the coupling constants it is believed they all converge to (e&eb) a single value at very high energies | |
| $G_{20}$ : Category one of unification with the electroweak interaction, but from the observed running of the coupling constants it is believed they all converge to (e&eb) a single value at very high energies<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of strong interaction<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>In the current epoch the strong interaction is not unified with the electroweak interaction, but from the observed running of the coupling constants it is believed they all converge to (e&eb) a single value at very high energies | |
| $G_{24}$ : Category one of **strong interaction is not unified with the electroweak interaction, but from the observed running of the coupling constants it is believed they all converge;** single value at very high energies<br><br>$G_{25}$ : Category two of SAS | |

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| $G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of single value at very high energies **;strong interaction is not unified with the electroweak interaction, but from the observed running of the coupling constants it is believed they all converge**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Phenomenology at lower energies in (eb) quantum chromodynamics is not completely understood due to (e) the difficulties of managing such a theory with (e&eb) a strong coupling | |
| $G_{28}$ : Category one of Phenomenology at lower energies<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of quantum chromodynamics is not completely understood due to (e) the difficulties of managing such a theory with (e&eb) a strong coupling<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Phenomenology at lower energies in quantum chromodynamics is not completely understood due to (e) the difficulties of managing such a theory with (e&eb) a strong coupling | |
| $G_{32}$ : Category one of difficulties of managing such a theory with (e&eb) a strong coupling<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of Phenomenology at lower energies in quantum chromodynamics is not completely understood<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Phenomenology at lower energies in quantum chromodynamics is not completely understood due to (e) the difficulties of managing such a theory with (e&eb) a strong coupling | |

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| $G_{36}$ : Category one of **Phenomenology at lower energies in quantum chromodynamics;** strong coupling<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of strong coupling **;Phenomenology at lower energies in quantum chromodynamics**<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>QCD Confines At Low Energy<br>We assume the proposition and give a model | |
| $G_{40}$ : Category one of **QCD Confines;** Low Energy<br>$G_{41}$ : Category two of SAS<br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of Low Energy **;QCD Confines**<br>$T_{41}$ : Category two of SAS<br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br>Proof that QCD confines at (eb) low energy is a mathematical problem of great relevance, and an award has been proposed by the Clay Mathematics Institute for whoever is also able to show that the Yang–Mills theory has (e) a mass gap and its existence.<br>**Again the proposition has been assumed and model given which gives the prediction and stability of the Yang Mills Theory with mass gap** | |
| $G_{44}$ : Category one of mass gap<br>$G_{45}$ : Category two of SAS<br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of Yang–Mills theory<br>$T_{45}$ : Category two of SAS<br>$T_{46}$ : Category three of SAS | |

## SECTION A(27)

## Yang Mills Theory: Mathematical Overview

### INTRODUCTION—VARIABLES USED

**Source: Wikipedia**

(1) Yang–Mills theories are a special example of gauge theory with (e&eb) a non-abelian symmetry group given by the Lagrangian $\mathcal{L}_{\text{gf}} = -\frac{1}{2}\operatorname{Tr}(F^2) = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}$ with the generators of (e) the Lie algebra corresponding to (e&eb) the F-quantities (the curvature or field-strength form) satisfying $\operatorname{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}, \quad [T^a, T^b] = if^{abc}T^c$ and the covariant derivative defined as $D_\mu = I\partial_\mu - igT^aA^a_\mu$ where I is the identity for the group generators, $A^a_\mu$ is the vector potential, and g is the coupling constant. In four dimensions, the coupling constant g is a pure number and for a SU(N) group one has $a, b, c = 1 \ldots N^2 - 1.$

(2) The relation $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$ can be derived by the commutator $[D_\mu, D_\nu] = -igT^aF^a_{\mu\nu}.$

(3) The field has the property of being self-interacting and equations of motion that one obtains are said to be semilinear, as (e) nonlinearities are both with and without derivatives.

(4) This means that one can manage this theory only by (e) perturbation theory, with small nonlinearities.

(5) Note that the transition between "upper" ("contravariant") and "lower" ("covariant") vector or tensor components is (=) trivial for a indices (e.g. $f^{abc} = f_{abc}$), whereas for (e) μ and ν it is nontrivial, corresponding e.g. to (e&eb) the usual Lorentz signature, $\eta_{\mu\nu} = \operatorname{diag}(+ - - -)$.

(6) From the given Lagrangian one can derive the equations of motion given by $\partial^\mu F^a_{\mu\nu} + gf^{abc}A^{\mu b}F^c_{\mu\nu} = 0.$

(7) Putting $F_{\mu\nu} = T^aF^a_{\mu\nu}$, these can be rewritten as $(D^\mu F_{\mu\nu})^a = 0.$

(8) A Bianchi identity holds $(D_\mu F_{\nu\kappa})^a + (D_\kappa F_{\mu\nu})^a + (D_\nu F_{\kappa\mu})^a = 0$ which is equivalent to the Jacobi identity $[D_\mu, [D_\nu, D_\kappa]] + [D_\kappa, [D_\mu, D_\nu]] + [D_\nu, [D_\kappa, D_\mu]] = 0$ since (e) $[D_\mu, F^a_{\nu\kappa}] = D_\mu F^a_{\nu\kappa}.$

(9) Define the dual strength tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, then the Bianchi identity can be

rewritten as $D_\mu \tilde{F}^{\mu\nu} = 0.$

**(10)** A source $J^a_\mu$ enters into the equations of motion as $\partial^\mu F^a_{\mu\nu} + g f^{abc} A^{b\mu} F^c_{\mu\nu} = -J^a_\nu.$

**(11)** Note that the currents must properly change under (e&eb) gauge group transformations.

**NOTATION**

**Module One**

Yang–Mills theories are a special example of gauge theory with (e&eb) a non-abelian symmetry group given by the Lagrangian $\mathcal{L}_{\text{gf}} = -\frac{1}{2}\,\text{Tr}(F^2) = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu}$ with the generators of (e) the Lie algebra corresponding to (e&eb) the F-quantities (the curvature or field-strength form) satisfying $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \quad [T^a, T^b] = i f^{abc} T^c$ and the covariant derivative defined as $D_\mu = I\partial_\mu - igT^a A^a_\mu$ where I is the identity for the group generators, $A^a_\mu$ is the vector potential, and g is the coupling constant. In four dimensions, the coupling constant g is a pure number and for a SU(N) group one has $a, b, c = 1 \ldots N^2 - 1.$

$G_{13}$ : Category one of **Yang–Mills theories are a special example of gauge theory;** non-abelian symmetry group given by the Lagrangian $\mathcal{L}_{\text{gf}} = -\frac{1}{2}\,\text{Tr}(F^2) = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu}$

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of non-abelian symmetry group given by the Lagrangian $\mathcal{L}_{\text{gf}} = -\frac{1}{2}\,\text{Tr}(F^2) = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu}$ ;**Yang–Mills theories are a special example of gauge theory**

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2}\,\text{Tr}(F^2) = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu}$$

$G_{16}$ : Category one of first term on RHS (one slowly eats up the other)

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

$T_{16}$ : Category one ofsecond term on RHS

$T_{17}$ : Category two of SAS

$T_{18}$ : Category three of SAS

**Module three**

Yang–Mills theories are a special example of gauge theory with (e&eb) a non-abelian symmetry group given by the Lagrangian $\mathcal{L}_{\text{gf}} = -\frac{1}{2}\,\text{Tr}(F^2) = -\frac{1}{4}F^{a\mu\nu}\,F^{a}_{\mu\nu}$ with the generators of (e) the Lie algebra corresponding to (e&eb) the F-quantities (the curvature or field-strength form) satisfying $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \quad [T^a, T^b] = if^{abc}T^c$ and the covariant derivative defined as $D_\mu = I\partial_\mu - igT^a A^a_\mu$ where I is the identity for the group generators, $A^a_\mu$ is the vector potential, and g is the coupling constant.

In four dimensions, the coupling constant g is a pure number and for a SU(N) group one has $a, b, c = 1 \ldots N^2 - 1.$

$G_{20}$ : Category one of Lie algebra corresponding to (e&eb) the F-quantities (the curvature or field-strength form) satisfying $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \quad [T^a, T^b] = if^{abc}T^c$ and the covariant derivative defined as $D_\mu = I\partial_\mu - igT^a A^a_\mu$ where I is the identity for the group generators, $A^a_\mu$ is the vector potential, and g is the coupling constant.

$G_{21}$ : Category two of SAS

$G_{22}$ : Category three of SAS

$T_{20}$ : Category one of Yang–Mills theories are a special example of gauge theory with (e&eb) a non-abelian symmetry group given by the Lagrangian $\mathcal{L}_{\text{gf}} = -\frac{1}{2}\,\text{Tr}(F^2) = -\frac{1}{4}F^{a\mu\nu}\,F^{a}_{\mu\nu}$ with the generators

$T_{21}$ : Category two of SAS

$T_{22}$ : Category three of SAS

**Module four**

Yang–Mills theories are a special example of gauge theory with (e&eb) a non-abelian symmetry group given by the Lagrangian $\mathcal{L}_{\text{gf}} = -\frac{1}{2}\,\text{Tr}(F^2) = -\frac{1}{4}F^{a\mu\nu}\,F^{a}_{\mu\nu}$ with the generators of the Lie algebra corresponding to (e&eb) the F-quantities (the curvature or field-strength form) satisfying $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \quad [T^a, T^b] = if^{abc}T^c$ and the covariant derivative defined as

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| $D_\mu = I\partial_\mu - igT^a A^a_\mu$ where I is the identity for the group generators, $A^a_\mu$ is the vector potential, and g is the coupling constant | |
| $G_{24}$ : Category one of Yang–Mills theories are a special example of gauge theory with (e&eb) a non-abelian symmetry group given by the Lagrangian $\mathcal{L}_{\text{gf}} = -\frac{1}{2}\,\text{Tr}(F^2) = -\frac{1}{4}F^{a\mu\nu}\,F^a_{\mu\nu}$ with the generators of the Lie algebra corresponding; F-quantities (the curvature or field-strength form) satisfying $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \quad [T^a, T^b] = if^{abc}T^c$ and the covariant derivative defined as $D_\mu = I\partial_\mu - igT^a A^a_\mu$ where I is the identity for the group generators, $A^a_\mu$ is the vector potential, and g is the coupling constant<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of F-quantities (the curvature or field-strength form) satisfying $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \quad [T^a, T^b] = if^{abc}T^c$ and the covariant derivative defined as $D_\mu = I\partial_\mu - igT^a A^a_\mu$ where I is the identity for the group generators, $A^a_\mu$ is the vector potential, and g is the coupling constant;Yang–Mills theories are a special example of gauge theory with (e&eb) a non-abelian symmetry group given by the Lagrangian $\mathcal{L}_{\text{gf}} = -\frac{1}{2}\,\text{Tr}(F^2) = -\frac{1}{4}F^{a\mu\nu}\,F^a_{\mu\nu}$ with the generators of the Lie algebra corresponding<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Yang–Mills theories are a special example of gauge theory with (e&eb) a non-abelian symmetry group given by the Lagrangian $\mathcal{L}_{\text{gf}} = -\frac{1}{2}\,\text{Tr}(F^2) = -\frac{1}{4}F^{a\mu\nu}\,F^a_{\mu\nu}$ with the generators of the Lie algebra corresponding to (e&eb) the F-quantities (the curvature or field-strength form) satisfying $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \quad [T^a, T^b] = if^{abc}T^c$ and the covariant derivative defined as $D_\mu = I\partial_\mu - igT^a A^a_\mu$ where I is the identity for the group generators, $A^a_\mu$ is the vector potential, and g is the coupling constant<br><br>$D_\mu = I\partial_\mu - igT^a A^a_\mu$ | |
| $G_{28}$ : Category one of first term on RHS | |

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| $G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of second term on RHS<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \quad [T^a, T^b] = i f^{abc} T^c$$ | |
| $G_{32}$ : Category one of LHS of the first equation<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of RHS of the first equation<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>The field has the property of being self-interacting and equations of motion that one obtains are said to be semilinear, as nonlinearities are both with and without derivatives. | |
| $G_{36}$ : Category one of field has the property of being self-interacting and equations of motion that one obtains are said to be semilinear,<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of nonlinearities are both with and without derivatives.<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>This field has the property of being self-interacting and equations of motion that one obtains are said to be semilinear, as nonlinearities are both with and without derivatives means that one can manage this theory only by (e) perturbation theory, with small nonlinearities | |

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| $G_{40}$ : Category one of field has the property of being self-interacting and equations of motion that one obtains are said to be semilinear, as nonlinearities are both with and without derivatives<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of one can manage this theory only by (e) perturbation theory, with small nonlinearities<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>one can manage this theory only by (e) perturbation theory, with small nonlinearities | |
| $G_{44}$ : Category one of perturbation theory, with small nonlinearities<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of one can manage this theory<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)},$<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$<br>$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$<br>$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}$ $(a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,<br>$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$<br>$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)},$<br>$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)},$<br>$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)},$<br>are Dissipation coefficients | |

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| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14},t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14},t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14},t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G,t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G,t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14},t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G,t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17},t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17},t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17},t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17},t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}),t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |

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| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a'_{24})^{(4)} + (a''_{24})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a'_{25})^{(4)} + (a''_{25})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a'_{26})^{(4)} + (a''_{26})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b'_{24})^{(4)} - (b''_{24})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b'_{25})^{(4)} - (b''_{25})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b'_{26})^{(4)} - (b''_{26})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a''_{24})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b''_{24})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a'_{28})^{(5)} + (a''_{28})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a'_{29})^{(5)} + (a''_{29})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a'_{30})^{(5)} + (a''_{30})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b'_{28})^{(5)} - (b''_{28})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b'_{29})^{(5)} - (b''_{29})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b'_{30})^{(5)} - (b''_{30})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a''_{28})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b''_{28})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a'_{32})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b'_{32})^{(6)} - (b''_{32})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b'_{33})^{(6)} - (b''_{33})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b'_{34})^{(6)} - (b''_{34})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a''_{32})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b'_{36})^{(7)} - (b''_{36})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |

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| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)}\ \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}\ \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}\ \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}\ \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}\ \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}\ \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)}\ \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}\ \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}\ \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}\ \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}\ \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}\ \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)}\ \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}\ \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}\ \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}\ \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}\ \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}\ \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3 | |

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| $+(a_{20}^{''})^{(3,3,)}(T_{21},t)$ , $+(a_{21}^{''})^{(3,3,)}(T_{21},t)$ , $+(a_{22}^{''})^{(3,3,)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{38}^{''})^{(7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3<br>$+(a_{40}^{''})^{(8,8)}(T_{41},t)$ $+(a_{41}^{''})^{(8,8)}(T_{41},t)$ $+(a_{42}^{''})^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3<br>$+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ , $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ , $+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\boxed{-(b_{15}^{''})^{(1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | *60* |
| Where $\boxed{-(b_{13}^{''})^{(1)}(G,t)}$ , $\boxed{-(b_{14}^{''})^{(1)}(G,t)}$ , $\boxed{-(b_{15}^{''})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)}$ , $\boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)}$ , $\boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)}$ , $\boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)}$ , $\boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)}$ , $\boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)}$ , $\boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)}$ , $\boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)}$ , $\boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)}$ , $\boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)}$ , $\boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}$ , $\boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}$ , $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3 | |

$-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{16}}{dt}=(a_{16})^{(2)}G_{17}-\begin{bmatrix}(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,)}(T_{14},t) & +(a_{20}^{''})^{(3,3,3)}(T_{21},t)\\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)\\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9)}(T_{45},t)\end{bmatrix}G_{16} \quad (61)$$

$$\frac{dG_{17}}{dt}=(a_{17})^{(2)}G_{16}-\begin{bmatrix}(a_{17}^{'})^{(2)}+(a_{17}^{''})^{(2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,)}(T_{14},t) & +(a_{21}^{''})^{(3,3,3)}(T_{21},t)\\ +(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)\\ +(a_{37}^{''})^{(7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9)}(T_{45},t)\end{bmatrix}G_{17} \quad (62)$$

$$\frac{dG_{18}}{dt}=(a_{18})^{(2)}G_{17}-\begin{bmatrix}(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17},t) & +(a_{15}^{''})^{(1,1,)}(T_{14},t) & +(a_{22}^{''})^{(3,3,3)}(T_{21},t)\\ +(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)\\ +(a_{38}^{''})^{(7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9)}(T_{45},t)\end{bmatrix}G_{18} \quad (63)$$

Where $+(a_{16}^{''})^{(2)}(T_{17},t)$, $+(a_{17}^{''})^{(2)}(T_{17},t)$, $+(a_{18}^{''})^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3

$+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3

$+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3

$$\frac{dT_{16}}{dt}=(b_{16})^{(2)}T_{17}-\begin{bmatrix}(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t)\\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)\\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t)\end{bmatrix}T_{16} \quad (64)$$

$$\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b'_{17})^{(2)}\boxed{-(b''_{17})^{(2)}(G_{19},t)} & \boxed{-(b''_{14})^{(1,1,)}(G,t)} & \boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{17} \quad (65)$$

$$\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b'_{18})^{(2)}\boxed{-(b''_{18})^{(2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,)}(G,t)} & \boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{18} \quad (66)$$

where $\boxed{-(b''_{16})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{17})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{18})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,)}(G,t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{44})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2 and 3

$$\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a'_{20})^{(3)}\boxed{+(a''_{20})^{(3)}(T_{21},t)} & \boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{20} \quad (67)$$

$$\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a'_{21})^{(3)}\boxed{+(a''_{21})^{(3)}(T_{21},t)} & \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{21} \quad (68)$$

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a'_{22})^{(3)}\boxed{+(a''_{22})^{(3)}(T_{21},t)} & \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{22} \quad (69)$$

$\boxed{+(a''_{20})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

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| $+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)}\,-(b_{20}^{''})^{(3)}(G_{23},t) & -(b_{16}^{''})^{(2,2,2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,1,)}(G,t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{20}$ | *70* |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)}\,-(b_{21}^{''})^{(3)}(G_{23},t) & -(b_{17}^{''})^{(2,2,2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,1,)}(G,t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21}$ | *71* |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b_{22}^{'})^{(3)}\,-(b_{22}^{''})^{(3)}(G_{23},t) & -(b_{18}^{''})^{(2,2,2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,1,)}(G,t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{22}$ | *72* |
| $-(b_{20}^{''})^{(3)}(G_{23},t)$, $-(b_{21}^{''})^{(3)}(G_{23},t)$, $-(b_{22}^{''})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*<br>$-(b_{16}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*<br>$-(b_{13}^{''})^{(1,1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition* | |

*coefficients for category 1, 2 and 3*

$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$ $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a_{24}^{'})^{(4)} & +(a_{24}^{''})^{(4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,)}(T_{29},t) & +(a_{32}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a_{25}^{'})^{(4)} & +(a_{25}^{''})^{(4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,)}(T_{29},t) & +(a_{33}^{''})^{(6,6)}(T_{33},t) \\ & +(a_{14}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a_{26}^{'})^{(4)} & +(a_{26}^{''})^{(4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,)}(T_{29},t) & +(a_{34}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a_{24}^{''})^{(4)}(T_{25},t)$, $(a_{25}^{''})^{(4)}(T_{25},t)$, $(a_{26}^{''})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a_{28}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3

$+(a_{32}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3

$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$ *are fourth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$ *are fifth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$ *are sixth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$ *are seventh augmentation coefficients for category* 1, 2 *and* 3

$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$ *are eighth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

| | |
|---|---|
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b'_{24})^{(4)}\boxed{-(b''_{24})^{(4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1)}(G,t)} & \boxed{-(b''_{16})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{24}$ | 76 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b'_{25})^{(4)}\boxed{-(b''_{25})^{(4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1)}(G,t)} & \boxed{-(b''_{17})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{25}$ | 77 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b'_{26})^{(4)}\boxed{-(b''_{26})^{(4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{26}$ | 78 |
| *Where* $\boxed{-(b''_{24})^{(4)}(G_{27},t)}, \boxed{-(b''_{25})^{(4)}(G_{27},t)}, \boxed{-(b''_{26})^{(4)}(G_{27},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b''_{28})^{(5,5,)}(G_{31},t)}, \boxed{-(b''_{29})^{(5,5,)}(G_{31},t)}, \boxed{-(b''_{30})^{(5,5,)}(G_{31},t)}$ *are second detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b''_{32})^{(6,6,)}(G_{35},t)}, \boxed{-(b''_{33})^{(6,6,)}(G_{35},t)}, \boxed{-(b''_{34})^{(6,6,)}(G_{35},t)}$ *are third detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b''_{13})^{(1,1,1,1)}(G,t)}, \boxed{-(b''_{14})^{(1,1,1,1)}(G,t)}, \boxed{-(b''_{15})^{(1,1,1,1)}(G,t)}$<br>*are fourth detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b''_{16})^{(2,2,2,2)}(G_{19},t)}, \boxed{-(b''_{17})^{(2,2,2,2)}(G_{19},t)}, \boxed{-(b''_{18})^{(2,2,2,2)}(G_{19},t)}$<br>*are fifth detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b''_{20})^{(3,3,3,3)}(G_{23},t)}, \boxed{-(b''_{21})^{(3,3,3,3)}(G_{23},t)}, \boxed{-(b''_{22})^{(3,3,3,3)}(G_{23},t)}$<br>*are sixth detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b''_{36})^{(7,7,7,7,7)}(G_{39},t)}, \boxed{-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)}, \boxed{-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)}$<br>*are seventh detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b''_{40})^{(8,8,8,8,8)}(G_{43},t)}, \boxed{-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)}, \boxed{-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)}$<br>*are eighth detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b''_{46})^{(9,9,9,9)}(G_{47},t)}, \boxed{-(b''_{45})^{(9,9,9,9)}(G_{47},t)}, \boxed{-(b''_{44})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a'_{28})^{(5)}\boxed{+(a''_{28})^{(5)}(T_{29},t)} & \boxed{+(a''_{24})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{32})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{28}$ | 79 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a'_{29})^{(5)}\boxed{+(a''_{29})^{(5)}(T_{29},t)} & \boxed{+(a''_{25})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{33})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{29}$ | 80 |

$$\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[\begin{array}{ccc} (a'_{30})^{(5)} & +(a''_{30})^{(5)}(T_{29},t) & +(a''_{26})^{(4,4,)}(T_{25},t) \\ +(a''_{34})^{(6,6,6)}(T_{33},t) & +(a''_{15})^{(1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2)}(T_{17},t) \\ +(a''_{22})^{(3,3,3,3,3)}(T_{21},t) & +(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t) \\ +(a''_{46})^{(9,9,9,9,9)}(T_{45},t) & & \end{array}\right] G_{30} \quad (81)$$

*Where* $+(a''_{28})^{(5)}(T_{29},t)$, $+(a''_{29})^{(5)}(T_{29},t)$, $+(a''_{30})^{(5)}(T_{29},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

*And* $+(a''_{24})^{(4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,)}(T_{25},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3

$+(a''_{32})^{(6,6,6)}(T_{33},t)$, $+(a''_{33})^{(6,6,6)}(T_{33},t)$, $+(a''_{34})^{(6,6,6)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3

$+(a''_{16})^{(2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3

$+(a''_{20})^{(3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3

$+(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t)$ , $+(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3

$+(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3

$+(a''_{46})^{(9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9)}(T_{45},t)$, $+(a''_{44})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[\begin{array}{ccc} (b'_{28})^{(5)} & -(b''_{28})^{(5)}(G_{31},t) & -(b''_{24})^{(4,4,)}(G_{27},t) \\ -(b''_{32})^{(6,6,6)}(G_{35},t) & -(b''_{13})^{(1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2)}(G_{19},t) \\ -(b''_{20})^{(3,3,3,3,3)}(G_{23},t) & -(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t) \\ -(b''_{44})^{(9,9,9,9,9)}(G_{47},t) & & \end{array}\right] T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[\begin{array}{ccc} (b'_{29})^{(5)} & -(b''_{29})^{(5)}(G_{31},t) & -(b''_{25})^{(4,4,)}(G_{27},t) \\ -(b''_{33})^{(6,6,6)}(G_{35},t) & -(b''_{14})^{(1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2)}(G_{19},t) \\ -(b''_{21})^{(3,3,3,3,3)}(G_{23},t) & -(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t) \\ -(b''_{45})^{(9,9,9,9,9)}(G_{47},t) & & \end{array}\right] T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[\begin{array}{ccc} (b'_{30})^{(5)} & -(b''_{30})^{(5)}(G_{31},t) & -(b''_{26})^{(4,4,)}(G_{27},t) \\ -(b''_{34})^{(6,6,6)}(G_{35},t) & -(b''_{15})^{(1,1,1,1,1,)}(G,t) & -(b''_{18})^{(2,2,2,2,2)}(G_{19},t) \\ -(b''_{22})^{(3,3,3,3,3)}(G_{23},t) & -(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t) \\ -(b''_{46})^{(9,9,9,9,9)}(G_{47},t) & & \end{array}\right] T_{30} \quad (84)$$

*where* $-(b''_{28})^{(5)}(G_{31},t)$ , $-(b''_{29})^{(5)}(G_{31},t)$ , $-(b''_{30})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{24})^{(4,4,)}(G_{27},t)$, $-(b''_{25})^{(4,4,)}(G_{27},t)$, $-(b''_{26})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3

$-(b''_{32})^{(6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3

| | |
|---|---|
| $-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3<br>$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3<br>$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3<br>$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3<br>$-(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3<br>$-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3 | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a_{32}^{'})^{(6)} & +(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,4,)}(T_{25},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{32}$ | *85* |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a_{33}^{'})^{(6)} & +(a_{33}^{''})^{(6)}(T_{33},t) & +(a_{29}^{''})^{(5,5,5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,4,)}(T_{25},t) \\ & +(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{33}$ | *86* |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \begin{bmatrix} (a_{34}^{'})^{(6)} & +(a_{34}^{''})^{(6)}(T_{33},t) & +(a_{30}^{''})^{(5,5,5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,4,)}(T_{25},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34}$ | *87* |
| $+(a_{32}^{''})^{(6)}(T_{33},t)$, $+(a_{33}^{''})^{(6)}(T_{33},t)$, $+(a_{34}^{''})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{28}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients<br>$+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients<br>$+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients<br>$+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients | |

| | |
|---|---|
| $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients | |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b_{32}^{'})^{(6)} & -(b_{32}^{''})^{(6)}(G_{35},t) & -(b_{28}^{''})^{(5,5,5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,4,)}(G_{27},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{32}$ | *88* |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b_{33}^{'})^{(6)} & -(b_{33}^{''})^{(6)}(G_{35},t) & -(b_{29}^{''})^{(5,5,5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,4,)}(G_{27},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{33}$ | *89* |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b_{34}^{'})^{(6)} & -(b_{34}^{''})^{(6)}(G_{35},t) & -(b_{30}^{''})^{(5,5,5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,4,)}(G_{27},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t) & \\ -(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{34}$ | *90* |
| | |
| $-(b_{32}^{''})^{(6)}(G_{35},t)$, $-(b_{33}^{''})^{(6)}(G_{35},t)$, $-(b_{34}^{''})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3 $-(b_{28}^{''})^{(5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3 $-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3 $-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3 $-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3 $-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3 $-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3 $-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3 $-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3 | |

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a'_{36})^{(7)}\boxed{+(a''_{36})^{(7)}(T_{37},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a'_{37})^{(7)}\boxed{+(a''_{37})^{(7)}(T_{37},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a'_{38})^{(7)}\boxed{+(a''_{38})^{(7)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad (93)$$

Where $\boxed{(a''_{36})^{(7)}(T_{37},t)}$, $\boxed{(a''_{37})^{(7)}(T_{37},t)}$, $\boxed{(a''_{38})^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = \quad (94)$$

$$(b_{36})^{(7)}T_{37} - \begin{bmatrix} (b'_{36})^{(7)}\boxed{-(b''_{36})^{(7)}(G_{39},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{37}}{dt} =$$

$$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b'_{37})^{(7)}\boxed{-(b''_{37})^{(7)}(G_{39},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} =$$

$$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b'_{38})^{(7)}\boxed{-(b''_{38})^{(7)}(G_{39},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a'_{40})^{(8)}\boxed{+(a''_{40})^{(8)}(T_{41},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a'_{41})^{(8)}\boxed{+(a''_{41})^{(8)}(T_{41},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a'_{42})^{(8)}\boxed{+(a''_{42})^{(8)}(T_{41},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{40})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b'_{40})^{(8)}\boxed{-(b''_{40})^{(8)}(G_{43},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} =$$

$$(b_{41})^{(8)}T_{40} - \begin{bmatrix} (b'_{41})^{(8)}\boxed{-(b''_{41})^{(8)}(G_{43},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b'_{42})^{(8)}\boxed{-(b''_{42})^{(8)}(G_{43},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} =$$

$$(b_{45})^{(9)}T_{44}-\begin{bmatrix}(b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}\\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}\\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}\end{bmatrix}T_{14}$$

$$\frac{dT_{46}}{dt}=$$

$$(b_{46})^{(9)}T_{45}-\begin{bmatrix}(b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}\\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}\\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}\end{bmatrix}T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)},(a_i^{'})^{(1)},(a_i^{''})^{(1)},(b_i)^{(1)},(b_i^{'})^{(1)},(b_i^{''})^{(1)}>0,\quad i,j=13,14,15$ *97*

The functions $(a_i^{''})^{(1)},(b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

$(a_i^{''})^{(1)}(T_{14},t)\le(p_i)^{(1)}\le(\hat{A}_{13})^{(1)}$

$(b_i^{''})^{(1)}(G,t)\le(r_i)^{(1)}\le(b_i^{'})^{(1)}\le(\hat{B}_{13})^{(1)}$

$lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t)=(p_i)^{(1)}$ 98

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| $\lim_{G\to\infty}(b_i'')^{(1)}(G,t) = (r_i)^{(1)}$<br><br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br><br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(1)}(T_{14}', t) - (a_i'')^{(1)}(T_{14}, t)\rvert \le (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}'\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\lvert(b_i'')^{(1)}(G', t) - (b_i'')^{(1)}(G, t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G'\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(1)}(T_{14}', t)$ and $(a_i'')^{(1)}(T_{14}, t)$ . $(T_{14}', t)$ and $(T_{14}, t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i'')^{(1)}(T_{14}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i'')^{(1)}(T_{14}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i')^{(1)}, (b_i)^{(1)}, (b_i')^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i, j = 16,17,18$ | |
| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17}, t) \le (p_i)^{(2)} \le (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19}, t) \le (r_i)^{(2)} \le (b_i')^{(2)} \le (\hat{B}_{16})^{(2)}$ | 103 |

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| $\lim_{T_2\to\infty}(a_i^{''})^{(2)}(T_{17},t)=(p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty}(b_i^{''})^{(2)}\big((G_{19}),t\big)=\ (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)}$ :<br>Where $\boxed{(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)},(p_i)^{(2)},\ (r_i)^{(2)}}$ are positive constants and $\boxed{i=16{,}17{,}18}$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'},t)-(a_i^{''})^{(2)}(T_{17},t)\rvert\le(\hat{k}_{16})^{(2)}\lvert T_{17}-T_{17}^{'}\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'},t)-(b_i^{''})^{(2)}\big((G_{19}),t\big)\rvert<(\hat{k}_{16})^{(2)}\lvert\lvert(G_{19})-(G_{19})^{'}\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'},t)$ and$(a_i^{''})^{(2)}(T_{17},t)$ . $(T_{17}^{'},t)$ and $(T_{17},t)$ are points belonging to the interval $\big[(\hat{k}_{16})^{(2)},(\hat{M}_{16})^{(2)}\big]$. It is to be noted that $(a_i^{''})^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)}=1$ then the function $(a_i^{''})^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\ ,\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}<1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)},(\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)},(\hat{A}_{16})^{(2)}and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)},(a_i^{'})^{(2)},(b_i)^{(2)},(b_i^{'})^{(2)},(p_i)^{(2)},\ (r_i)^{(2)},i=16{,}17{,}18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)}+(a_i^{'})^{(2)}+\ (\hat{A}_{16})^{(2)}+\ (\hat{P}_{16})^{(2)}\ (\hat{k}_{16})^{(2)}]<1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[\ (b_i)^{(2)}+(b_i^{'})^{(2)}+\ (\hat{B}_{16})^{(2)}+\ (\hat{Q}_{16})^{(2)}\ (\hat{k}_{16})^{(2)}]<1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)},(a_i^{'})^{(3)},(a_i^{''})^{(3)},(b_i)^{(3)},(b_i^{'})^{(3)},(b_i^{''})^{(3)}>0,\quad i,j=20{,}21{,}22$<br>The functions $(a_i^{''})^{(3)},(b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(3)},\ (\mathrm{r_i})^{(3)}$: | 112 |

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| $(a_i^{''})^{(3)}(T_{21},t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23},t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(3)}(T_{21},t) = (p_i)^{(3)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'},t) - (a_i^{''})^{(3)}(T_{21},t)\rvert \leq (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'},t) - (b_i^{''})^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'},t)$ and $(a_i^{''})^{(3)}(T_{21},t)$ . $(T_{21}^{'},t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)},$ are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$: | 117 |

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| $(a_i^{''})^{(4)}(T_{25},t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}((G_{27}),t) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(4)}(T_{25},t) = (p_i)^{(4)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(4)}((G_{27}),t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'},t) - (a_i^{''})^{(4)}(T_{25},t)\rvert \leq (\hat{k}_{24})^{(4)}\lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'},t) - (b_i^{''})^{(4)}((G_{27}),t)\rvert < (\hat{k}_{24})^{(4)}\lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)}t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'},t)$ and$(a_i^{''})^{(4)}(T_{25},t)$ . $(T_{25}^{'},t)$ and $(T_{25},t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)}(\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)} + (b_i^{'})^{(4)} + (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}(\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded. | 122 |

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| **Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br>$(a_i^{''})^{(5)}(T_{29},t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br>$(b_i^{''})^{(5)}\big((G_{31}),t\big) \leq (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(5)}(T_{29},t) = (p_i)^{(5)}$<br>$\lim_{G\to\infty}(b_i^{''})^{(5)}(G_{31},t) = (r_i)^{(5)}$<br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'},t) - (a_i^{''})^{(5)}(T_{29},t)\rvert \leq (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'},t) - (b_i^{''})^{(5)}\big((G_{31}),t\big)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'},t)$ and$(a_i^{''})^{(5)}(T_{29},t)$ . $(T_{29}^{'},t)$ and $(T_{29},t)$ are points belonging to the interval $\left[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\right]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$<br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$ | 126 |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$ | 127 |

The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:

$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$

$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$

$lim_{T_2 \to \infty}(a_i^{''})^{(6)}(T_{33}, t) = (p_i)^{(6)}$

$\lim_{G \to \infty}(b_i^{''})^{(6)}((G_{35}), t) = (r_i)^{(6)}$

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**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :

Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$

They satisfy Lipschitz condition:

$|(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)| \leq (\hat{k}_{32})^{(6)}|T_{33} - T_{33}^{'}|e^{-(\hat{M}_{32})^{(6)}t}$

$|(b_i^{''})^{(6)}((G_{35})^{'}, t) - (b_i^{''})^{(6)}((G_{35}), t)| < (\hat{k}_{32})^{(6)}||(G_{35}) - (G_{35})^{'}||e^{-(\hat{M}_{32})^{(6)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and$(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $\left[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\right]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :

$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants

$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$

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**Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :

There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities

$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$

$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$

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Where we suppose

(CCCCCC) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i,j = 36,37,38$ 131

(DDDDDD) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}$, $(r_i)^{(7)}$:

$$(a_i^{''})^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$$

$$(b_i^{''})^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$$

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*(EEEEEE)* $\lim_{T_2 \to \infty} (a_i^{''})^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(FFFFFF)*

$\lim_{G \to \infty} (b_i^{''})^{(7)}\big((G_{39}), t\big) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

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They satisfy Lipschitz condition:

$$|(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)} |T_{37} - T_{37}^{'}| e^{-(\hat{M}_{36})^{(7)} t}$$

$$|(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}\big((G_{39}), t\big)| < (\hat{k}_{36})^{(7)} ||(G_{39}) - (G_{39})^{'}|| e^{-(\hat{M}_{36})^{(7)} t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'}, t)$ and $(a_i^{''})^{(7)}(T_{37}, t)$ . $(T_{37}^{'}, t)$ and $(T_{37}, t)$ are points belonging to the interval $\big[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}\big]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

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**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :

(GGGGGG) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$, are positive constants

$$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$$

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**Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :

(HHHHHH) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities

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| $\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i')^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i')^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i')^{(8)}, (a_i'')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (b_i'')^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i'')^{(8)}, (b_i'')^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i'')^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i'')^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i')^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty}(a_i'')^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty}(b_i'')^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(8)}(T_{41}', t) - (a_i'')^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)}\lvert T_{41} - T_{41}'\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 142 |
| $\lvert(b_i'')^{(8)}((G_{43})', t) - (b_i'')^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)}\lvert\lvert(G_{43}) - (G_{43})'\rvert\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(8)}(T_{41}', t)$ and $(a_i'')^{(8)}(T_{41}, t)$ . $(T_{41}', t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i'')^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i'')^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ : | |

There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$

Satisfy the inequalities

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1 \quad (145)$$

$$\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1 \quad (146)$$

Where we suppose

$$(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i,j = 44,45,46 \quad (146A)$$

The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:

$$(a_i'')^{(9)}(T_{45},t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$$

$$(b_i'')^{(9)}(G_{47},t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$$

$$\lim_{T_2\to\infty}(a_i'')^{(9)}(T_{45},t) = (p_i)^{(9)}$$

$$\lim_{G\to\infty}(b_i'')^{(9)}(G_{47},t) = (r_i)^{(9)}$$

**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :

Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$

They satisfy Lipschitz condition:

$$|(a_i'')^{(9)}(T_{45}',t) - (a_i'')^{(9)}(T_{45},t)| \le (\hat{k}_{44})^{(9)}|T_{45} - T_{45}'|e^{-(\hat{M}_{44})^{(9)}t}$$

$$|(b_i'')^{(9)}((G_{47})',t) - (b_i'')^{(9)}((G_{47}),t)| < (\hat{k}_{44})^{(9)}||(G_{47}) - (G_{47})'||e^{-(\hat{M}_{44})^{(9)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}',t)$ and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i'')^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants

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| $\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$ | |
| **Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :<br><br>There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}, i = 44,45,46,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$<br><br>$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$ | |
| **Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>Definition of $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 147 |
| **Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$<br><br>$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $G_i(0) = G_i^0 > 0$<br><br>$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | 148 |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br><br>$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions | 151 |

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| **Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le \left(\hat{P}_{13}\right)^{(1)}$ , $T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le \left(\hat{P}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{16})^{(2)}$, $T_i^0 \le (\hat{Q}_{16})^{(2)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[ (a_{18})^{(2)} G_{17}(s_{(16)}) - \left( (a_{18}')^{(2)} + (a_{18}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{18}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[ (b_{16})^{(2)} T_{17}(s_{(16)}) - \left( (b_{16}')^{(2)} - (b_{16}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{16}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[ (b_{17})^{(2)} T_{16}(s_{(16)}) - \left( (b_{17}')^{(2)} - (b_{17}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[ (b_{18})^{(2)} T_{17}(s_{(16)}) - \left( (b_{18}')^{(2)} - (b_{18}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{18}(s_{(16)}) \right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |

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| **Proof:**<br><br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{20})^{(3)}$ , $T_i^0 \leq (\hat{Q}_{20})^{(3)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br><br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a_{20}')^{(3)} + a_{20}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a_{21}')^{(3)} + (a_{21}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a_{22}')^{(3)} + (a_{22}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b_{20}')^{(3)} - (b_{20}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b_{21}')^{(3)} - (b_{21}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b_{22}')^{(3)} - (b_{22}'')^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br><br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{24})^{(4)}$ , $T_i^0 \leq (\hat{Q}_{24})^{(4)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br><br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a_{24}')^{(4)} + a_{24}'')^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a_{25}')^{(4)} + (a_{25}'')^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a_{26}')^{(4)} + (a_{26}'')^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |

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| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[(b_{24})^{(4)} T_{25}(s_{(24)}) - \left((b_{24}')^{(4)} - (b_{24}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{24}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[(b_{25})^{(4)} T_{24}(s_{(24)}) - \left((b_{25}')^{(4)} - (b_{25}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b_{26}')^{(4)} - (b_{26}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{28})^{(5)}$ , $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a_{28}')^{(5)} + a_{28}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a_{29}')^{(5)} + (a_{29}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a_{30}')^{(5)} + (a_{30}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b_{28}')^{(5)} - (b_{28}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b_{29}')^{(5)} - (b_{29}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b_{30}')^{(5)} - (b_{30}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{32})^{(6)}$ , $T_i^0 \le (\hat{Q}_{32})^{(6)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By | 164 |

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| $\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a'_{32})^{(6)} + a''_{32})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b'_{32})^{(6)} - (b''_{32})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b'_{33})^{(6)} - (b''_{33})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b'_{34})^{(6)} - (b''_{34})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \leq (\hat{P}_{36})^{(7)}$, $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a'_{36})^{(7)} + a''_{36})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b'_{36})^{(7)} - (b''_{36})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b'_{37})^{(7)} - (b''_{37})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b'_{38})^{(7)} - (b''_{38})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |

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| **Proof:**<br><br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{40})^{(8)}$ , $T_i^0 \leq (\hat{Q}_{40})^{(8)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| By<br><br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{40})^{(8)} + a''_{40})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{40}(s_{(40)}) \right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[ (a_{41})^{(8)} G_{40}(s_{(40)}) - \left( (a'_{41})^{(8)} + (a''_{41})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{41}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[ (a_{42})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{42})^{(8)} + (a''_{42})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{42}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[ (b_{40})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{40})^{(8)} - (b''_{40})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{40}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[ (b_{41})^{(8)} T_{40}(s_{(40)}) - \left( (b'_{41})^{(8)} - (b''_{41})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{41}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[ (b_{42})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{42})^{(8)} - (b''_{42})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{42}(s_{(40)}) \right] ds_{(40)}$<br><br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br><br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[ (a_{44})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{44}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[ (a_{45})^{(9)} G_{44}(s_{(44)}) - \left( (a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{45}(s_{(44)}) \right] ds_{(44)}$ | |

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| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{\mathrm{T}}_{46}(\mathrm{t}) = \mathrm{T}_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[((\hat{P}_{13})^{(1)} + G_{14}^0) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that<br>$(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left[((\hat{P}_{16})^{(2)} + G_{17}^0) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} \left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$<br>$\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}} \left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$ | 171 |

From which it follows that

$$(G_{20}(t)-G_{20}^{0})e^{-(\bar{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^{0})e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^{0}}{G_{21}^{0}}\right)}+(\hat{P}_{20})^{(3)}\right] \quad (172)$$

Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$

The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that

$$G_{24}(t) \le G_{24}^{0}+\int_{0}^{t}\left[(a_{24})^{(4)}\left(G_{25}^{0}+(\hat{P}_{24})^{(4)}e^{(\bar{M}_{24})^{(4)}s_{(24)}}\right)\right]ds_{(24)} =$$

$$(1+(a_{24})^{(4)}t)G_{25}^{0}+\frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left(e^{(\bar{M}_{24})^{(4)}t}-1\right) \quad (173)$$

From which it follows that

$$(G_{24}(t)-G_{24}^{0})e^{-(\bar{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^{0})e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^{0}}{G_{25}^{0}}\right)}+(\hat{P}_{24})^{(4)}\right] \quad (174)$$

$(G_i^0)$ is as defined in the statement of theorem 4

The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{28}(t) \le G_{28}^{0}+\int_{0}^{t}\left[(a_{28})^{(5)}\left(G_{29}^{0}+(\hat{P}_{28})^{(5)}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\right)\right]ds_{(28)} =$$

$$(1+(a_{28})^{(5)}t)G_{29}^{0}+\frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)}t}-1\right)$$

From which it follows that

$$(G_{28}(t)-G_{28}^{0})e^{-(\bar{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^{0})e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^{0}}{G_{29}^{0}}\right)}+(\hat{P}_{28})^{(5)}\right] \quad (175)$$

$(G_i^0)$ is as defined in the statement of theorem 5

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{32}(t) \le G_{32}^{0}+\int_{0}^{t}\left[(a_{32})^{(6)}\left(G_{33}^{0}+(\hat{P}_{32})^{(6)}e^{(\bar{M}_{32})^{(6)}s_{(32)}}\right)\right]ds_{(32)} =$$

$$(1+(a_{32})^{(6)}t)G_{33}^{0}+\frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left(e^{(\bar{M}_{32})^{(6)}t}-1\right) \quad (176)$$

From which it follows that

$$(G_{32}(t)-G_{32}^{0})e^{-(\bar{M}_{32})^{(6)}t} \le \frac{(a_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^{0})e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^{0}}{G_{33}^{0}}\right)}+(\hat{P}_{32})^{(6)}\right] \quad (177)$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}$ , $G_{26}, T_{24}, T_{25}, T_{26}$

(w) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 178

$$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} \left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$$
$$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$$

From which it follows that

$$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left[\left((\hat{P}_{36})^{(7)} + G_{37}^0\right) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} \left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$$
$$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$$ 180

From which it follows that 181

$$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left[\left((\hat{P}_{40})^{(8)} + G_{41}^0\right) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 8

Analogous inequalities hold also for $G_{41}$ , $G_{42}, T_{40}, T_{41}, T_{42}$

The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} \left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$$
$$\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left(e^{(\hat{M}_{44})^{(9)} t} - 1\right)$$

From which it follows that

$$(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left[\left((\hat{P}_{44})^{(9)} + G_{45}^0\right) e^{\left(-\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0}\right)} + (\hat{P}_{44})^{(9)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 9

Analogous inequalities hold also for $G_{45}$ , $G_{46}, T_{44}, T_{45}, T_{46}$

It is now sufficient to take $\frac{(a_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\widehat{M}_{13})^{(1)}} < 1$ and to choose

$(\widehat{P}_{13})^{(1)}$ and $(\widehat{Q}_{13})^{(1)}$ large to have

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$$\frac{(a_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}\left[(\widehat{P}_{13})^{(1)} + \left((\hat{P}_{13})^{(1)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{13})^{(1)}$$

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$$\frac{(b_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{13})^{(1)}\right] \leq (\hat{Q}_{13})^{(1)}$$

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In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric

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$$d\left(\left(G^{(1)}, T^{(1)}\right), \left(G^{(2)}, T^{(2)}\right)\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{13})^{(1)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{13})^{(1)}t}\}$$

Indeed if we denote

**Definition of** $\tilde{G}, \tilde{T}$ : $(\tilde{G}, \tilde{T}) = \mathcal{A}^{(1)}(G, T)$

It results

$$\left|\tilde{G}_{13}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{13})^{(1)}\left|G_{14}^{(1)} - G_{14}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\,ds_{(13)} +$$

$$\int_0^t\{(a'_{13})^{(1)}\left|G_{13}^{(1)} - G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} +$$

$$(a''_{13})^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right)\left|G_{13}^{(1)} - G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}} +$$

$$G_{13}^{(2)}\left|(a''_{13})^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right) - (a''_{13})^{(1)}\left(T_{14}^{(2)}, s_{(13)}\right)\right|\ e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$$

Where $s_{(13)}$ represents integrand that is integrated over the interval [0, t]

From the hypotheses it follows

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$$\left|G^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}t} \leq$$
$$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)}, T^{(1)};\ G^{(2)}, T^{(2)}\right)\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

**Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate

condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 2:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$

From 19 to 24 it results

$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(1)} - (a_i^{''})^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$

$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(1)}t)} > 0$ for t > 0

**Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$ : 187

**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if

$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a_{14}^{'})^{(1)}G_{14}$ and by integrating

$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$

In the same way , one can obtain

$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}^{'})^{(1)}$

If $G_{14}\ or\ G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively.

**Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}\ and\ G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. 188

**Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t), t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$. 189

**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :

Indeed let $t_1$ be so that for $t > t_1$

$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t), t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$

Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to

$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results

$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right),\quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.
The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t), t) = (b_{15}^{'})^{(1)}$

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| We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ $and$ $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)}, (T_{19})^{(1)}), ((G_{19})^{(2)}, (T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)}\left\|G_{17}^{(1)} - G_{17}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\,ds_{(16)} +$<br>$\int_0^t\{(a_{16}')^{(2)}\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{-(\widehat{M}_{16})^{(2)}s_{(16)}} +$<br>$(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}} +$<br>$G_{16}^{(2)}\left\|(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}'')^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\|\ e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}')^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right)d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |

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| **Remark 6:** The fact that we supposed $(a_{16}'')^{(2)}$ and $(b_{16}'')^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ and $(\widehat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(2)}$ and $(b_i'')^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(2)} - (a_i'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(2)}t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $((\widehat{M}_{16})^{(2)})_1, ((\widehat{M}_{16})^{(2)})_2$ and $((\widehat{M}_{16})^{(2)})_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq ((\widehat{M}_{16})^{(2)})_1 - (a_{17}')^{(2)}G_{17}$ and by integrating<br><br>$G_{17} \leq ((\widehat{M}_{16})^{(2)})_2 = G_{17}^0 + 2(a_{17})^{(2)}((\widehat{M}_{16})^{(2)})_1/(a_{17}')^{(2)}$<br><br>In the same way , one can obtain<br><br>$G_{18} \leq ((\widehat{M}_{16})^{(2)})_3 = G_{18}^0 + 2(a_{18})^{(2)}((\widehat{M}_{16})^{(2)})_2/(a_{18}')^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | 201 |
| **Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t), t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t), t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded. | 205 |

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| The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right)=(b_{18}')^{(2)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}<1$ and to choose<br>$(\widehat{P}_{20})^{(3)}$ and $(\widehat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[(\widehat{P}_{20})^{(3)}+\left((\hat{P}_{20})^{(3)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{20})^{(3)}\right]\leq(\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | 211 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}},\widetilde{T_{23}}:(\widetilde{(G_{23})},\widetilde{(T_{23})})=\mathcal{A}^{(3)}((G_{23}),(T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{20})^{(3)}\left\|G_{21}^{(1)}-G_{21}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)}+$<br>$\int_0^t\{(a_{20}')^{(3)}\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$(a_{20}'')^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$G_{20}^{(2)}\left\\|(a_{20}'')^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)-(a_{20}'')^{(3)}\left(T_{21}^{(2)},s_{(20)}\right)\right\\|\ e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)}-G_{23}{}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}t}\leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)}+(a_{20}')^{(3)}+(\widehat{A}_{20})^{(3)}+(\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)},(T_{23})^{(1)};(G_{23})^{(2)},(T_{23})^{(2)})\right)$ | 214 |

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| And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ *and* $(\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)}t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}\ and\ G_{22}$ . indeed if<br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)}G_{21}$ and by integrating<br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br>In the same way , one can obtain<br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br>If $G_{21}\ or\ G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}\ and\ G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t), t\right)) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br>Indeed let $t_3$ be so that for $t > t_3$<br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t), t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results | 220 |

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| $T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right), \quad t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded. The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right) = \mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br><br>It results<br><br>$\left\|\tilde{G}_{24}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{24})^{(4)}\left\|G_{25}^{(1)} - G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)} +$<br><br>$\int_0^t\{(a_{24}^{'})^{(4)}\left\|G_{24}^{(1)} - G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}} +$<br><br>$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)} - G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}} +$<br><br>$G_{24}^{(2)}\left\|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right) - (a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\| \; e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br><br>Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)} - (G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t} \leq$ | 226 |

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| $\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};(G_{27})^{(2)},(T_{27})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]}\geq 0$<br>$T_i(t)\geq T_i^0 e^{\left(-(b_i^{'})^{(4)}t\right)}>0$ for t > 0 | 228 |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1,\left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ :<br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if<br>$G_{24}<(\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt}\leq\left((\widehat{M}_{24})^{(4)}\right)_1-(a_{25}^{'})^{(4)}G_{25}$ and by integrating<br>$G_{25}\leq\left((\widehat{M}_{24})^{(4)}\right)_2=G_{25}^0+2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a_{25}^{'})^{(4)}$<br>In the same way , one can obtain<br>$G_{26}\leq\left((\widehat{M}_{24})^{(4)}\right)_3=G_{26}^0+2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a_{26}^{'})^{(4)}$<br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t),t))=(b_{25}^{'})^{(4)}$ then $T_{25}\to\infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t>t_4$<br>$(b_{25})^{(4)}-(b_i^{''})^{(4)}((G_{27})(t),t)<\varepsilon_4, T_{24}(t)>(m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt}\geq(a_{25})^{(4)}(m)^{(4)}-\varepsilon_4 T_{25}$ which leads to | 232 |

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1-e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right), \quad t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42

Analogous inequalities hold also for $G_{29}, G_{30}, T_{28}, T_{29}, T_{30}$

It is now sufficient to take $\frac{(a_i)^{(5)}}{(\bar{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\bar{M}_{28})^{(5)}} < 1$ and to choose

$(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ large to have

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$$\frac{(a_i)^{(5)}}{(\bar{M}_{28})^{(5)}}\left[(\hat{P}_{28})^{(5)} + \left((\hat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{28})^{(5)}$$

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$$\frac{(b_i)^{(5)}}{(\bar{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{28})^{(5)}\right] \leq (\hat{Q}_{28})^{(5)}$$

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In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric

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$$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{28})^{(5)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{28})^{(5)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{31}}), (\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right) = \mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$

It results

$$\left|\tilde{G}_{28}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{28})^{(5)}\left|G_{29}^{(1)} - G_{29}^{(2)}\right|e^{-(\bar{M}_{28})^{(5)}s_{(28)}}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)} +$$

$$\int_0^t\{(a_{28}^{'})^{(5)}\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\bar{M}_{28})^{(5)}s_{(28)}}e^{-(\bar{M}_{28})^{(5)}s_{(28)}} +$$

$$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\bar{M}_{28})^{(5)}s_{(28)}}e^{(\bar{M}_{28})^{(5)}s_{(28)}} +$$

$$G_{28}^{(2)}\left|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right) - (a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right| \; e^{-(\bar{M}_{28})^{(5)}s_{(28)}}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$$

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| Where $s_{(28)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on it follows | |
| $\left\|(G_{31})^{(1)} - (G_{31})^{(2)}\right\| e^{-(\widehat{M}_{28})^{(5)}t} \leq$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)} + (a'_{28})^{(5)} + (\widehat{A}_{28})^{(5)} + (\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right) d\left(((G_{31})^{(1)}, (T_{31})^{(1)};\ (G_{31})^{(2)}, (T_{31})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a''_{28})^{(5)}$ and $(b''_{28})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(5)}$ and $(b''_i)^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(5)} - (a''_i)^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(5)}t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}\ and\ G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a'_{29})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a'_{29})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a'_{30})^{(5)}$<br><br>If $G_{29}\ or\ G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}\ and\ G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(5)}((G_{31})(t), t)) = (b'_{29})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ : | 242 |

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| Indeed let $t_5$ be so that for $t > t_5$<br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t), t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t), t\right) = (b_{30}^{'})^{(5)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br>Analogous inequalities hold also for $G_{33}, G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\bar{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\bar{M}_{32})^{(6)}} < 1$ and to choose<br>$(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\bar{M}_{32})^{(6)}}\left[(\hat{P}_{32})^{(6)} + \left((\hat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\bar{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{32})^{(6)}\right] \leq (\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)}, (T_{35})^{(1)}), ((G_{35})^{(2)}, (T_{35})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}), (\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}), (\widetilde{T_{35}})\right) = \mathcal{A}^{(6)}\left((G_{35}), (T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{32})^{(6)}\left|G_{33}^{(1)} - G_{33}^{(2)}\right|e^{-(\bar{M}_{32})^{(6)}s_{(32)}}e^{(\bar{M}_{32})^{(6)}s_{(32)}}\,ds_{(32)} +$<br>$\int_0^t\{(a_{32}^{'})^{(6)}\left|G_{32}^{(1)} - G_{32}^{(2)}\right|e^{-(\bar{M}_{32})^{(6)}s_{(32)}}e^{-(\bar{M}_{32})^{(6)}s_{(32)}} +$ | 247 |

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| $(a_{32}^{''})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)\left\|G_{32}^{(1)} - G_{32}^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}} +$<br>$G_{32}^{(2)}\left\|(a_{32}^{''})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right) - (a_{32}^{''})^{(6)}\left(T_{33}^{(2)}, s_{(32)}\right)\right\| \; e^{-(\widehat{M}_{32})^{(6)}s_{(32)}} e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\} ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | |
| $\left\|(G_{35})^{(1)} - (G_{35})^{(2)}\right\| e^{-(\widehat{M}_{32})^{(6)}t} \leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)} + (a_{32}^{'})^{(6)} + (\widehat{A}_{32})^{(6)} + (\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right) d\left(\left((G_{35})^{(1)}, (T_{35})^{(1)}; \; (G_{35})^{(2)}, (T_{35})^{(2)}\right)\right)$<br>And analogous inequalities for $G_i \; and \; T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a_{32}^{''})^{(6)}$ and $(b_{32}^{''})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t} \; and \; (\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i = 32{,}33{,}34$ depend only on $T_{33}$ and respectively on $(G_{35})(and \; not \; on \; t)$ and hypothesis can replaced by a usual Lipschitz condition. | 249 |
| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0 \; and \; T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(6)} - (a_i^{''})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(6)}t\right)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2 \; and \; \left((\widehat{M}_{32})^{(6)}\right)_3$ :<br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33} \; and \; G_{34}$ . indeed if<br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}^{'})^{(6)}G_{33}$ and by integrating<br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1/(a_{33}^{'})^{(6)}$<br>In the same way , one can obtain<br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2/(a_{34}^{'})^{(6)}$<br>If $G_{33} \; or \; G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33} \; and \; G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |

**Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$. 253

**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :

Indeed let $t_6$ be so that for $t > t_6$

$(b_{33})^{(6)} - (b_i^{''})^{(6)}\big((G_{35})(t),t\big) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$

Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to 254

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right), \quad t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.
The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

Analogous inequalities hold also for $G_{37}, G_{38}, T_{36}, T_{37}, T_{38}$ 255

It is now sufficient to take $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ and to choose
$(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ large to have

$$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[(\hat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)} \quad 256$$

$$\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)} \quad 257$$

In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric 258

$$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right) =$$
$$\sup_i\{\max_{t\in\mathbb{R}_+}|G_i^{(1)}(t) - G_i^{(2)}(t)|e^{-(\hat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}|T_i^{(1)}(t) - T_i^{(2)}(t)|e^{-(\hat{M}_{36})^{(7)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{39})}, \widetilde{(T_{39})}$ : $\left(\widetilde{(G_{39})}, \widetilde{(T_{39})}\right) = \mathcal{A}^{(7)}((G_{39}),(T_{39}))$
It results

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| $\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{36})^{(7)} \left\|G_{37}^{(1)}-G_{37}^{(2)}\right\| e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\, ds_{(36)} +$ $\int_0^t \{(a_{36}^{'})^{(7)} \left\|G_{36}^{(1)}-G_{36}^{(2)}\right\| e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} +$ $(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\| e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} e^{(\widehat{M}_{36})^{(7)}s_{(36)}} +$ $G_{36}^{(2)}\left\|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right)-(a_{36}^{''})^{(7)}\left(T_{37}^{(2)}, s_{(36)}\right)\right\| \; e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\} ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on it follows | |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\| e^{-(\widehat{M}_{36})^{(7)}t} \leq$ $\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}^{'})^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right) d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)}t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2$ *and* $\left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}$ *and* $G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)}G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1/(a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}^{'})^{(7)}$ | 262 |

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| If $G_{37}$ *or* $G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}$ *and* $G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t),t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right)$, $t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t),t\right) = (b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ and to choose<br>$(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\left[(\hat{P}_{40})^{(8)} + \left((\hat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{40})^{(8)}\right] \leq (\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{40})^{(8)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{40})^{(8)}t}\}$ | 269 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})}, \widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})}, \widetilde{(T_{43})}\right) = \mathcal{A}^{(8)}((G_{43}), (T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{40})^{(8)} \left\|G_{41}^{(1)} - G_{41}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}}\, ds_{(40)} +$<br>$\int_0^t \{(a'_{40})^{(8)} \left\|G_{40}^{(1)} - G_{40}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} +$<br>$(a''_{40})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}} +$<br>$G_{40}^{(2)} \left\|(a''_{40})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right) - (a''_{40})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)\right\| \; e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}}\} ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)} - (G_{43})^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} t} \leq$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)} + (a'_{40})^{(8)} + (\widehat{A}_{40})^{(8)} + (\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right) d\left(((G_{43})^{(1)}, (T_{43})^{(1)};\ (G_{43})^{(2)}, (T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a''_{40})^{(8)}$ and $(b''_{40})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)} t}$ *and* $(\widehat{Q}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(8)}$ and $(b''_i)^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(8)} - (a''_i)^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(8)} t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ *and* $G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a'_{41})^{(8)} G_{41}$ and by integrating | 276 |

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| $G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1/(a_{41}')^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2/(a_{42}')^{(8)}$<br><br>If $G_{41}$ *or* $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | |
| **Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}$ *and* $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(8)}((G_{43})(t),t)) = (b_{41}')^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i'')^{(8)}\left((G_{43})(t),t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$ | 278 |
| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right)$, $t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded. The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}'')^{(8)}\left((G_{43})(t),t(t),t\right) = (b_{42}')^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$ | |
| In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself | |
| The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric | |

$$d\left(((G_{47})^{(1)},(T_{47})^{(1)}),((G_{47})^{(2)},(T_{47})^{(2)})\right)=$$
$$sup_i\{\max_{t\in\mathbb{R}_+}|G_i^{(1)}(t)-G_i^{(2)}(t)|e^{-(\widehat{M}_{44})^{(9)}t},\max_{t\in\mathbb{R}_+}|T_i^{(1)}(t)-T_i^{(2)}(t)|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{47}}),(\widetilde{T_{47}}):\left((\widetilde{G_{47}}),(\widetilde{T_{47}})\right)=\mathcal{A}^{(9)}((G_{47}),(T_{47}))$
It results

$$|\tilde{G}_{44}^{(1)}-\tilde{G}_i^{(2)}|\le\int_0^t(a_{44})^{(9)}|G_{45}^{(1)}-G_{45}^{(2)}|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}ds_{(44)}+$$
$$\int_0^t\{(a_{44}^{'})^{(9)}|G_{44}^{(1)}-G_{44}^{(2)}|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$(a_{44}^{''})^{(9)}(T_{45}^{(1)},s_{(44)})|G_{44}^{(1)}-G_{44}^{(2)}|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$G_{44}^{(2)}|(a_{44}^{''})^{(9)}(T_{45}^{(1)},s_{(44)})-(a_{44}^{''})^{(9)}(T_{45}^{(2)},s_{(44)})|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$|(G_{47})^{(1)}-G^{(2)}|e^{-(\widehat{M}_{44})^{(9)}t}\le$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}((a_{44})^{(9)}+(a_{44}^{'})^{(9)}+(\widehat{A}_{44})^{(9)}+(\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)})d\left(((G_{47})^{(1)},(T_{47})^{(1)};\ (G_{47})^{(2)},(T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}\ and\ (\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i=44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

From 99 to 44 it results

$$G_i\ (t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(9)}-(a_i^{''})^{(9)}(T_{45}(s_{(44)}),s_{(44)})\}ds_{(44)}\right]}\ge 0$$

$$T_i\ (t)\ge T_i^0e^{(-(b_i^{'})^{(9)}t)}>0\quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1,\left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44}<(\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt}\le\left((\widehat{M}_{44})^{(9)}\right)_1-(a_{45}^{'})^{(9)}G_{45}$ and by integrating

$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}')^{(9)}$

In the same way , one can obtain

$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}')^{(9)}$

If $G_{45}$ *or* $G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}$ *and* $G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(9)}\left((G_{47})(t),t\right) = (b_{45}')^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i'')^{(9)}\left((G_{47})(t),t\right) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}'')^{(9)}\left((G_{47})(t),t\right) = (b_{46}')^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}')^{(1)} + (a_{14}')^{(1)} - (a_{13}'')^{(1)}(T_{14},t) + (a_{14}'')^{(1)}(T_{14},t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b_{13}')^{(1)} + (b_{14}')^{(1)} - (b_{13}'')^{(1)}(G,t) - (b_{14}'')^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$ , $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$ , $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)}, , (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$ , $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$ , $(\bar{u}_2)^{(1)} < 0$ the roots of the equations

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| $(a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$ | |
| **Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (v_0)^{(1)}$ :-<br><br>If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by<br>$(m_2)^{(1)} = (v_0)^{(1)}, (m_1)^{(1)} = (v_1)^{(1)},\ if\ (v_0)^{(1)} < (v_1)^{(1)}$<br><br>$(m_2)^{(1)} = (v_1)^{(1)}, (m_1)^{(1)} = (\bar{v}_1)^{(1)}, if\ (v_1)^{(1)} < (v_0)^{(1)} < (\bar{v}_1)^{(1)},$<br><br>and $\boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$<br><br>$(m_2)^{(1)} = (v_1)^{(1)}, (m_1)^{(1)} = (v_0)^{(1)},\ if\ (\bar{v}_1)^{(1)} < (v_0)^{(1)}$ | 283 |
| and analogously<br><br>$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)},\ if\ (u_0)^{(1)} < (u_1)^{(1)}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)},$<br><br>and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$<br><br>are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br><br>where $(p_i)^{(1)}$ is defined by equation<br><br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le\right.$<br><br>$\left.\frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a'_{15})^{(1)})}\left[e^{(S_1)^{(1)}t} - e^{-(a'_{15})^{(1)}t}\right] + G_{15}^0 e^{-(a'_{15})^{(1)}t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)} T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)} - (b'_{15})^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b'_{15})^{(1)}t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)}t} \le T_{15}(t) \le$ | 289 |

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| $\frac{(a_{15})^{(1)}T_{13}^{0}}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t}-e^{-(R_2)^{(1)}t}\right]+T_{15}^{0}e^{-(R_2)^{(1)}t}$ | |
| **Definition of** $(S_1)^{(1)},(S_2)^{(1)},(R_1)^{(1)},(R_2)^{(1)}$:- <br> Where $(S_1)^{(1)}=(a_{13})^{(1)}(m_2)^{(1)}-(a_{13}^{'})^{(1)}$ <br> $(S_2)^{(1)}=(a_{15})^{(1)}-(p_{15})^{(1)}$ <br> $(R_1)^{(1)}=(b_{13})^{(1)}(\mu_2)^{(1)}-(b_{13}^{'})^{(1)}$ <br> $(R_2)^{(1)}=(b_{15}^{'})^{(1)}-(r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** <br> **Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ : <br> $(\sigma_1)^{(2)},(\sigma_2)^{(2)},(\tau_1)^{(2)},(\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)}\le-(a_{16}^{'})^{(2)}+(a_{17}^{'})^{(2)}-(a_{16}^{''})^{(2)}(T_{17},t)+(a_{17}^{''})^{(2)}(T_{17},t)\le-(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)}\le-(b_{16}^{'})^{(2)}+(b_{17}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19}),t)-(b_{17}^{''})^{(2)}((G_{19}),t)\le-(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)},(\nu_2)^{(2)},(u_1)^{(2)},(u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)}>0$ , $(\nu_2)^{(2)}<0$ and respectively $(u_1)^{(2)}>0$ , $(u_2)^{(2)}<0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}(\nu^{(2)})^{2}+(\sigma_1)^{(2)}\nu^{(2)}-(a_{16})^{(2)}=0$ | 297 |
| and $(b_{14})^{(2)}(u^{(2)})^{2}+(\tau_1)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)},,(\bar{\nu}_2)^{(2)},(\bar{u}_1)^{(2)},(\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)}>0$ , $(\bar{\nu}_2)^{(2)}<0$ and respectively $(\bar{u}_1)^{(2)}>0$ , $(\bar{u}_2)^{(2)}<0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}(\nu^{(2)})^{2}+(\sigma_2)^{(2)}\nu^{(2)}-(a_{16})^{(2)}=0$ | 301 |
| and $(b_{17})^{(2)}(u^{(2)})^{2}+(\tau_2)^{(2)}u^{(2)}-(b_{16})^{(2)}=0$ | 302 |
| **Definition of** $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)},(m_2)^{(2)},(\mu_1)^{(2)},(\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)}=(\nu_0)^{(2)},(m_1)^{(2)}=(\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)}<(\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)}=(\nu_1)^{(2)},(m_1)^{(2)}=(\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)}<(\nu_0)^{(2)}<(\bar{\nu}_1)^{(2)}$, | 306 |

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| and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)},$ ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)},$ ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)},$<br>and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)},$ ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)} t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)} t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)} t}\right] + G_{18}^0 e^{-(S_2)^{(2)} t} \le G_{18}(t) \le \right.$<br>$\left.\frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a'_{18})^{(2)})}\left[e^{(S_1)^{(2)} t} - e^{-(a'_{18})^{(2)} t}\right] + G_{18}^0 e^{-(a'_{18})^{(2)} t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b'_{18})^{(2)})}\left[e^{(R_1)^{(2)} t} - e^{-(b'_{18})^{(2)} t}\right] + T_{18}^0 e^{-(b'_{18})^{(2)} t} \le T_{18}(t) \le$<br>$\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)} + (r_{16})^{(2)} + (R_2)^{(2)})}\left[e^{((R_1)^{(2)} + (r_{16})^{(2)})t} - e^{-(R_2)^{(2)} t}\right] + T_{18}^0 e^{-(R_2)^{(2)} t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a'_{16})^{(2)}$<br>$(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b'_{16})^{(2)}$<br>$(R_2)^{(2)} = (b'_{18})^{(2)} - (r_{18})^{(2)}$ | 318 |

**Behavior of the solutions** 319

**Theorem 3:** If we denote and define

**Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ :

$(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying

$$-(\sigma_2)^{(3)} \le -(a'_{20})^{(3)} + (a'_{21})^{(3)} - (a''_{20})^{(3)}(T_{21}, t) + (a''_{21})^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$$

$$-(\tau_2)^{(3)} \le -(b'_{20})^{(3)} + (b'_{21})^{(3)} - (b''_{20})^{(3)}(G_{23}, t) - (b''_{21})^{(3)}\big((G_{23}), t\big) \le -(\tau_1)^{(3)}$$

**Definition of** $(v_1)^{(3)}, (v_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ : 320

By $(v_1)^{(3)} > 0$ , $(v_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$ , $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}\big(v^{(3)}\big)^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\big(u^{(3)}\big)^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and

By $(\bar{v}_1)^{(3)} > 0$ , $(\bar{v}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$ , $(\bar{u}_2)^{(3)} < 0$ the

roots of the equations $(a_{21})^{(3)}\big(v^{(3)}\big)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\big(u^{(3)}\big)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$

**Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :- 321

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (v_0)^{(3)}, (m_1)^{(3)} = (v_1)^{(3)}$, ***if*** $(v_0)^{(3)} < (v_1)^{(3)}$

$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (\bar{v}_1)^{(3)}$ , ***if*** $(v_1)^{(3)} < (v_0)^{(3)} < (\bar{v}_1)^{(3)}$,

and $\boxed{(v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}}}$

$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (v_0)^{(3)}$, ***if*** $(\bar{v}_1)^{(3)} < (v_0)^{(3)}$

and analogously 322

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$ , ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T^0_{20}}{T^0_{21}}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$$G^0_{20}e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G^0_{20}e^{(S_1)^{(3)}t}$$

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| $(p_i)^{(3)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)}-(p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)}G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)}-(p_{20})^{(3)}-(S_2)^{(3)})}\left[e^{((S_1)^{(3)}-(p_{20})^{(3)})t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \le G_{22}(t) \le \right.$ $\left.\frac{(a_{22})^{(3)}G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)}-(a'_{22})^{(3)})}[e^{(S_1)^{(3)}t} - e^{-(a'_{22})^{(3)}t}] + G_{22}^0 e^{-(a'_{22})^{(3)}t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)}+(r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)}+(r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)}T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)}-(b'_{22})^{(3)})}\left[e^{(R_1)^{(3)}t} - e^{-(b'_{22})^{(3)}t}\right] + T_{22}^0 e^{-(b'_{22})^{(3)}t} \le T_{22}(t) \le$ $\frac{(a_{22})^{(3)}T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)}+(r_{20})^{(3)}+(R_2)^{(3)})}\left[e^{((R_1)^{(3)}+(r_{20})^{(3)})t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$ | 327 |
| **Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- <br> Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$ <br> $(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$ <br> $(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$ <br> $(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$ | 328 |
| **Behavior of the solutions of equation** <br> **Theorem:** If we denote and define <br> **Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ : <br> $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying <br> $-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$ <br> $-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : <br> By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}(v^{(4)})^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ <br> and $(b_{25})^{(4)}(u^{(4)})^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and | 329 |
| **Definition of** $(\bar{v}_1)^{(4)},, (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : | 330 |

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| By $(\bar{\nu}_1)^{(4)} > 0$ , $(\bar{\nu}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$ , $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)}$, $(\mu_2)^{(4)}$, $(\nu_0)^{(4)}$ :-<br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)}$, $(\mu_2)^{(4)}$ by<br>$(m_2)^{(4)} = (\nu_0)^{(4)}, (m_1)^{(4)} = (\nu_1)^{(4)}$, ***if*** $(\nu_0)^{(4)} < (\nu_1)^{(4)}$<br>$(m_2)^{(4)} = (\nu_1)^{(4)}, (m_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ , ***if*** $(\nu_4)^{(4)} < (\nu_0)^{(4)} < (\bar{\nu}_1)^{(4)}$,<br>and $\boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$<br>$(m_2)^{(4)} = (\nu_4)^{(4)}, (m_1)^{(4)} = (\nu_0)^{(4)}$, ***if*** $(\bar{\nu}_4)^{(4)} < (\nu_0)^{(4)}$ | |
| and analogously<br>$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b'_{26})^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b'_{26})^{(4)}t}\right] + T_{26}^0 e^{-(b'_{26})^{(4)}t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)} + (r_{24})^{(4)} + (R_2)^{(4)})}\left[e^{((R_1)^{(4)} + (r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:- | 337 |

Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a'_{24})^{(4)}$

$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$

$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b'_{24})^{(4)}$

$(R_2)^{(4)} = (b'_{26})^{(4)} - (r_{26})^{(4)}$

**Behavior of the solutions of equation** 338

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :

$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying

$-(\sigma_2)^{(5)} \le -(a'_{28})^{(5)} + (a'_{29})^{(5)} - (a''_{28})^{(5)}(T_{29}, t) + (a''_{29})^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$

$-(\tau_2)^{(5)} \le -(b'_{28})^{(5)} + (b'_{29})^{(5)} - (b''_{28})^{(5)}((G_{31}), t) - (b''_{29})^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$

**Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ : 339

By $(v_1)^{(5)} > 0$ , $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$ , $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$
and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and

**Definition of** $(\bar{v}_1)^{(5)}, , (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ : 340

By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$
and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$

**Definition of** $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :-

If we define $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}$ by

$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$

$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,
and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$

$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$

and analogously 341

$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$

$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,

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| and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}, \boldsymbol{if}\ (\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)} t}$<br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b'_{30})^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b'_{30})^{(5)} t}\right] + T_{30}^0 e^{-(b'_{30})^{(5)} t} \le T_{30}(t) \le$<br>$\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a'_{28})^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b'_{28})^{(5)}$<br>$(R_2)^{(5)} = (b'_{30})^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :<br>$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying<br>$-(\sigma_2)^{(6)} \le -(a'_{32})^{(6)} + (a'_{33})^{(6)} - (a''_{32})^{(6)}(T_{33}, t) + (a''_{33})^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$ | 349 |

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| $-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}((G_{35}),t) - (b''_{33})^{(6)}((G_{35}),t) \le -(\tau_1)^{(6)}$ | |
| **Definition of** $(v_1)^{(6)}, (v_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, v^{(6)}, u^{(6)}$ :<br><br>By $(v_1)^{(6)} > 0$ , $(v_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_1)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\left(u^{(6)}\right)^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{v}_1)^{(6)},, (\bar{v}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br><br>By $(\bar{v}_1)^{(6)} > 0$ , $(\bar{v}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_2)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\left(u^{(6)}\right)^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}$ , $(m_2)^{(6)}$ , $(\mu_1)^{(6)}, (\mu_2)^{(6)}, (v_0)^{(6)}$ :-<br><br>If we define $(m_1)^{(6)}$ , $(m_2)^{(6)}$ , $(\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br><br>$(m_2)^{(6)} = (v_0)^{(6)}, (m_1)^{(6)} = (v_1)^{(6)}$, ***if*** $(v_0)^{(6)} < (v_1)^{(6)}$<br><br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (\bar{v}_6)^{(6)}$ , ***if*** $(v_1)^{(6)} < (v_0)^{(6)} < (\bar{v}_1)^{(6)}$,<br>and $\boxed{(v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$<br><br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (v_0)^{(6)}$, ***if*** $(\bar{v}_1)^{(6)} < (v_0)^{(6)}$ | 351 |
| and analogously<br><br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br><br>$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$<br><br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G^0_{32}e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)}G^0_{32}}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G^0_{34}e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |

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|---|---|
| $\boxed{T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)}T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)}-(b'_{34})^{(6)})}\left[e^{(R_1)^{(6)}t} - e^{-(b'_{34})^{(6)}t}\right] + T_{34}^0 e^{-(b'_{34})^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)}T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t} - e^{-(R_2)^{(6)}t}\right] + T_{34}^0 e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:- Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a'_{32})^{(6)}$ $(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$ $(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b'_{32})^{(6)}$ $(R_2)^{(6)} = (b'_{34})^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ :<br><br>$(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying<br><br>$-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$<br><br>$-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}((G_{39}), t) - (b''_{37})^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)}, (v_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, v^{(7)}, u^{(7)}$ :<br><br>By $(v_1)^{(7)} > 0, (v_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}\left(u^{(7)}\right)^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)}, , (\bar{v}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ :<br><br>By $(\bar{v}_1)^{(7)} > 0, (\bar{v}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}\left(u^{(7)}\right)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$<br>**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (v_0)^{(7)}$ :-<br><br>If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by | 362 |

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|---|---|
| $(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)},$ ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ , ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)},$<br>and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)},$ ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)},$ ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$ , ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)},$<br>and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)},$ ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)} t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | 364 |
| $\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)} t}$ | 365 |
| $(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})} \left[ e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)} t} \right] + G_{38}^0 e^{-(S_2)^{(7)} t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a_{38}')^{(7)})} [e^{(S_1)^{(7)} t} - e^{-(a_{38}')^{(7)} t}] + G_{38}^0 e^{-(a_{38}')^{(7)} t})$ | 366 |
| $\boxed{T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b_{38}')^{(7)})} \left[ e^{(R_1)^{(7)} t} - e^{-(b_{38}')^{(7)} t} \right] + T_{38}^0 e^{-(b_{38}')^{(7)} t} \le T_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)} + (r_{36})^{(7)} + (R_2)^{(7)})} \left[ e^{((R_1)^{(7)} + (r_{36})^{(7)})t} - e^{-(R_2)^{(7)} t} \right] + T_{38}^0 e^{-(R_2)^{(7)} t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:- | 370 |

Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a'_{36})^{(7)}$

$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$

$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b'_{36})^{(7)}$

$(R_2)^{(7)} = (b'_{38})^{(7)} - (r_{38})^{(7)}$

**Behavior of the solutions of equation** 371

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :

$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying

$-(\sigma_2)^{(8)} \le -(a'_{40})^{(8)} + (a'_{41})^{(8)} - (a''_{40})^{(8)}(T_{41}, t) + (a''_{41})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$

$-(\tau_2)^{(8)} \le -(b'_{40})^{(8)} + (b'_{41})^{(8)} - (b''_{40})^{(8)}((G_{43}), t) - (b''_{41})^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$

**Definition of** $(\nu_1)^{(8)}, (\nu_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, \nu^{(8)}, u^{(8)}$ : 372

By $(\nu_1)^{(8)} > 0, (\nu_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}(\nu^{(8)})^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$
and $(b_{41})^{(8)}(u^{(8)})^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(8)}, , (\bar{\nu}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :

By $(\bar{\nu}_1)^{(8)} > 0, (\bar{\nu}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0, (\bar{u}_2)^{(8)} < 0$ the

roots of the equations $(a_{41})^{(8)}(\nu^{(8)})^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$

and $(b_{41})^{(8)}(u^{(8)})^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$

**Definition of** $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (\nu_0)^{(8)}$ :-

If we define $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by

$(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)}$, ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$

$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$, ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)}$,

and $\boxed{(\nu_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$

| | |
|---|---|
| $(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)},\ \boldsymbol{if}\ (\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},\ \boldsymbol{if}\ (u_0)^{(8)} < (u_1)^{(8)}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}, \boldsymbol{if}\ (u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}, \boldsymbol{if}\ (\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)} t}$<br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \le G_{42}(t) \le \right.$<br>$\left.\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)} t} - e^{-(a'_{42})^{(8)} t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)} t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b'_{42})^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b'_{42})^{(8)} t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)} t} \le T_{42}(t) \le$<br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$ | 381 |

$(R_2)^{(8)} = (b_{42}^{'})^{(8)} - (r_{42})^{(8)}$

**Behavior of the solutions of equation 37 to 92** 382

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$$-(\sigma_2)^{(9)} \le -(a_{44}^{'})^{(9)} + (a_{45}^{'})^{(9)} - (a_{44}^{''})^{(9)}(T_{45}, t) + (a_{45}^{''})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$$

$$-(\tau_2)^{(9)} \le -(b_{44}^{'})^{(9)} + (b_{45}^{'})^{(9)} - (b_{44}^{''})^{(9)}\big((G_{47}), t\big) - (b_{45}^{''})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$$

**Definition of** $(v_1)^{(9)}, (v_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, v^{(9)}, u^{(9)}$ :

By $(v_1)^{(9)} > 0$ , $(v_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(v^{(9)}\big)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{v}_1)^{(9)}, , (\bar{v}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{v}_1)^{(9)} > 0$ , $(\bar{v}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(v^{(9)}\big)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (v_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (v_0)^{(9)}, (m_1)^{(9)} = (v_1)^{(9)}$, ***if*** $(v_0)^{(9)} < (v_1)^{(9)}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (\bar{v}_1)^{(9)}$ , ***if*** $(v_1)^{(9)} < (v_0)^{(9)} < (\bar{v}_1)^{(9)}$,

and $\boxed{(v_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)} = (v_1)^{(9)}, (m_1)^{(9)} = (v_0)^{(9)}$, ***if*** $(\bar{v}_1)^{(9)} < (v_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^{0}e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^{0}e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^{0}e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}}G_{44}^{0}e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^{0}e^{-(S_2)^{(9)}t} \le G_{46}(t) \le \right.$$

$$\left.\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}^{'})^{(9)})}\left[e^{(S_1)^{(9)}t}-e^{-(a_{46}^{'})^{(9)}t}\right]+\ G_{46}^{0}e^{-(a_{46}^{'})^{(9)}t}\right)$$

$$\boxed{T_{44}^{0}e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^{0}e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}}T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^{0}}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}^{'})^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}^{'})^{(9)}t}\right]+T_{46}^{0}e^{-(b_{46}^{'})^{(9)}t} \le T_{46}(t) \le$$

$$\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a_{44}^{'})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b_{44}^{'})^{(9)}$

$(R_2)^{(9)} = (b_{46}^{'})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain

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$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a_{13}^{'})^{(1)} - (a_{14}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14},t)\right) - (a_{14}^{''})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (\nu_1)^{(1)} < (\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t) \geq \frac{(\nu_1)^{(1)}+(C)^{(1)}(\nu_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_0)^{(1)}\right)t\right]}}{1+(C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(\nu_1)^{(1)}-(\nu_0)^{(1)}}{(\nu_0)^{(1)}-(\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\nu_1)^{(1)}$

In the same manner , we get 384

$$\nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{\nu}_1)^{(1)}-(\nu_0)^{(1)}}{(\nu_0)^{(1)}-(\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\bar{\nu}_1)^{(1)}$

If $0 < (\nu_1)^{(1)} < (\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{\nu}_1)^{(1)}$ we find like in the previous case, 385

$$(\nu_1)^{(1)} \leq \frac{(\nu_1)^{(1)}+(C)^{(1)}(\nu_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_2)^{(1)}\right)t\right]}}{1+(C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_2)^{(1)}\right)t\right]}} \leq \nu^{(1)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$ if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \le \frac{d\nu^{(2)}}{dt} \le -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \ge \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \le \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \le \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \le \nu^{(2)}(t) \le$$

| | |
|---|---|
| $\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$ | |
| If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br><br>$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :-<br><br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right) - (a_{21}^{''})^{(3)}(T_{21},t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| | 400 |

From which one obtains

For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

In the same manner , we get

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$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

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If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case,

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

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If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$ if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

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In the same manner , we get

$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

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$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (C)^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}'')^{(5)} = (b_{29}'')^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then

$(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(v_1)^{(5)}$ and $(\bar{v}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain 412

$$\frac{dv^{(6)}}{dt} = (a_{32})^{(6)} - \left((a'_{32})^{(6)} - (a'_{33})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right) - (a''_{33})^{(6)}(T_{33},t)v^{(6)} - (a_{33})^{(6)}v^{(6)}$$

**Definition of** $v^{(6)}$ :- $\boxed{v^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_2)^{(6)}v^{(6)} - (a_{32})^{(6)}\right) \le \frac{dv^{(6)}}{dt} \le -\left((a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_1)^{(6)}v^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(6)}, (v_0)^{(6)}$ :-

For $0 < \boxed{(v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}} < (v_1)^{(6)} < (\bar{v}_1)^{(6)}$

$$v^{(6)}(t) \ge \frac{(v_1)^{(6)} + (C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)} - (v_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)} - (v_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(v_1)^{(6)} - (v_0)^{(6)}}{(v_0)^{(6)} - (v_2)^{(6)}}}$$

it follows $(v_0)^{(6)} \le v^{(6)}(t) \le (v_1)^{(6)}$

In the same manner , we get 413

$$v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)} + (\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{v}_1)^{(6)} - (v_0)^{(6)}}{(v_0)^{(6)} - (\bar{v}_2)^{(6)}}}$$

From which we deduce $(v_0)^{(6)} \le v^{(6)}(t) \le (\bar{v}_1)^{(6)}$

If $0 < (v_1)^{(6)} < (v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}} < (\bar{v}_1)^{(6)}$ we find like in the previous case, 414

$$(v_1)^{(6)} \le \frac{(v_1)^{(6)} + (C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)} - (v_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)} - (v_2)^{(6)}\right)t\right]}} \le v^{(6)}(t) \le$$

$$\frac{(\bar{v}_1)^{(6)} + (\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}} \le (\bar{v}_1)^{(6)}$$

If $0 < (v_1)^{(6)} \le (\bar{v}_1)^{(6)} \le \boxed{(v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$ , we obtain 415

$$(v_1)^{(6)} \le v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)} + (\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)} - (\bar{v}_2)^{(6)}\right)t\right]}} \le (v_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}$, $\boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

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$$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$$

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If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain

$$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(C)^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

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**Definition of** $u^{(7)}(t)$ :-

$$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}, \quad \boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}'')^{(7)} = (a_{37}'')^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}'')^{(7)} = (b_{37}'')^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right) - (a_{41}^{''})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \leq \frac{dv^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(8)}, (v_0)^{(8)}$ :-

For $0 < \boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (v_1)^{(8)} < (\bar{v}_1)^{(8)}$

$$v^{(8)}(t) \geq \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(v_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (v_2)^{(8)}}}$$

it follows $(v_0)^{(8)} \leq v^{(8)}(t) \leq (v_1)^{(8)}$

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In the same manner , we get

$$v^{(8)}(t) \leq \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{v}_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (\bar{v}_2)^{(8)}}}$$

From which we deduce $(v_0)^{(8)} \leq v^{(8)}(t) \leq (\bar{v}_8)^{(8)}$

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If $0 < (v_1)^{(8)} < (v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{v}_1)^{(8)}$ we find like in the previous case,

$$(v_1)^{(8)} \leq \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}} \leq v^{(8)}(t) \leq$$

$$\frac{(\bar{v}_1)^{(8)} + (C)^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \leq (\bar{v}_1)^{(8)}$$

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If $0 < (v_1)^{(8)} \leq (\bar{v}_1)^{(8)} \leq \boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain

$$(v_1)^{(8)} \leq v^{(8)}(t) \leq \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \leq (v_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

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**Definition of** $v^{(8)}(t)$ :-

$(m_2)^{(8)} \leq v^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{v^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $v^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)}-(\nu_0)^{(9)}}{(\nu_0)^{(9)}-(\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \le \frac{(\nu_1)^{(9)}+(C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}} \le \nu^{(9)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \le (\bar{\nu}_1)^{(9)} \le \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(\nu_1)^{(9)} \le \nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \le \nu^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \le u^{(9)}(t) \le (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)}G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

| | |
|---|---|
| $(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,<br>$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$<br>$with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$ | 433 |

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| $(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$ | 436 A |

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| $(a'_{44})^{(9)}(a'_{45})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a'_{45})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b'_{44})^{(9)}(r_{45})^{(9)} - (b'_{45})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b'_{13})^{(1)} - (b''_{13})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b'_{14})^{(1)} - (b''_{14})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b'_{15})^{(1)} - (b''_{15})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b'_{16})^{(2)} - (b''_{16})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b'_{17})^{(2)} - (b''_{17})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b'_{18})^{(2)} - (b''_{18})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a'_{20})^{(3)} + (a''_{20})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a'_{21})^{(3)} + (a''_{21})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a'_{22})^{(3)} + (a''_{22})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b'_{20})^{(3)} - (b''_{20})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b'_{21})^{(3)} - (b''_{21})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b'_{22})^{(3)} - (b''_{22})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) +$ | 485 |

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| $(a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | |
| **Proof:**<br><br>(w) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |

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| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492<br>A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations | 498 |

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| $G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |
| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 501 |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br><br>$\left[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G) + (b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)\right]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that $\varphi$ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that | 502 |

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| there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*})=0$ | |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br>$\varphi(G_{19})=(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)}-(b_{16})^{(2)}(b_{17})^{(2)}-$<br>$\left[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19})+(b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})\right]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19})=0$ | 503 |
| Where in $(G_{19})(G_{16},G_{17},G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*})=0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br>$\varphi(G_{23})=(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)}-(b_{20})^{(3)}(b_{21})^{(3)}-$<br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23})+(b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23})=0$<br>Where in $G_{23}(G_{20},G_{21},G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*})=0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br>$\varphi(G_{27})=(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)}-(b_{24})^{(4)}(b_{25})^{(4)}-$<br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27})+(b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27})=0$<br>Where in $(G_{27})(G_{24},G_{25},G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*})=0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31})=(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)}-(b_{28})^{(5)}(b_{29})^{(5)}-$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31})+(b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right]+(b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31})=0$<br>Where in $(G_{31})(G_{28},G_{29},G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*})=0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35})=(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)}-(b_{32})^{(6)}(b_{33})^{(6)}-$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35})+(b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right]+(b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35})=0$<br>Where in $(G_{35})(G_{32},G_{33},G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0)>0$ , $\varphi(\infty)<0$ it follows that | 508 |

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| there exists a unique $G_{33}^{*}$ such that $\varphi(G_{35}{}^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi(G_{39}{}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi(G_{43}{}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0\,, \varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0\,, T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0\,, T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |

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| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^*$ given by $\varphi((G_{23})^*) = 0$ , $T_{21}^*$ given by $f(T_{21}^*) = 0$ and<br><br>$G_{20}^* = \frac{(a_{20})^{(3)} G_{21}^*}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^*)]}$ , $G_{22}^* = \frac{(a_{22})^{(3)} G_{21}^*}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^*)]}$<br><br>$T_{20}^* = \frac{(b_{20})^{(3)} T_{21}^*}{[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}{}^*)]}$ , $T_{22}^* = \frac{(b_{22})^{(3)} T_{21}^*}{[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}{}^*)]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br><br>$G_{25}^*$ given by $\varphi(G_{27}) = 0$ , $T_{25}^*$ given by $f(T_{25}^*) = 0$ and<br><br>$G_{24}^* = \frac{(a_{24})^{(4)} G_{25}^*}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}^*)]}$ , $G_{26}^* = \frac{(a_{26})^{(4)} G_{25}^*}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}^*)]}$ | 516 |
| $T_{24}^* = \frac{(b_{24})^{(4)} T_{25}^*}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27})^*)]}$ , $T_{26}^* = \frac{(b_{26})^{(4)} T_{25}^*}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27})^*)]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br><br>$G_{29}^*$ given by $\varphi((G_{31})^*) = 0$ , $T_{29}^*$ given by $f(T_{29}^*) = 0$ and<br><br>$G_{28}^* = \frac{(a_{28})^{(5)} G_{29}^*}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^*)]}$ , $G_{30}^* = \frac{(a_{30})^{(5)} G_{29}^*}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^*)]}$ | 518 |
| $T_{28}^* = \frac{(b_{28})^{(5)} T_{29}^*}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^*)]}$ , $T_{30}^* = \frac{(b_{30})^{(5)} T_{29}^*}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^*)]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br><br>$G_{33}^*$ given by $\varphi((G_{35})^*) = 0$ , $T_{33}^*$ given by $f(T_{33}^*) = 0$ and<br><br>$G_{32}^* = \frac{(a_{32})^{(6)} G_{33}^*}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^*)]}$ , $G_{34}^* = \frac{(a_{34})^{(6)} G_{33}^*}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^*)]}$ | 520 |
| $T_{32}^* = \frac{(b_{32})^{(6)} T_{33}^*}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^*)]}$ , $T_{34}^* = \frac{(b_{34})^{(6)} T_{33}^*}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^*)]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br><br>$G_{37}^*$ given by $\varphi((G_{39})^*) = 0$ , $T_{37}^*$ given by $f(T_{37}^*) = 0$ and | 522 |

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| $G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)} (T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)} (T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)} ((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)} ((G_{39})^{*})]}$ | |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)} (T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)} (T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)} ((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)} ((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br><br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br><br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)} (T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)} (T_{45}^{*})]}$<br><br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)} ((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)} ((G_{47})^{*})]}$ | 523 A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:**_Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}} (T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j} (G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)} \mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*} \mathbb{T}_{14}$ | 525 |

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| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^{*}\mathbb{G}_{j}\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^{*}\mathbb{G}_{j}\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^{*}\mathbb{G}_{j}\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 532 |
| $\frac{\partial(a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial(b_i^{''})^{(2)}}{\partial G_j}((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^{*}\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | 540 |

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| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)} \ , \ \frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^*\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^*\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^*\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^*\mathbb{G}_j\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^*\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^*\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ *and* $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)} \ , \ \frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^*) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right)$ | 554 |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^*) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^*\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^*\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^*\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |

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| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{37})^{(7)}}{\partial T_{37}}(T^*_{37}) = (q_{37})^{(7)}$ , $\frac{\partial(b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a'_{36})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G^*_{36}\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a'_{37})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G^*_{37}\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. | |

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| Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \ , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}((G_{43})^*) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^*\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^*\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b_{41}^{'})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T_{41}^*\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b_{42}^{'})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T_{42}^*\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS** **Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. **Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |

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| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $(\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\{((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)})$ $\left[\left(((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)})(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\right]$ $\left(((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)})s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\right)$ $+\left(((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)})(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\right)$ $\left(((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)})s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\right)$ $\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $\left((\lambda)^{(1)}\right)^2 + \left((b'_{13})^{(1)} + (b'_{14})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $+\left(\left((\lambda)^{(1)}\right)^2 + \left((a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$ $+\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G^*_{13}\right)$ $\left(((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)})s_{(14),(15)}T^*_{14} + (b_{14})^{(1)}s_{(13),(15)}T^*_{13}\right)\} = 0$ $+$ $(\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)}\{((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)})$ $\left[\left(((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)})(q_{17})^{(2)}\mathrm{G}^*_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^*_{16}\right)\right]$ $\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(17)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^*_{17}\right)$ $+\left(((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)})(q_{16})^{(2)}\mathrm{G}^*_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^*_{17}\right)$ $\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(16)}\mathrm{T}^*_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^*_{16}\right)$ $\left((\lambda)^{(2)}\right)^2 + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ $\left((\lambda)^{(2)}\right)^2 + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\right)$ | |

$$+\left(\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\right)\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left(\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\right)$$

$$+\left(\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^2+\left((b_{20}')^{(3)}+(b_{21}')^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}')^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}')^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}')^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}')^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}')^{(4)}+(a_{25}')^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^2+\left((b_{24}')^{(4)}+(b_{25}')^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^2+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\}\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^2+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G^*_{33}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G^*_{32}\right)$$

$$\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T^*_{33}+(b_{33})^{(6)}s_{(32),(34)}T^*_{32}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b'_{38})^{(7)}-(r_{38})^{(7)}\}\{\left((\lambda)^{(7)}+(a'_{38})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(q_{36})^{(7)}G^*_{36}\right)\right]$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T^*_{37}+(b_{37})^{(7)}s_{(36),(37)}T^*_{37}\Big)$$

$$+\left(\left((\lambda)^{(7)}+(a'_{37})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G^*_{36}+(a_{36})^{(7)}(q_{37})^{(7)}G^*_{37}\right)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T^*_{37}+(b_{37})^{(7)}s_{(36),(36)}T^*_{36}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^*_{36}\right)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^*_{37}+(b_{37})^{(7)}s_{(36),(38)}T^*_{36}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^*_{40}\right)\right]$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^*_{41}+(b_{41})^{(8)}s_{(40),(41)}T^*_{41}\Big)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^*_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^*_{41}\right)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^*_{41}+(b_{41})^{(8)}s_{(40),(40)}T^*_{40}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\Big(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^*_{40}\right)$$

$$\Big(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^*_{41}+(b_{41})^{(8)}s_{(40),(42)}T^*_{40}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\Big(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^*_{44}\Big)\right]$$

$$\Big(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^*_{45}+(b_{45})^{(9)}s_{(44),(45)}T^*_{45}\Big)$$

$$+\Big(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^*_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\Big)$$

$$\Big(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45}+(b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\Big(\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\Big(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45}+(b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

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# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i(x)\, xi$ where $x_i \in R^n_i$, $x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i(x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i(\mu)$ of the $A_i(x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k \tag{23}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to

replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 23. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon\ C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \ i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0}$

$\overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow *x.t restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set *K* is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** *x.t* is defined for all $t \geq 0$

**A4:** *x.t* is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\mathcal{A}(\tilde{A}), C \backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \longrightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) in\ ||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population *i* is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^{t}_{0} (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\to\infty}{limsup} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v = au_i(x) + w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |B_i(t,x)u_i(x)|(a + \alpha + exp(-\beta t)|w|)$. Hence, $\limsup_{t\to\infty} \frac{1}{t} In\, B_i(t,x)\, v| \leq$

$$\limsup_{t\to\infty} \frac{1}{t} In\, B_i(t,x)\, u_i(x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\, |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\, u_i(x)| = In\, |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\, u_i(x.t)||\, B_i(t,x)\, u_i(x)|$$

$$= \text{In } b_i(s,t.x)\, b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x - bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

$$=$$

$$= (A_i(x.t)u_i(x.s), u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s), u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } | (x.t)_i |$$

$$\leq \limsup_{t\to\infty} \frac{In\ \gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x) =$ Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of

irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x)u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$ where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ *and* $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as $n\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\le \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\le \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x)u_i^n(x), u_i^n(x)\rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)|d\mu(x) + |\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\,d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\,ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i\,dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\,d\,v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak$^*$ limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \{ \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i \epsilon I}{if\ i \notin I}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that \quad \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t$ −invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently *K* is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s =$

$s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_s$. If $r(x) = s(x), then\ x \in M_r$. Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume $x \mapsto A(x)$ is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists $j \in \{1,\ldots,k\}$ such that $M_i$ ----- $\{x \in C : x_l \gg 0 \text{ for } l \neq j\}$ Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_i M_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, *$\lambda_i(\mu) > 0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, ..., x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda(0)$, which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i(0) > 0$ for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the

system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j \ (x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1{}^1, \ldots, E_1{}^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i{}_2(E_1{}^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j \ x_2^j \ - \alpha^j \ x_3^j) + \sum d_1^{jk} \ x_1^k \ - e_1^j \ x_1^j$$

$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > 1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$and $\lambda_3(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j, \qquad i = 1,\ldots,k;\ j = 1,\ldots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i\, a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i\, x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \qquad i = 1, \ldots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \ldots, k$$

It has been shown [12,ch.16.1,16.2] That for $k = 3$,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is

the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1-y) - b(N)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*′ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\curlywedge_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\curlywedge_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\curlywedge_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + K p\, x_i x_i - 1}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions

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# Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a

qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and

orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t\geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a\geq 0)$ at time $t$ so that we have

(1.67) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t\geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.68) $D\rho_i(a, t) = \limsup_{h\to 0} \frac{\rho_i(a+h,t+h)-\rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.69) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0,' & (1.4) \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s\epsilon\,[-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s\in[-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1, \varphi_2, f_1, f_2, b_1, b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for

each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as* $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 23. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C(R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C(R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i(a, P_1', P_2') - f_i(a, P_1'', P_2'')| \leq d_i\ [||P_1' - P_1'', P_2' - P_2''||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\partial P_1, \partial f_1/\partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\partial P_1, \partial f_2/\partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C(R^+ K_1 \times K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C(R^+ K_2 \times K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$|b_i(a, P_1', P_2') - b_i(a, P_1'', P_2'')| \leq \beta_i\ [||P_1' - P_1''|| + ||P_2' - P_2''||]$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2)) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, \text{R}^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[-\int_0^\infty f_i(a + s, P(t + S))ds\right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[-\int_o^x f_1(a + s, p(t + s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0)\exp\left[-\int_o^t f_1(a - t + s, p_1(s))\, ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

*(2.3)* $p_i(a,t) = B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0,t), i = 1,2, t > 0.$$

Define $M_i$ *and* $L_i$ $(i = 1,2)$ as follows:

$$M_i(a,t,p_i(t)) = \exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a,t,p_i(t)) = \exp[-\int_o^t f_1(a+s, p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

$$(2.6)\ p_i(t) = \int_o^t B_1(a) M_1(a,t,p_i(t))da + \int_o^\infty \varphi_i(a,0)\ L_i(a,t,p_i(t)da,$$

$$t > 0, i = 1,2,$$

$$(2.7)\ p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a,s)da, s \in [-\tau, 0], i = 1,2,$$

$$(2.8)\ \beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.(a,t,P_i(t))da$$

$$+\int_0^\infty b_i(a+t, P_i(t))\varphi_i(a,0)L_i(a,t,P_i(t))da$$

$$i = 1, 2, t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t) \phi(0) d_i^*] t \quad for \ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T], R^+), \quad B_i \in C([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in C([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau, T]$. Then for any

$(x, y) \in C^+[-\tau, T] \times C^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau, T], \quad C_T^2 = C^+[-\tau, T] \times C^+[-\tau, T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for $i = 1,2$.

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \pi C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \quad P_t = \ (P_{1t}, P_{2t}) \in C^2_{t*} \mid \|P_t - \Phi_t\|_{t*} \le r^*$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)\, da \ for\ s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)\, da \ for\ s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Phi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Phi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^* \right] \quad t^*$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Phi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Phi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Phi(P_t^{(1)}) - \Phi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Phi(P^{(1)}(t)) - \Phi(P^{(2)}(t)| \le \left| \int_0^t B_1^{(1)}(a) \left\{ M_1(a, t, P_t^{(1)}) - M_1(a, t, P_t^{(2)}) \right\} da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a, t, P_t^{(2)}) da \right|$$

$$+ \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \; for \; x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \| ds\right) da$$

$$\leq d_1 \beta_1^* \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o)\left[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds\right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \| t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\text{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right]\right] d\sigma$$

$$+ \int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^{\infty} \varphi(\sigma,o) b_1\left(\sigma+a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma,a,P_t^{(1)}(a)\right)-L_1\left(\sigma,a,P_t^{(2)}(a)\right)\right]d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right)-b_1\left(\sigma+a, P_t^{(2)}(a)\right)\right| \le B_1 \,/\!/\, P_t^{(1)}-P_t^{(2)} /\!/ \text{ for } a\,\varepsilon[o,t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \le \beta_1 \Phi_1(0)||P_t^{(1)}-P_t^{(2)}||_t^{*}+\Phi_1(0)\beta_1 d_1||P_t^{(1)}-P_t^{(2)}||_t^{*a},$$

$$|R_2(a)|=[\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a))-b_1(a-\sigma, P_t^{(2)}(a))\}M_1(\sigma, a, P_t^{(1)}(a))$$

$$+b_1(a,-\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a))-M_1(a-\sigma, P_t^{(2)}(a))\}]d\sigma]$$

$$+b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a))-M_1(a-\sigma, P_t^{(2)}(a))\}]d\sigma]$$

$$\le \int_0^a [\beta_1^{*}\Phi_1(0)\, e^{\delta 1^a}\beta_1 \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}+\beta_1 d_1 \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}(a-o)]do$$

$$\le \beta_1^{*}\Phi_1(0)\, e^{\delta 1^a} \| P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\{\beta_1 a+d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a)\right|$$

$$\le \int_o^a \left| B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma)\right| B_1^{*d}\sigma$$

$$+\Phi_1(0) /\!/ P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\left\{B_1+B_2 d_1 a+B_1^{*}\left(B_2 a+d_1 a^2/2\right)e^{\delta 1^a}\right\}$$

$$\le \beta_1^{*}\int_o^a \left| B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma)\right| do+\left(r^{*}+/\!/\Phi_t /\!/_{t*}\right)\| P_t^{(1)}-P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a)\to \beta_1$ as $a\to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a)\right| \le \left(r^{*}+/\!/\Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)}-P_t^{(2)} /\!/_{t^*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))\text{-}b_1(a\text{-}\sigma, P_t^{(1)}(a))\, M_1(\sigma, a, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + /\!/ \Phi_t /\!/_{t*}\right) /\!/ \; P_t^{(1)}\text{-}P_t^{(2)} /\!/_{t*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right) /\!/ \leq (r^{*} + /\!/ \Phi_t /\!/_{t*}) /\!/ \; P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^{*})$$

As $t^{*} \to o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^{*}, P_t^{*}$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp\, P_2^{*}(a)}{da} + f_i(\mathrm{a}, P_1^{*}, P_2^{*})\, P_i^{*}(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^{*} = \int_o^{\infty} P_i^{*}(a)da, i = 1,\ 2,$$

$$(3,3)\ P_i^{*} = \int_o^{\infty} b_i(a,\ P_1^{*},\ P_2^{*}) P_i^{*}(a)da,\ i = 1.\ \ 2$$

All solutions of (3.1)are of the from

$$(3.4)\ P_i^{*}(a) = P_i^{*}(o) \exp\left[ - \int_o^{\infty} f_i\,(s, P_1^{*}, P_2^{*})ds \right], i = 1, \quad 2, a \geq 0,$$

Which together with (3.3) lead to

$$1=\int_o^{\infty} b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^{\infty} b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $$F_{i.}\ (P_1,P_2) = \int_o^{\infty} b_i(a,P_1,P_2)\exp\left[-\int_o^{\infty} f_i\,(s,P_1,P_2)ds\right]da,$$

$$i = 1,\ 2, \quad P_1 \geq 0, \quad P_2 \geq 0,$$

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_5$. *there exists a positive constant c for which*

$$F_1(\mathrm{C},0) < F_{2.}(\mathrm{C},0),$$

(3.8) $$F_1(0.\mathrm{C}) < F_{2.}(0,\mathrm{C}),$$

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $$F_1(P_1^*,P_2^*) = 1 = F_2(P_1^*,P_2^*)$$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $$P_2^*(a) = [P_2^*\exp[-f_i\,(s,P_1^*,P_2^*)ds])/(\int_o^{\infty}\exp[-\int_o^a f_i\,(s,P_1^*,P_2^*)ds]da),$$

$$i = 1,\ 2, \quad a \geq 0,$$

Proof Consider the elements of the surfaces defined by

$$\mathrm{Z}= F_1(P_1,P_2), z = F_2(P_1,P_2),\ P_1 \geq 0, P_2 \geq 0,$$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i/\partial P_j < 0\ and$ , by ($A_2$), $\partial b_i/\partial P_j \leq 0$ for all

$$P_1 \geq 0, P_2 \geq 0, \quad i = 1,\ 2,$$

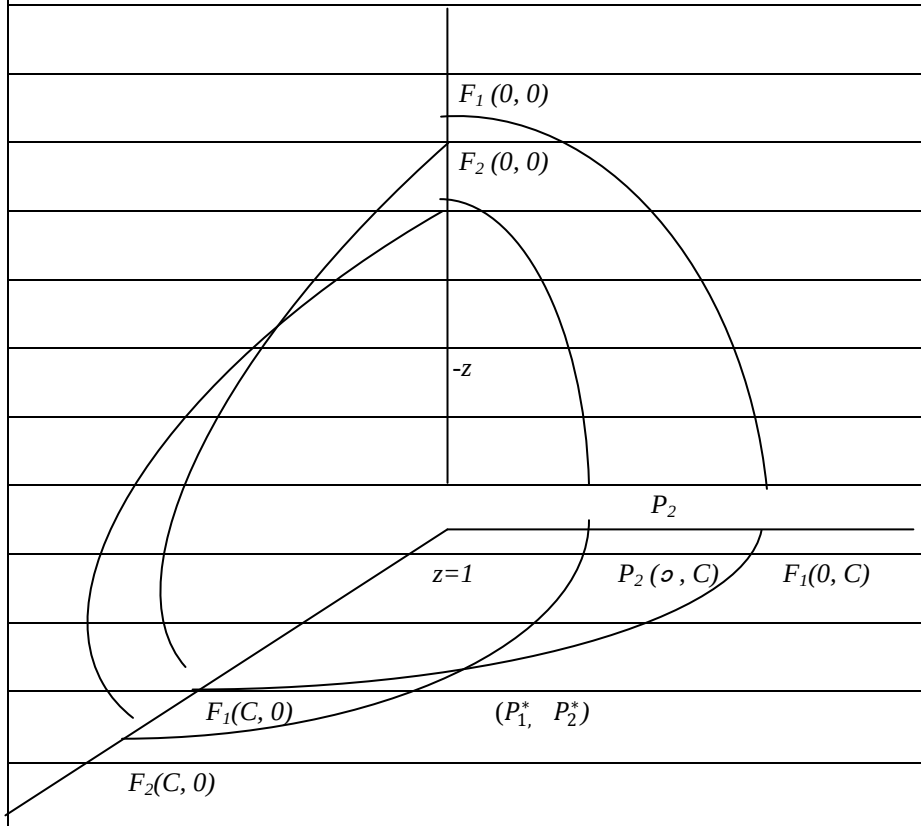
$j = 1,\ 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= 1, $P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$$F_1(P_1, P_2 = 1 \; and \; F_2(P_1, P_2) = 1, \; F_1(P_1, P_2 \geq 0,$$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility 1< $F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p_1^*, p_2^*)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p_1^*, p_2^*$, we let

$$(3.11) \quad \begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \; i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*}\, p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square 1}^{2} \frac{\partial b_i}{\partial p_{\square}^*}\, p_{i\square}(t)\right\} da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_{\square}(a)\exp[\lambda t], \quad \square(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty \square(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d\square}{da} + \lambda\square = -f_i\left(a, p_1^*, p_2^*\right)\square(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*}\, p_{i\square}^*\right\},$$

(3.15)

$$\square(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\square(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*}\, p_{i\square}^*\right\} da,$$

Where $i = 1, 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \quad p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\Delta_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\Delta_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\Delta_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\Delta_i(a) = \{\Delta_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$

$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Delta_i\right\} e^{\lambda s} ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\Delta_i(a)\left[p_i^*(a)/p_i^*(0)c_i \exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Delta_i\right\} da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a}\left\{\int_0^a \left(\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Delta_i\right) e^{\lambda s} ds\right\} da\right]$$

$$- p_i^*(a) e^{-\lambda a}\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* \Delta_i\right\} e^{\lambda s} ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\Delta_1(a)$ and $\Delta_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^* \exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^* \exp(-\lambda\tau) + (\lambda - A_{22})p_2^* = g_{21}(\lambda)p_1^* \exp(-\lambda\tau) + g_{22}(\lambda)p_2^*$$

Where

(3.22) $A_{ij} = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_j^*} - \frac{\partial f_i}{\partial p_j^*}\right]da,$

(3.23) $g_{ij}(\lambda) = [p_i^*(0)c_i]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\,da\right\}$
$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_j^*} - b_i(a, p_1^*, p_2^*)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)\right]da\right\}$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*}\,e^{\lambda s}\,ds\right)da,\quad i, j = 1, 2,$$

(3.24) $\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*),\quad i, = 1, 2.$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix}\lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda))\end{bmatrix} = 0$

Or equivalently

(3.25) $\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = [A_{21}g_{12}(\lambda) + A_{12}g_{21}(\lambda) + g_{12}(\lambda)g_{21}(\lambda)]\exp[-2\lambda\tau]$

$-A_{11}g_{22}(\lambda) - A_{22}g_{11}(\lambda) - g_{11}(\lambda)g_{22}(\lambda)$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i, = 1, 2.)\ (see(A_1)\ and\ (A_2))$;

$A_7.\ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\},$

$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$

$\| S_2(0) \| = |A_{11}g_{12}(0)| + |A_{12}g_{21}(0)| + |g_{12}(0)g_{21}(0)|$

$+ |A_{11}g_{22}(0)| + |A_{22}g_{11}(0)| + |g_{11}(0)g_{22}(0)|$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i(a, t) = \phi_i(a) \exp[\lambda t] \to 0$ as $t \to \infty$ for $i = 1, 2$ and $a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda$ $\mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$

$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \left|$

$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$

$\geq [(|\lambda| - P)^2 + (q)]^2$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \{4|(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| | - |A_{11} + A_{22} + \| S_1(0) \| |^2\}/2,$

It will now follow, from (A) and (3.26), $F_1(\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder

concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power

obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy

withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation

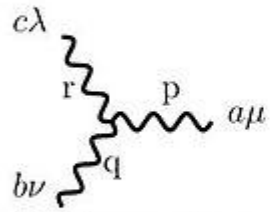
from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

| **SECTION A(28)**<br><br>**Feynman Diagrams** | |
|---|---|
| **INTRODUCTION—VARIABLES USED** | |

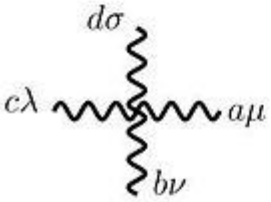
 




gluon propagator: $D^{ab}_{\mu\nu}(p) = \frac{-i\delta^{ab}}{p^2+i0}\left[\eta_{\mu\nu} - \frac{(1-\xi)p_\mu p_\nu}{p^2+i0}\right]$



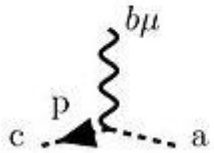

3–gluon vertex: $\Gamma^{abc}_{\mu\nu\lambda}(p,q,r) = -gf^{abc}[(p-q)_\lambda\eta_{\mu\nu} + (q-r)_\mu\eta_{\nu\lambda} + (r-p)_\nu\eta_{\mu\lambda}]$




4–gluon vertex: $\Gamma^{abcd}_{\mu\nu\lambda\sigma} = -ig^2 f^{abe}f^{cde}(\eta_{\mu\lambda}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\lambda}) - ig^2 f^{ace}f^{bde}(\eta_{\mu\nu}\eta_{\sigma\lambda} - \eta_{\mu\sigma}\eta_{\nu\lambda}) - ig^2 f^{ade}f^{bce}(\eta_{\mu\nu}\eta_{\sigma\lambda} - \eta_{\mu\lambda}\eta_{\nu\sigma})$




ghost propagator: $C^{ab}(p) = \frac{i\delta^{ab}}{p^2+i0}$




$\bar{c}cg$ – vertex: $\Gamma^{abc}(p) = gf^{abc}p_\mu$

**NOTATION**

**Module One**

$G_{13}$ : Category one of b (nu)

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of a(mu)

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|---|---|
| $T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two** | |
| $G_{16}$ : Category one of RHS of the first equation<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of LHS of the first equation<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three** | |
| $G_{20}$ : Category one of RHS of the first equation<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of LHS of the first equation<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four** | |
| $G_{24}$ : Category one of RHS of the first equation<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of LHS of the first equation<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five** | |
| $G_{28}$ : Category one of RHS of the first equation<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of LHS of the first equation | |

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| $T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six** | |
| $G_{32}$ : Category one of c (lambda)<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of a(nu)<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven** | |
| $G_{36}$ : Category one of c(lambda) +b(nu)<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of a(mu)<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight** | |
| $G_{40}$ : Category one of d(sugma)<br>$G_{41}$ : Category two of SAS<br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of b(mu)<br>$T_{41}$ : Category two of SAS<br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br>**All the three categories can be taken as equivalent.** | |
| $G_{44}$ : Category one of a | |

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| $G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of b (ghost propagator)<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION A(29)

## Renormalizability

**INTRODUCTION—VARIABLES USED**

**Source: Wikipedia**

The theory is renormalizable and corrections are finite at any order of perturbation theory.

(1) For quantum electrodynamics the ghost field decouples because (e) the gauge group is abelian.
(2) This can be seen from the coupling between the gauge field and (e&eb) the ghost field that is (=) $\bar{c}^a f^{abc} \partial_\mu A^{b\mu} c^c$.
(3) For the abelian case, all the structure constants $f^{abc}$ are (=) zero and so there is (=) no coupling.
(4) In the non-abelian case, the ghost field appears as (=) a useful way to rewrite the quantum field theory without (e) physical consequences on (e&eb) the observables of the theory such as (e) cross sections or decay rates.
(5) One of the most important results obtained for (e) Yang–Mills theory is asymptotic freedom. This result can be obtained by (e) assuming that the coupling constant g is (=) small (so small nonlinearities), as for high energies, and applying perturbation theory.
(6) The relevance of this result is due to (e) the fact that a Yang–Mills theory that describes *(eb) strong interaction and (e&eb) asymptotic freedom permits (eb) proper treatment of experimental results coming from (e) deep inelastic scattering.
(7) To obtain the behavior of the Yang–Mills theory at high energies, and so to prove (eb) asymptotic freedom, one applies (e&eb) perturbation theory assuming (=) a small coupling.
**(8)** This is verified a posterioriin the ultraviolet limit. In the opposite limit, the infrared limit, the situation is (=) the opposite, as the coupling is (=) too large for perturbation theory to be (=) reliable.
**(9)** Most of the difficulties that research meets are (=) just managing the theory at (eb) low energies

**NOTATION**

**Module One**

For quantum electrodynamics the ghost field decouples because (e) the gauge group is abelian

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| $G_{13}$ : Category one of gauge group is abelian<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of quantum electrodynamics the ghost field decouples<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>This can be seen from the coupling between the gauge field and (e&eb) the ghost field that is (=) $\bar{c}^a f^{abc} \partial_\mu A^{b\mu} c^c$. | |
| $G_{16}$ : Category one of **gauge field;** ghost field that is $\bar{c}^a f^{abc} \partial_\mu A^{b\mu} c^c$.<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of ghost field that is (=) $\bar{c}^a f^{abc} \partial_\mu A^{b\mu} c^c$ .; **gauge field**<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>, all the structure constants $f^{abc}$ for the abelian case are zero and so there is (=) no coupling | |
| $G_{20}$ : Category one of all the structure constants $f^{abc}$ for the abelian case<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of zero and so there is (=) no coupling<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four** | |
| $G_{24}$ : Category one of all the structure constants $f^{abc}$ for the abelian case are zero<br>$G_{25}$ : Category two of SAS | |

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| $G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of no coupling<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>the ghost field in the non-abelian case appears as (=) a useful way to rewrite the quantum field theory without (e) physical consequences on (e&eb) the observables of the theory such as (e) cross sections or decay rates | |
| $G_{28}$ : Category one of ghost field in the non-abelian case appears<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of useful way to rewrite the quantum field theory without (e) physical consequences on (e&eb) the observables of the theory such as (e) cross sections or decay rates<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>the ghost field in the non-abelian case appears as a useful way to rewrite the quantum field theory without (e) physical consequences on (e&eb) the observables of the theory such as (e) cross sections or decay rates | |
| $G_{32}$ : Category one of physical consequences on (e&eb) the observables of the theory such as (e) cross sections or decay rates<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of ghost field in the non-abelian case appears as a useful way to rewrite the quantum field theory<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br>the ghost field in the non-abelian case appears as a useful way to rewrite the quantum field theory without physical consequences on (e&eb) the observables of the theory such as (e) cross sections or decay rates | |
| $G_{36}$ : Category one of **ghost field in the non-abelian case appears as a useful way to rewrite the quantum field theory without physical consequences;** observables of the theory such as (e) cross sections | |

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| or decay rates<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of observables of the theory such as (e) cross sections or decay rates **;ghost field in the non-abelian case appears as a useful way to rewrite the quantum field theory without physical consequences**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>One of the most important results obtained for (e) **Yang–Mills theory** is asymptotic freedom.<br><br>This result can be obtained by (e) assuming that the coupling constant g is (=) small (so small nonlinearities), as for high energies, and applying perturbation theory | |
| $G_{40}$ : Category one of **Yang–Mills theory;** asymptotic freedom<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of asymptotic freedom **;Yang–Mills theory**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>This result can be obtained by (e) assuming that the coupling constant g is (=) small (so small nonlinearities), as for high energies, and applying perturbation theory | |
| $G_{44}$ : Category one of assuming that the coupling constant g is small (so small nonlinearities), as for high energies, and applying perturbation theory<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of result can be obtained<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION A(29)-1

## Theory Is Renormalizable And Corrections Are Finite At Any Order Of Perturbation Theory

### INTRODUCTION—VARIABLES USED

**Source: Wikipedia**

(1) The theory is renormalizable and corrections are finite at any order of perturbation theory.
(2) The relevance of this result is due to (e) the fact that a Yang–Mills theory that describes *(eb) strong interaction and (e&eb) asymptotic freedom permits (eb) proper treatment of experimental results coming from (e) deep inelastic scattering.
(3) To obtain the behavior of the Yang–Mills theory at high energies, and so to prove (eb) asymptotic freedom, one applies (e&eb) perturbation theory assuming (=) a small coupling.
**(4)** This is verified a posterioriin the ultraviolet limit. In the opposite limit, the infrared limit, the situation is (=) the opposite, as the coupling is (=) too large for perturbation theory to be (=) reliable.
**(5)** Most of the difficulties that research meets are (=) just managing the theory at (eb) low energies

**The light bound states of supersymmetric SU(2) Yang-Mills theory Georg Bergner, Pietro Giudice, Istvan Montvay, Gernot Münster, Stefano Piemonte (Submitted on 22 Dec 2015)**

(1) Supersymmetry provides (eb) a well-established theoretical framework for (e) extensions of the standard model of particle physics and the general understanding of quantum field theories.
(2) Authors summarise here investigations of N=1 supersymmetric Yang-Mills theory with (e&eb) SU (2) gauge symmetry using (e) the non-perturbative first-principles method of (e) numerical lattice simulations.
(3) The strong interactions of gluons and (e&eb) their superpartners, the gluinos, lead to (eb) confinement, and a spectrum of bound states including (e) glueballs, mesons, and gluino-glueballs emerges (eb) at low energies.
(4) For unbroken supersymmetry these particles have to be (=) arranged in supermultiplets of equal masses.
(5) In lattice simulations supersymmetry can only be recovered in (eb) the continuum limit since it is explicitly broken by (e) the discretisation.
(6) Authors present the first continuum extrapolation of the mass spectrum of (e) supersymmetric Yang-Mills theory.
(7) The results are consistent with the formation of supermultiplets and the absence of (e) non-perturbative sources of supersymmetry breaking.
(8) Investigations also indicate (eb) that numerical lattice simulations can be applied to (e&eb) non-trivial supersymmetric theories. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) DOI:10.1007/JHEP03(2016)080 Report number: MS-TP-15-29, DESY 15-221 Cite as: arXiv:1512.07014 [hep-lat] (or arXiv:1512.07014v1 [hep-lat] for this

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| version) | |
| **NOTATION** | |
| **Module One**<br>The theory is renormalizable and corrections are finite at any order of perturbation theory | |
| $G_{13}$ : Category one of theory is renormalizable and corrections<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of finite at any order of perturbation theory<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>The relevance of this result is due to (e) the fact that a Yang–Mills theory that describes *(eb) strong interaction and (e&eb) asymptotic freedom permits (eb) proper treatment of experimental results coming from (e) deep inelastic scattering | |
| $G_{16}$ : Category one of fact that a Yang–Mills theory that describes *(eb) strong interaction and (e&eb) asymptotic freedom permits (eb) proper treatment of experimental results coming from (e) deep inelastic scattering<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of relevance of this result<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>The relevance of this result is due to the fact that a Yang–Mills theory that describes *(eb) strong interaction and (e&eb) asymptotic freedom permits (eb) proper treatment of experimental results coming from (e) deep inelastic scattering | |
| $G_{20}$ : Category one of relevance of this result is due to the fact that a Yang–Mills theory<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of) strong interaction and (e&eb) asymptotic freedom permits (eb) proper treatment of experimental results coming from (e) deep inelastic scattering<br>$T_{21}$ : Category two of SAS | |

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| $T_{22}$ : Category three of SAS | |
| **Module four**<br><br>The relevance of this result is due to the fact that a Yang–Mills theory that describes strong interaction and (e&eb) asymptotic freedom permits (eb) proper treatment of experimental results coming from (e) deep inelastic scattering | |
| $G_{24}$ : Category one of **relevance of this result is due to the fact that a Yang–Mills theory that describes strong interaction;** asymptotic freedom permits (eb) proper treatment of experimental results coming from (e) deep inelastic scattering<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of asymptotic freedom permits (eb) proper treatment of experimental results coming from (e) deep inelastic scattering **;relevance of this result is due to the fact that a Yang–Mills theory that describes strong interaction**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>The relevance of this result is due to the fact that a Yang–Mills theory that describes strong interaction and asymptotic freedom permits (eb) proper treatment of experimental results coming from (e) deep inelastic scattering | |
| $G_{28}$ : Category one of relevance of this result is due to the fact that a Yang–Mills theory that describes strong interaction and asymptotic freedom<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of proper treatment of experimental results coming from deep inelastic scattering<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>To obtain the behavior of the Yang–Mills theory at high energies, and so to prove (eb) asymptotic freedom, one applies (e&eb) perturbation theory assuming (=) a small coupling | |
| $G_{32}$ : Category one of perturbation theory assuming a small coupling<br><br>$G_{33}$ : Category two of SAS | |

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| $G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of behavior of the Yang–Mills theory at high energies, and so to prove asymptotic freedom<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>This is verified a posterioriin the ultraviolet limit<br><br>. In the opposite limit, the infrared limit, the situation is (=) the opposite, as the coupling is (=) too large for perturbation theory to be (=) reliable | |
| $G_{36}$ : Category one of infrared limit, the situation is the opposite, as the coupling<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of too large for perturbation theory to be reliable<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight** | |
| $G_{40}$ : Category one of Yang Mills theory; low energies<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of low energies; Yang Mills theory<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine** | |
| $G_{44}$ : Category one of infrared limit; **perturbation theory**<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |

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| $T_{44}$ : Category one of **perturbation theory** ;infrared limit<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION A(29)-2**<br><br>**Light Bound States Of Supersymmetric SU(2) Yang-Mills Theory** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**The light bound states of supersymmetric SU(2) Yang-Mills theory Georg Bergner, Pietro Giudice, Istvan Montvay, Gernot Münster, Stefano Piemonte (Submitted on 22 Dec 2015)**<br><br>(1) Supersymmetry provides (eb) a well-established theoretical framework for (e) extensions of the standard model of particle physics and the general understanding of quantum field theories.<br>(2) Authors summarise here investigations of N=1 supersymmetric Yang-Mills theory with (e&eb) SU (2) gauge symmetry using (e) the non-perturbative first-principles method of (e) numerical lattice simulations.<br>(3) The strong interactions of gluons and (e&eb) their superpartners, the gluinos, lead to (eb) confinement, and a spectrum of bound states including (e) glueballs, mesons, and gluino-glueballs emerges (eb) at low energies.<br>(4) For unbroken supersymmetry these particles have to be (=) arranged in supermultiplets of equal masses.<br>(5) In lattice simulations supersymmetry can only be recovered in (eb) the continuum limit since it is explicitly broken by (e) the discretisation.<br>(6) Authors present the first continuum extrapolation of the mass spectrum of (e) supersymmetric Yang-Mills theory.<br>**(7)** The results are consistent with the formation of supermultiplets and the absence of (e) non-perturbative sources of supersymmetry breaking.<br>**(8)** Investigations also indicate (eb) that numerical lattice simulations can be applied to (e&eb) non-trivial supersymmetric theories. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) DOI:10.1007/JHEP03(2016)080 Report number: MS-TP-15-29, DESY 15-221 Cite as: arXiv:1512.07014 [hep-lat] (or arXiv:1512.07014v1 [hep-lat] for this version) | |
| **NOTATION** | |
| **Module One**<br>Supersymmetry provides (eb) a well-established theoretical framework for (e) extensions of the standard model of particle physics and the general understanding of quantum field theories | |
| $G_{13}$ : Category one of Supersymmetry | |

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| $G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of well-established theoretical framework for (e) extensions of the standard model of particle physics and the general understanding of quantum field theories<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>Supersymmetry provides a well-established theoretical framework for (e) extensions of the standard model of particle physics and the general understanding of quantum field theories | |
| $G_{16}$ : Category one of extensions of the standard model of particle physics and the general understanding of quantum field theories<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of Supersymmetry provides a well-established theoretical framework<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>Authors summarise here investigations of N=1 supersymmetric Yang-Mills theory with (e&eb) SU (2) gauge symmetry using (e) the non-perturbative first-principles method of (e) numerical lattice simulations | |
| $G_{20}$ : Category one of **N=1 supersymmetric Yang-Mills theory;** SU (2) gauge symmetry using (e) the non-perturbative first-principles method of (e) numerical lattice simulations<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of SU (2) gauge symmetry using (e) the non-perturbative first-principles method of (e) numerical lattice simulations **;N=1 supersymmetric Yang-Mills theory**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Authors summarise here investigations of N=1 supersymmetric Yang-Mills theory with SU (2) gauge symmetry using (e) the non-perturbative first-principles method of (e) numerical lattice simulations | |
| $G_{24}$ : Category one of non-perturbative first-principles method of (e) numerical lattice simulations | |

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|---|---|
| $G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of N=1 supersymmetric Yang-Mills theory with SU (2) gauge symmetry<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>non-perturbative first-principles method of (e) numerical lattice simulations | |
| $G_{28}$ : Category one of numerical lattice simulations<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of non-perturbative first-principles method<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>The strong interactions of gluons and (e&eb) their superpartners, the gluinos, lead to (eb) confinement, and a spectrum of bound states including (e) glueballs, mesons, and gluino-glueballs emerges (eb) at low energies | |
| $G_{32}$ : Category one of **strong interactions of gluons;** their superpartners, the gluinos, lead to (eb) confinement, and a spectrum of bound states including (e) glueballs, mesons, and gluino-glueballs emerges (eb) at low energies<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of their superpartners, the gluinos, lead to (eb) confinement, and a spectrum of bound states including (e) glueballs, mesons, and gluino-glueballs emerges (eb) at low energies **;strong interactions of gluons**<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br>The strong interactions of gluons and their superpartners, the gluinos, lead to (eb) confinement, and a spectrum of bound states including (e) glueballs, mesons, and gluino-glueballs emerges (eb) at low energies | |

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| $G_{36}$ : Category one of strong interactions of gluons and their superpartners, the gluinos<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of confinement, and a spectrum of bound states including (e) glueballs, mesons, and gluino-glueballs emerges (eb) at low energies<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The strong interactions of gluons and their superpartners, the gluinos, lead to confinement, and a spectrum of bound states including (e) glueballs, mesons, and gluino-glueballs emerges (eb) at low energies | |
| $G_{40}$ : Category one of glueballs, mesons, and gluino-glueballs emerges (eb) at low energies<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of strong interactions of gluons and their superpartners, the gluinos, lead to confinement, and a spectrum of bound states including<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The strong interactions of gluons and their superpartners, the gluinos, lead to confinement, and a spectrum of bound states including glueballs, mesons, and gluino-glueballs emerges (eb) at low energies | |
| $G_{44}$ : Category one of strong interactions of gluons and their superpartners, the gluinos, lead to confinement, and a spectrum of bound states including glueballs, mesons, and gluino-glueballs<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of low energies<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION A(30)

## Propagators

### INTRODUCTION—VARIABLES USED

**Source: Wikipedia**

(1) That is the interesting case, being inherent to (e) the description of hadronic matter and, more generally, to (e) the entire observed bound states of (e) gluons and quarks and their confinement (see hadrons).
(2) The most used method to study the theory in this limit is to try to solve it (e) on computers (see lattice gauge theory).
(3) In this case, large computational resources are needed to be (=) sure the correct limit of (e) infinite volume (smaller lattice spacing) is obtained.
(4) This is the limit the results must be compared with. Smaller spacing and (e&eb) larger coupling are not independent of each other, and larger computational resources are needed for (e) each.
(5) As of today, the situation appears somewhat satisfactory for (e) the hadronic spectrum and the computation of (e) the gluon and ghost propagators, but the glueball and hybrids spectra are yet a questioned matter in view of the experimental observation of such exotic states.
(6) Indeed, the σ resonance [6][7] is not seen in (eb) any of such lattice computations and contrasting interpretations have been put forward. This is a hotly debated issue.
(7) In order to understand the behavior of the theory at large and small momenta, a key quantity is (=) the propagator.
(8) For a Yang–Mills theory we have to consider both the gluon and (e&eb) the ghost propagators.
**(9)** At large momenta (ultraviolet limit), the quess completely settled with (e&eb) the discovery of (e) the asymptotic freedom.[8][9]

### NOTATION

### Module One

**The light bound states of supersymmetric SU(2) Yang-Mills theory Georg Bergner, Pietro Giudice, Istvan Montvay, Gernot Münster, Stefano Piemonte (Submitted on 22 Dec 2015)**

(1) For unbroken supersymmetry these particles have to be (=) arranged in supermultiplets of equal masses.
(2) In lattice simulations supersymmetry can only be recovered in (eb) the continuum limit since it is explicitly broken by (e) the discretisation.
(3) Authors present the first continuum extrapolation of the mass spectrum of (e) supersymmetric Yang-Mills theory.
(4) The results are consistent with the formation of supermultiplets and the absence of (e) non-perturbative sources of supersymmetry breaking
(5) Investigations also indicate (eb) that numerical lattice simulations can be applied to (e&eb) non-trivial supersymmetric theories. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) DOI:10.1007/JHEP03(2016)080 Report number: MS-TP-15-29, DESY 15-221 Cite as: arXiv:1512.07014 [hep-lat] (or arXiv:1512.07014v1 [hep-lat] for this version)

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| For unbroken supersymmetry these particles have to be (=) arranged in supermultiplets of equal masses | |
| $G_{13}$ : Category one of these particles for unbroken supersymmetry<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of arranged in supermultiplets of equal masses<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>supersymmetry in lattice simulations can only be recovered in (eb) the continuum limit since it is explicitly broken by (e) the discretisation | |
| $G_{16}$ : Category one of supersymmetry in lattice simulations can only be recovered<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of continuum limit since it is explicitly broken by (e) the discretisation<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>supersymmetry in lattice simulations can only be recovered in the continuum limit since it is explicitly broken by (e) the discretisation | |
| $G_{20}$ : Category one of discretisation<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of supersymmetry in lattice simulations can only be recovered in the continuum limit since it is explicitly broken<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>Authors present the first continuum extrapolation of the mass spectrum of (e) supersymmetric Yang-Mills theory | |
| $G_{24}$ : Category one of supersymmetric Yang-Mills theory | |

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| $G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of first continuum extrapolation of the mass spectrum<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>The results are consistent with the formation of supermultiplets and the absence of (e) non-perturbative sources of supersymmetry breaking | |
| $G_{28}$ : Category one of **formation of supermultiplets;** absence of non-perturbative sources of supersymmetry breaking<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of absence of (e) non-perturbative sources of supersymmetry breaking **;formation of supermultiplets**<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>Investigations also indicate (eb) that numerical lattice simulations can be applied to (e&eb) non-trivial supersymmetric theories | |
| $G_{32}$ : Category one of **numerical lattice simulations;** non-trivial supersymmetric theories<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of non-trivial supersymmetric theories; **numerical lattice simulations**<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br>That is the interesting case, being inherent to the description of hadronic matter and, more generally, to (e) the entire observed bound states of (e) gluons and quarks and their confinement (see hadrons) | |
| $G_{36}$ : Category one of gluons and quarks and their confinement (see hadrons) | |

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| $G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of description of hadronic matter and, more generally, to (e) the entire observed bound states<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>In this case, large computational resources are needed to be (=) sure the correct limit of (e) infinite volume (smaller lattice spacing) is obtained | |
| $G_{40}$ : Category one of, large computational resources are needed<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of sure the correct limit of infinite volume (smaller lattice spacing) is obtained<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>For a Yang–Mills theory we have to consider both the gluon and (e&eb) the ghost propagators | |
| $G_{44}$ : Category one of **gluon ;** ghost propagators<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of ghost propagators **;gluon**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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# Quantum Query Complexity

## INTRODUCTION—VARIABLES USED

**On exact quantum query complexity Ashley Montanaro, Richard Jozsa, Graeme Mitchison**

(1) Authors present several families of (e) total Boolean functions which have (e) exact quantum query complexity which is (=) a constant multiple (between 1/2 and 2/3) of (e) their classical query complexity, and show (eb) those optimal quantum algorithms for these functions cannot be obtained by (e) simply computing parities of pairs of bits.
(2) Authors also characterise the model of nonadaptive exact quantum query complexity in terms of (e&eb) coding theory and completely characterise (eb) the query complexity of symmetric Boolean functions in this context.
(3) These results were originally inspired by numerically solving (e) the semidefinite programs characterizing (eb) quantum query complexity for (e) small problem sizes.
(4) Authors include numerical results giving the optimal success probabilities achievable by (e) quantum algorithms computing all Boolean functions on up to 4 bits, and all symmetric Boolean functions on up to 6 bits. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1111.0475 [quant-ph] (or arXiv:1111.0475v2 [quant-ph] for this version)

**An introduction to measurement based quantum computation Richard Jozsa**

**(5)** In the formalism of measurement based quantum computation start with a given fixed entangled state of many qubits and perform computation by applying a sequence of measurements to (e&eb) designated qubits in designated bases.
**(6)** The choice of basis for later measurements may depend on earlier measurement outcomes and the final result of the computation is determined from (e) the classical data of all the measurement outcomes.
**(7)** This is in contrast to the more familiar gate array model in which computational steps are (=) unitary operations, developing (eb) a large entangled state prior to some final measurements for (e) the output.
**(8)** Two principal schemes of measurement based computation are (=) teleportation quantum computation (TQC) and the so-called cluster model or one-way quantum computer (1WQC).
**(9)** They will describe these schemes and show how they are able to perform (e&eb) universal quantum computation.
**(10)** Authors will outline various possible relationships between the models which serve to clarify their workings. We will also discuss possible novel computational benefits of the measurement based models compared to the gate array model, especially issues of parallelisability of algorithms. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: quant-ph/0508124 (or arXiv:quant-ph/0508124v2 for this version

## NOTATION

### Module One

Authors present several families of (e) total Boolean functions which have (e) exact quantum query complexity which is (=) a constant multiple (between 1/2 and 2/3) of (e) their classical query complexity, and show (eb) those optimal quantum algorithms for these functions cannot be obtained by (e) simply computing parities of pairs of bits.

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| $G_{13}$ : Category one of exact quantum query complexity which is (=) a constant multiple (between 1/2 and 2/3) of (e) their classical query complexity, and show (eb) those optimal quantum algorithms for these functions cannot be obtained by (e) simply computing parities of pairs of bits.<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of Boolean functions<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>Authors present several families of total Boolean functions which have exact quantum query complexity which is (=) a constant multiple (between 1/2 and 2/3) of (e) their classical query complexity, and show (eb) those optimal quantum algorithms for these functions cannot be obtained by (e) simply computing parities of pairs of bits. | |
| $G_{16}$ : Category one of several families of total Boolean functions which have exact quantum query complexity<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one ofconstant multiple (between 1/2 and 2/3) of (e) their classical query complexity, and show (eb) those optimal quantum algorithms for these functions cannot be obtained by (e) simply computing parities of pairs of bits<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>Authors present several families of total Boolean functions which have exact quantum query complexity which is a constant multiple (between 1/2 and 2/3) of (e) their classical query complexity and show (eb) those optimal quantum algorithms for these functions cannot be obtained by (e) simply computing parities of pairs of bits.<br><br>In the case of analysis of the rsources necessary and corresponding linear stability analysis each and every statement could be taken as what it is one gobbling up the other, or the other obfuscating the other. There need not be restriction upon the number of and type of categories and both could be added to in once resource faction. | |
| $G_{20}$ : Category one of their classical query complexity, and show (eb) those optimal quantum algorithms for these functions cannot be obtained by (e) simply computing parities of pairs of bits.<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |

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| $T_{20}$ : Category one of several families of total Boolean functions which have exact quantum query complexity which is a constant multiple (between 1/2 and 2/3)<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Authors present several families of total Boolean functions which have exact quantum query complexity which is a constant multiple (between 1/2 and 2/3) of their classical query complexity and show those optimal quantum algorithms for these functions cannot be obtained by (e) simply computing parities of pairs of bits | |
| $G_{24}$ : Category one of simply computing parities of pairs of bits<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of several families of total Boolean functions which have exact quantum query complexity which is a constant multiple (between 1/2 and 2/3) of their classical query complexity and show those optimal quantum algorithms for these functions cannot be obtained<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Authors also characterise the model of nonadaptive exact quantum query complexity in terms of (e&eb) coding theory and completely characterise (eb) the query complexity of symmetric Boolean functions in this context | |
| $G_{28}$ : Category one of **characterise the model of nonadaptive exact quantum query complexity;** coding theory and completely characterise (eb) the query complexity of symmetric Boolean functions in this context<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of coding theory and completely characterise (eb) the query complexity of symmetric Boolean functions in this context; **characterise the model of nonadaptive exact quantum query complexity**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Authors also characterise the model of nonadaptive exact quantum query complexity in terms of (e&eb) coding theory and completely characterise the query complexity of symmetric Boolean functions in this | |

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| context | |
| $G_{32}$ : Category one of **characterise the model of nonadaptive exact quantum query complexity in terms of coding theory and completely characterise;** symmetric Boolean functions in this context<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of symmetric Boolean functions in this context **;characterise the model of nonadaptive exact quantum query complexity in terms of coding theory and completely characterise**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>These results were originally inspired by numerically solving (e) the semidefinite programs characterizing (eb) quantum query complexity for (e) small problem sizes<br><br>Note here semi definite programs characterizing quantum query complexity for small problem sizes is the only resource. Fronm module to module such resource varies and there might be one or two factors wanting to obfuscate iof not obliterate such a resource. We can find the linear stability of the situational significance therefor. (See Gopalaswamy: We shall dilate on the topic in next papers). | |
| $G_{36}$ : Category one of semidefinite programs characterizing (eb) quantum query complexity for (e) small problem sizes<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of results were originally inspired by numerically solving<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>These results were originally inspired by numerically solving the semidefinite programs characterizing (eb) quantum query complexity for small problem sizes | |
| $G_{40}$ : Category one of originally inspired by numerically solving the semidefinite programs characterizing<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |

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| $T_{40}$ : Category one of quantum query complexity for small problem sizes<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Authors include numerical results giving the optimal success probabilities achievable by (e) quantum algorithms computing all Boolean functions on up to 4 bits, and all symmetric Boolean functions on up to 6 bits.<br><br>Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1111.0475 [quant-ph] (or arXiv:1111.0475v2 [quant-ph] for this version) | |
| $G_{44}$ : Category one of quantum algorithms computing all Boolean functions on up to 4 bits, and all symmetric Boolean functions on up to 6 bits.<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of numerical results giving the optimal success probabilities achievable<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION TWO**<br><br>**Measurement Based Quantum Computation** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**An introduction to measurement based quantum computation Richard Jozsa**<br><br>**(1)** In the formalism of measurement based quantum computation start with a given fixed entangled state of many qubits and perform computation by applying a sequence of measurements to (e&eb) designated qubits in designated bases.<br>**(2)** The choice of basis for later measurements may depend on earlier measurement outcomes and the final result of the computation is determined from (e) the classical data of all the measurement outcomes.<br>**(3)** This is in contrast to the more familiar gate array model in which computational steps are (=) unitary operations, developing (eb) a large entangled state prior to some final measurements for (e) the output.<br>**(4)** Two principal schemes of measurement based computation are (=) teleportation quantum computation (TQC) and the so-called cluster model or one-way quantum computer (1WQC).<br>**(5)** They will describe these schemes and show how they are able to perform (e&eb) universal quantum computation. | |

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| **(6)** Authors will outline various possible relationships between the models which serve to clarify their workings.<br>**(7)** They will also discuss possible novel computational benefits of the measurement based models compared to (e&eb) the gate array model, especially issues of parallelisability of algorithms. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: quant-ph/0508124 (or arXiv:quant-ph/0508124v2 for this version | |
| **NOTATION** | |
| **Module One**<br>In the formalism of **measurement based quantum computation start with a given fixed entangled state of many qubits and perform computation by applying a sequence of measurements** to (e&eb) designated qubits in designated bases | |
| $G_{13}$ : Category one of **measurement based quantum computation start with a given fixed entangled state of many qubits and perform computation by applying a sequence of measurements;** designated qubits in designated bases<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of designated qubits in designated bases **;measurement based quantum computation start with a given fixed entangled state of many qubits and perform computation by applying a sequence of measurements**<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>The choice of basis for later measurements may depend on earlier measurement outcomes and the final result of the computation is determined from (e) the classical data of all the measurement outcomes | |
| $G_{16}$ : Category one of classical data of all the measurement outcomes<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of choice of basis for later measurements may depend on earlier measurement outcomes and the final result of the computation is determined<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>This is in contrast to the more familiar gate array model in which computational steps are (=) unitary operations, developing (eb) a large entangled state prior to some final measurements for (e) the output | |

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| $G_{20}$ : Category one of gate array model computational steps<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of unitary operations, developing (eb) a large entangled state prior to some final measurements for (e) the output<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>This is in contrast to the more familiar gate array model in which computational steps are unitary operations, developing (eb) a large entangled state prior to some final measurements for the output | |
| $G_{24}$ : Category one of gate array model in which computational steps are unitary operations<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of large entangled state prior to some final measurements for the output<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Two principal schemes of measurement based computation are (=) teleportation quantum computation (TQC) and the so-called cluster model or one-way quantum computer (1WQC) | |
| $G_{28}$ : Category one of principal schemes of measurement based computation<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of teleportation quantum computation (TQC) and the so-called cluster model or one-way quantum computer (1WQC<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>They will describe these schemes and show how they are able to perform (e&eb) universal quantum computation | |

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| $G_{32}$ : Category one of **these schemes;** universal quantum computation<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of universal quantum computation **;these schemes**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>They will also discuss possible novel computational benefits of the measurement based models compared to (e&eb) the gate array model, especially issues of parallelisability of algorithms<br><br>. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: quant-ph/0508124 (or arXiv:quant-ph/0508124v2 for this version | |
| $G_{36}$ : Category one of **measurement based models;** gate array model, especially issues of parallelisability of algorithms<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of gate array model, especially issues of parallelisability of algorithms; **measurement based models**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight** | |
| $G_{40}$ : Category one of parallelisability of algorithms; **gate array models**<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of **gate array models;** parallelisability of algorithms<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine** | |
| $G_{44}$ : Category one of parallelisability of algorithms; **measurement based models** | |

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| $G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of **measurement based models** ;parallelisability of algorithms<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION THREE

## Confining String In An Analytically Solvable 3D Model

### INTRODUCTION—VARIABLES USED

**fine structure of the confining string in an analytically solvable 3D model Davide Vadacchino, Michele Caselle, Roberto Pellegrini, Marco Panero**

(1) In U (1) lattice gauge theory in (eb) three spacetime dimensions, confinement can be analytically shown to persist at all values of (e) the coupling.
(2) Furthermore, the explicit predictions for the dependence of string tension σ and mass gap m0 on (=) the coupling allows one to (eb) tune their ratio at will.
(3) These features, and the possibility of obtaining high-precision numerical results via (e&eb) an exact duality map to (e&eb) a spin model, make this theory an ideal laboratory to test the effective string description of (e) confining flux tubes.
(4) In this contribution, authors discuss investigation of next-to-leading-order corrections to (e) the confining potential and of the finite-temperature behavior of (e) the flux tube width.
(5) Data provide (eb) a very stringent test of (e) the theoretical predictions for these quantities and allow to (eb) test their dependence on the m0/σ√ ratio. Subjects: High Energy Physics - Lattice (hep-lat)Journal reference: PoS (LATTICE 2013) 373Report number: IFT-UAM/CSIC-13-104Cite as: arXiv:1311.4071 [hep-lat] (or arXiv:1311.4071v1 [hep-lat] for this version)

**Undersaturation of quarks at early stages of relativistic nuclear collisions: the hot glue initial scenario and its observable signatures H. Stoecker, M. Beitel, T.S. Biró, L.P. Csernai, K. Gallmeister, M.I. Gorenstein, C. Greiner, I.N. Mishustin, M. Panero, S. Raha, L.M. Satarov, S. Schramm, F. Senzel, B. Sinha, J. Steinheimer, J. Struckmeier, V. Vovchenko, Z. Xu, K. Zhou, P. Zhuang**

**(6)** The early stage of high multiplicity nuclear collisions is represented by (e) nearly quark less, hot, deconfined pure gluon plasma.
**(7)** This new scenario should be characterized by (e) suppression of high pT photons and dileptons as

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| well as by (e) reduced baryon to meson ratios.<br>**(8)** Authors present the numerical results for central Pb+Pb collisions at the LHC energies by using (e) the ideal Bjorken hydrodynamics with (e&eb)time-dependent quark fugacity.<br>**(9)** It is shown that about 25\% of final total entropy is generated during (e&eb)the hydrodynamic evolution of chemically under (e&eb) saturated quark-gluon plasma. Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex); High Energy Physics - Lattice (hep-lat); Nuclear Experiment (nucl-ex); Nuclear Theory (nucl-th) Cite as: arXiv:1509.07682 [hep-ph] (or arXiv:1509.07682v3 [hep-ph] for this version) | |
| **NOTATION** | |
| **Module One**<br>In U (1) lattice gauge theory in (eb) three spacetime dimensions, confinement can be analytically shown to persist at all values of (e) the coupling | |
| $G_{13}$ : Category one of U (1) lattice gauge theory<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of three spacetime dimensions, confinement can be analytically shown to persist at all values of (e) the coupling<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>In U (1) lattice gauge theory in three spacetime dimensions, confinement can be analytically shown to persist at all values of the coupling | |
| $G_{16}$ : Category one of **U (1) lattice gauge theory in three spacetime dimensions, confinement can be analytically shown to persist;** all values of the coupling<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of all values of the coupling **; U (1) lattice gauge theory in three spacetime dimensions, confinement can be analytically shown to persist**<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>Furthermore, the explicit predictions for the dependence of string tension σ and mass gap m0 on (=) the coupling allows one to (eb) tune their ratio at will | |
| $G_{20}$ : Category one of **explicit predictions for the dependence of string tension σ and mass gap m0;** coupling allows one to (eb) tune their ratio at will | |

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| $G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of coupling allows one to (eb) tune their ratio at will **;explicit predictions for the dependence of string tension σ and mass gap m0**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Furthermore, the explicit predictions for the dependence of string tension σ and mass gap m0 on the coupling allows one to (eb) tune their ratio at will | |
| $G_{24}$ : Category one of explicit predictions for the dependence of string tension σ and mass gap m0 on the coupling<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of tune their ratio at will<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>These features, and the possibility of obtaining high-precision numerical results via (e&eb) an exact duality map to (e&eb) a spin model, make this theory an ideal laboratory to test the effective string description of (e) confining flux tubes | |
| $G_{28}$ : Category one of **features, and the possibility of obtaining high-precision numerical results** exact duality map to (e&eb) a spin model, make this theory an ideal laboratory to test the effective string description of (e) confining flux tubes**;**<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of exact duality map to (e&eb) a spin model, make this theory an ideal laboratory to test the effective string description of (e) confining flux tubes **;features, and the possibility of obtaining high-precision numerical results**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>These features, and the possibility of obtaining high-precision numerical results via an exact duality map to | |

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| (e&eb) a spin model, make this theory an ideal laboratory to test the effective string description of (e) confining flux tubes | |
| $G_{32}$ : Category one of **These features, and the possibility of obtaining high-precision numerical results via an exact duality map;** spin model, make this theory an ideal laboratory to test the effective string description of (e) confining flux tubes<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of spin model, make this theory an ideal laboratory to test the effective string description of (e) confining flux tubes **;These features, and the possibility of obtaining high-precision numerical results via an exact duality map**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>These features, and the possibility of obtaining high-precision numerical results via an exact duality map to a spin model, make this theory an ideal laboratory to test the effective string description of (e) confining flux tubes | |
| $G_{36}$ : Category one of confining flux tubes<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of These features, and the possibility of obtaining high-precision numerical results via an exact duality map to a spin model, make this theory an ideal laboratory to test the effective string description<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>In this contribution, authors discuss investigation of next-to-leading-order corrections to (e) the confining potential and of the finite-temperature behavior of (e) the flux tube width | |
| $G_{40}$ : Category one of confining potential and of the finite-temperature behavior of (e) the flux tube width<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |

$T_{40}$ : Category one of next-to-leading-order corrections

$T_{41}$ : Category two of SAS

$T_{42}$ : Category three of SAS

**Module Nine**

In this contribution, authors discuss investigation of next-to-leading-order corrections to the confining potential and of the finite-temperature behavior of (e) the flux tube width

$G_{44}$ : Category one of flux tube width

$G_{45}$ : Category two of SAS

$G_{46}$ : Category three of SAS

$T_{44}$ : Category one of next-to-leading-order corrections to the confining potential and of the finite-temperature behavior

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

## SECTION FOUR

## Quarks At Early Stages Of Relativistic Nuclear Collisions

**INTRODUCTION—VARIABLES USED**

(1) Data provide (eb) a very stringent test of (e) the theoretical predictions for these quantities and allow to (eb) test their dependence on the m0/σ√ ratio. Subjects: High Energy Physics - Lattice (hep-lat)Journal reference: PoS (LATTICE 2013) 373Report number: IFT-UAM/CSIC-13-104Cite as: arXiv:1311.4071 [hep-lat] (or arXiv:1311.4071v1 [hep-lat] for this version)

**Undersaturation of quarks at early stages of relativistic nuclear collisions: the hot glue initial scenario and its observable signatures H. Stoecker, M. Beitel, T.S. Biró, L.P. Csernai, K. Gallmeister, M.I. Gorenstein, C. Greiner, I.N. Mishustin, M. Panero, S. Raha, L.M. Satarov, S. Schramm, F. Senzel, B. Sinha, J. Steinheimer, J. Struckmeier, V. Vovchenko, Z. Xu, K. Zhou, P. Zhuang**

**(2)** The early stage of high multiplicity nuclear collisions is represented by (e) nearly quark less, hot, deconfined pure gluon plasma.

**(3)** This new scenario should be characterized by (e) suppression of high pT photons and dileptons as well as by (e) reduced baryon to meson ratios.

**(4)** Authors present the numerical results for central Pb+Pb collisions at the LHC energies by using (e) the ideal Bjorken hydrodynamics with (e&eb)time-dependent quark fugacity.

**(5)** It is shown that about 25\% of final total entropy is generated during (e&eb )the hydrodynamic evolution of chemically under (e&eb) saturated quark-gluon plasma. Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex); High Energy Physics - Lattice (hep-lat); Nuclear Experiment (nucl-ex); Nuclear Theory (nucl-th) Cite as:

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| arXiv:1509.07682 [hep-ph] (or arXiv:1509.07682v3 [hep-ph] for this version) | |
| **NOTATION** | |
| **Module One**<br>Data provide a very stringent test of the theoretical predictions for these quantities and allow to (eb) test their dependence on the m0/σ√ ratio.<br>Subjects: High Energy Physics - Lattice (hep-lat)Journal reference: PoS (LATTICE 2013) 373Report number: IFT-UAM/CSIC-13-104Cite as: arXiv:1311.4071 [hep-lat] (or arXiv:1311.4071v1 [hep-lat] for this version) | |
| $G_{13}$ : Category one of Data<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of very stringent test of the theoretical predictions for these quantities and allow to (eb) test their dependence on the m0/σ√ ratio.<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>Data provide a very stringent test of the theoretical predictions for these quantities and allow to (eb) test their dependence on the m0/σ√ ratio.<br>Subjects: High Energy Physics - Lattice (hep-lat)Journal reference: PoS (LATTICE 2013) 373Report number: IFT-UAM/CSIC-13-104Cite as: arXiv:1311.4071 [hep-lat] (or arXiv:1311.4071v1 [hep-lat] for this version) | |
| $G_{16}$ : Category one of Data provide a very stringent test of the theoretical predictions for these quantities<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of test their dependence on the m0/σ√ ratio.<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>The early stage of high multiplicity nuclear collisions is represented by (e) nearly quark less, hot, deconfined pure gluon plasma | |
| $G_{20}$ : Category one of **early stage of high multiplicity nuclear collisions;** nearly quark less, hot, deconfined pure gluon plasma | |

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| $G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of nearly quark less, hot, deconfined pure gluon plasma; **early stage of high multiplicity nuclear collisions**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>This new scenario should be characterized by (e) suppression of high pT photons and dileptons as well as by (e) reduced baryon to meson ratios | |
| $G_{24}$ : Category one of suppression of high pT photons and dileptons as well as by (e) reduced baryon to meson ratios<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of new scenario should be characterized<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>This new scenario should be characterized by suppression of high pT photons and dileptons as well as by (e) reduced baryon to meson ratios | |
| $G_{28}$ : Category one of **reduced baryon to meson ratios;** new scenario should be characterized by suppression of high pT photons and dilepton<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of new scenario should be characterized by suppression of high pT photons and dileptons; **reduced baryon to meson ratios**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Authors present the numerical results for central Pb+Pb collisions at the LHC energies by using (e) the ideal Bjorken hydrodynamics with (e&eb)time-dependent quark fugacity | |

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| $G_{32}$ : Category one of ideal Bjorken hydrodynamics with (e&eb)time-dependent quark fugacity<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of numerical results for central Pb+Pb collisions at the LHC energies<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>ideal Bjorken hydrodynamics with (e&eb)time-dependent quark fugacity | |
| $G_{36}$ : Category one of **ideal Bjorken hydrodynamics;** time-dependent quark fugacity<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of time-dependent quark fugacity **;ideal Bjorken hydrodynamics**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>It is shown that about 25\% of final total entropy is generated during (e&eb) the hydrodynamic evolution of chemically under (e&eb) saturated quark-gluon plasma.<br><br>Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex); High Energy Physics - Lattice (hep-lat); Nuclear Experiment (nucl-ex); Nuclear Theory (nucl-th) Cite as: arXiv:1509.07682 [hep-ph] (or arXiv:1509.07682v3 [hep-ph] for this version) | |
| $G_{40}$ : Category one of**25\% of final total entropy is generated;** hydrodynamic evolution of chemically under (e&eb) saturated quark-gluon plasma.<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of hydrodynamic evolution of chemically under (e&eb) saturated quark-gluon plasma. ;**25\% of final total entropy is generated**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |

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| **Module Nine**<br><br>It is shown that about 25\% of final total entropy is generated during the hydrodynamic evolution of chemically under (e&eb) saturated quark-gluon plasma.<br><br>Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex); High Energy Physics - Lattice (hep-lat); Nuclear Experiment (nucl-ex); Nuclear Theory (nucl-th) Cite as: arXiv:1509.07682 [hep-ph] (or arXiv:1509.07682v3 [hep-ph] for this version) | |
| $G_{44}$ : Category one of**25\% of final total entropy is generated during the hydrodynamic evolution of chemically;** saturated quark-gluon plasma<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of saturated quark-gluon plasma **;25\% of final total entropy is generated during the hydrodynamic evolution of chemically**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION FIVE**<br><br>**Hagedorn Spectrum And Equation Of State Of Yang-Mills Theories** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Hagedorn spectrum and equation of state of Yang-Mills theories Michele Caselle, Alessandro Nada, Marco Panero**<br><br>(1) Authors present a novel lattice calculation of (e) the equation of state of (e) SU (2) Yang-Mills theory in (eb) the confining phase.<br>(2) They show that a gas of massive, non-interacting glueballs describes (eb) remarkably well the results, provided (e) that a bosonic closed-string model is used to derive (eb) an exponentially growing Hagedorn spectrum for the heavy glueball states with (e&eb) no free parameters<br>(3) This effective model can be applied to (e&eb) SU (3) Yang-Mills theory and (e&eb) the theoretical prediction agrees nicely with the lattice (=) results reported by Bors\'anyi et al. in JHEP 07 (2012) 056. Subjects: High Energy Physics - Lattice (hep-lat) Cite as: arXiv: 1509.06905 [hep-lat] (or arXiv:1509.06905v1 [hep-lat] for this version) | |

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| **Hagedorn spectrum and thermodynamics of SU(2) and SU(3) Yang-Mills theories Michele Caselle, Alessandro Nada, Marco Panero**<br><br>**(4)** Authors present a high-precision lattice calculation of (e) the equation of state in the confining phase of (e) SU (2) Yang-Mills theory.<br>**(5)** Theyshow that the results are described very well by (e) a gas of massive, non-interacting glueballs, provided (e) one assumes an exponentially growing Hagedorn spectrum.<br>**(6)** The latter can be derived within (eb) an effective bosonic closed-string model, leading to (eb) a parameter-free theoretical prediction, which is in perfect agreement with (=) our lattice results.<br>**(7)** Furthermore, when applied to (e&eb) SU (3) Yang-Mills theory, this effective model accurately describes (eb) the lattice results reported by Bors\'anyi et al. in JHEP 07 (2012) 056. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 07 (2015) 143 DOI: 10.1007/JHEP07(2015)143 Cite as: arXiv:1505.01106 [hep-lat] (or arXiv:1505.01106v2 [hep-lat] for this version) | |
| **NOTATION** | |
| **Module One**<br>Authors present a novel lattice calculation of the equation of state of (e) SU (2) Yang-Mills theory in (eb) the confining phase | |
| $G_{13}$ : Category one of SU (2) Yang-Mills theory in (eb) the confining phase<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of lattice calculation of the equation of state<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>Authors present a novel lattice calculation of the equation of state of SU (2) Yang-Mills theory in (eb) the confining phase | |
| $G_{16}$ : Category one of lattice calculation of the equation of state of SU (2) Yang-Mills theory<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of confining phase<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>They show that a gas of massive, non-interacting glueballs describes (eb) remarkably well the results, provided (e) that a bosonic closed-string model is used to derive (eb) an exponentially growing Hagedorn | |

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| spectrum for the heavy glueball states with (e&eb) no free parameters | |
| $G_{20}$ : Category one of gas of massive, non-interacting glueballs<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one ofremarkably well the results, provided (e) that a bosonic closed-string model is used to derive (eb) an exponentially growing Hagedorn spectrum for the heavy glueball states with (e&eb) no free parameters<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>They show that a gas of massive, non-interacting glueballs describes remarkably well the results, provided (e) that a bosonic closed-string model is used to derive (eb) an exponentially growing Hagedorn spectrum for the heavy glueball states with (e&eb) no free parameters | |
| $G_{24}$ : Category one of bosonic closed-string model is used to derive (eb) an exponentially growing Hagedorn spectrum for the heavy glueball states with (e&eb) no free parameters<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of gas of massive, non-interacting glueballs describes remarkably well the results<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>They show that a gas of massive, non-interacting glueballs describes remarkably well the results, provided that a bosonic closed-string model is used to derive (eb) an exponentially growing Hagedorn spectrum for the heavy glueball states with (e&eb) no free parameters | |
| $G_{28}$ : Category one of gas of massive, non-interacting glueballs describes remarkably well the results, provided that a bosonic closed-string model<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of exponentially growing Hagedorn spectrum for the heavy glueball states with (e&eb) no free parameters<br><br>$T_{29}$ : Category two of SAS | |

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| $T_{30}$ : Category three of SAS | |
| **Module six**<br><br>They show that a gas of massive, non-interacting glueballs describes remarkably well the results, provided that a bosonic closed-string model is used to derive an exponentially growing Hagedorn spectrum for the heavy glueball states with (e&eb) no free parameters | |
| $G_{32}$ : Category one of **gas of massive, non-interacting glueballs describes remarkably well the results, provided that a bosonic closed-string model is used to derive an exponentially growing Hagedorn spectrum;** heavy glueball states with (e&eb) no free parameters<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of heavy glueball states with (e&eb) no free parameters **;gas of massive, non-interacting glueballs describes remarkably well the results, provided that a bosonic closed-string model is used to derive an exponentially growing Hagedorn spectrum**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>They show that a gas of massive, non-interacting glueballs describes remarkably well the results, provided that a bosonic closed-string model is used to derive an exponentially growing Hagedorn spectrum for the heavy glueball states with (e&eb) no free parameters | |
| $G_{36}$ : Category one of **gas of massive, non-interacting glueballs describes remarkably well the results, provided that a bosonic closed-string model is used to derive an exponentially growing Hagedorn spectrum for the heavy glueball states;** no free parameters<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of no free parameters **; gas of massive, non-interacting glueballs describes remarkably well the results, provided that a bosonic closed-string model is used to derive an exponentially growing Hagedorn spectrum for the heavy glueball states**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>This effective model can be applied to (e&eb) SU (3) Yang-Mills theory and (e&eb) the theoretical prediction agrees nicely with the lattice (=) results reported by Bors\'anyi et al. in JHEP 07 (2012) 056. | |

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| Subjects: High Energy Physics - Lattice (hep-lat) Cite as: arXiv: 1509.06905 [hep-lat] (or arXiv:1509.06905v1 [hep-lat] for this version) | |
| $G_{40}$ : Category one of **effective model;** SU (3) Yang-Mills theory and (e&eb) the theoretical prediction agrees nicely with the lattice (=) results reported by Bors\'anyi et al. in JHEP 07 (2012) 056.<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of SU (3) Yang-Mills theory and (e&eb) the theoretical prediction agrees nicely with the lattice (=) results reported by Bors\'anyi et al. in JHEP 07 (2012) 056. ; **effective model**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>This effective model can be applied to SU (3) Yang-Mills theory and (e&eb) the theoretical prediction agrees nicely with the lattice (=) results reported by Bors\'anyi et al. in JHEP 07 (2012) 056 | |
| $G_{44}$ : Category one of effective model can be applied to SU (3) Yang-Mills theory and (e&eb) the theoretical prediction<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of results reported by Bors\'anyi et al. in JHEP 07 (2012) 056<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)},$<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$<br>$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$<br>$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}$ $(a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,<br>$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$ | |

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| $(a_{36}^{'})^{(7)}, (a_{37}^{'})^{(7)}, (a_{38}^{'})^{(7)}, (b_{36}^{'})^{(7)}, (b_{37}^{'})^{(7)}, (b_{38}^{'})^{(7)},$ $(a_{40}^{'})^{(8)}, (a_{41}^{'})^{(8)}, (a_{42}^{'})^{(8)}, (b_{40}^{'})^{(8)}, (b_{41}^{'})^{(8)}, (b_{42}^{'})^{(8)},$ $(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)},$ are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}, t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}, t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21}, t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23}, t) =$ First detritions factor | |

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| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25}, t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}, t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}), t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}), t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}), t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25}, t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}), t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29}, t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}, t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}), t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}), t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}), t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29}, t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}), t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33}, t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}, t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}), t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}), t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}), t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33}, t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}, t)\right]G_{37}$ | *38* |

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|---|---|
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{matrix}(a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)}\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}\end{matrix}\right]G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{matrix}(a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)}\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}\end{matrix}\right]G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[\begin{matrix}(a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)}\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}\end{matrix}\right]G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for | |

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|---|---|
| category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3<br>$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3<br>$\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\boxed{-(b_{15}^{''})^{(1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | *60* |
| Where $\boxed{-(b_{13}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)}$ are sixth detrition coefficients for | |

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| category 1, 2 and 3 $\boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[\begin{array}{ccc} (a_{16}^{'})^{(2)} & \boxed{+(a_{16}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)} \quad \boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)} \end{array}\right] G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[\begin{array}{ccc} (a_{17}^{'})^{(2)} & \boxed{+(a_{17}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)} \quad \boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)} \end{array}\right] G_{17}$ | *62* |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[\begin{array}{ccc} (a_{18}^{'})^{(2)} & \boxed{+(a_{18}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)} \quad \boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)} \end{array}\right] G_{18}$ | *63* |
| Where $\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)}$ are first augmentation coefficients for category 1, 2 and 3 $\boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)}$ are second augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)}$ are seventh augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)}$ are eight augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)}$ are ninth augmentation coefficient for category 1, 2 and 3 | |

$$\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b'_{16})^{(2)}\boxed{-(b''_{16})^{(2)}(G_{19},t)} & \boxed{-(b''_{13})^{(1,1,)}(G,t)} & \boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{16} \quad (64)$$

$$\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b'_{17})^{(2)}\boxed{-(b''_{17})^{(2)}(G_{19},t)} & \boxed{-(b''_{14})^{(1,1,)}(G,t)} & \boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{17} \quad (65)$$

$$\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b'_{18})^{(2)}\boxed{-(b''_{18})^{(2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,)}(G,t)} & \boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{18} \quad (66)$$

where $\boxed{-(b''_{16})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{17})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{18})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,)}(G,t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{44})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2 and 3

$$\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a'_{20})^{(3)}\boxed{+(a''_{20})^{(3)}(T_{21},t)} & \boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{20} \quad (67)$$

$$\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a'_{21})^{(3)}\boxed{+(a''_{21})^{(3)}(T_{21},t)} & \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{21} \quad (68)$$

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a'_{22})^{(3)}\boxed{+(a''_{22})^{(3)}(T_{21},t)} & \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{22} \quad (69)$$

$\boxed{+(a''_{20})^{(3)}(T_{21},t)}, \boxed{+(a''_{21})^{(3)}(T_{21},t)}, \boxed{+(a''_{22})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)}, \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)}, \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)}, \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)}, \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)}$ are third augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)}, \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)}, \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)}, \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)}, \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)}, \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)}, \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)}, \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)}, \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)}, \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)}, \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)}$ are eight augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)}, \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)}, \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1, 2 and 3

$$\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b'_{20})^{(3)}\boxed{-(b''_{20})^{(3)}(G_{23},t)} & \boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{13})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{20} \quad (70)$$

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b'_{21})^{(3)}\boxed{-(b''_{21})^{(3)}(G_{23},t)} & \boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{14})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)}\boxed{-(b''_{22})^{(3)}(G_{23},t)} & \boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{22} \quad (72)$$

$\boxed{-(b''_{20})^{(3)}(G_{23},t)}, \boxed{-(b''_{21})^{(3)}(G_{23},t)}, \boxed{-(b''_{22})^{(3)}(G_{23},t)}$ *are first detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)}, \boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)}, \boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)}$ *are second detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{13})^{(1,1,1,)}(G,t)}, \boxed{-(b''_{14})^{(1,1,1,)}(G,t)}, \boxed{-(b''_{15})^{(1,1,1,)}(G,t)}$ *are third detrition coefficients for category 1,2 and 3*

| | |
|---|---|
| $-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*<br>$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$ $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a_{24}^{'})^{(4)} & +(a_{24}^{''})^{(4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,)}(T_{29},t) & +(a_{32}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24}$ | *73* |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a_{25}^{'})^{(4)} & +(a_{25}^{''})^{(4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,)}(T_{29},t) & +(a_{33}^{''})^{(6,6)}(T_{33},t) \\ & +(a_{14}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25}$ | *74* |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a_{26}^{'})^{(4)} & +(a_{26}^{''})^{(4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,)}(T_{29},t) & +(a_{34}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26}$ | *75* |

$(a_{24}^{''})^{(4)}(T_{25},t)$, $(a_{25}^{''})^{(4)}(T_{25},t)$, $(a_{26}^{''})^{(4)}(T_{25},t)$ are first augmentation coefficients category 1, 2 3

$+(a_{28}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,)}(T_{29},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,)}(T_{33},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1, 2 and 3

$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3

$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1, 2 and 3

$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b_{24}^{'})^{(4)} & -(b_{24}^{''})^{(4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,)}(G_{35},t) \\ & -(b_{13}^{''})^{(1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3)}(G_{23},t) \\ & -(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b_{25}^{'})^{(4)} & -(b_{25}^{''})^{(4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,)}(G_{35},t) \\ & -(b_{14}^{''})^{(1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3)}(G_{23},t) \\ & -(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b_{26}^{'})^{(4)} & -(b_{26}^{''})^{(4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,)}(G_{35},t) \\ & -(b_{15}^{''})^{(1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3)}(G_{23},t) \\ & -(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9)}(G_{47},t) \end{bmatrix} T_{26} \quad (78)$$

Where $-(b_{24}^{''})^{(4)}(G_{27},t)$, $-(b_{25}^{''})^{(4)}(G_{27},t)$, $-(b_{26}^{''})^{(4)}(G_{27},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,)}(G_{31},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,)}(G_{35},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2 and 3

| | |
|---|---|
| $-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 $-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3 $-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a_{28}^{'})^{(5)} & +(a_{28}^{''})^{(5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,)}(T_{25},t) & +(a_{32}^{''})^{(6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a_{29}^{'})^{(5)} & +(a_{29}^{''})^{(5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,)}(T_{25},t) & +(a_{33}^{''})^{(6,6,6)}(T_{33},t) \\ & +(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a_{30}^{'})^{(5)} & +(a_{30}^{''})^{(5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,)}(T_{25},t) & +(a_{34}^{''})^{(6,6,6)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ & +(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{30}$ | *81* |
| *Where* $+(a_{28}^{''})^{(5)}(T_{29},t)$, $+(a_{29}^{''})^{(5)}(T_{29},t)$, $+(a_{30}^{''})^{(5)}(T_{29},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3 *And* $+(a_{24}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,)}(T_{25},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3 $+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3 $+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3 $+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3 $+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3 $+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3 $+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3 $+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b'_{28})^{(5)} & -(b''_{28})^{(5)}(G_{31},t) & -(b''_{24})^{(4,4,)}(G_{27},t) & -(b''_{32})^{(6,6,6)}(G_{35},t) \\ -(b''_{13})^{(1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b'_{29})^{(5)} & -(b''_{29})^{(5)}(G_{31},t) & -(b''_{25})^{(4,4,)}(G_{27},t) & -(b''_{33})^{(6,6,6)}(G_{35},t) \\ -(b''_{14})^{(1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b'_{30})^{(5)} & -(b''_{30})^{(5)}(G_{31},t) & -(b''_{26})^{(4,4,)}(G_{27},t) & -(b''_{34})^{(6,6,6)}(G_{35},t) \\ -(b''_{15})^{(1,1,1,1,1,)}(G,t) & -(b''_{18})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{30} \quad (84)$$

*where* $-(b''_{28})^{(5)}(G_{31},t)$ , $-(b''_{29})^{(5)}(G_{31},t)$ , $-(b''_{30})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{24})^{(4,4,)}(G_{27},t)$, $-(b''_{25})^{(4,4,)}(G_{27},t)$, $-(b''_{26})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3

$-(b''_{32})^{(6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b''_{13})^{(1,1,1,1,1)}(G,t)$, $-(b''_{14})^{(1,1,1,1,1)}(G,t)$ , $-(b''_{15})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3

$-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3

$-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)$, $-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)$, $-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3

$-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3

$-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3

$-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a'_{32})^{(6)} & +(a''_{32})^{(6)}(T_{33},t) & +(a''_{28})^{(5,5,5)}(T_{29},t) & +(a''_{24})^{(4,4,4,)}(T_{25},t) \\ +(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a'_{33})^{(6)} & +(a''_{33})^{(6)}(T_{33},t) & +(a''_{29})^{(5,5,5)}(T_{29},t) & +(a''_{25})^{(4,4,4,)}(T_{25},t) \\ +(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)} G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)} T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)} T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)} T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{28})^{(5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)} & +(a_{36}^{''})^{(7)}(T_{37},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)} & +(a_{37}^{''})^{(7)}(T_{37},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)} & +(a_{38}^{''})^{(7)}(T_{37},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15} \quad (93)$$

Where $(a_{36}^{''})^{(7)}(T_{37},t)$, $(a_{37}^{''})^{(7)}(T_{37},t)$, $(a_{38}^{''})^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation

coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3 $+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}(G_{39},t) & -(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$ 94

$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}-(b_{37}^{''})^{(7)}(G_{39},t) & -(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}(G_{39},t) & -(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3 $-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3 $-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3 $-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}^{'})^{(8)} & +(a_{40}^{''})^{(8)}(T_{41},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13}$$

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$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}^{'})^{(8)} & +(a_{41}^{''})^{(8)}(T_{41},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}^{'})^{(8)} & +(a_{42}^{''})^{(8)}(T_{41},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$$

Where $+(a_{40}^{''})^{(8)}(T_{41},t)$, $+(a_{41}^{''})^{(8)}(T_{41},t)$, $+(a_{42}^{''})^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & -(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & -(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & -(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3



$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[\begin{matrix} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)}\ \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\ \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\ \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\ \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{matrix}\right] G_{13}$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[\begin{matrix} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)}\ \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\ \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\ \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\ \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{matrix}\right] G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[\begin{matrix} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)}\ \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\ \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\ \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\ \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{matrix}\right] G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$\frac{dT_{44}}{dt} =$

$$(b_{44})^{(9)}T_{45} - \left[\begin{array}{ccc} (b_{44}^{'})^{(9)}\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[\begin{array}{ccc} (b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[\begin{array}{ccc} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$$

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

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| **Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br>$(a_i^{''})^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$<br>$(b_i^{''})^{(1)}(G,t) \le (r_i)^{(1)} \le (b_i^{'})^{(1)} \le (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$<br>$\lim_{G\to\infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$<br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'},t) - (a_i^{''})^{(1)}(T_{14},t)\rvert \le (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br>$\lvert(b_i^{''})^{(1)}(G^{'},t) - (b_i^{''})^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G^{'}\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i^{'})^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i^{'})^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i^{'})^{(2)}, (a_i^{''})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (b_i^{''})^{(2)} > 0, \quad i,j = 16,17,18$ | |

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| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17},t) \le (p_i)^{(2)} \le (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19},t) \le (r_i)^{(2)} \le (b_i')^{(2)} \le (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2\to\infty}(a_i'')^{(2)}(T_{17},t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty}(b_i'')^{(2)}((G_{19}),t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $\boxed{(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}}$ are positive constants and $\boxed{i = 16,17,18}$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(2)}(T_{17}',t) - (a_i'')^{(2)}(T_{17},t)\rvert \le (\hat{k}_{16})^{(2)}\lvert T_{17} - T_{17}'\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i'')^{(2)}((G_{19})',t) - (b_i'')^{(2)}((G_{19}),t)\rvert < (\hat{k}_{16})^{(2)}\lvert\lvert(G_{19}) - (G_{19})'\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}',t)$ and $(a_i'')^{(2)}(T_{17},t)$ . $(T_{17}',t)$ and $(T_{17},t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}]$ . It is to be noted that $(a_i'')^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i'')^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i')^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ | 111 |

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| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(3)} (T_{21}, t) = (p_i)^{(3)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(3)} (G_{23}, t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'}, t) - (a_i^{''})^{(3)}(T_{21}, t)\rvert \leq (\hat{k}_{20})^{(3)} \lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)} t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'}, t) - (b_i^{''})^{(3)}(G_{23}, t)\rvert < (\hat{k}_{20})^{(3)} \lvert\lvert G_{23} - G_{23}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)} t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'}, t)$ and $(a_i^{''})^{(3)}(T_{21}, t)$ . $(T_{21}^{'}, t)$ And $(T_{21}, t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$ | 116 |

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| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24{,}25{,}26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}\big((G_{27}), t\big) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(4)}\big((G_{27}), t\big) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24{,}25{,}26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\rvert \leq (\hat{k}_{24})^{(4)} \lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)} t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'}, t) - (b_i^{''})^{(4)}\big((G_{27}), t\big)\rvert < (\hat{k}_{24})^{(4)} \lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)} t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $\big[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\big]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24{,}25{,}26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |

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| $\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i^{'})^{(4)} +\ (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}\big((G_{31}), t\big) \leq\ (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(5)}\,(T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(5)}\,(G_{31}, t) =\ (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'}, t) - (b_i^{''})^{(5)}\big((G_{31}), t\big)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $\big[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\big]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br><br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities | 126 |

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| $\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(6)}(T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(6)}((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)}\lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})^{'}, t) - (b_i^{''})^{(6)}((G_{35}), t)\rvert < (\hat{k}_{32})^{(6)}\lvert\lvert(G_{35}) - (G_{35})^{'}\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and$(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants | 130 |

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| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$ | |
| Where we suppose | |
| (IIIIII) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i, j = 36,37,38$<br><br>(JJJJJJ) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$:<br><br>$(a_i^{''})^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$<br><br>$(b_i^{''})^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$ | 131 |
| *(KKKKKK)* $\lim_{T_2 \to \infty} (a_i^{''})^{(7)}(T_{37}, t) = (p_i)^{(7)}$<br>*(LLLLLL)*<br>$\lim_{G \to \infty} (b_i^{''})^{(7)}((G_{39}), t) = (r_i)^{(7)}$<br><br>**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :<br><br>Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, \ (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$ | 132 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)\rvert \leq (\hat{k}_{36})^{(7)} \lvert T_{37} - T_{37}^{'}\rvert e^{-(\hat{M}_{36})^{(7)} t}$<br><br>$\lvert(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}((G_{39}), t)\rvert < (\hat{k}_{36})^{(7)} \lvert\lvert(G_{39}) - (G_{39})^{'}\rvert\rvert e^{-(\hat{M}_{36})^{(7)} t}$ | 133 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'}, t)$ and $(a_i^{''})^{(7)}(T_{37}, t)$ . $(T_{37}^{'}, t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :<br><br>(MMMMMM) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants | 134 |

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| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :<br><br>(NNNNNN) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}, (r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})', t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})'\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would | |

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| be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i, j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45}, t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47}, t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2 \to \infty} (a_i'')^{(9)}(T_{45}, t) = (p_i)^{(9)}$<br>$\lim_{G \to \infty} (b_i'')^{(9)}(G_{47}, t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}', t) - (a_i'')^{(9)}(T_{45}, t)\rvert \le (\hat{k}_{44})^{(9)} \lvert T_{45} - T_{45}' \rvert e^{-(\hat{M}_{44})^{(9)} t}$<br>$\lvert(b_i'')^{(9)}((G_{47})', t) - (b_i'')^{(9)}((G_{47}), t)\rvert < (\hat{k}_{44})^{(9)} \lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}', t)$ | |

and $(a_i^{''})^{(9)}(T_{45},t)$ . $(T_{45}^{'},t)$ and $(T_{45},t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)},(\widehat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i^{''})^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{44})^{(9)}=1$ then the function $(a_i^{''})^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$ :

$(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\ ,\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}<1$$

**Definition of** $(\hat{P}_{44})^{(9)},(\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},(\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)},(a_i^{'})^{(9)},(b_i)^{(9)},(b_i^{'})^{(9)},(p_i)^{(9)},\ (r_i)^{(9)},i=44,45,46,$ satisfy the inequalities

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[(a_i)^{(9)}+(a_i^{'})^{(9)}+\ (\hat{A}_{44})^{(9)}+\ (\hat{P}_{44})^{(9)}\ (\hat{k}_{44})^{(9)}]<1$$

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[\ (b_i)^{(9)}+(b_i^{'})^{(9)}+\ (\hat{B}_{44})^{(9)}+\ (\hat{Q}_{44})^{(9)}\ \ (\hat{k}_{44})^{(9)}]<1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0)\ ,T_i(0)$ :

$$G_i(t)\leq\ \left(\hat{P}_{13}\right)^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{G_i(0)=G_i^0>0}$$

$$T_i(t)\leq\ (\hat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{T_i(0)=T_i^0>0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0)\ ,T_i(0)$

$$G_i(t)\leq\ (\hat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\leq\ (\hat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t)\leq\ (\hat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\leq\ (\hat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

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| **Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$ | 153 B |

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| $T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |

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| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br><br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| By<br><br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} G_{21}(s_{(20)}) - \left((a_{20}')^{(3)} + a_{20}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{20}(s_{(20)})\right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[(a_{21})^{(3)} G_{20}(s_{(20)}) - \left((a_{21}')^{(3)} + (a_{21}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}')^{(3)} - (b_{21}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br><br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{28})^{(5)}$, $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[ (b_{29})^{(5)} T_{28}(s_{(28)}) - \left( (b'_{29})^{(5)} - (b''_{29})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[ (b_{30})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{30})^{(5)} - (b''_{30})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{30}(s_{(28)}) \right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** | |

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| Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[ (a_{32})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{32}(s_{(32)}) \right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[ (a_{33})^{(6)} G_{32}(s_{(32)}) - \left( (a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[ (a_{34})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{34}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[ (b_{32})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{32}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[ (b_{33})^{(6)} T_{32}(s_{(32)}) - \left( (b'_{33})^{(6)} - (b''_{33})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[ (b_{34})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{34})^{(6)} - (b''_{34})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{34}(s_{(32)}) \right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |

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| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}^{'})^{(8)} + a_{40}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^{0} + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^{0} + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^{0} + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{13}(t) \leq G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \leq \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}$ , $G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \leq G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that | 170 |

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| $(G_{16}(t)-G_{16}^0)e^{-(\bar{M}_{16})^{(2)}t} \le \frac{(a_{16})^{(2)}}{(\bar{M}_{16})^{(2)}}\left[((\hat{P}_{16})^{(2)}+G_{17}^0)e^{\left(-\frac{(\hat{P}_{16})^{(2)}+G_{17}^0}{G_{17}^0}\right)}+(\hat{P}_{16})^{(2)}\right]$ | |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0+(\hat{P}_{20})^{(3)}e^{(\bar{M}_{20})^{(3)}s_{(20)}}\right)\right] ds_{(20)} =$<br>$(1+(a_{20})^{(3)}t)G_{21}^0 + \frac{(a_{20})^{(3)}(\hat{P}_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left(e^{(\bar{M}_{20})^{(3)}t}-1\right)$ | 171 |
| From which it follows that<br>$(G_{20}(t)-G_{20}^0)e^{-(\bar{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that<br>$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\bar{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$<br>$(1+(a_{24})^{(4)}t)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left(e^{(\bar{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that<br>$(G_{24}(t)-G_{24}^0)e^{-(\bar{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$<br>$(1+(a_{28})^{(5)}t)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)}t}-1\right)$ | |
| From which it follows that<br>$(G_{28}(t)-G_{28}^0)e^{-(\bar{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 5 | 175 |
| The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious | 176 |

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| that<br>$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$<br>$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$ | |
| From which it follows that<br>$(G_{32}(t) - G_{32}^0) e^{-(\hat{M}_{32})^{(6)} t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[\left((\hat{P}_{32})^{(6)} + G_{33}^0\right) e^{\left(-\frac{(\hat{P}_{32})^{(6)} + G_{33}^0}{G_{33}^0}\right)} + (\hat{P}_{32})^{(6)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 6<br>Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$ | 177 |
| (x) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)}\left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$<br>$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$ | 178 |
| From which it follows that<br>$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{P}_{36})^{(7)} + G_{37}^0\right) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 7 | |
| The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)}\left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$<br>$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$ | 180 |
| From which it follows that<br>$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[\left((\hat{P}_{40})^{(8)} + G_{41}^0\right) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)}\left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$ | |

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| $\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$ | |
| From which it follows that<br>$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t}\leq\frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{P}_{44})^{(9)}+G_{45}^{0}\right)e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ and to choose<br>$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{13})^{(1)}\left\|G_{14}^{(1)}-G_{14}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$<br>$\int_0^t\{(a_{13}')^{(1)}\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{-(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$G_{13}^{(2)}\left\\|(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}'')^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right\\|\ e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$ | |

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| Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | |
| $\left\|G^{(1)} - G^{(2)}\right\| e^{-(\widehat{M}_{13})^{(1)}t} \leq$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right) d\left(\left(G^{(1)}, T^{(1)};\ G^{(2)}, T^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(1)}$ and $(b''_i)^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(1)} - (a''_i)^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1 / (a'_{14})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2 / (a'_{15})^{(1)}$<br><br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(1)}(G(t), t)) = (b'_{14})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ : | 189 |

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| Indeed let $t_1$ be so that for $t > t_1$<br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ $and$ $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |

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| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br>$\int_0^t \{(a_{16}')^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$G_{16}^{(2)} \left\|(a_{16}'')^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}'')^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}')^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right) d\left(\left((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)}\right)\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}'')^{(2)}$ and $(b_{16}'')^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(2)}$ and $(b_i'')^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(2)} - (a_i'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(2)} t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}')^{(2)} G_{17}$ and by integrating<br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}')^{(2)}$<br>In the same way , one can obtain | 201 |

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| $G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$, $G_{18}$ and $G_{16}$, $G_{17}$ respectively. | |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$. The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}$, $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}} < 1$ and to choose<br><br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$, $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{20})^{(3)}t}\}$ | 211 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :\left( \widetilde{(G_{23})}, \widetilde{(T_{23})} \right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)} \left\|G_{21}^{(1)} - G_{21}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\, ds_{(20)} +$<br><br>$\int_0^t \{(a_{20}^{'})^{(3)} \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$G_{20}^{(2)} \left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}^{''})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\} ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} t} \leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a_{20}^{'})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)} (\widehat{k}_{20})^{(3)}\right) d\left(\left((G_{23})^{(1)}, (T_{23})^{(1)};\ (G_{23})^{(2)}, (T_{23})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ *and* $(\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)} (T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)} t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating | 217 |

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| $G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}')^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}')^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(3)}\left((G_{23})(t),t\right) = (b_{21}')^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i'')^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}'')^{(3)}\left((G_{23})(t),t\right) = (b_{22}')^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}\left[(\hat{P}_{24})^{(4)} + \left((\hat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}}\left[\left((\hat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{24})^{(4)}\right] \leq (\hat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$ | 225 |

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{24})^{(4)}\left|G_{25}^{(1)}-G_{25}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$$

$$\int_0^t\{(a_{24}^{'})^{(4)}\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$G_{24}^{(2)}\left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)}-(G_{27})^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}t}\le$$
$$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows

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**Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}\ and\ (\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

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**Remark 17:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

it results

$$G_i\ (t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]}\ge 0$$

$$T_i\ (t)\ge T_i^0e^{\left(-(b_i^{'})^{(4)}t\right)}>0\quad \text{for t} > 0$$

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**Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1,\left((\widehat{M}_{24})^{(4)}\right)_2\ and\ \left((\widehat{M}_{24})^{(4)}\right)_3$ :

**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if

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| $G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a'_{25})^{(4)} G_{25}$ and by integrating<br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a'_{25})^{(4)}$<br>In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a'_{26})^{(4)}$<br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t),t)) = (b_{25}^{'})^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i^{''})^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded.<br>The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + \left((\widehat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\widehat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{28})^{(5)}\right] \leq (\widehat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying | |

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| Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}((G_{31}),(T_{31}))$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)}+$<br>$\int_0^t\{(a_{28}^{'})^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\le$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}^{'})^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}^{''})^{(5)}$ and $(b_{28}^{''})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(5)}-(a_i^{''})^{(5)}(T_{29}(s_{(28)}),s_{(28)})\}ds_{(28)}\right]}\ge 0$ | 239 |

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| $T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(5)}t)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}^{'})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1/(a_{29}^{'})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2/(a_{30}^{'})^{(5)}$<br><br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t),t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\,\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |

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| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{32})^{(6)}\right]\leq(\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}ds_{(32)}+$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a''_{32})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a'_{32})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(6)}$ and $(b_i'')^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})$($and\ not\ on\ t$) and hypothesis can replaced by a usual Lipschitz condition. | 249 |

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| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(6)} - (a_i'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(6)} t)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}')^{(6)} G_{33}$ and by integrating<br><br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}')^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}')^{(6)}$<br><br>If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(6)}((G_{35})(t), t)) = (b_{33}')^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i'')^{(6)}\left((G_{35})(t), t\right) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty} (b_{34}'')^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}')^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have | 255 |

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| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)}+\left((\hat{P}_{36})^{(7)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{36})^{(7)}\right]\leq(\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}')^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}'')^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}')^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}'')^{(7)}$ and $(b_{36}'')^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$. | 260 |

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| If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2 / (a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t), t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t), t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right),\ \ t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded. | 265 |

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| The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t),t\right)=(b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}<1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)}+\left((\hat{P}_{40})^{(8)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{40})^{(8)}\right]\leq(\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}ds_{(40)}+$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\leq$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |

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| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(8)}-(a_i^{''})^{(8)}(T_{41}(s_{(40)}),s_{(40)})\}ds_{(40)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2\ and\ \left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)}G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1/(a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2/(a_{42}^{'})^{(8)}$<br><br>If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}\,((G_{43})(t),t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t),t\right) < \varepsilon_8, T_{40}\,(t) > (m)^{(8)}$ | 278 |

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| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.<br>The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}^{'})^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$ | |
| In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying 39,35,36 into itself | |
| The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{47})^{(1)}, (T_{47})^{(1)}), ((G_{47})^{(2)}, (T_{47})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \left(\widetilde{(G_{47})}, \widetilde{(T_{47})}\right) = \mathcal{A}^{(9)}\left((G_{47}), (T_{47})\right)$<br>It results<br><br>$\left\|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$<br>$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right) - (a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right| \; e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$<br><br>Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on 45,46,47,28 and 29 it follows | |
| $\left|(G_{47})^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t} \leq$<br>$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)}, (T_{47})^{(1)}; (G_{47})^{(2)}, (T_{47})^{(2)})\right)$ | |

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$

From 99 to 44 it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(9)}t\right)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)}G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t), t\right) < \varepsilon_9, T_{44}\,(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded. The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b''_{46})^{(9)}\left((G_{47})(t),t\right) = (b'_{46})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a'_{13})^{(1)} + (a'_{14})^{(1)} - (a''_{13})^{(1)}(T_{14}, t) + (a''_{14})^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b'_{13})^{(1)} + (b'_{14})^{(1)} - (b''_{13})^{(1)}(G,t) - (b''_{14})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,

and $\boxed{(u_0)^{(1)} = \frac{T^0_{13}}{T^0_{14}}}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$

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| are defined | |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le \right.$<br>$\left.\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a'_{15})^{(1)})}[e^{(S_1)^{(1)}t} - e^{-(a'_{15})^{(1)}t}] + G_{15}^0 e^{-(a'_{15})^{(1)}t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b'_{15})^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b'_{15})^{(1)}t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)}t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$<br>$(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ :<br>$(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a'_{16})^{(2)} + (a'_{17})^{(2)} - (a''_{16})^{(2)}(T_{17}, t) + (a''_{17})^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b'_{16})^{(2)} + (b'_{17})^{(2)} - (b''_{16})^{(2)}((G_{19}), t) - (b''_{17})^{(2)}((G_{19}), t) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(v_1)^{(2)}, (v_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |

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| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\left(u^{(2)}\right)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}$,, $(\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\left(u^{(2)}\right)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously $(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$ $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities $G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)}t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le\right.$ | 312 |

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| $\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_2)^{(2)}((S_1)^{(2)}-(a_{18}')^{(2)})}[e^{(S_1)^{(2)}t}-e^{-(a_{18}')^{(2)}t}]+G_{18}^{0}e^{-(a_{18}')^{(2)}t})$ | |
| $\boxed{T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}}T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le\frac{1}{(\mu_2)^{(2)}}T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)}T_{16}^{0}}{(\mu_1)^{(2)}((R_1)^{(2)}-(b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t}-e^{-(b_{18}')^{(2)}t}\right]+T_{18}^{0}e^{-(b_{18}')^{(2)}t}\le T_{18}(t)\le$ $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)},(S_2)^{(2)},(R_1)^{(2)},(R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)}=(a_{16})^{(2)}(m_2)^{(2)}-(a_{16}')^{(2)}$ $(S_2)^{(2)}=(a_{18})^{(2)}-(p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)}=(b_{16})^{(2)}(\mu_2)^{(1)}-(b_{16}')^{(2)}$ $(R_2)^{(2)}=(b_{18}')^{(2)}-(r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$: $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)}\le-(a_{20}')^{(3)}+(a_{21}')^{(3)}-(a_{20}'')^{(3)}(T_{21},t)+(a_{21}'')^{(3)}(T_{21},t)\le-(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)}\le-(b_{20}')^{(3)}+(b_{21}')^{(3)}-(b_{20}'')^{(3)}(G_{23},t)-(b_{21}'')^{(3)}((G_{23}),t)\le-(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)},(v_2)^{(3)},(u_1)^{(3)},(u_2)^{(3)}$: By $(v_1)^{(3)}>0,(v_2)^{(3)}<0$ and respectively $(u_1)^{(3)}>0,(u_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_1)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_1)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ and By $(\bar{v}_1)^{(3)}>0,(\bar{v}_2)^{(3)}<0$ and respectively $(\bar{u}_1)^{(3)}>0,(\bar{u}_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_2)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_2)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ | 320 |
| **Definition of** $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$:- | 321 |

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$, ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,

and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously

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$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$$

$(p_i)^{(3)}$ is defined by equation

$$\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t} \quad (323)$$

$$\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$$

$$\left. \frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right) \quad (324)$$

$$\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}} \quad (325)$$

$$\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t} \quad (326)$$

$$\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$$

$$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t} \quad (327)$$

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- (328)

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$$

$$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$$

$$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}\big((G_{27}), t\big) - (b''_{25})^{(4)}\big((G_{27}), t\big) \le -(\tau_1)^{(4)}$$

**Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : 329

By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$
and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{v}_1)^{(4)} > 0$, $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$
and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (v_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}, \ \textbf{\textit{if}}\ (v_0)^{(4)} < (v_1)^{(4)}$$

$$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}, \textbf{\textit{if}}\ (v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)},$$

and $\boxed{(v_0)^{(4)} = \frac{G^0_{24}}{G^0_{25}}}$

$$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}, \ \textbf{\textit{if}}\ (\bar{v}_4)^{(4)} < (v_0)^{(4)}$$

and analogously 331

$$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}, \ \textbf{\textit{if}}\ (u_0)^{(4)} < (u_1)^{(4)}$$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}, \textbf{\textit{if}}\ (u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)},$$

and $\boxed{(u_0)^{(4)} = \frac{T^0_{24}}{T^0_{25}}}$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \textbf{\textit{if}}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)} \text{ where } (u_1)^{(4)}, (\bar{u}_1)^{(4)}$$

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| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le\right.$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}')^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}')^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)}t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(\nu_1)^{(5)}, (\nu_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, \nu^{(5)}, u^{(5)}$ :<br>By $(\nu_1)^{(5)} > 0$, $(\nu_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}(\nu^{(5)})^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)} = 0$ | 339 |

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| and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ :<br><br>By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)},(v_0)^{(5)}$ :-<br><br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)}$ by<br><br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br><br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br><br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |

$\frac{(b_{30})^{(5)}T_{28}^{0}}{(\mu_1)^{(5)}((R_1)^{(5)}-(b_{30}^{'})^{(5)})}\left[e^{(R_1)^{(5)}t}-e^{-(b_{30}^{'})^{(5)}t}\right]+T_{30}^{0}e^{-(b_{30}^{'})^{(5)}t}\le T_{30}(t)\le$

$\frac{(a_{30})^{(5)}T_{28}^{0}}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t}-e^{-(R_2)^{(5)}t}\right]+T_{30}^{0}e^{-(R_2)^{(5)}t}$ 347

**Definition of** $(S_1)^{(5)},(S_2)^{(5)},(R_1)^{(5)},(R_2)^{(5)}$:- 348

Where $(S_1)^{(5)}=(a_{28})^{(5)}(m_2)^{(5)}-(a_{28}^{'})^{(5)}$

$(S_2)^{(5)}=(a_{30})^{(5)}-(p_{30})^{(5)}$

$(R_1)^{(5)}=(b_{28})^{(5)}(\mu_2)^{(5)}-(b_{28}^{'})^{(5)}$

$(R_2)^{(5)}=(b_{30}^{'})^{(5)}-(r_{30})^{(5)}$

**Behavior of the solutions of equation** 349

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ four constants satisfying

$-(\sigma_2)^{(6)}\le-(a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}-(a_{32}^{''})^{(6)}(T_{33},t)+(a_{33}^{''})^{(6)}(T_{33},t)\le-(\sigma_1)^{(6)}$

$-(\tau_2)^{(6)}\le-(b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(b_{32}^{''})^{(6)}\left((G_{35}),t\right)-(b_{33}^{''})^{(6)}\left((G_{35}),t\right)\le-(\tau_1)^{(6)}$

**Definition of** $(\nu_1)^{(6)},(\nu_2)^{(6)},(u_1)^{(6)},(u_2)^{(6)},\nu^{(6)},u^{(6)}$ : 350

By $(\nu_1)^{(6)}>0$ , $(\nu_2)^{(6)}<0$ and respectively $(u_1)^{(6)}>0$ , $(u_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_1)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2+(\tau_1)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$ and

**Definition of** $(\bar{\nu}_1)^{(6)},,(\bar{\nu}_2)^{(6)},(\bar{u}_1)^{(6)},(\bar{u}_2)^{(6)}$ : 351

By $(\bar{\nu}_1)^{(6)}>0$ , $(\bar{\nu}_2)^{(6)}<0$ and respectively $(\bar{u}_1)^{(6)}>0$ , $(\bar{u}_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_2)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2+(\tau_2)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$

**Definition of** $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)},(\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)}$ by

$(m_2)^{(6)}=(\nu_0)^{(6)},(m_1)^{(6)}=(\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)}<(\nu_1)^{(6)}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)}<(\nu_0)^{(6)}<(\bar{\nu}_1)^{(6)}$,

and $\boxed{(\nu_0)^{(6)}=\frac{G_{32}^{0}}{G_{33}^{0}}}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)}<(\nu_0)^{(6)}$

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| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},\ \boldsymbol{if}\ (u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}, \boldsymbol{if}\ (u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)},$<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}, \boldsymbol{if}\ (\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le\right.$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b_{34}')^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b_{34}')^{(6)} t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$<br>$(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ : | |

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| $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying<br><br>$-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$<br><br>$-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}((G_{39}), t) - (b''_{37})^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ :<br><br>By $(\nu_1)^{(7)} > 0, (\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ :<br><br>By $(\bar{\nu}_1)^{(7)} > 0, (\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$<br>**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-<br><br>If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by<br><br>$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,<br>and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | 362 |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,<br>and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities | 364 |

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| $G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^{0}e^{(S_1)^{(7)}t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | |
| $\frac{1}{(m_7)^{(7)}} G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^{0}e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^{0}e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}^{'})^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}^{'})^{(7)}t}] + G_{38}^{0}e^{-(a_{38}^{'})^{(7)}t})$ | 366 |
| $\boxed{T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)}T_{36}^{0}}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}^{'})^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}^{'})^{(7)}t}\right] + T_{38}^{0}e^{-(b_{38}^{'})^{(7)}t} \le T_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)}T_{36}^{0}}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^{0}e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}^{'})^{(7)}$<br><br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br><br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}^{'})^{(7)}$<br><br>$(R_2)^{(7)} = (b_{38}^{'})^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br><br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br><br>$-(\sigma_2)^{(8)} \le -(a_{40}^{'})^{(8)} + (a_{41}^{'})^{(8)} - (a_{40}^{''})^{(8)}(T_{41}, t) + (a_{41}^{''})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br><br>$-(\tau_2)^{(8)} \le -(b_{40}^{'})^{(8)} + (b_{41}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}), t) - (b_{41}^{''})^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |
| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : | 372 |

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| By $(\nu_1)^{(8)} > 0\,, (\nu_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0\,, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$ and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(8)},, (\bar{\nu}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :<br><br>By $(\bar{\nu}_1)^{(8)} > 0\,, (\bar{\nu}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0\,, (\bar{u}_2)^{(8)} < 0$ the<br><br>roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$<br><br>and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$<br><br>**Definition of** $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (\nu_0)^{(8)}$ :-<br><br>If we define $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by<br><br>$(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)},$ ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$<br><br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)},$<br>and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br><br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)},$ ***if*** $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br><br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},$ ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)},$ ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)}t}$<br><br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$ | 376 |

| | |
|---|---|
| $\left(\frac{(a_{42})^{(8)}G_{40}^{0}}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t}-e^{-(S_2)^{(8)}t}\right]+G_{42}^{0}e^{-(S_2)^{(8)}t}\leq G_{42}(t)\leq \frac{(a_{42})^{(8)}G_{40}^{0}}{(m_2)^{(8)}((S_1)^{(8)}-(a_{42}^{'})^{(8)})}[e^{(S_1)^{(8)}t}-e^{-(a_{42}^{'})^{(8)}t}]+G_{42}^{0}e^{-(a_{42}^{'})^{(8)}t}\right)$ | 377 |
| $\boxed{T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}}T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq \frac{1}{(\mu_2)^{(8)}}T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^{0}}{(\mu_1)^{(8)}((R_1)^{(8)}-(b_{42}^{'})^{(8)})}\left[e^{(R_1)^{(8)}t}-e^{-(b_{42}^{'})^{(8)}t}\right]+T_{42}^{0}e^{-(b_{42}^{'})^{(8)}t}\leq T_{42}(t)\leq$ $\frac{(a_{42})^{(8)}T_{40}^{0}}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t}-e^{-(R_2)^{(8)}t}\right]+T_{42}^{0}e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)},(S_2)^{(8)},(R_1)^{(8)},(R_2)^{(8)}$:-<br>Where $(S_1)^{(8)}=(a_{40})^{(8)}(m_2)^{(8)}-(a_{40}^{'})^{(8)}$<br>$(S_2)^{(8)}=(a_{42})^{(8)}-(p_{42})^{(8)}$<br>$(R_1)^{(8)}=(b_{40})^{(8)}(\mu_2)^{(8)}-(b_{40}^{'})^{(8)}$<br>$(R_2)^{(8)}=(b_{42}^{'})^{(8)}-(r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)}\leq -(a_{44}^{'})^{(9)}+(a_{45}^{'})^{(9)}-(a_{44}^{''})^{(9)}(T_{45},t)+(a_{45}^{''})^{(9)}(T_{45},t)\leq -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)}\leq -(b_{44}^{'})^{(9)}+(b_{45}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47}),t)-(b_{45}^{''})^{(9)}((G_{47}),t)\leq -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(\nu_1)^{(9)},(\nu_2)^{(9)},(u_1)^{(9)},(u_2)^{(9)},\nu^{(9)},u^{(9)}$ :<br>By $(\nu_1)^{(9)}>0$ , $(\nu_2)^{(9)}<0$ and respectively $(u_1)^{(9)}>0$ , $(u_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_1)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$<br>and $(b_{45})^{(9)}(u^{(9)})^2+(\tau_1)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(9)},,(\bar{\nu}_2)^{(9)},(\bar{u}_1)^{(9)},(\bar{u}_2)^{(9)}$ :<br>By $(\bar{\nu}_1)^{(9)}>0$ , $(\bar{\nu}_2)^{(9)}<0$ and respectively $(\bar{u}_1)^{(9)}>0$ , $(\bar{u}_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_2)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$ | |

and $(b_{45})^{(9)}\left(u^{(9)}\right)^2+(\tau_2)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$

**Definition of** $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)},(\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)}$ by

$(m_2)^{(9)}=(\nu_0)^{(9)},(m_1)^{(9)}=(\nu_1)^{(9)},$ ***if*** $(\nu_0)^{(9)}<(\nu_1)^{(9)}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)}<(\nu_0)^{(9)}<(\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)}=\frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\nu_0)^{(9)},$ ***if*** $(\bar{\nu}_1)^{(9)}<(\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)}=(u_0)^{(9)},(\mu_1)^{(9)}=(u_1)^{(9)},$ ***if*** $(u_0)^{(9)}<(u_1)^{(9)}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)}<(u_0)^{(9)}<(\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)}=\frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(u_0)^{(9)},$ ***if*** $(\bar{u}_1)^{(9)}<(u_0)^{(9)}$ where $(u_1)^{(9)},(\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{44}(t)\le G_{44}^0e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{45}(t)\le\frac{1}{(m_2)^{(9)}}G_{44}^0e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)}G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^0e^{-(S_2)^{(9)}t}\le G_{46}(t)\le\right.$$

$$\left.\frac{(a_{46})^{(9)}G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}')^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}')^{(9)}t}]+\ G_{46}^0e^{-(a_{46}')^{(9)}t}\right)$$

$$\boxed{T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le\frac{1}{(\mu_2)^{(9)}}T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}')^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}')^{(9)}t}\right]+T_{46}^0e^{-(b_{46}')^{(9)}t}\le T_{46}(t)\le$$

$$\frac{(a_{46})^{(9)}T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^0e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain (383)

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right) - (a''_{14})^{(1)}(T_{14}, t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get (384)

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{v}_1)^{(1)}$ we find like in the previous case, (385)

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)}G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)}T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$$

If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain 394

$$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(2)}(t)$ :- 395

$$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}, \quad \boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$$

In a completely analogous way, we obtain 396

**Definition of** $u^{(2)}(t)$ :-

$$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}, \quad \boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$$

| | |
|---|---|
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$ if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| From which one obtains<br><br>For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$<br><br>$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)} + (C)^{(3)}(\nu_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}$ , $\boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\nu_2)^{(3)}}}$<br><br>it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$ | 400 |
| In the same manner , we get<br><br>$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)} + (\bar{C})^{(3)}(\bar{\nu}_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\bar{\nu}_2)^{(3)}}}$<br><br>**Definition of** $(\bar{\nu}_1)^{(3)}$ :-<br><br>From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$ | 401 |
| If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, | 402 |

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

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$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right) - (a_{25}^{''})^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\quad \boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

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$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} =$

$(\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b''_{24})^{(4)} = (b''_{25})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a'_{28})^{(5)} - (a'_{29})^{(5)} + (a''_{28})^{(5)}(T_{29}, t)\right) - (a''_{29})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G^0_{28}}{G^0_{29}}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\quad \boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get 413

$$v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{v}_1)^{(6)}-(v_0)^{(6)}}{(v_0)^{(6)}-(\bar{v}_2)^{(6)}}}$$

From which we deduce $(v_0)^{(6)} \le v^{(6)}(t) \le (\bar{v}_1)^{(6)}$

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If $0 < (v_1)^{(6)} < (v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{v}_1)^{(6)}$ we find like in the previous case,

$$(v_1)^{(6)} \le \frac{(v_1)^{(6)}+(C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}} \le v^{(6)}(t) \le$$

$$\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (\bar{v}_1)^{(6)}$$

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If $0 < (v_1)^{(6)} \le (\bar{v}_1)^{(6)} \le \boxed{(v_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(v_1)^{(6)} \le v^{(6)}(t) \le \frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}} \le (v_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$$(m_2)^{(6)} \le v^{(6)}(t) \le (m_1)^{(6)}, \quad \boxed{v^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)} \le u^{(6)}(t) \le (\mu_1)^{(6)}, \quad \boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(v_1)^{(6)} = (\bar{v}_1)^{(6)}$ if in addition $(v_0)^{(6)} = (v_1)^{(6)}$ then $v^{(6)}(t) = (v_0)^{(6)}$ and as a consequence $G_{32}(t) = (v_0)^{(6)}G_{33}(t)$ **this also defines** $(v_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(v_1)^{(6)}$ and $(\bar{v}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right) - (a_{37}^{''})^{(7)}(T_{37},t)v^{(7)} - (a_{37})^{(7)}v^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

$$\nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \leq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}} \leq \nu^{(7)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\bar{\nu}_1)^{(7)}$$

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If $0 < (\nu_1)^{(7)} \leq (\bar{\nu}_1)^{(7)} \leq \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain

$$(\nu_1)^{(7)} \leq \nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \leq \nu^{(7)}(t) \leq (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

419

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :- 420

$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)} G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain 421

$$\frac{d\nu^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right) - (a_{41}^{''})^{(8)}(T_{41}, t)\nu^{(8)} - (a_{41})^{(8)}\nu^{(8)}$$

**Definition of** $\nu^{(8)}$ :- $\boxed{\nu^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right) \leq \frac{d\nu^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)}-(\nu_0)^{(8)}}{(\nu_0)^{(8)}-(\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \le \nu^{(8)}(t) \le (\bar{\nu}_8)^{(8)}$

If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case, 423

$$(\nu_1)^{(8)} \le \frac{(\nu_1)^{(8)}+(C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}} \le \nu^{(8)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\bar{\nu}_1)^{(8)}$$

If $0 < (\nu_1)^{(8)} \le (\bar{\nu}_1)^{(8)} \le \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$, we obtain 424

$$(\nu_1)^{(8)} \le \nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \le \nu^{(8)}(t) \le (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \le u^{(8)}(t) \le (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)}G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)}T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

**Proof** : From 99,20,44,22,23,44 we obtain 424 A

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right) - (a''_{45})^{(9)}(T_{45},t)v^{(9)} - (a_{45})^{(9)}v^{(9)}$$

**Definition of** $v^{(9)}$ :- $\boxed{v^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)}\right) \le \frac{dv^{(9)}}{dt} \le -\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(9)}, (v_0)^{(9)}$ :-

For $0 < \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (v_1)^{(9)} < (\bar{v}_1)^{(9)}$

$$v^{(9)}(t) \ge \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(v_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (v_2)^{(9)}}}$$

it follows $(v_0)^{(9)} \le v^{(9)}(t) \le (v_9)^{(9)}$

In the same manner , we get

$$v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{v}_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (\bar{v}_2)^{(9)}}}$$

From which we deduce $(v_0)^{(9)} \le v^{(9)}(t) \le (\bar{v}_1)^{(9)}$

If $0 < (v_1)^{(9)} < (v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}} < (\bar{v}_1)^{(9)}$ we find like in the previous case,

$$(v_1)^{(9)} \le \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}} \le v^{(9)}(t) \le$$

$$\frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (\bar{v}_1)^{(9)}$$

If $0 < (v_1)^{(9)} \le (\bar{v}_1)^{(9)} \le \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$ , we obtain

$$(v_1)^{(9)} \le v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (v_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(9)}(t)$ :-

$$(m_2)^{(9)} \le v^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{v^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}$, $\boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ 430

$with\ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations 431

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$

| | |
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| $with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | |
| We can prove the following<br><br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br><br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br><br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br><br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$ | 435 |

| | |
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| $with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |

| | |
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| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27})\big)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\big((G_{27})\big)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\big((G_{27})\big)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |

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| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |

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| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(x) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) +$ | 489 |

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| $(a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that | 496 |

there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations

$$G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]} \quad , \quad G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$$

497

**Definition and uniqueness of** $T_{25}^{*}$ :-

After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations

$$G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]} \quad , \quad G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$$

498

**Definition and uniqueness of** $T_{29}^{*}$ :-

After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations

$$G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]} \quad , \quad G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$$

499

**Definition and uniqueness of** $T_{33}^{*}$ :-

After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations

$$G_{32}=\frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]} \quad , \quad G_{34}=\frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$$

500

**Definition and uniqueness of** $T_{37}^{*}$ :-

After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*})=0$. With this value , we obtain from the three first equations

$$G_{36}=\frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]} \quad , \quad G_{38}=\frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$$

501

**Definition and uniqueness of** $T_{41}^{*}$ :-

After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*})=0$. With this value , we obtain from the three first equations

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| $G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G) + (b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*}) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19}) + (b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br>$[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ | 506 |

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| is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution | 511 |

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| $G_{14}^{*}$ given by $\varphi(G^{*})=0$ , $T_{14}^{*}$ given by $f(T_{14}^{*})=0$ and<br>$G_{13}^{*}=\frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*}=\frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br>$T_{13}^{*}=\frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*}=\frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br>Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*})=0$ , $T_{17}^{*}$ given by $f(T_{17}^{*})=0$ and | 512 |
| $G_{16}^{*}=\frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*}=\frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*}=\frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*}=\frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*})=0$ , $T_{21}^{*}$ given by $f(T_{21}^{*})=0$ and<br>$G_{20}^{*}=\frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*}=\frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br>$T_{20}^{*}=\frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*}=\frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^{*}$ given by $\varphi(G_{27})=0$ , $T_{25}^{*}$ given by $f(T_{25}^{*})=0$ and<br>$G_{24}^{*}=\frac{(a_{24})^{(4)}G_{25}^{*}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*}=\frac{(a_{26})^{(4)}G_{25}^{*}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*}=\frac{(b_{24})^{(4)}T_{25}^{*}}{[(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*}=\frac{(b_{26})^{(4)}T_{25}^{*}}{[(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}((G_{27})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*})=0$ , $T_{29}^{*}$ given by $f(T_{29}^{*})=0$ and<br>$G_{28}^{*}=\frac{(a_{28})^{(5)}G_{29}^{*}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*}=\frac{(a_{30})^{(5)}G_{29}^{*}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |

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| $T_{28}^{*} = \frac{(b_{28})^{(5)}T_{29}^{*}}{[(b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)}T_{29}^{*}}{[(b_{30}^{'})^{(5)}-(b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br>$G_{32}^{*} = \frac{(a_{32})^{(6)}G_{33}^{*}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)}G_{33}^{*}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)}T_{33}^{*}}{[(b_{32}^{'})^{(6)}-(b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)}T_{33}^{*}}{[(b_{34}^{'})^{(6)}-(b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br>$G_{36}^{*} = \frac{(a_{36})^{(7)}G_{37}^{*}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)}G_{37}^{*}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br>$T_{36}^{*} = \frac{(b_{36})^{(7)}T_{37}^{*}}{[(b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)}T_{37}^{*}}{[(b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br>$G_{40}^{*} = \frac{(a_{40})^{(8)}G_{41}^{*}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)}G_{41}^{*}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br>$T_{40}^{*} = \frac{(b_{40})^{(8)}T_{41}^{*}}{[(b_{40}^{'})^{(8)}-(b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)}T_{41}^{*}}{[(b_{42}^{'})^{(8)}-(b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)}G_{45}^{*}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)}G_{45}^{*}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)}T_{45}^{*}}{[(b_{44}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)}T_{45}^{*}}{[(b_{46}^{'})^{(9)}-(b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | 523 A |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^*) = (q_{14})^{(1)} \quad , \frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^*) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)}G_{13}^*\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^*\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^*\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^*\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^*\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^*\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)} \quad , \frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |

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| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)} \;,\; \frac{\partial(b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_{j}\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_{j}\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_{j}\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ *and* $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 548 |

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| $\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^{*}) = (q_{25})^{(4)}$ , $\frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^{*}) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^{*}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^{*}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^{*}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^{*}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^{*}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^{*}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^{*}) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^{*}) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^{*}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^{*}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^{*}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^{*}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^{*}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^{*}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 563 |

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| **Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^*\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^*\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^*\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^*\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}((G_{39})^{**}) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |

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| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^{*}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^{*}\mathbb{G}_{j}\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^{*}\mathbb{G}_{j}\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^{*}\mathbb{G}_{j}\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^{*}) = (q_{41})^{(8)} \quad , \frac{\partial(b_i^{''})^{(8)}}{\partial G_j}\left((G_{43})^{*}\right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^{*}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^{*}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^{*}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b_{40}^{'})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T_{40}^{*}\mathbb{G}_{j}\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b_{41}^{'})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T_{41}^{*}\mathbb{G}_{j}\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b_{42}^{'})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T_{42}^{*}\mathbb{G}_{j}\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |

| | |
|---|---|
| | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \ \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b_{44}^{'})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T_{44}^*\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b_{45}^{'})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T_{45}^*\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b_{46}^{'})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T_{46}^*\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)$<br>$\left[\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(q_{13})^{(1)}G_{13}^*\right)\Big]$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T_{14}^* + (b_{14})^{(1)}s_{(13),(14)}T_{14}^*\Big)$<br>$+\left(\left((\lambda)^{(1)} + (a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^* + (a_{13})^{(1)}(q_{14})^{(1)}G_{14}^*\right)$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^* + (b_{14})^{(1)}s_{(13),(13)}T_{13}^*\Big)$<br>$\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$<br>$\left((\lambda)^{(1)}\right)^2 + \left((b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$<br>$+\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$<br>$+\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^*\right)$<br>$\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^* + (b_{14})^{(1)}s_{(13),(15)}T_{13}^*\Big)\} = 0$ | |

$$+$$

$$((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)})\{((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)})$$

$$\left[\left(((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)})(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)\right]$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(17)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^{*}_{17}\right)$$

$$+\left(((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)})(q_{16})^{(2)}\mathrm{G}^{*}_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(16)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^{*}_{16}\right)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left(\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(18)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(18)}\mathrm{T}^{*}_{16}\right)\} = 0$$

$$+$$

$$((\lambda)^{(3)} + (b'_{22})^{(3)} - (r_{22})^{(3)})\{((\lambda)^{(3)} + (a'_{22})^{(3)} + (p_{22})^{(3)})$$

$$\left[\left(((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)})(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)\right]$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(21)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(21)}T^{*}_{21}\right)$$

$$+\left(((\lambda)^{(3)} + (a'_{21})^{(3)} + (p_{21})^{(3)})(q_{20})^{(3)}G^{*}_{20} + (a_{20})^{(3)}(q_{21})^{(1)}G^{*}_{21}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(20)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(20)}T^{*}_{20}\right)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((b'_{20})^{(3)} + (b'_{21})^{(3)} - (r_{20})^{(3)} + (r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left(\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(22)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(22)}T^{*}_{20}\right)\} = 0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\right)\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\right)\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\right)\}=0$$

$$+$$

$$\left(\lambda\right)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\right)\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)\right]$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(41)}T^{*}_{41}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^{*}_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^{*}_{41}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(40)}T^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$+\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(42)}T^{*}_{40}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\right)\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)\right]$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(45)}T^{*}_{45}\right)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^{*}_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^{*}_{45}\right)$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(44)}T^{*}_{44}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^{2}+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$+\left(\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)$$

$\left(\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\right)\} = 0$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i \in R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i\ X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor

bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k \tag{24}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 24. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,...x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone $\{ x_i \epsilon C_i: \prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i(x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \in I, x \in K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\,\delta\,(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\mathcal{A}(\tilde{A}), C\backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population *i* is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int_0^t (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \epsilon R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v=au_i(x)+w$ with $a \epsilon R$ and $w \epsilon F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a| B_i(t,x)u_i(x)|+ |B_i(t,x)w| \leq | B_i(t,x)u_i(x)|(a+ \alpha+ exp(-\beta t)| w|)$. Hence, $\limsup_{t\to\infty} \frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\limsup_{t\to\infty} \frac{1}{t} In\ B_i(t,x)\ u_i(x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)|-|B_i(t,x)w|$$

$$\geq | B_i(t,x)u_i(x)|(a- \alpha+ exp(-\beta t)| w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In } b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x)$ =0 implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

=

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \le 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \le \gamma$ for all $t \ge 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \operatorname{In} |B_i(t,x)x_i| = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \operatorname{In} |\, (x.t)_i\, |$$

$$\le \operatorname{limsup}_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0. If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\ge 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\operatorname{limsup}_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x) =$ Integral $h(x.t)d\,\mu(x)$for all $t \ge 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure*. $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x) \rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1 \le j \le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |\mathrm{h}(x.t) - h(x^n.t)|d\mu_n(x)| + |\int_c h(x^n.t)|d\mu_n(x) - \int_c \mathrm{h}(x)d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x)d\mu_n(x) - \int_c h(x)d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $\mathrm{h}_i^n(x) = \langle \mathrm{A}_i^n(x)\mathrm{u}_i^n(x), \mathrm{u}_i^n(x)\rangle$ where $\mathrm{u}_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $\mathrm{A}_i^n(x) \to \mathrm{A}_i(x)$ uniformly for $x \in N_1((\mathrm{A}))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $\mathrm{u}_i^n(x) \to \mathrm{u}_i(x)$ converges uniformly for $x \in N_1(\Lambda(\mathrm{A}))$. Given $\in > 0$, choose $n$ sufficiently large so that $|\mathrm{h}_i^n(x) - \mathrm{h}_i(x)| \leq \in$ for all $x \in N_1(\Lambda(\mathrm{A}))$ and $|\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$\lambda_i(\mu) \leq \int_c |\mathrm{h}_i(x) - \mathrm{h}_i^n(x)|d\mu(x) + |\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)|$

$+ \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)$

$\leq \quad 2\in$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\, \mathrm{ds}$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument

implies that $\mu$ is $x.t$ invariant. Define $h_i$: C → R by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

(6) $\min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$ $\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \begin{cases} \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} & if\ i \epsilon I \\ if\ i \notin I \end{cases}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set $K$ if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$?then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$?and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume x* $\mapsto$ *A(x) is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists j* $\epsilon$ { 1,….,k} *such that $M_i$ ----- { $x \in C : x_l \gg 0$ for $l \neq j$} Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_iM_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----{1,…,k} such that $\mu(---_i \in_I C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures μ with support in ∂C, $\lambda_i(\mu)>0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $$\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j \qquad i = 1,..,k \qquad j=1,\ldots,m$$

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge

to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j$ ($x^j_i$) = 0 i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product ($E_1^1$ ,…,$E_1^m$), and the matrix $A^2$ ($E_1$) is close to the diagonal matrix diag($f^i_2(E_1{}^j)$)$^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f_2^j (E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j (E^j_1)>0$ and $\max_j f_2^j (E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial$C is given by two equilibria {0,$E_1$}, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j (E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$

$\frac{dx_2^j}{dt} = x_2^j(1 - \alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1 - \beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > 1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria $E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$ and $\lambda_3(E_1)$ are close to $\max_j 1 - a^j$ and $\max_j 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1 - a^j$ and $\max_j 1 - \beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1 - a^j) > -\max_j(1 - \beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \ \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i\left(r_i - \sum_s a_{is}x_s\right), \qquad i = 1,\dots,k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$$

It has been shown [12,ch.16.1,16.2] That for *k* =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \ \ with\ \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing

holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & O \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left[ \beta(1 - y - z)\frac{P}{\epsilon + P} - r - m(1 - y) - b(N)\ y \right]$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*′ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + Kp\, x_i x_i - 1}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords

Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation

of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1$ *(t)* and $P_2$ *(t)* denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial

alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.70) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.71) $$D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.72) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_{1,}$ $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_{i,}$ $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic ($as$ $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 24. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, \text{R}^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1, 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) \quad P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^\infty f_i(a + s, P(t + S)) ds \right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[ - \int_o^x f_1(a + s, p(t + s)) ds \right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0) \exp\left[ - \int_o^t f_1(a - t + s, p_1(s)) ds \right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

(2.4) $B_i(t) = p_i(0, t), i = 1,2, t > 0.$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

(2.5) $L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(a+s, p_i(s))ds]$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da+\int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$$t> 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \epsilon[-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.(a, t, P_i(t))da$

$$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$$

$$i = 1, 2, t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t) \left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t) \phi(0) d_i^*] t \quad for\ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T],R^+), \quad B_i \in C([0,T],R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau,T] = \{f \in C([-\tau,T],R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau,T]$. Then for any

$(x, y) \in C^+[-\tau,T] \times C^+[-\tau,T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau,T], \quad C_T^2 = C^+[-\tau,T] \times C^+[-\tau,T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau,T]$ *and* $t \in [0,T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$B: C_T^2 \to C_T^2$ *and* $\Pi C_T^2 \to C_T^2$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = P_t \left\{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right)$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Pi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Pi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Pi(P_t^{(1)}) - \Pi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Pi(P^{(1)}(t)) - \Pi(P^{(2)}(t)| \le \int_0^t B_1^{(1)}(a) \left\{ M_1(a, t, P_t^{(1)}) - M_1(a, t, P_t^{(2)}) \right\} da$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left( \int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_* ds \right) da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o) \left[ \int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds \right] da$$

$$\leq d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \|_t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$=\int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\mathrm{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right]\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \, /\!/ P_t^{(1)} - P_t^{(2)} /\!/ \text{ for } a \, \varepsilon [o, t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \leq \beta_1 \Phi_1(0) || P_t^{(1)} - P_t^{(2)} ||_t^* + \Phi_1(0) \beta_1 d_1 || P_t^{(1)} - P_t^{(2)} ||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) \, e^{\delta 1^a} \beta_1 \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) \, e^{\delta 1^a} \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| B_1^{*d}\sigma$$

$$+ \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \left\{ B_1 + B_2 d_1 a + B_1^* \left(B_2 a + d_1 a^2/2\right) e^{\delta 1^a} \right\}$$

$$\leq \beta_1^* \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| do + \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \rightarrow \beta_1$ as $a \rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{g_1(a) + 0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a)) - b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a)) do \right|$$

$$\left| B_1^* \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} 0(a) as\ a \rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi (P_t^{(1)} - \pi \left(P_t^{(2)}\right) /\!/ \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^*)$$

As $t^* \rightarrow o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp P_2^*(a)}{da} + f_i(a, P_1^*, P_2^*) P_i^*(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^* = \int_o^\infty P_i^*(a) da, i = 1,\ 2,$$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i=1.\ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s,P_1^*,P_2^*)ds\right], i=1,\quad 2, a\geq 0,$

Which together with (3.3) lead to

$$1=\int_o^\infty b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^\infty b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1,P_2) = \int_o^\infty b_i(a,P_1,P_2)\exp\left[-\int_o^\infty f_i(s,P_1,P_2)ds\right]da,$

$i=1,\ 2,\quad P_1\geq 0,\quad P_2\geq 0,$

**And suppose the following hold:**

(3.7) $A_{4.}\quad F_{i.}\ (0,\ 0)>1,\ i=1,2$:

$A_{5.}$ *there exists a positive constant c for which*

$$F_1(C,0)< F_{2.}(C,0),$$

(3.8) $$F_1(0.C)< F_{2.}(0,C),$$

Then there exists a unique pair $(P_1^*,P_2^*)$ of real numbers $P_1^*>o,$

$P_2^*>0$ such that

(3,9) $F_1(P_1^*,P_2^*)=1=F_2(P_1^*,P_2^*)$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a)=[P_2^*\exp[-f_i(s,P_1^*,P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s,P_1^*,P_2^*)ds]da),$

$i=1,2,\quad a\geq 0,$

Proof Consider the elements of the surfaces defined by

Z= $F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0,$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

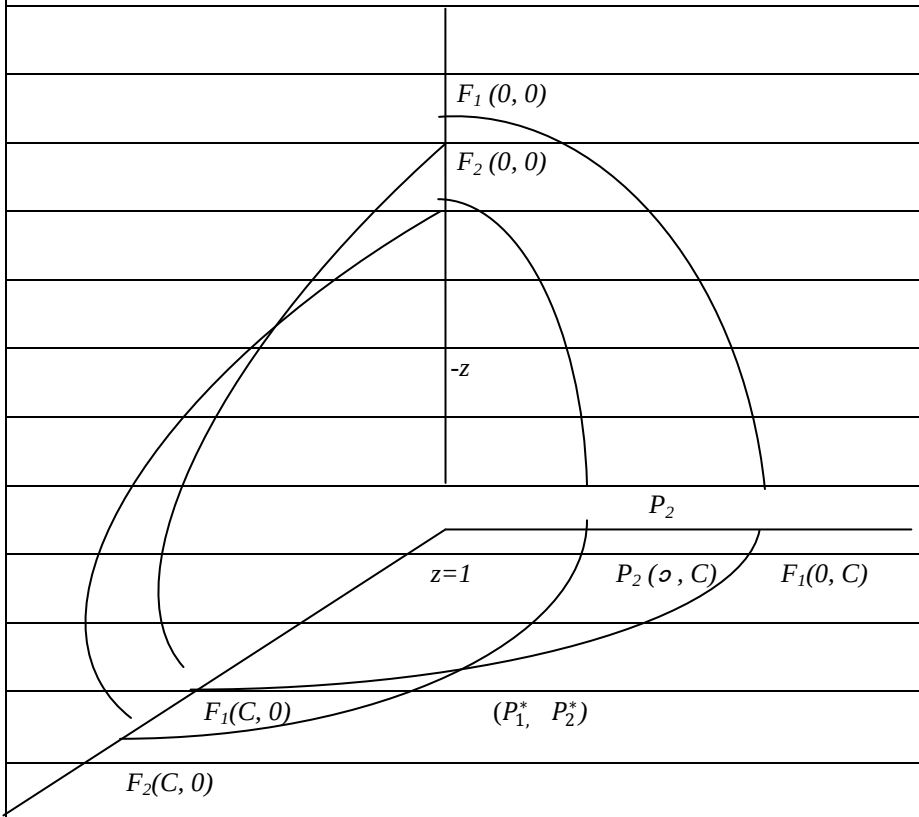
$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0,$ These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, F_1(P_1, P_2 \geq 0,$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11)
$$\begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square=1}^{2} \frac{\partial b_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u\square(a)\exp[\lambda t], \quad \square(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty \square(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d\square}{da} + \lambda\square = -f_i\left(a, p_1^*, p_2^*\right)\square(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_{i\square}^*\right\},$$

(3.15)

$$\square(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\square(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*} p_{i\square}^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^{*} = p_{1}^{*}, \quad p_{12}^{*} = p_{2}^{*} \exp\{-\lambda\tau\},$$

$$p_{21}^{*} = p_{1}^{*}, \exp\{-\lambda\tau\}, \ p_{22}^{*} = p_{2}^{*}.$$

If we integrate the first of (3.15) and use the fact that $\Omega_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\Omega_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\Omega_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\Omega_i(a) = \{\Omega_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$

$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\Omega_i\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\Omega_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\Omega_i\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\Omega_i\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\Omega_i\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)}\int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}da, \quad i = 1,\ 2,$$

Now supplying $\Omega_1(a)$ and $\Omega_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^*\exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^*\exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^* \exp(-\lambda\tau) + (\lambda - A_{22})\, p_2^* = g_{21}(\lambda)\, p_1^* \exp(-\lambda\tau) + g_{22}(\lambda)\, p_2^*$$

Where

(3.22) $$A_{i} = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_{}^*} - \frac{\partial f_i}{\partial p_{}^*}\right]da,$$

(3.23) $$g_{i}(\lambda) = \left[p_i^*(0)c_i\right]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\, da\right\}$$
$$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_{}^*} - b_i(a, p_1^*, p_2^*)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{}^*}\, e^{\lambda s}\, ds\right)\right]da\right\}$$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{}^*}\, e^{\lambda s}\, ds\right)da, \quad i = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*), \quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})\, e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$$S_2(\lambda, \tau) = \left[A_{21}\, g_{12}(\lambda) + A_{12}\, g_{21}(\lambda) + g_{12}(\lambda)\, g_{21}(\lambda)\right]\exp[-2\lambda\tau]$$

$$-A_{11}\, g_{22}(\lambda) - A_{22}\, g_{11}(\lambda) - g_{11}(\lambda)\, g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6$. $0 < \beta_i^* < d_i^*$ $(i = 1, 2.)$ $(see(A_1) and (A_2))$;

$A_7$. $\{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - \left| F_2(\lambda, \tau) \right|$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \,| - |A_{11} + A_{22} + \| S_1(0) \| \,|^2 \right\}/2,$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational

ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the

Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have

obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is:

individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## SECTION SIX

## Equation Of State Of G(2) Gauge Theory

### INTRODUCTION—VARIABLES USED

**Exceptional thermodynamics: The equation of state of G(2) gauge theory Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini**

(1) Authors present a lattice study of the equation of state in Yang-Mills theory based on (e) the exceptional G (2) gauge group.
(2) As is well-known, at zero temperature this theory shares (e&eb) many qualitative features with (e&eb) real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e&*eb) large distances.
(3) In agreement with previous works, authors show that at finite temperature this theory features (eb) a first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation.

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| (4) They also show that the equilibrium thermodynamic observables in (eb) the deconfined phase bear striking quantitative similarities with (=) those found in SU (N) gauge theories:<br>(5) In particular, these quantities exhibit (eb) nearly perfect proportionality to the number of gluon degrees of freedom, and the trace anomaly reveals (eb) a characteristic quadratic dependence on the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions).<br>(6) Authors compare lattice data with analytical predictions from (e) effective models, and discuss their implications for (e&eb) the deconfinement mechanism and high-temperature properties of strongly interacting, non-supersymmetric gauge theories.<br>(7) Results give strong evidence for the conjecture that the thermal deconfining transition is governed by (e&eb) a universal mechanism, common to all simple gauge groups. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03(2015)057 Report number: DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv:1409.8305 [hep-lat] (or arXiv:1409.8305v2 [hep-lat] for this version) | |
| **NOTATION** | |
| **Module One**<br>Authorspresent a lattice study of the equation of state in Yang-Mills theory based on (e) the exceptional G (2) gauge group | |
| $G_{13}$ : Category one of exceptional G (2) gauge group<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of lattice study of the equation of state in Yang-Mills theory<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>As is well-known, at zero temperature this theory shares (e&eb) many qualitative features with (e&eb) real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e&*eb) large distances | |
| $G_{16}$ : Category one of **zero temperature this theory;** many qualitative features with (e&eb) real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e&*eb) large distances<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of many qualitative features with (e&eb) real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e&*eb) large distances; **zero temperature this theory** | |

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| $T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>As is well-known, at zero temperature this theory shares many qualitative features with (e&eb) real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e&*eb) large distances | |
| $G_{20}$ : Category one of **zero temperature this theory shares many qualitative features;** real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e&*eb) large distances<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e&*eb) large distances; **zero temperature this theory shares many qualitative features**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>As is well-known, at zero temperature this theory shares many qualitative features with real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e&*eb) large distances | |
| $G_{24}$ : Category one of absence of colored states in the spectrum and dynamical string breaking at (e&*eb) large distances<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of zero temperature this theory shares many qualitative features with real-world QCD<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>As is well-known, at zero temperature this theory shares many qualitative features with real-world QCD, including the absence of colored states in the spectrum and dynamical string breaking at (e&*eb) large distances | |
| $G_{28}$ : Category one of **zero temperature this theory shares many qualitative features with real-world QCD, including the absence of colored states in the spectrum and dynamical string breaking;** large distances | |

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| $G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of large distances **;zero temperature this theory shares many qualitative features with real-world QCD, including the absence of colored states in the spectrum and dynamical string breaking**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>In agreement with previous works, authors show that at finite temperature this theory features (eb) a first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation | |
| $G_{32}$ : Category one of theory features at finite temperature<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>In agreement with previous works, authors show that at finite temperature this theory features a first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation | |
| $G_{36}$ : Category one of semi-classical computation<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of finite temperature this theory features a first-order deconfining phase transition, whose nature can be studied<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>They also show that the equilibrium thermodynamic observables in (eb) the deconfined phase bear striking | |

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| quantitative similarities with (=) those found in SU (N) gauge theories | |
| $G_{40}$ : Category one of equilibrium thermodynamic observables<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of deconfined phase bear striking quantitative similarities with (=) those found in SU (N) gauge theories<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>They also show that the equilibrium thermodynamic observables in the deconfined phase bear striking quantitative similarities with (=) those found in SU (N) gauge theories | |
| $G_{44}$ : Category one of equilibrium thermodynamic observables in the deconfined phase bear striking quantitative similarities<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of those found in SU (N) gauge theories<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION SEVEN**<br><br>**Trace Anomaly Reveals A Characteristic Quadratic Dependence On The Temperature** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Exceptional thermodynamics: The equation of state of G(2) gauge theory Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini** | |

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| (1) In particular, these quantities exhibit (eb) nearly perfect proportionality to the number of gluon degrees of freedom, and the trace anomaly reveals (eb) a characteristic quadratic dependence on the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions).<br>(2) Authors compare lattice data with analytical predictions from (e) effective models, and discuss their implications for (e&eb) the deconfinement mechanism and high-temperature properties of strongly interacting, non-supersymmetric gauge theories.<br>(3) Results give strong evidence for the conjecture that the thermal deconfining transition is governed by (e&eb) a universal mechanism, common to all simple gauge groups. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03(2015)057 Report number: DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv:1409.8305 [hep-lat] (or arXiv:1409.8305v2 [hep-lat] for this version) | |
| **NOTATION** | |
| **Module One**<br>In particular, these quantities exhibit (eb) nearly perfect proportionality to the number of gluon degrees of freedom, and the trace anomaly reveals (eb) a characteristic quadratic dependence on the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions) | |
| $G_{13}$ : Category one of these quantities<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of nearly perfect proportionality to the number of gluon degrees of freedom, and the trace anomaly reveals (eb) a characteristic quadratic dependence on the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions)<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>In particular, these quantities exhibit nearly perfect proportionality to the number of gluon degrees of freedom, and the trace anomaly reveals (eb) a characteristic quadratic dependence on the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions) | |
| $G_{16}$ : Category one of these quantities exhibit nearly perfect proportionality to the number of gluon degrees of freedom, and the trace anomaly<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of characteristic quadratic dependence on the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions)<br>$T_{17}$ : Category two of SAS | |

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| $T_{18}$ : Category three of SAS | |
| **Module three**<br><br>In particular, these quantities exhibit nearly perfect proportionality to the number of gluon degrees of freedom, and the trace anomaly reveals a characteristic quadratic dependence on the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions) | |
| $G_{20}$ : Category one of these quantities exhibit nearly perfect proportionality to the number of gluon degrees of freedom, and the trace anomaly reveals a characteristic quadratic dependence on the temperature<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of SU (N) Yang-Mills theories (both in four and in three spacetime dimensions)<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Authors compare lattice data with analytical predictions from (e) effective models, and discuss their implications for (e&eb) the deconfinement mechanism and high-temperature properties of strongly interacting, non-supersymmetric gauge theories | |
| $G_{24}$ : Category one of **lattice data;** analytical predictions from (e) effective models, and discuss their implications for (e&eb) the deconfinement mechanism and high-temperature properties of strongly interacting, non-supersymmetric gauge theories<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one ofanalytical predictions from (e) effective models, and discuss their implications for (e&eb) the deconfinement mechanism and high-temperature properties of strongly interacting, non-supersymmetric gauge theories; **lattice data**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Authors compare lattice data with analytical predictions from effective models, and discuss their implications for (e&eb) the deconfinement mechanism and **high-temperature properties of strongly interacting, non-supersymmetric gauge theories** | |
| $G_{28}$ : Category one of **lattice data with analytical predictions from effective models;** deconfinement mechanism and high-temperature properties of strongly interacting, non-supersymmetric gauge theories<br><br>$G_{29}$ : Category two of SAS | |

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| $G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of deconfinement mechanism and high-temperature properties of strongly interacting, non-supersymmetric gauge theories; **lattice data with analytical predictions from effective models**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six** | |
| $G_{32}$ : Category one of deconfinement mechanism; **high-temperature properties of strongly interacting, non-supersymmetric gauge theories**<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of **high-temperature properties of strongly interacting, non-supersymmetric gauge theories;** deconfinement mechanism<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Authors compare lattice data with analytical predictions from effective models, and discuss their implications for (e&eb) the deconfinement mechanism and high-temperature properties of strongly interacting, non-supersymmetric gauge theories | |
| $G_{36}$ : Category one of **lattice data with analytical predictions from effective models;** high-temperature properties of strongly interacting, non-supersymmetric gauge theories<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of high-temperature properties of strongly interacting, non-supersymmetric gauge theories **;lattice data with analytical predictions from effective models**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Results give strong evidence for the conjecture that the thermal deconfining transition is governed by (e&eb) a universal mechanism, common to all simple gauge groups.<br><br>Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03(2015)057 Report number: | |

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| DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv:1409.8305 [hep-lat] (or arXiv:1409.8305v2 [hep-lat] for this version) | |
| $G_{40}$ : Category one of Results<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of evidence for the conjecture that the thermal deconfining transition is governed by (e&eb) a universal mechanism, common to all simple gauge groups<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Results give strong evidence for the conjecture that the thermal deconfining transition is governed by (e&eb) a universal mechanism, common to all simple gauge groups.<br><br>Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03(2015)057 Report number: DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv:1409.8305 [hep-lat] (or arXiv:1409.8305v2 [hep-lat] for this version) | |
| $G_{44}$ : Category one of **thermal deconfining transition;** universal mechanism, common to all simple gauge groups<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of universal mechanism, common to all simple gauge groups; **thermal deconfining transition**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION EIGHT**<br><br>**Donald Hoffman Uses Evolutionary Game Theory To Show That Our Perceptions Of** | |

# An Independent Reality Must Be Illusions

## INTRODUCTION—VARIABLES USED

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) In the context of two illustrative examples from supersymmetric quantum mechanics authors show that the semiclassical analysis of the path integral requires (e) complexification of the configuration space and action, and the inclusion of (e) complex saddle points, even when the parameters in the action are real.

(2) Authors find new exact complex saddles, and show that without their contribution the semiclassical expansion is in conflict with (e&eb) basic properties such as (=) the positive semidefiniteness of (e) the spectrum, as well as (=) constraints of supersymmetry.

(3) Generic saddles are not only complex, but also possibly multivalued and even singular. This is in contrast to instanton solutions, which are real, smooth, and single valued

(4) The multivaluedness of the action can be interpreted as (=) a hidden topological angle, quantized in units of (e) $\pi$ in supersymmetric theories. The general ideas also apply to nonsupersymmetric theories. Received 12 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.011601 © 2016 American Physical Society

(5) The Evolutionary Argument Against Reality The cognitive scientist Donald Hoffman uses evolutionary game theory to show that our perceptions of an independent reality must be (=) illusions.[No Caption]David McNew for Quanta Magazine

(6) About Quanta: Quanta Magazine's mission is to enhance public understanding of research developments in mathematics and the physical and life sciences. Quanta articles do not necessarily represent the views of the Simons Foundation. Learn more By Amanda Gefter April 21, 2016

(7) As we go about our daily lives, we tend to assume that our perceptions — sights, sounds, textures, tastes — are (=) an accurate portrayal of the real world. Sure, when we stop and think about it — or when we find ourselves fooled by (e) a perceptual illusion — we realize with a jolt that what we perceive is (=) never the world directly, but rather our brain's best guess at what that world is (=) like, a kind of internal simulation of (e) an external reality. The Evolutionary Argument Against Reality Quanta magazine

(8) Still, we bank on the fact that our simulation is (=) a reasonably decent one. If it wasn't, wouldn't evolution have weeded us out by (e) now?

**(9)** The true reality might be forever beyond our reach, but surely our senses give (e&eb) us at least an inkling of what it's really like.

## NOTATION

### Module One

In the context of two illustrative examples from supersymmetric quantum mechanics authors show that the semiclassical analysis of the path integral requires (e) complexification of the configuration space and action, and the inclusion of (e) complex saddle points, even when the parameters in the action are real

$G_{13}$ : Category one of complexification of the configuration space and action, and the inclusion of (e)

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| complex saddle points, even when the parameters in the action are real<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of semiclassical analysis of the path integral<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>In the context of two illustrative examples from supersymmetric quantum mechanics authors show that the semiclassical analysis of the path integral requires complexification of the configuration space and action, and the inclusion of (e) complex saddle points, even when the parameters in the action are real | |
| $G_{16}$ : Category one of complex saddle points, even when the parameters in the action are real<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of semiclassical analysis of the path integral requires complexification of the configuration space and action, and the inclusion<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>In the context of two illustrative examples from supersymmetric quantum mechanics authors show that the semiclassical analysis of the path integral requires complexification of the configuration space and action, and the inclusion of complex saddle points, even when the parameters in the action are real | |
| $G_{20}$ : Category one of parameters in the action are real<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of semiclassical analysis of the path integral requires complexification of the configuration space and action, and the inclusion of complex saddle points<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Authors find new exact complex saddles, and show that without their contribution the semiclassical expansion is in conflict with (e&eb) basic properties such as (=) the positive semidefiniteness of (e) the spectrum, as well as (=) constraints of supersymmetry | |

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|---|---|
| $G_{24}$ : Category one of **new exact complex saddles, and show that without their contribution the semiclassical expansion;** basic properties such as (=) the positive semidefiniteness of (e) the spectrum, as well as (=) constraints of supersymmetry<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of basic properties such as (=) the positive semidefiniteness of (e) the spectrum, as well as (=) constraints of supersymmetry **;new exact complex saddles, and show that without their contribution the semiclassical expansion**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Authors find new exact complex saddles, and show that without their contribution the semiclassical expansion is in conflict with basic properties such as (=) the positive semidefiniteness of the spectrum, as well as (=) constraints of supersymmetry | |
| $G_{28}$ : Category one of exact complex saddles, and show that without their contribution the semiclassical expansion is in conflict with basic properties<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of positive semidefiniteness of the spectrum, as well as constraints of supersymmetry<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Generic saddles are not only complex, but also possibly multivalued and even singular.<br><br>This is in contrast to instanton solutions, which are real, smooth, and single valued | |
| $G_{32}$ : Category one of Generic saddle<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of complex, but also possibly multivalued and even singular.<br><br>$T_{33}$ : Category two of<br><br>$T_{34}$ : Category three of | |

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| **Module seven** | |
| $G_{36}$ : Category one of **Generic saddles are not only complex, but also possibly multivalued and even singular;** instanton solutions, which are real, smooth, and single valued<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of instanton solutions, which are real, smooth, and single valued **;Generic saddles are not only complex, but also possibly multivalued and even singular**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The multivaluedness of the action can be interpreted as (=) a hidden topological angle, quantized in units of (e) π in supersymmetric theories.<br><br>The general ideas also apply to nonsupersymmetric theories.<br><br>Received 12 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.011601 © 2016 American Physical Society | |
| $G_{40}$ : Category one of multivaluedness of the action can be interpreted<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of hidden topological angle, quantized in units of (e) π in supersymmetric theories.<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The multivaluedness of the action can be interpreted as a hidden topological angle, quantized in units of (e) π in supersymmetric theories | |
| $G_{44}$ : Category one of π in supersymmetric theories<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |

$T_{44}$ : Category one of multivaluedness of the action can be interpreted as a hidden topological angle, quantized in units

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

## SECTION NINE

## Evolutionary Argument Against Reality

### INTRODUCTION—VARIABLES USED

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) The Evolutionary Argument Against Reality The cognitive scientist Donald Hoffman uses evolutionary game theory to show that our perceptions of an independent reality must be (=) illusions.[No Caption]David McNew for Quanta Magazine

(2) About Quanta: Quanta Magazine's mission is to enhance public understanding of research developments in mathematics and the physical and life sciences. Quanta articles do not necessarily represent the views of the Simons Foundation. Learn more By Amanda Gefter April 21, 2016

(3) As we go about our daily lives, we tend to assume that our perceptions — sights, sounds, textures, tastes — are (=) an accurate portrayal of the real world. Sure, when we stop and think about it — or when we find ourselves fooled by (e) a perceptual illusion — we realize with a jolt that what we perceive is (=) never the world directly, but rather our brain's best guess at what that world is (=) like, a kind of internal simulation of (e) an external reality. The Evolutionary Argument Against Reality Quanta magazine

**(4)** Still, we bank on the fact that our simulation is (=) a reasonably decent one. If it wasn't, wouldn't evolution have weeded us out by (e) now?

**(5)** The true reality might be forever beyond our reach, but surely our senses give (e&eb) us at least an inkling of what it's really like.

### NOTATION

### Module One

The Evolutionary Argument Against Reality The cognitive scientist Donald Hoffman uses evolutionary game theory to show that our perceptions of an independent reality must be (=) illusions

.[No Caption]David McNew for Quanta Magazine

| | |
|---|---|
| $G_{13}$ : Category one of evolutionary game theory<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of our perceptions of an independent reality must be (=) illusions<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>The Evolutionary Argument Against Reality The cognitive scientist Donald Hoffman uses evolutionary game theory to show that our perceptions of an independent reality must be (=) illusions<br>.[No Caption]David McNew for Quanta Magazine | |
| $G_{16}$ : Category one of evolutionary game theory to show that our perceptions of an independent reality<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of illusions<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>As we go about our daily lives, we tend to assume that our perceptions — sights, sounds, textures, tastes — are (=) an accurate portrayal of the real world.<br>Sure, when we stop and think about it — or when we find ourselves fooled by (e) a perceptual illusion — we realize with a jolt that what we perceive is (=) never the world directly, but rather our brain's best guess at what that world is (=) like, a kind of internal simulation of (e) an external reality.<br>The Evolutionary Argument Against Reality Quanta magazine | |
| $G_{20}$ : Category one of we tend to assume that our perceptions — sights, sounds, textures, tastes<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of accurate portrayal of the real world<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |

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| **Module four**<br><br>Sure, when we stop and think about it — or when we find ourselves fooled by (e) a perceptual illusion — we realize with a jolt that what we perceive is (=) never the world directly, but rather our brain's best guess at what that world is (=) like, a kind of internal simulation of (e) an external reality | |
| $G_{24}$ : Category one ofperceptual illusion — we realize with a jolt that what we perceive is (=) never the world directly, but rather our brain's best guess at what that world is (=) like, a kind of internal simulation of (e) an external reality<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of when we find ourselves fooled<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Sure, when we stop and think about it — or when we find ourselves fooled by a perceptual illusion — we realize with a jolt that what we perceive is (=) never the world directly, but rather our brain's best guess at what that world is (=) like, a kind of internal simulation of (e) an external reality | |
| $G_{28}$ : Category one of when we find ourselves fooled by a perceptual illusion — we realize with a jolt that what we perceive<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of never the world directly, but rather our brain's best guess at what that world is (=) like, a kind of internal simulation of (e) an external reality<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Sure, when we stop and think about it — or when we find ourselves fooled by a perceptual illusion — we realize with a jolt that what we perceive is never the world directly, but rather our brain's best guess at what that world is (=) like, a kind of internal simulation of (e) an external reality | |
| $G_{32}$ : Category one of when we find ourselves fooled by a perceptual illusion — we realize with a jolt that what we perceive is never the world directly, but rather our brain's best guess at what that world<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |

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| $T_{32}$ : Category one of kind of internal simulation of (e) an external reality<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Sure, when we stop and think about it — or when we find ourselves fooled by a perceptual illusion — we realize with a jolt that what we perceive is never the world directly, but rather our brain's best guess at what that world is like, a kind of internal simulation of (e) an external reality | |
| $G_{36}$ : Category one of external reality<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of when we find ourselves fooled by a perceptual illusion — we realize with a jolt that what we perceive is never the world directly, but rather our brain's best guess at what that world is like, a kind of internal simulation<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Still, we bank on the fact that our simulation is (=) a reasonably decent one.<br><br>If it wasn't, wouldn't evolution have weeded us out by (e) now? | |
| $G_{40}$ : Category one of we bank on the fact that our simulation<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of reasonably decent one<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>If it wasn't, wouldn't evolution have weeded us out by (e) now? | |
| $G_{44}$ : Category one of If it wasn't<br><br>$G_{45}$ : Category two of SAS | |

| | |
|---|---|
| $G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of evolution would have weeded us out by now<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION TEN

## Hard Problem

### INTRODUCTION—VARIABLES USED

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) Not so, says Donald D. Hoffman, a professor of cognitive science at the University of California, Irvine. Hoffman has spent the past three decades studying perception, artificial intelligence, evolutionary game theory and the brain, and his conclusion is a dramatic one: The world presented to us by our perceptions is (=) nothing like reality.

(2) What's more, he says, we have evolution itself to thank for (e) this magnificent illusion, as it maximizes evolutionary fitness by driving truth to (e&eb) extinction.

(3) Getting at questions about the nature of reality, and disentangling the observer from (e) the observed, is (=) an endeavor that straddles the boundaries of neuroscience and fundamental physics.

(4) On one side you'll find researchers scratching their chins raw trying to understand how a three-pound lump of gray matter obeying nothing more than the ordinary laws of physics can give rise to (eb) first-person conscious experience. This is the aptly named "hard problem."

(5) On the other side are quantum physicists, marveling at the strange fact that quantum systems don't seem to be (=) definite objects localized in space until (e) we come along to observe them.

(6) Experiment after experiment has shown — defying common sense (?) — That if we assume that the particles that make up ordinary objects have (e) an objective, observer-independent existence, (eb) we get the wrong answers.

(7) The central lesson of quantum physics is clear: There are (=) no public objects sitting out there in (eb) some preexisting space.

(8) As the physicist John Wheeler put it, "Useful as it is under ordinary circumstances to say that the world exists 'out there' independent of (e) us, that view can no longer be upheld."

**(9)** So while neuroscientists struggle to understand how there can be such a thing as (=) a first-person reality, quantum physicists have to grapple with (e&eb) the mystery of how there can be anything but a first-person reality.

| NOTATION | |
|---|---|
| **Module One**<br>Not so, says Donald D. Hoffman, a professor of cognitive science at the University of California, Irvine. Hoffman has spent the past three decades studying perception, artificial intelligence, evolutionary game theory and the brain, and his conclusion is a dramatic one: The<br><br>world presented to us by our perceptions is (=) nothing like reality | |
| $G_{13}$ : Category one of world presented to us by our perceptions<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of nothing like reality<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>What's more, he says, we have evolution itself to thank for this magnificent illusion, as evolution maximizes evolutionary fitness by driving truth to (e&eb) extinction | |
| $G_{16}$ : Category one of evolution maximizes evolutionary fitness by driving truth to (e&eb) extinction<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of evolution itself to thank for this magnificent illusion<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>What's more, he says, we have evolution itself to thank for this magnificent illusion, as evolution maximizes evolutionary fitness by driving truth to (e&eb) extinction | |
| $G_{20}$ : Category one of**evolution itself to thank for this magnificent illusion, as evolution maximizes evolutionary fitness by driving truth ;** extinction<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of extinction **;evolution itself to thank for this magnificent illusion, as evolution maximizes evolutionary fitness by driving truth**<br>$T_{21}$ : Category two of SAS | |

| | |
|---|---|
| $T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Getting at questions about the nature of reality, and disentangling the observer from (e) the observed, is (=) an endeavor that straddles the boundaries of neuroscience and fundamental physics | |
| $G_{24}$ : Category one of observed, is (=) an endeavor that straddles the boundaries of neuroscience and fundamental physics<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of disentangling the observer<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>On one side you'll find researchers scratching their chins raw trying to understand how a three-pound lump of gray matter obeying nothing more than the ordinary laws of physics can give rise to (eb) first-person conscious experience.<br><br>This is the aptly named "hard problem." | |
| $G_{28}$ : Category one of three-pound lump of gray matter obeying nothing more than the ordinary laws of physics<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of first-person conscious experience<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>On the other side are quantum physicists, marveling at the strange fact that quantum systems don't seem to be (=) definite objects localized in space until (e) we come along to observe them | |
| $G_{32}$ : Category one of quantum systems don't seem<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |

| | |
|---|---|
| $T_{32}$ : Category one of definite objects localized in space until (e) we come along to observe them<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>On the other side are quantum physicists, marveling at the strange fact that quantum systems don't seem to be definite objects localized in space until (e) we come along to observe them | |
| $G_{36}$ : Category one of we come along to observe them<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of quantum systems don't seem to be definite objects localized in space<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Experiment after experiment has shown — defying common seen (?) — That if we assume that the particles that make up ordinary objects have (e) an objective, observer-independent existence, (eb) we get the wrong answers | |
| $G_{40}$ : Category one of assume that the particles that make up ordinary objects have an objective, observer-independent existence<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of we get the wrong answers<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The central lesson of quantum physics is clear: There are (=) no public objects sitting out there in (eb) some preexisting space | |
| $G_{44}$ : Category one of no public objects sitting out there<br><br>$G_{45}$ : Category two of SAS | |

| | |
|---|---|
| $G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of some preexisting space<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION ELEVEN

## Evolution Has Shaped Us With Perceptions That Allow Us To Survive

### INTRODUCTION—VARIABLES USED

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) In short, all roads lead back to (e&eb) the observer. And that's where you can find Hoffman — straddling the boundaries, attempting a mathematical model of the observer, trying to get at the reality behind the illusion. Quanta Magazine caught up with him to find out more.
(2) QUANTA MAGAZINE: People often use Darwinian evolution as an argument that our perceptions accurately reflect (e&eb) reality. They say, "Obviously we must be latching onto (e&eb) reality in some way because otherwise we would have been wiped out (e) a long time ago. If I think I'm seeing a palm tree but it's really a tiger, I'm in trouble."
(3) Evolution has shaped us with (e&eb) perceptions that allow us to survive.
(4) But part of that involves hiding from (e) us the stuff we don't need to know.
(5) And that's (=) pretty much all of reality, whatever reality might be."
(6) DONALD HOFFMAN: Right. The classic argument is that those of our ancestors who saw more accurately had a competitive advantage over (e&eb) those who saw less accurately and thus were (=) more likely to pass on their genes that coded for (e) those more accurate perceptions, so after thousands of generations we can be quite confident that we're (=) the offspring of those who saw accurately, and so we see accurately.
(7) That sounds very plausible. But I think it is (=) utterly false. It misunderstands the fundamental fact about evolution, which is (=) that it's about fitness functions — mathematical functions that describe (eb) how well a given strategy achieves (eb) the goals of (e) survival and reproduction.
(8) The mathematical physicist Chetan Prakash proved a theorem that I devised that says: According to evolution by(e) natural selection, an organism that sees reality as it is will never be (=) more fit than an organism of (e) equal complexity that sees none of reality but is (=) just tuned to fitness. Never.
**(9)** You've done computer simulations to show this. Can you give an example?

### NOTATION

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| **Module One**<br>QUANTA MAGAZINE: People often use Darwinian evolution as an argument that our perceptions accurately reflect (e&eb) reality.<br><br>They say, "Obviously we must be latching onto (e&eb) reality in some way because otherwise we would have been wiped out (e) a long time ago. If I think I'm seeing a palm tree but it's really a tiger, I'm in trouble." | |
| $G_{13}$ : Category one of **Darwinian evolution as an argument that our perceptions accurately reflect;** reality.<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of reality.; **Darwinian evolution as an argument that our perceptions accurately reflect**<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>They say, "Obviously we must be latching onto reality in some way because (e) otherwise we would have been wiped out (e) a long time ago.<br><br>If I think I'm seeing a palm tree but it's really a tiger, I'm in trouble." | |
| $G_{16}$ : Category one of otherwise we would have been wiped out a long time ago<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of we must be latching onto reality in some way<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>Evolution has shaped us with (e&eb) perceptions that allow us to survive | |
| $G_{20}$ : Category one of **Evolution has shaped us;** perceptions that allow us to survive<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of perceptions that allow us to survive; **Evolution has shaped us**<br><br>$T_{21}$ : Category two of SAS | |

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| $T_{22}$ : Category three of SAS | |
| **Module four**<br><br>But part of that involves hiding from (e) us the stuff we don't need to know<br><br>***In fact it is this ignorance that is making us to survive****(italics mine)* | |
| $G_{24}$ : Category one of us the stuff we don't need to know<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of part of that involves hiding<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>And that's (=) pretty much all of reality, whatever reality might be | |
| $G_{28}$ : Category one of whatever that is hidden<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of pretty much all of reality, whatever reality might be<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>DONALD HOFFMAN: Right.<br><br>The classic argument is that those of our ancestors who saw more accurately had a competitive advantage over (e&eb) those who saw less accurately and thus were (=) more likely to pass on their genes that coded for (e) those more accurate perceptions, so after thousands of generations we can be quite confident that we're (=) the offspring of those who saw accurately, and so we see accurately | |
| $G_{32}$ : Category one of **classic argument is that those of our ancestors who saw more accurately had a competitive advantage;** those who saw less accurately and thus were (=) more likely to pass on their genes that coded for (e) those more accurate perceptions, so after thousands of generations we can be quite confident that we're (=) the offspring of those who saw accurately, and so we see accurately<br><br>$G_{33}$ : Category two of SAS | |

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| $G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of those who saw less accurately and thus were (=) more likely to pass on their genes that coded for (e) those more accurate perceptions, so after thousands of generations we can be quite confident that we're (=) the offspring of those who saw accurately, and so we see accurately;**classic argument is that those of our ancestors who saw more accurately had a competitive advantage**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>The classic argument is that those of our ancestors who saw more accurately had a competitive advantage over those who saw less accurately and thus were (=) more likely to pass on their genes that coded for (e) those more accurate perceptions, so after thousands of generations we can be quite confident that we're (=) the offspring of those who saw accurately, and so we see accurately | |
| $G_{36}$ : Category one of classic argument is that those of our ancestors who saw more accurately had a competitive advantage over those who saw less accurately and thus were<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one ofmore likely to pass on their genes that coded for (e) those more accurate perceptions, so after thousands of generations we can be quite confident that we're (=) the offspring of those who saw accurately, and so we see accurately<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The classic argument is that those of our ancestors who saw more accurately had a competitive advantage over those who saw less accurately and thus were more likely to pass on their genes that coded for (e) those more accurate perceptions, so after thousands of generations we can be quite confident that we're (=) the offspring of those who saw accurately, and so we see accurately | |
| $G_{40}$ : Category one of those more accurate perceptions, so after thousands of generations we can be quite confident that we're the offspring of those who saw accurately, and so we see accurately<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of classic argument is that those of our ancestors who saw more accurately had a competitive advantage over those who saw less accurately and thus were more likely to pass on their genes that coded | |

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| $T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>That sounds very plausible.<br><br>But I think it is (=) utterly false.<br><br>It misunderstands the fundamental fact about evolution, which is (=) that it's about fitness functions — mathematical functions that describe how well a given strategy achieves (eb) the goals of (e) survival and reproduction | |
| $G_{44}$ : Category one of mathematical functions describe how well a given strategy<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of goals of survival and reproduction<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION TWELVE

## Fitness Function Happens To Align With The True Structure In Reality

### INTRODUCTION—VARIABLES USED

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) Suppose in reality there's (=) a resource, like water, and you can quantify how much of it there is (=) in an objective order — very little water, medium amount of water, a lot of water.

**(2)** Now suppose your fitness function is (=) linear, so a little water gives (eb) you a little fitness, medium water gives you (eb) medium fitness, and lots of water gives you (eb) lots of fitness — in that case, the organism that sees the truth about the water in the world can win, but only because (e) the **fitness function happens to align with (e&eb) the true structure in reality.**

(3) Generically, in the real world, that will never be the case.

(4) Something much more natural is (=) a bell curve — say, too little water (eb) you die of thirst, but too much water (eb) you drown, and only somewhere in between is (=) good for survival.

(5) Now the fitness function doesn't (e) match the structure in the real world.

(6) And that's enough to send truth to (e&eb) extinction.

(7) For example, an organism tuned to fitness might see small and large quantities of some resource as, say (=), red, to indicate low fitness, whereas they might see intermediate quantities as green, to indicate high fitness. Its perceptions will be tuned to (e) fitness, but not to truth.

**(8)** It won't see any distinction between small and large — it only sees (=) red — even though such a distinction exists in (eb) reality.

**(9)** But how can seeing a false reality be (=) beneficial to an organism's survival? (Quanta)

**NOTATION**

**Module One**

The mathematical physicist Chetan Prakash proved a theorem that I devised that says:

According to evolution by natural selection, an organism that sees reality as it is will never be (=) more fit than an organism of (e) equal complexity that sees none of reality but is (=) just tuned to fitness.

Never.

$G_{13}$ : Category one of evolution by natural selection

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of an organism that sees reality as it is will never be (=) more fit than an organism of (e) equal complexity that sees none of reality but is (=) just tuned to fitness.

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

According to evolution by natural selection, an organism that sees reality as it is will never be more fit than an organism of (e) equal complexity that sees none of reality but is (=) just tuned to fitness

$G_{16}$ : Category one of equal complexity that sees none of reality but is (=) just tuned to fitness

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

$T_{16}$ : Category one of evolution by natural selection, an organism that sees reality as it is will never be more fit than an organism

$T_{17}$ : Category two of SAS

$T_{18}$ : Category three of SAS

**Module three**

According to evolution by natural selection, an organism that sees reality as it is will never be more fit than an organism of equal complexity that sees none of reality but is (=) just tuned to fitness

| | |
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| $G_{20}$ : Category one of evolution by natural selection, an organism that sees reality as it is will never be more fit than an organism of equal complexity that sees none of reality<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of just tuned to fitness<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Now suppose your fitness function is (=) linear, so a little water gives (eb) you a little fitness, medium water gives you (eb) medium fitness, and lots of water gives you (eb) lots of fitness — in that case, the organism that sees the truth about the water in the world can win, but only because (e) the<br><br>**fitness function happens to align with (e&eb) the** true structure in reality | |
| $G_{24}$ : Category one of **fitness function happens to align;** true structure in reality<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of true structure in reality; **fitness function happens to align**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Now the fitness function doesn't match the structure in the real world | |
| $G_{28}$ : Category one of match the structure in the real world<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of **fitness function**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>And that's enough to send truth to (e&eb) extinction | |

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| $G_{32}$ : Category one of fitness function doesn't match the structure in the real world; **extinction of truth**<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of **extinction of truth** ; fitness function doesn't match the structure in the real world<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br>For example, an organism tuned to fitness might see small and large quantities of some resource as, say (=), red, to indicate low fitness, whereas they might see intermediate quantities as green, to indicate high fitness. Its perceptions will be tuned to (e) fitness, but not to truth | |
| $G_{36}$ : Category one of fitness, but not to truth<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of perceptions will be tuned<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>It won't see any distinction between small and large — it only sees (=) red — even though such a distinction exists in (eb) reality | |
| $G_{40}$ : Category one of It won't see any distinction between small and large — it only sees red<br>$G_{41}$ : Category two of SAS<br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of even though such a distinction exists in reality<br>$T_{41}$ : Category two of SAS<br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br>But how can seeing a false reality be (=) beneficial to an organism's survival | |

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| $G_{44}$ : Category one of seeing a false reality<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of beneficial to an organism's survival<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION THIRTEEN

## If You Had To Spend All That Time Figuring Blue Icon Out The Tiger Would Eat You

### INTRODUCTION—VARIABLES USED

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) There's a metaphor that's only been available to us in the past 30 or 40 years, and that's the desktop interface. Suppose there's a blue rectangular icon on (eb) the lower right corner of your computer's desktop — does that mean that the file itself is (=) blue and rectangular and lives in (eb) the lower right corner of your computer? Of course not.
(2) But those are the only things that can be asserted about anything on (eb) the desktop — it has color, position and shape.
(3) Those are (=) the only categories available to you, and yet none of them are (=) true about the file itself or anything in the computer.
(4) They couldn't possibly be true. That's an interesting thing. You could not form (eb) a true description of the innards of the computer if your entire view of reality was confined to (e&eb) the desktop. And yet the desktop is useful.
(5) That blue rectangular icon guides (e&eb) my behavior, and it hides (e) a complex reality that I don't need to know. That's the key idea.
(6) Evolution has shaped us with (e&eb) perceptions that allow us to survive.
(7) They guide adaptive behaviors. But part of that involves hiding from (e) us the stuff we don't need to know.
**(8)** And that's (=) pretty much all of reality, whatever reality might be.
**(9)** If you had to spend all that time figuring blue icon out, (eb) the tiger would eat you.

### NOTATION

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| **Module One**<br>There's a metaphor that's only been available to us in the past 30 or 40 years, and that's the desktop interface.<br><br>Suppose there's a blue rectangular icon on (eb) the lower right corner of your computer's desktop — does that mean that the file itself is (=) blue and rectangular and lives in (eb) the lower right corner of your computer? Of course not. | |
| $G_{13}$ : Category one of file itself<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of blue and rectangular and lives in (eb) the lower right corner of your computer? Of course not<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>But those are the only things that can be asserted about anything on (eb) the desktop — it has color, position and shape | |
| $G_{16}$ : Category one of those are the only things that can be asserted about anything<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of desktop — it has color, position and shape<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>Those are the only categories available to you, and yet none of them are (=) true about the file itself or anything in the computer | |
| $G_{20}$ : Category one of Those are the only categories available to you, and yet none of them<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of true about the file itself or anything in the computer<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |

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| **Module four**<br><br>They couldn't possibly be true.<br><br>That's an interesting thing.<br><br>You could not form a true description of the innards of the computer if your entire view of reality was confined to (e&eb) the desktop.<br><br>And yet the desktop is useful | |
| $G_{24}$ : Category one of your entire view of reality was confined to the desktop<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of You could not form a true description of the innards of the computer<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>That blue rectangular icon guides (e&eb) my behavior, and it hides (e) a complex reality that I don't need to know.<br><br>That's the key idea | |
| $G_{28}$ : Category one of **blue rectangular icon;** my behavior, and it hides (e) a complex reality that I don't need to know.<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of my behavior, and it hides (e) a complex reality that I don't need to know.; **blue rectangular icon**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>That blue rectangular icon guides my behavior, and it hides (e) a complex reality that I don't need to know | |
| $G_{32}$ : Category one of **blue rectangular icon;** my behavior, and it hides (e) a complex reality that I don't need to know<br><br>$G_{33}$ : Category two of SAS | |

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| $G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of my behavior, and it hides (e) a complex reality that I don't need to know; **blue rectangular icon**<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br>That blue rectangular icon guides my behavior, and it hides (e) a complex reality that I don't need to know | |
| $G_{36}$ : Category one of complex reality that I don't need to know<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of blue rectangular icon guides my behavior, and it hides<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>Evolution has shaped us with (e&eb) perceptions that allow us to survive | |
| $G_{40}$ : Category one of **Evolution has shaped us;** perceptions that allow us to survive<br>$G_{41}$ : Category two of SAS<br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of perceptions that allow us to survive; **Evolution has shaped us**<br>$T_{41}$ : Category two of SAS<br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br>They guide adaptive behaviors.<br>But part of that involves hiding from (e) us the stuff we don't need to know | |
| $G_{44}$ : Category one of **Evolution;** adaptive behaviors<br>$G_{45}$ : Category two of SAS<br>$G_{46}$ : Category three of SAS | |

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| $T_{44}$ : Category one of adaptive behaviors **;Evolution**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION FOURTEEN**<br><br>**Everything We See Is One Big Illusion** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**<br><br>(1) So everything we see is (=) one big illusion?<br>(2) We've been shaped to have (e) perceptions that keep us alive, so we have to take them seriously.<br>(3) If I see something that I think of as a snake, (eb) I don't pick it up.<br>(4) If I see a train, (eb) I don't step in front of it.<br>(5) I've evolved (eb) these symbols to keep me alive, so I have to take them seriously.<br>(6) But it's a logical flaw to think that if we have to take it seriously, we also have to take it literally.<br>(7) If snakes aren't snakes and trains aren't trains, what are (=) they?<br>(8) Snakes and trains, like the particles of physics, have (e) no objective, observer-independent features.<br>**(9)** The snake I see is a description created by (e) my sensory system to inform me of the fitness consequences of my actions. | |
| **NOTATION** | |
| **Module One**<br>So everything we see is (=) one big illusion?<br><br>We assume the proposition and give a model | |
| $G_{13}$ : Category one of everything we see<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of big illusion | |

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| $T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>We've been shaped to have (e) perceptions that keep us alive, so we have to take them seriously | |
| $G_{16}$ : Category one of perceptions that keep us alive, so we have to take them seriously<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of We've been shaped<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>If I see something that I think of as a snake, (eb) I don't pick it up | |
| $G_{20}$ : Category one of If I see something that I think of as a snake<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of I don't pick it up<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>If I see a train, (eb) I don't step in front of it | |
| $G_{24}$ : Category one of If I see a train<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of I don't step in front of it (generally I=human beings)<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>I've evolved (eb) these symbols to keep me alive, so I have to take them seriously | |

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| $G_{28}$ : Category one of human beings have evolved<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of these symbols to keep them alive, so they have to take them seriously<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>But it's a logical flaw to think that if we have to take it seriously, we also have to take it literally | |
| $G_{32}$ : Category one of it's a logical flaw to think that if we have to take it seriously,<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of we also have to take it literally<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br>Snakes and trains, like the particles of physics, have (e) no objective, observer-independent features. | |
| $G_{36}$ : Category one of no objective, observer-independent features<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of Snakes and trains, like the particles of physics<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>The snake I see is a description created by (e) my sensory system to inform me of the fitness consequences of my actions | |

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| $G_{40}$ : Category one of snake I see<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of description created by (e) my sensory system to inform me of the fitness consequences of my actions<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The snake I see is a description created by (e) my sensory system to inform me of the fitness consequences of my actions | |
| $G_{44}$ : Category one of my sensory system to inform me of the fitness consequences of my actions<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of snake I see is a description<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION FIFTEEN

## Evolution Shapes Acceptable Solutions, Not Optimal Ones

**INTRODUCTION—VARIABLES USED**

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) Evolution shapes (e&eb) acceptable solutions, not (e) optimal ones.
(2) A snake is an acceptable solution to (e) the problem of telling me how to act in (e&eb) a situation.
(3) My snakes and trains are (=) my mental representations; your snakes and trains are (=) your mental representations.
(4) How did you first become interested in these ideas?
(5) As a teenager, I was very interested in the question “Are we machines?” My reading of the science

suggested (eb) that we are.

(6) But my dad was a minister, and at church they were saying we're (=) not.

(7) So I decided I needed to figure it out for myself. It's sort of an important personal question — if I'm a machine, (eb) I would like to find that out!

**(8)** And if I'm not, I'd like to know, what is that special magic beyond the machine? So eventually in the 1980s I went to the artificial intelligence lab at MIT and worked on (e&eb) machine perception.

**(9)** The field of vision research was enjoying a newfound success in developing mathematical models for (e) specific visual abilities

**NOTATION**

**Module One**

Evolution shapes (e&eb) acceptable solutions, not (e) optimal ones

$G_{13}$ : Category one of **Evolution;** acceptable solutions, not (e) optimal ones

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of acceptable solutions, not (e) optimal ones **;Evolution**

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

Evolution shapes acceptable solutions, not (e) optimal ones

$G_{16}$ : Category one of optimal ones

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

$T_{16}$ : Category one of Evolution shapes acceptable solutions

$T_{17}$ : Category two of SAS

$T_{18}$ : Category three of SAS

**Module three**

A snake is an acceptable solution to (e) the problem of telling me how to act in (e&eb) a situation

$G_{20}$ : Category one of problem of telling me how to act in (e&eb) a situation

$G_{21}$ : Category two of SAS

$G_{22}$ : Category three of SAS

$T_{20}$ : Category one of snake is an acceptable solution

$T_{21}$ : Category two of SAS

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| $T_{22}$ : Category three of SAS | |
| **Module four**<br>A snake is an acceptable solution tothe problem of telling me how to act in (e&eb) a situation | |
| $G_{24}$ : Category one of **snake is an acceptable solution tothe problem of telling me how to act;** situation<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of situation **;snake is an acceptable solution tothe problem of telling me how to act**<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>My snakes and trains are (=) my mental representations;<br>your snakes and trains are (=) your mental representations | |
| $G_{28}$ : Category one of My snakes and trains<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of my mental representations<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>your snakes and trains are (=) your mental representations | |
| $G_{32}$ : Category one of your snakes and trains; **your mental representations**<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of **your mental representations** ;your snakes and trains<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven** | |

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| As a teenager, I was very interested in the question "Are we machines?" My reading of the science suggested (eb) that we are | |
| $G_{36}$ : Category one of we<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of machines<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>And if I'm not, I'd like to know, what is that special magic beyond the machine? So eventually in the 1980s I went to the artificial intelligence lab at MIT and worked on (e&eb) machine perception | |
| $G_{40}$ : Category one of human beings;**machine perception**<br>$G_{41}$ : Category two of SAS<br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of **machine perception;** human beings<br>$T_{41}$ : Category two of SAS<br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br>The field of vision research was enjoying a newfound success in developing mathematical models for (e) specific visual abilities | |
| $G_{44}$ : Category one of specific visual abilities<br>$G_{45}$ : Category two of SAS<br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of mathematical models<br>$T_{45}$ : Category two of SAS<br>$T_{46}$ : Category three of SAS | |

## SECTION SIXTEEN

## Mathematical Model Of Consciousness

### INTRODUCTION—VARIABLES USED

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) I noticed that they seemed to share a common mathematical structure, so I thought it might be possible to write down a formal structure for (e) observation that encompassed all of them, perhaps all possible modes of observation.
(2) I was inspired in part by Alan Turing. When he invented the Turing machine, he was trying to come up with a notion of computation, and instead of putting bells and whistles on it, he said, let's get the simplest, most pared down mathematical description that could possibly work. And that simple formalism is (=) the foundation for the science of computation. So I wondered, could I provide a similarly simple formal foundation for (e) the science of observation?
(3) A mathematical model of consciousness: That's right. My intuition was, there are (=) conscious experiences.
(4) I have (e) pains, tastes, smells, all my sensory experiences, moods, emotions and so forth.
(5) So I'm just going to say: One part of this consciousness structure is (=) a set of all possible experiences.
(6) When I'm having (e) an experience, based on (e) that experience I may want to change (e&eb) what I'm doing.
(7) So I need to have a collection of possible actions I can take and a decision strategy that, given my experiences, allows me (eb) to change how I'm acting.
**(8)** That's the basic idea of (e) the whole thing.
**(9)** I have a space X of experiences, a space G of actions, and an algorithm D that lets me choose (e&eb) a new action given my experiences

### NOTATION

**Module One**

I noticed that they seemed to share a common mathematical structure, so I thought it might be possible to write down a formal structure for (e) observation that encompassed all of them, perhaps all possible modes of observation

$G_{13}$ : Category one of observation that encompassed all of them, perhaps all possible modes of observation

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of formal structure

$T_{14}$ : Category two of SAS

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| $T_{15}$ : Category three of SAS | |
| **Module Two**<br>A mathematical model of consciousness: That's right.<br>My intuition was, there are (=) conscious experiences | |
| $G_{16}$ : Category one of there<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of conscious experiences<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>I have (e) pains, tastes, smells, all my sensory experiences, moods, emotions and so forth. | |
| $G_{20}$ : Category one of pains, tastes, smells, all my sensory experiences, moods, emotions and so forth<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of human beings<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>So I'm just going to say: One part of this consciousness structure is (=) a set of all possible experiences | |
| $G_{24}$ : Category one of part of this consciousness structure<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of set of all possible experiences<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>When I'm having (e) an experience, based on (e) that experience I may want to change (e&eb) what I'm | |

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| doing | |
| $G_{28}$ : Category one of experience, based on (e) that experience they may want to change (e&eb) what I'm doing<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of human beings<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>When human beings having an experience, based on (e) that experience they might want to change (e&eb) what they are doing | |
| $G_{32}$ : Category one of that experience they might want to change (e&eb) what they are doing<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of human beings having an experience<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>When human beings having an experience, based on that experience they might want to change (e&eb) what they are doing | |
| $G_{36}$ : Category one of **human beings having an experience, based on that experience they might want to change;** what they are doing<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of what they are doing **;human beings having an experience, based on that experience they might want to change**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |

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| **Module eight**<br><br>So I need to have a collection of possible actions I can take and a decision strategy that, given my experiences, allows me (eb) to change how I'm acting | |
| $G_{40}$ : Category one of I need to have a collection of possible actions I can take and a decision strategy that, given my experiences,<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of change how I'm acting<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>I have a space X of experiences, a space G of actions, and an algorithm D that lets me choose (e&eb) a new action given my experiences | |
| $G_{44}$ : Category one of **I have a space X of experiences, a space G of actions, and an algorithm D that lets me choose;** new action given my experiences<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of new action given my experiences **;I have a space X of experiences, a space G of actions, and an algorithm D that lets me choose**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION SEVENTEEN**<br><br>**World Is Just Another Conscious Agents** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview** | |

(1) Then I posited a W for a world, which is (=) also a probability space.
(2) Somehow the world affects my (e&eb) perceptions, so there's (=) a perception map P from (e) the world to my experiences, and when I act, (e&eb) I change the world, so there's (=) a map A from the space of actions to (e&eb) the world. That's the entire structure. Six elements. The claim is: This is (=) the structure of consciousness. I put that out there so people have something to shoot at.
(3) But if there's a W, are you saying there is (=) an external world?
(4) Here's the striking thing about that. I can pull the W out of (e) the model and stick a conscious agent in its place and get (eb) a circuit of conscious agents.
(5) In fact, you can have (e) whole networks of (e) arbitrary complexity.
(6) And that's (=) the world. David McNew for Quanta Magazine Video: Donald Hoffman explains how our perceptions have evolved to become like a computer interface.
(7) The world is just other conscious agents?
(8) I call it (=) conscious realism: Objective reality is (=) just conscious agents, just points of view.
**(9)** Interestingly, I can take two conscious agents and have them interact (e&eb) and the mathematical structure of (e) that interaction also satisfies (eb) the definition of (e) a conscious agent.

**NOTATION**

**Module One**

Then I posited a W for a world, which is (=) also a probability space

$G_{13}$ : Category one of posit a W for a world

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of probability space

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

Somehow the world affects my perceptions, so there's (=) a perception map P from (e) the world to my experiences, and when I act, (e&eb) I change the world, so there's (=) a map A from the space of actions to (e&eb) the world.

That's the entire structure. Six elements.

The claim is: This is (=) the structure of consciousness. I put that out there so people have something to shoot at.

$G_{16}$ : Category one of **world;** perceptions

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

$T_{16}$ : Category one of perceptions **;world**

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| $T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>Somehow the world affects my perceptions, so there's (=) a perception map P from (e) the world to my experiences, and when I act, (e&eb) I change the world, so there's (=) a map A from the space of actions to (e&eb) the world.<br><br>That's the entire structure. Six elements.<br><br>The claim is: This is (=) the structure of consciousness. I put that out there so people have something to shoot at. | |
| $G_{20}$ : Category one of world affects my perceptions<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of there's (=) a perception map P from (e) the world to my experiences, and when I act, (e&eb) I change the world, so there's (=) a map A from the space of actions to (e&eb) the world.<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>There's (=) a perception map P from the world to my experiences, and when I act, (e&eb) I change the world, so there's (=) a map A from the space of actions to (e&eb) the world. | |
| $G_{24}$ : Category one of perception map P from the world; my experiences, and when I act, (e&eb) I change the world, so there's (=) a map A from the space of actions to (e&eb) the world.<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of my experiences, and when I act, (e&eb) I change the world, so there's (=) a map A from the space of actions to (e&eb) the world.;perception map P from the world<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>There's a perception map P from the world to my experiences, and when I act, (e&eb) I change the world, so there's (=) a map A from the space of actions to (e&eb) the world. | |
| $G_{28}$ : Category one of perception map P from the world to my experiences, and when I act | |

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| $G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of I change the world, so there's (=) a map A from the space of actions to (e&eb) the world.<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>There's a perception map P from the world to my experiences, and when I act, I change the world, so there's a map A from the space of actions to the world. | |
| $G_{32}$ : Category one of **perception map P from the world to my experiences, and when I act, I change the world, so there's a map A from the space of actions;** world.<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of world.; **perception map P from the world to my experiences, and when I act, I change the world, so there's a map A from the space of actions**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>The claim is: This is (=) the **structure of consciousness**. I put that out there so people have something to shoot at. | |
| $G_{36}$ : Category one of perception map P from the world to my experiences, and when I act, I change the world, so there's a map A from the space of actions to the world; **structure of consciousness**<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of **structure of consciousness;** perception map P from the world to my experiences, and when I act, I change the world, so there's a map A from the space of actions to the world.<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight** | |

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| Here's the striking thing about that.<br><br>I can pull the W out of (e) the model and stick a conscious agent in its place and get (eb) a circuit of conscious agents | |
| $G_{40}$ : Category one of model and stick a conscious agent in its place and get (eb) a circuit of conscious agents<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of I can pull the W<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Human beings can pull the W out of the model and stick a conscious agent in its place and get (eb) a circuit of conscious agents | |
| $G_{44}$ : Category one of Human beings can pull the W out of the model and stick a conscious agent in its place<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of circuit of conscious agents<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION EIGHTEEN**<br><br>**World Is Just Other Conscious Agents?** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**<br><br>(1) In fact, you can have (e) whole networks of (e) arbitrary complexity.<br>(2) And that's (=) the world. David McNew for Quanta Magazine Video: Donald Hoffman explains | |

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| how our perceptions have evolved to become like a computer interface.<br>(3) The world is just other conscious agents?<br>**(4)** I call it (=) conscious realism: Objective reality is (=) just conscious agents, just points of view.<br>**(5)** Interestingly, I can take two conscious agents and have them interact (e&eb) and the mathematical structure of (e) that interaction also satisfies (eb) the definition of (e) a conscious agent. | |
| **NOTATION** | |
| **Module One**<br>In fact, you can have (e) whole networks of (e) arbitrary complexity. | |
| $G_{13}$ : Category one of whole networks of (e) arbitrary complexity<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of you can<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>In fact, you can havewhole networks of (e) arbitrary complexity. | |
| $G_{16}$ : Category one of arbitrary complexity<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of you can havewhole networks<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>And that's (=) the world. David McNew for Quanta Magazine Video:<br>Donald Hoffman explains how our perceptions have evolved to become like a computer interface | |
| $G_{20}$ : Category one of perceptions have evolved<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of computer interface<br>$T_{21}$ : Category two of SAS | |

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| $T_{22}$ : Category three of SAS | |
| **Module four**<br>The world is just other conscious agents | |
| $G_{24}$ : Category one of world<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of conscious realism<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>Objective reality is (=) just conscious agents, just points of view. | |
| $G_{28}$ : Category one of Objective reality<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of just conscious agents, just points of view<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>Interestingly, I can take two conscious agents and have them interact (e&eb) and the mathematical structure of (e) that interaction also satisfies (eb) the definition of (e) a conscious agent | |
| $G_{32}$ : Category one of conscious agent A; conscious agent B<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of conscious agent A; conscious agent A<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven** | |

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| mathematical structure of (e) that interaction also satisfies (eb) the definition of (e) a conscious agent | |
| $G_{36}$ : Category one of interaction also satisfies (eb) the definition of (e) a conscious agent<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of mathematical structure<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>mathematical structure of that interaction also satisfies (eb) the definition of (e) a conscious agent | |
| $G_{40}$ : Category one of mathematical structure of that interaction<br>$G_{41}$ : Category two of SAS<br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of definition of a conscious agent<br>$T_{41}$ : Category two of SAS<br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br>**A strict experimental test of macroscopic realism in a superconducting flux qubit George C. Knee, Kosuke Kakuyanagi, Mao-Chuang Yeh, Yuichiro Matsuzaki, Hiraku Toida, Hiroshi Yamaguchi, Shiro Saito, Anthony J. Leggett, William J. Munro (Submitted on 14 Jan 2016 (v1), last revised 15 Jan 2016 (this version, v2))**<br>(1) Macroscopic realism is the name for a class of modifications to quantum theory that allow macroscopic objects to be described in (eb) a measurement-independent fashion, while largely preserving (eb) a fully quantum mechanical description of the microscopic world.<br>(2) Objective collapse theories are examples which attempt to provide (eb) a physical mechanism for wavefunction collapse, and thereby aim to solve (e) the quantum measurement problem.<br>(3) Here authors describe and implement an experimental protocol capable of constraining theories of (e) this class, and show how it is related to (e&eb) the original Leggett-Garg test, yet more noise tolerant and conceptually transparent.<br>(4) By conducting a set of simple 'prepare, shuffle, measure' tests in a superconducting flux qubit, authors rule out (by over 83 standard deviations) those theories which would deny (e) coherent superpositions of 170 nA currents over a 9 ns timescale.<br>(5) Further, they address the 'clumsiness loophole' by determining classical disturbance in a set of control experiments. Subjects: Quantum Physics (quant-ph) Cite as:arXiv:1601.03728 [quant-ph] | |

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| (or arXiv:1601.03728v2 [quant-ph] for this version)<br><br>Macroscopic realism is the name for a class of modifications to quantum theory that allow macroscopic objects to be described in (eb) a measurement-independent fashion, while largely preserving (eb) a fully quantum mechanical description of the microscopic world | |
| $G_{44}$ : Category one of Macroscopic realism is the name for a class of modifications to quantum theory that allow macroscopic objects to be described<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of measurement-independent fashion, while largely preserving (eb) a fully quantum mechanical description of the microscopic world<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION NINETEEN**<br><br>**Macroscopic Realism In A Superconducting Flux Qubit** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**A strict experimental test of macroscopic realism in a superconducting flux qubit George C. Knee, Kosuke Kakuyanagi, Mao-Chuang Yeh, Yuichiro Matsuzaki, Hiraku Toida, Hiroshi Yamaguchi, Shiro Saito, Anthony J. Leggett, William J. Munro (Submitted on 14 Jan 2016 (v1), last revised 15 Jan 2016 (this version, v2))**<br><br>(1) Objective collapse theories are examples which attempt to provide (eb) a physical mechanism for wavefunction collapse, and thereby aim to solve (e) the quantum measurement problem.<br>(2) Here authors describe and implement an experimental protocol capable of constraining theories of (e) this class, and show how it is related to (e&eb) the original Leggett-Garg test, yet more noise tolerant and conceptually transparent.<br>(3) By conducting a set of simple 'prepare, shuffle, measure' tests in a superconducting flux qubit, authors rule out (by over 83 standard deviations) those theories which would deny (e) coherent superpositions of 170 nA currents over a 9 ns timescale.<br>**(4)** Further, they address the 'clumsiness loophole' by determining classical disturbance in a set of control experiments. Subjects: Quantum Physics (quant-ph) Cite as:arXiv:1601.03728 [quant-ph]<br>(or arXiv:1601.03728v2 [quant-ph] for this version) | |

| NOTATION | |
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| **Module One**<br>Objective collapse theories are examples which attempt to provide (eb) a physical mechanism for wavefunction collapse, and thereby aim to solve (e) the quantum measurement problem | |
| $G_{13}$ : Category one of Objective collapse theories are examples which attempt<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of physical mechanism for wavefunction collapse, and thereby aim to solve (e) the quantum measurement problem<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>Objective collapse theories are examples which attempt to provide a physical mechanism for wavefunction collapse, and thereby aim to solve (e) the quantum measurement problem | |
| $G_{16}$ : Category one of the quantum measurement problem<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of Objective collapse theories are examples which attempt to provide a physical mechanism for wavefunction collapse, and thereby aim<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>Here authors describe and implement an experimental protocol capable of constraining theories of (e) this class, and show how it is related to (e&eb) the original Leggett-Garg test, yet more noise tolerant and conceptually transparent | |
| $G_{20}$ : Category one of class, and show how it is related to (e&eb) the original Leggett-Garg test, yet more noise tolerant and conceptually transparent<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of experimental protocol capable of constraining theories<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |

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| **Module four**<br><br>Here authors describe and implement an experimental protocol capable of constraining theories of this class, and show how it is related to (e&eb) the original Leggett-Garg test, yet more noise tolerant and conceptually transparent | |
| $G_{24}$ : Category one of **constraining theories of this class;** original Leggett-Garg test, yet more noise tolerant and conceptually transparent<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of original Leggett-Garg test, yet more noise tolerant and conceptually transparent; **constraining theories of this class**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>By conducting a set of simple 'prepare, shuffle, measure' tests in a superconducting flux qubit, authors rule out (by over 83 standard deviations) those theories which would deny (e) coherent superpositions of 170 nA currents over a 9 ns timescale. | |
| $G_{28}$ : Category one of set of simple 'prepare, shuffle, measure' tests in a superconducting flux qubit<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of rule out (by over 83 standard deviations) those theories which would deny (e) coherent superpositions of 170 nA currents over a 9 ns timescale.<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>By conducting a set of simple 'prepare, shuffle, measure' tests in a superconducting flux qubit, authors rule out (by over 83 standard deviations) those theories which would deny (e) coherent superpositions of 170 nA currents over a 9 ns timescale. | |
| $G_{32}$ : Category one of coherent superpositions of 170 nA currents over a 9 ns timescale.<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of simple 'prepare, shuffle, measure' tests in a superconducting flux qubit, authors rule | |

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| out (by over 83 standard deviations) those theories<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Further, they address the 'clumsiness loophole' by determining classical disturbance in a set of control experiments.<br><br>Subjects: Quantum Physics (quant-ph) Cite as:arXiv:1601.03728 [quant-ph] (or arXiv:1601.03728v2 [quant-ph] for this version) | |
| $G_{36}$ : Category one of **'clumsiness loophole';** determination of classical disturbance in a set of control experiments.<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of determination of classical disturbance in a set of control experiments; **'clumsiness loophole'**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>**Chapter The South China Sea and Asian Regionalism Volume 24 of the series Springer Briefs in Environment, Security, Development and Peace pp 15-25 Date: 15 January 2016 Critical Realism and the Morphogenetic Approach Thanh-Dam Truong , Karim Knio**<br><br>(1) This chapter argues that that Critical Realism, a philosophy of science, when applied in combination with (e&eb) the morphogenetic approach in historical and social research, can contribute (eb)to a deeper understanding of social transformation and help to disentangle the structure–agency relations in the maritime disputes in the South China Sea.<br>(2) The ways in which a society (or a group of people) understands maritime space, adopts practices of demarcating borders, and negotiates disputes, cannot be taken as given. Making them subject to historical and social analysis is both scientifically significant and politically relevant, especially with respect to the role of self-reflexivity in social science research into peaceful transformation. Keywords Critical Realism Morphogenetic approach Social transformation Structure–agency relations Self-reflexivity Maritime disputes | |
| $G_{40}$ : Category one of **Critical Realism;** morphogenetic approach in historical and social research, can contribute to a deeper understanding of social transformation and help to disentangle the structure–agency relations in the maritime disputes in the South China Sea<br><br>$G_{41}$ : Category two of SAS | |

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| $G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of morphogenetic approach in historical and social research, can contribute to a deeper understanding of social transformation and help to disentangle the structure–agency relations in the maritime disputes in the South China Sea;**Critical Realism**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Social transformation Structure–agency relations Self-reflexivity Maritime disputes | |
| $G_{44}$ : Category one of **Social transformation Structure;** agency relations Self-reflexivity Maritime disputes<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of agency relations Self-reflexivity Maritime disputes **;Social transformation Structure**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION NINETEEN**<br><br>**A Quantum Equivalence Principle**<br><br>**Heat As An Inertial Force** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Heat as an inertial force: A quantum equivalence principle K. Thanjavur, W. Israel (Submitted on 17 Jan 2016 (v1), last revised 31 Jan 2016 (this version, v2))**<br><br>(1) The firewall was introduced into (e*eb) black hole evaporation scenarios as a dues ex machina designed to break (e) entanglements and preserve unitarity (Almheiri et.al, 2013).<br>(2) Here authors show that the firewall actually exists and does break (e) entanglements, but only in the context of a virtual reality for (e) observers stationed near the horizon, which are following (e&eb)the long-term evolution of the hole.<br>(3) These observers are heated by (e) acceleration radiation at (eb) the Unruh temperature and see pair | |

creation at (eb) the horizon as a high-energy phenomenon.

(4) The objective reality is very different. Authors argue that Hawking pair creation is (=) entirely a low-energy process in (eb) which entanglements never arise.

(5) The Hawking particles materialize as (=) low-energy excitations with (e&eb) typical wavelength considerably larger than (e) the black hole radius.

**(6)** They thus emerge into (e&eb) a very non-uniform environment inimical to (e) entanglement-formation. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as:arXiv:1601.04319 [gr-qc] (or arXiv:1601.04319v2 [gr-qc] for this version)

**NOTATION**

**Module One**

The firewall was introduced into (e*eb) black hole evaporation scenarios as a dues ex machina designed to break (e) entanglements and preserve unitarity (Almheiri et.al, 2013)

$G_{13}$ : Category one of **firewall was introduced** black hole evaporation scenarios as a dues ex machina designed to break (e) entanglements and preserve unitarity (Almheiri et.al, 2013)**;**

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of black hole evaporation scenarios as a dues ex machina designed to break (e) entanglements and preserve unitarity (Almheiri et.al, 2013);**firewall was introduced**

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

The firewall was introduced into black hole evaporation scenarios as a dues ex machina designed to break (e) entanglements and preserve unitarity (Almheiri et.al, 2013)

$G_{16}$ : Category one of entanglements and preserve unitarity (Almheiri et.al, 2013)

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

$T_{16}$ : Category one of firewall was introduced into black hole evaporation scenarios as a dues ex machina designed

$T_{17}$ : Category two of SAS

$T_{18}$ : Category three of SAS

**Module three**

Here authors show that the firewall actually exists and does break (e) entanglements, but only in the context of a virtual reality for (e) observers stationed near the horizon, which are following (e&eb)the long-term evolution of the hole

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| $G_{20}$ : Category one of entanglements, but only in the context of a virtual reality for (e) observers stationed near the horizon, which are following (e&eb)the long-term evolution of the hole<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of firewall actually exists and does break<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Here authors show that the firewall actually exists and does break entanglements, but only in the context of a virtual reality for (e) observers stationed near the horizon, which are following (e&eb)the long-term evolution of the hole | |
| $G_{24}$ : Category one of observers stationed near the horizon, which are following (e&eb)the long-term evolution of the hole<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of firewall actually exists and does break entanglements, but only in the context of a virtual reality<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Here authors show that the firewall actually exists and does break entanglements, but only in the context of a virtual reality for observers stationed near the horizon, which are following (e&eb)the long-term evolution of the hole | |
| $G_{28}$ : Category one of **firewall actually exists and does break entanglements, but only in the context of a virtual reality for observers stationed near the horizon;** long-term evolution of the hole<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of long-term evolution of the hole; **firewall actually exists and does break entanglements, but only in the context of a virtual reality for observers stationed near the horizon**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |

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| **Module six**<br><br>These observers are heated by (e) acceleration radiation at (eb) the Unruh temperature and see pair creation at (eb) the horizon as a high-energy phenomenon | |
| $G_{32}$ : Category one of acceleration radiation at (eb) the Unruh temperature and see pair creation at (eb) the horizon as a high-energy phenomenon<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of observers are heated<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>These observers are heated by acceleration radiation at (eb) the Unruh temperature and see pair creation at (eb) the horizon as a high-energy phenomenon | |
| $G_{36}$ : Category one of observers are heated by acceleration radiation<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of Unruh temperature and see pair creation at (eb) the horizon as a high-energy phenomenon<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>These observers are heated by acceleration radiation at the Unruh temperature and see pair creation at (eb) the horizon as a high-energy phenomenon | |
| $G_{40}$ : Category one of observers are heated by acceleration radiation at the Unruh temperature and see pair creation<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of horizon as a high-energy phenomenon<br><br>$T_{41}$ : Category two of SAS | |

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| $T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The objective reality is very different<br><br>. Authors argue that Hawking pair creation is (=) entirely a low-energy process in (eb) which entanglements never arise | |
| $G_{44}$ : Category one of Hawking pair creation<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of entirely a low-energy process in (eb) which entanglements never arise<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION TWENTY TWO

## Amplification, Decoherence, and the Acquisition of Information

**INTRODUCTION—VARIABLES USED**

**Amplification, Decoherence, and the Acquisition of Information by Spin Environments Michael Zwolak, C. Jess Riedel, Wojciech H. Zurek (Submitted on 7 Mar 2016)**

(1) Quantum Darwinism recognizes the role of the environment as (=) a communication channel: Decoherence can selectively amplify (eb) information about the pointer states of (e) a system of interest (preventing access to complementary information about (e&eb)their superpositions) and can make (eb)records of this information accessible to (e) many observers.
(2) This redundancy explains (eb) the emergence of objective, classical reality in our quantum Universe.
(3) Here, authors demonstrate that the amplification of information in realistic spin environments can be quantified by (e) the quantum Chernoff information, which characterizes (eb) the distinguish ability of partial records in (eb) individual environment subsystems.
(4) Authors show that, except for (e) a set of initial states of (e) measure zero, the environment always acquires (e) redundant information.
(5) Moreover, the Chernoff information captures (e) the rich behavior of amplification in (eb) both finite and infinite spin environments, from quadratic growth of the redundancy to (e&eb) oscillatory behavior.

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| **(6)** These results will considerably simplify experimental testing of quantum Darwinism, e.g., using (e) nitrogen vacancies in diamond. Subjects: Quantum Physics (quant-ph) Report number:LAUR-14-25093 Cite as:arXiv: 1603.01916 [quant-ph] (or arXiv:1603.01916v1 [quant-ph] for this version) | |
| **NOTATION** | |
| **Module One**<br>Authors argue that Hawking pair creation is entirely a low-energy process in (eb) which entanglements never arise | |
| $G_{13}$ : Category one of Hawking pair creation is entirely a low-energy process<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of entanglements never arise<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>The Hawking particles materialize as (=) low-energy excitations with (e&eb) typical wavelength considerably larger than (e) the black hole radius | |
| $G_{16}$ : Category one of Hawking particles materialize<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of low-energy excitations with (e&eb) typical wavelength considerably larger than (e) the black hole radius<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>The Hawking particles materialize as low-energy excitations with (e&eb) typical wavelength considerably larger than (e) the black hole radius | |
| $G_{20}$ : Category one of **Hawking particles materialize as low-energy excitations;** typical wavelength considerably larger than (e) the black hole radius<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of typical wavelength considerably larger than (e) the black hole radius **;Hawking particles materialize as low-energy excitations** | |

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| $T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>The Hawking particles materialize as low-energy excitations with typical wavelength considerably larger than (e) the black hole radius | |
| $G_{24}$ : Category one of black hole radius<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of Hawking particles materialize as low-energy excitations with typical wavelength considerably larger<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>Quantum Darwinism recognizes the role of the environment as (=) a communication channel:<br>Decoherence can selectively amplify (eb) information about the pointer states of (e) a system of interest (preventing access to complementary information about (e&eb)their superpositions) and can make (eb)records of this information accessible to (e) many observers | |
| $G_{28}$ : Category one of Quantum Darwinism recognizes the role of the environment<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of communication channel<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>Decoherence can selectively amplify (eb) information about the pointer states of (e) a system of interest (preventing access to complementary information about (e&eb)their superpositions) and can make (eb)records of this information accessible to (e) many observers | |
| $G_{32}$ : Category one of Decoherence can selectively amplify<br>$G_{33}$ : Category two of SAS | |

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| $G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one ofinformation about the pointer states of (e) a system of interest (preventing access to complementary information about (e&eb)their superpositions) and can make (eb)records of this information accessible to (e) many observers<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Decoherence can selectively amplify information about the pointer states of a system of interest (preventing access to complementary information about (e&eb)their superpositions) and can make (eb)records of this information accessible to (e) many observers | |
| $G_{36}$ : Category one of Decoherence can selectively amplify information about the pointer states of a system of interest (preventing access to complementary information; their superpositions) and can make (eb)records of this information accessible to (e) many observers<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of their superpositions) and can make (eb)records of this information accessible to (e) many observers ;Decoherence can selectively amplify information about the pointer states of a system of interest (preventing access to complementary information<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Decoherence can selectively amplify information about the pointer states of a system of interest (preventing access to complementary information about their superpositions) and can make (eb)records of this information accessible to (e) many observers | |
| $G_{40}$ : Category one of Decoherence can selectively amplify information about the pointer states of a system of interest (preventing access to complementary information about their superpositions) and can make<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of information accessible to (e) many observers<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |

| **Module Nine**<br><br>Authors show that, except for (e) a set of initial states of (e) measure zero, the environment always acquires (e) redundant information | |
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| $G_{44}$ : Category one of measure zero, the environment always acquires (e) redundant information<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of set of initial states<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION TWENTY ONE

## When You Do A Split-Brain Operation, A Complete Transection Of The Corpus Callosum, You Get Clear Evidence Of Two Separate Consciousnesses

**INTRODUCTION—VARIABLES USED**

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) This mathematics is telling me something. I can take two minds, and they can generate (eb) a new, unified single mind.
(2) Here's a concrete example. We have two hemispheres in (eb) our brain.
(3) But when you do a split-brain operation, a complete transection of the corpus callosum, you get (eb) clear evidence of two separate consciousnesses.
(4) So it's not implausible that there is (=) a single conscious agent.
(5) And yet it's (=) also the case that there are two conscious agents there, and you can see that when (e) they're split.
(6) I didn't expect that, the mathematics forced me to recognize this. It suggests (eb) that I can take separate observers, put them together and create (eb) new observers, and keep doing this ad infinitum
(7) It's (=) conscious agents all the way down.
(8) If it's (=) conscious agents all the way down, all first-person points of view, what happens to science? Science has always been a (=) third-person description of the world.

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| **(9)** The idea that what we're doing is (=) measuring publicly accessible objects, the idea that objectivity results from (e) the fact that you and I can measure the same object in the exact same situation and get the same results — it's (=) very clear from quantum mechanics that that idea has to go. | |
| **NOTATION** | |
| **Module One**<br>This mathematics is telling me something.<br>I can take two minds, and they can new, unified single mind generate (eb) a new, unified single mind | |
| $G_{13}$ : Category one of I can take two minds, and they can generate<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of new, unified single mind<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>Here's a concrete example<br>. We have two hemispheres in (eb) our brain \ | |
| $G_{16}$ : Category one of We have two hemispheres<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of our brain **\e Two**<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>Here's a concrete example.<br>We have two hemispheres in (eb) our brain.<br>But when you do a split-brain operation, a complete transection of the corpus callosum, you get (eb) clear evidence of two separate consciousnesses | |
| $G_{20}$ : Category one of) clear evidence of two separate consciousnesses<br>$G_{21}$ : Category two of SAS | |

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| $G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of But when you do a split-brain operation, a complete transection of the corpus callosum<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>But when you do a split-brain operation, a complete transection of the corpus callosum, you get (eb) clear evidence of two separate consciousnesses | |
| $G_{24}$ : Category one of you do a split-brain operation, a complete transection of the corpus callosum<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of clear evidence of two separate consciousnesses<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Before that slicing happened, it seemed there was a single unified consciousness | |
| $G_{28}$ : Category one of<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of it seemed there was a single unified consciousness<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>So it's not implausible that there is (=) a single conscious agent | |
| $G_{32}$ : Category one of it's not implausible<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of single conscious agent | |

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| $T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>And yet it's also the case that there are two conscious agents there, and you can see that when (e) they're split | |
| $G_{36}$ : Category one of yet it's also the case that there are two conscious agents<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of you can see that when (e) they're split<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>And yet it's also the case that there are two conscious agents there, and you can see that when (e) they're split | |
| $G_{40}$ : Category one of they're split<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of yet it's also the case that there are two conscious agents there, and you can see<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The idea that what we're doing is (=) measuring publicly accessible objects, the idea that objectivity results from (e) the fact that you and I can measure the same object in the exact same situation and get the same results — it's (=) very clear from quantum mechanics that that idea has to go. | |
| $G_{44}$ : Category one of objective reality<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |

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| $T_{44}$ : Category one of fact that you and I can measure the same object in the exact same situation and get the same results<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION TWENTY TWO

## Physics Tells Us That There Are No Public Physical Objects

### INTRODUCTION—VARIABLES USED

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) Physics tells us that there are (=) no public physical objects.
(2) So what's going on? Here's how I think about it. I can talk to you about my headache and believe that I am communicating (e&eb) effectively with you, because (e) you've had your own headaches.
(3) The same thing is (=) true as apples and the moon and the sun and the universe.
(4) Just like you have your own headache, you have (e) your own moon.
(5) But I assume it's (=) relevantly similar to mine.
(6) That's an assumption that could be (=) false, but that's (=) the source of my communication, and that's the best we can do in terms of (e&eb) public physical objects and (e&eb) objective science.
(7) It doesn't seem like many people in neuroscience or philosophy of mind are thinking about fundamental physics. Do you think that's been a stumbling block for (e) those trying to understand consciousness? I think it has been (=) a stumbling block
**(8)** Not only are they ignoring the progress in fundamental physics, they are often explicit about it. They'll say openly that quantum physics is not relevant to (e) the aspects of brain function that are causally involved (e&eb) in consciousness.

### NOTATION

**Module One**

Physics tells us that there are (=) no public physical objects

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| $G_{13}$ : Category one of Physics tells us that there are<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of no public physical objects | |

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| $T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>So what's going on?<br>Here's how I think about it. I can talk to you about my headache and believe that I am communicating effectively with you, because (e) you've had your own headaches | |
| $G_{16}$ : Category one of you've had your own headaches<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of **I can talk to you about my headache and believe that I am communicating effectively with you**<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>The same thing is (=) true as apples and the moon and the sun and the universe | |
| $G_{20}$ : Category one of same thing<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of true as apples and the moon and the sun and the universe<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>Just like you have your own headache, you have your own moon | |
| $G_{24}$ : Category one of you have your own headache<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of you have your own moon<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |

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| **Module five**<br><br>But I assume it's (=) relevantly similar to mine. | |
| $G_{28}$ : Category one of assumption<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of relevantly similar to mine<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>That's an assumption that could be false, but that's the source of my communication, and that's the best we can do in terms of (e&eb) public physical objects and (e&eb) objective science | |
| $G_{32}$ : Category one of assumption that could be false<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of that's the source of my communication, and that's the best we can do in terms of (e&eb) public physical objects and (e&eb) objective science<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>That's an assumption that could be false, but that's the source of my communication, and that's the best we can do in terms of (e&eb) public physical objects and (e&eb) objective science | |
| $G_{36}$ : Category one of **That's an assumption that could be false, but that's the source of my communication, and that's the best we can do;** public physical objects and (e&eb) objective science<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of public physical objects and (e&eb) objective science **;That's an assumption that could be false, but that's the source of my communication, and that's the best we can do**<br><br>$T_{37}$ : Category two of SAS | |

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| $T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Non existent public physical objects and (e&eb) objective science | |
| $G_{40}$ : Category one of **Non existent public physical objects;** objective science<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of objective science; **Non existent public physical objects**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Not only are they ignoring the progress in fundamental physics, they are often explicit about it. They'll say openly that quantum physics is not relevant to the aspects of brain function that are causally involved (e&eb) in consciousness | |
| $G_{44}$ : Category one of **quantum physics is not relevant to the aspects of brain function that are causally involved;** consciousness<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of consciousness; **quantum physics is not relevant to the aspects of brain function that are causally involved**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

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| **SECTION TWENTY THREE**<br><br>**Neurons, Brains, Space … These Are Just Symbols We Use, They're Not Real** | |
| **INTRODUCTION—VARIABLES USED**<br><br>**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref:** | |

**arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) They are certain that it's got to be (=) classical properties of neural activity, which exist independent of (e) any observers — spiking rates, connection strengths at synapses, perhaps dynamical properties as well.
(2) These are (=) all very classical notions under Newtonian physics, where (e) time is absolute and objects exist absolutely
(3) And then [neuroscientists] are mystified as to why they don't make progress. They don't avail themselves of the incredible insights and breakthroughs that physics has made. Those insights are out there for us to use, and yet my field says, "We'll stick with Newton, thank you. We'll stay 300 years behind in our physics."
(4) I suspect they're reacting to (e) things like Roger Penrose and Stuart Hameroff model, where (e) you still have a physical brain, it's (=) still sitting in space, but supposedly it's (=) performing some quantum feat.
(5) In contrast, you're saying, "Look, quantum mechanics is telling us that we have (e) to question the very notions of 'physical things' sitting in (eb) 'space.'"
(6) I think that's (=) absolutely true.
(7) The neuroscientists are saying, "We don't need to invoke (e) those kinds of quantum processes, we don't (e) need quantum wave functions collapsing (eb) inside neurons, we can just use (e) classical physics to describe (eb) processes in the brain."
**(8)** I'm emphasizing the larger lesson of quantum mechanics: **Neurons, brains, space … these are (=) just symbols we use, they're (=) not real.**
**(9)** It's not that there's (=) a classical brain that does (eb) some quantum magic.

**NOTATION**

**Module One**

They are certain that it's got to be (=) classical properties of neural activity, which exist independent of (e) any observers — spiking rates, connection strengths at synapses, perhaps dynamical properties as well

$G_{13}$ : Category one of brain activities

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of classical properties of neural activity, which exist independent of (e) any observers — spiking rates, connection strengths at synapses, perhaps dynamical properties as well

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

They are certain that it's

Brain activities and consciousness activities got to be classical properties of neural activity, which exist independent of (e) any observers — spiking rates, connection strengths at synapses, perhaps dynamical properties as well

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| $G_{16}$ : Category one of observers — spiking rates, connection strengths at synapses, perhaps dynamical properties as well<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of Brain activities and consciousness activities got to be classical properties of neural activity, which exist independent<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>I suspect they're reacting to (e) things like Roger Penrose and Stuart Hameroff model, where (e) you still have a physical brain, it's (=) still sitting in space, but supposedly it's (=) performing some quantum feat | |
| $G_{20}$ : Category one of physical brain, it's (=) still sitting in space, but supposedly it's (=) performing some quantum feat<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of Roger Penrose and Stuart Hameroff model<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four** | |
| $G_{24}$ : Category one of physical brain; quantum feat<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of quantum feat; physical brain<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>In contrast, you're saying, "Look, quantum mechanics is telling us that we have (e) to question the very notions of 'physical things' sitting in (eb) 'space.' | |
| $G_{28}$ : Category one of physical things in space<br><br>$G_{29}$ : Category two of SAS | |

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| $G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of quantum mechanics<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six** | |
| $G_{32}$ : Category one of space<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of quantum mechanics<br>$T_{33}$ : Category two of<br>$T_{34}$ : Category three of | |
| **Module seven** | |
| $G_{36}$ : Category one of time in space<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of quantum mechanics<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>The neuroscientists are saying, "We don't need to invoke (e) those kinds of quantum processes, we don't (e) need quantum wave functions collapsing (eb) inside neurons, we can just use (e) classical physics to describe (eb) processes in the brain." | |
| $G_{40}$ : Category one of quantum wave functions collapsing inside neurons; processes in brain<br>$G_{41}$ : Category two of SAS<br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of processes in brain; quantum wave functions collapsing inside neurons<br>$T_{41}$ : Category two of SAS | |

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| $T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>I'm emphasizing the larger lesson of quantum mechanics:<br><br>**Neurons, brains, space … these are (=) just symbols we use, they're (=) not real** | |
| $G_{44}$ : Category one of **Neurons, brains, space**<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of **not real, just symbols we use**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION TWENTY FOUR

## It's That There's No Brain!

### INTRODUCTION—VARIABLES USED

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) It's that there's (=) no brain!
(2) Quantum mechanics says that classical objects — including brains — don't (e) exist.
(3) So this is a far more radical claim about (e&eb) the nature of reality and does not (e) involve the brain pulling off (eb) some tricky quantum computation.
(4) So even Penrose hasn't taken it far enough. But most of us, you know, we're (=) born realists. We're (=) born physicalists. This is a really, really hard one to let go of.
(5) To return to the question you started with as a teenager, are we (=) machines?
(6) The formal theory of conscious agents I've been developing is (=) computationally universal — in that sense, it's (=) a machine theory.
(7) And it's because (e) the theory is computationally universal that I can get all of (e) cognitive science and neural networks back out of it.
(8) Nevertheless, for now I don't think we are (=) machines — in part because (e) I distinguish between the mathematical representation and (e&eb) the thing being represented.

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| **(9)** As a conscious realist, I am postulating conscious experiences as (=) ontological primitives, the most basic ingredients of the world. | |
| **NOTATION** | |
| **Module One**<br>It's that there's (=) no brain | |
| $G_{13}$ : Category one of existence<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of brain<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>Quantum mechanics says that classical objects — including brains — don't (e) exist | |
| $G_{16}$ : Category one of existence<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of Quantum mechanics says that classical objects — including brains<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>So this is a far more radical claim about (e&eb) the nature of reality and does not (e) involve the brain pulling off (eb) some tricky quantum computation | |
| $G_{20}$ : Category one of quantum computation<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of brain<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>So even Penrose hasn't taken it far enough. | |

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| But most of us, you know, we're (=) born realists. We're (=) born physicalists<br><br>. This is a really, really hard one to let go of. | |
| $G_{24}$ : Category one of we<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of realists<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five** | |
| $G_{28}$ : Category one of we<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of physicalists<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>To return to the question you started with as a teenager, are we (=) machines | |
| $G_{32}$ : Category one of we<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of machines<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>The formal theory of conscious agents I've been developing is (=) computationally universal — in that sense, it's (=) a machine theory<br><br>And it's because (e) the theory is computationally universal that I can get all of (e) cognitive science and | |

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| neural networks back out of it. | |
| $G_{36}$ : Category one of theory is computationally universal<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of we can get all of cognitive science and neural networks back out of it.<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>Nevertheless, for now I don't think we are (=) machines — in part because (e) I distinguish between the mathematical representation and (e&eb) the thing being represented | |
| $G_{40}$ : Category one of **mathematical representation;** thing being represented<br>$G_{41}$ : Category two of SAS<br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of thing being represented; **mathematical representation**<br>$T_{41}$ : Category two of SAS<br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br>As a conscious realist, I am postulating conscious experiences as (=) ontological primitives, the most basic ingredients of the world | |
| $G_{44}$ : Category one of conscious experiences<br>$G_{45}$ : Category two of SAS<br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of ontological primitives, the most basic ingredients of the world<br>$T_{45}$ : Category two of SAS<br>$T_{46}$ : Category three of SAS | |

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| **SECTION TWENTY FIVE** | |

## Simulating Physics On Such A Fundamental Level Is More Or Less Equivalent To Simulating The Universe Itself.

### INTRODUCTION—VARIABLES USED

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) I'm claiming that experiences are (=) the real coin of the realm.
(2) The experiences of everyday life — my real feeling of (e) a headache, my real taste of (e) chocolate — that really is (=) the ultimate nature of reality.
(3) One of modern physics' most cherished ideas is (=) quantum chromodynamics, the theory that describes (eb) the strong nuclear force, how it binds (e&eb) quarks and gluons into protons and neutrons, how these form (eb) nuclei those themselves interact.
(4) This is (=) the universe at its most fundamental.
(5) So an interesting pursuit is to simulate (e&eb) quantum chromodynamics on a computer to (e) see what kind of complexity arises.
(6) The promise is that simulating physics on (e&eb) such a fundamental level is more or less equivalent to (=) simulating the universe itself.
(7) There are one or two challenges of course. The physics is mind-bogglingly complex and operates on (e&eb) a vanishingly small scale.
(8) So even using (e) the world's most powerful supercomputers, physicists have (e) only managed to simulate (e&eb) tiny corners of the cosmos just a few femtometers across. (A femtometer is 10^-15 metres.)
**(9)** That may not sound like much but the significant point is that the simulation is (=) essentially indistinguishable from the real thing (at least as far as we understand it). This is true. Once I saw a mob being chased by a sub inspector and the immediate conclusion, for having been living in a hostile environment is that police are behind me and some people are behind police who are helping me. But one thing was certain. Some people are out to get the blood out of me. If I choose to follow those who made sub inspector wielding stick to petrify mob then I am safe! This is one of the mental games some people play. There are there everywhere and they use it to their advantage

### NOTATION

**Module One**

I'm claiming that experiences are (=) the real coin of the realm

$G_{13}$ : Category one of experiences

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of real coin of the realm

$T_{14}$ : Category two of SAS

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| $T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>The experiences of everyday life — my real feeling of (e) a headache, my real taste of (e) chocolate — that really is (=) the ultimate nature of reality | |
| $G_{16}$ : Category one of experiences of everyday life — my real feeling of (e) a headache, my real taste of (e) chocolate — that really<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of ultimate nature of reality<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>One of modern physics' most cherished ideas is (=) quantum chromodynamics, the theory that describes (eb) the strong nuclear force, how it binds (e&eb) quarks and gluons into protons and neutrons, how these form (eb) nuclei those themselves interact | |
| $G_{20}$ : Category one of **strong nuclear force;** quarks and gluons into (e&eb) protons and neutrons, how these form (eb) nuclei those themselves interact<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of quarks and gluons into (e&eb) protons and neutrons, how these form (eb) nuclei those themselves interact; **strong nuclear force**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>quarks and gluons into (e&eb) protons and neutrons, how these form (eb) nuclei those themselves interact | |
| $G_{24}$ : Category one of **quarks and gluons;** protons and neutrons<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of protons and neutrons **;quarks and gluons**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |

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| **Module five**<br><br>The promise is that simulating physics on ) such a fundamental level is more or less equivalent to (=) simulating the universe itself | |
| $G_{28}$ : Category one of simulating physics on such a fundamental level<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of simulating the universe itself<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>**Published online: 2 March 2009 \| doi: 10.1038/nphys1202 Subject Category: Quantum physics Quantum Darwinism Wojciech Hubert Zurek**<br><br>(1) Quantum Darwinism describes (eb) the proliferation, in the environment, of multiple records of (e) selected states of a quantum system.<br>(2) It explains how the quantum fragility of a state of a single quantum system can lead to (eb) the classical robustness of states in their correlated multitude;<br>(3) shows how effective 'wave-packet collapse' arises as a result of (e) the proliferation throughout the environment of imprints of the state of the system; and provides (eb) a framework for the derivation of (e) Born's rule, which relates the probabilities of detecting states to (e&eb) their amplitudes.<br>(4) Taken together, these three advances mark considerable progress towards settling the quantum measurement problem.<br><br>Quantum Darwinism describes (eb) the proliferation, in the environment, of multiple records of (e) selected states of a quantum system | |
| $G_{32}$ : Category one of Quantum Darwinism<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of proliferation, in the environment, of multiple records of (e) selected states of a quantum system<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Quantum Darwinism describes the proliferation, in the environment, of multiple records of (e) selected states | |

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| of a quantum system | |
| $G_{36}$ : Category one of selected states of a quantum system<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of Quantum Darwinism describes the proliferation, in the environment, of multiple records<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>It explains how the quantum fragility of a state of a single quantum system can lead to (eb) the classical robustness of states in their correlated multitude | |
| $G_{40}$ : Category one of quantum fragility of a state of a single quantum system<br>$G_{41}$ : Category two of SAS<br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of classical robustness of states in their correlated multitude<br>$T_{41}$ : Category two of SAS<br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br>shows how effective 'wave-packet collapse' arises as a result of (e) the proliferation throughout the environment of imprints of the state of the system; and provides (eb) a framework for the derivation of (e) Born's rule, which relates the probabilities of detecting states to (e&eb) their amplitudes | |
| $G_{44}$ : Category one ofproliferation throughout the environment of imprints of the state of the system; and provides (eb) a framework for the derivation of (e) Born's rule, which relates the probabilities of detecting states to (e&eb) their amplitudes<br>$G_{45}$ : Category two of SAS<br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of effective 'wave-packet collapse'<br>$T_{45}$ : Category two of SAS<br>$T_{46}$ : Category three of SAS | |

## SECTION TWENTY SIX

## Quantum Darwinism

**INTRODUCTION—VARIABLES USED**

**NOTATION**

**Module One**

shows how effective 'wave-packet collapse' arises as a result of the proliferation throughout the environment of imprints of the state of the system; and provides (eb) a framework for the derivation of (e) Born's rule, which relates the probabilities of detecting states to (e&eb) their amplitudes

$G_{13}$ : Category one of'wave-packet collapse' arises as a result of the proliferation throughout the environment of imprints of the state of the system

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of framework for the derivation of (e) Born's rule, which relates the probabilities of detecting states to (e&eb) their amplitudes

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

shows how effective 'wave-packet collapse' arises as a result of the proliferation throughout the environment of imprints of the state of the system; and provides a framework for the derivation of (e) Born's rule, which relates the probabilities of detecting states to (e&eb) their amplitudes

$G_{16}$ : Category one of Born's rule, which relates the probabilities of detecting states to (e&eb) their amplitudes

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

$T_{16}$ : Category one of effective 'wave-packet collapse' arises as a result of the proliferation throughout the environment of imprints of the state of the system; and provides a framework for the derivation

$T_{17}$ : Category two of SAS

$T_{18}$ : Category three of SAS

**Module three**

Quantum Darwinism shows how effective 'wave-packet collapse' arises as a result of the proliferation throughout the environment of imprints of the state of the system; and provides a framework for the

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| derivation of Born's rule, which relates the probabilities of detecting states to their amplitudes | |
| $G_{20}$ : Category one of **Quantum Darwinism shows how effective 'wave-packet collapse' arises as a result of the proliferation throughout the environment of imprints of the state of the system; and provides a framework for the derivation of Born's rule, which relates the probabilities of detecting states;** their amplitudes<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of their amplitudes **;Quantum Darwinism shows how effective 'wave-packet collapse' arises as a result of the proliferation throughout the environment of imprints of the state of the system; and provides a framework for the derivation of Born's rule, which relates the probabilities of detecting states**<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four** | |
| $G_{24}$ : Category one of quantum measurement problem<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of Quantum Darwinism shows how effective 'wave-packet collapse' arises as a result of the proliferation throughout the environment of imprints of the state of the system; and provides a framework for the derivation of Born's rule, which relates the probabilities of detecting states to their amplitudes<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>$\sigma_x \sigma_p \geq \frac{\hbar}{2}$<br>Read the dove tailing explanation with greater than or equal to symbol. | |
| $G_{28}$ : Category one of LHS of the above equation<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of<br>$T_{29}$ : Category two of SAS | |

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| $T_{30}$ : Category three of SAS | |
| **Module six"** $$\hat{H}\|\psi_n(t)\rangle = i\hbar\frac{\partial}{\partial t}\|\psi_n(t)\rangle$$ | |
| $G_{32}$ : Category one of LHS of the above equation<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of RHS iof the above equation (RHS gobbles up LHS)<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven** $$\left(\beta mc^2 + c\left(\sum_{n=1}^{3}\alpha_n p_n\right)\right)\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t}$$ | |
| $G_{36}$ : Category one of LHS of the above equation<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of RHS iof the above equation (RHS gobbles up LHS)<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>**Quantum Darwinism from Wikipedia, the free encyclopedia**<br>(1) Quantum Darwinism is a theory claiming to explain the emergence of the classical world from (e) the quantum world as due to (e) a process of Darwinian natural selection; where (e) the many possible quantum states are selected against in favor of (e) a stable pointer state.<br>(2) It was proposed in 2003 by Wojciech Zurek and a group of collaborators including Ollivier, Poulin, Paz and Blume-Kohout. The development of the theory is due to (e) the integration of a number of Zurek's research topics pursued over the course of twenty-five years including: pointer states, einselection and decoherence.<br>(3) A study in 2010 is claimed to provide preliminary supporting evidence of quantum Darwinism with scars of a quantum dot "becoming (=) a family of mother-daughter states" indicating (eb) they | |

could "stabilize into multiple pointer states."[1]

(4) However, the claimed evidence is also subject to (e) the circularity criticism by Kastner (see Implications below). Basically, the de facto phenomenon of decoherence that underlies the claims of Quantum Darwinism may not really arise in a unitary-only dynamics. Thus, even if there is decoherence, this does not show that macroscopic pointer states naturally emerge without (e) some form of collapse

$G_{40}$ : Category one of quantum world as due to (e) a process of Darwinian natural selection; where (e) the many possible quantum states are selected against in favor of (e) a stable pointer state

$G_{41}$ : Category two of SAS

$G_{42}$ : Category three of SAS

$T_{40}$ : Category one of emergence of the classical world

$T_{41}$ : Category two of SAS

$T_{42}$ : Category three of SAS

**Module Nine**

Quantum Darwinism is a theory claiming to explain the emergence of the classical world from the quantum world as due to (e) a process of Darwinian natural selection; where (e) the many possible quantum states are selected against in favor of (e) a stable pointer state

$G_{44}$ : Category one of process of Darwinian natural selection

$G_{45}$ : Category two of SAS

$G_{46}$ : Category three of SAS

$T_{44}$ : Category one of emergence of the classical world from the quantum world

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

## SECTION TWENTY SEVEN

## Quantum States Are Selected Against In Favor Of A Stable Pointer State

**INTRODUCTION—VARIABLES USED**

**Quantum Darwinism from Wikipedia, the free encyclopedia: For references please see Wikipedia**

(1) Quantum Darwinism is a theory claiming to explain the emergence of the classical world from the

quantum world as due to a process of Darwinian natural selection; where (e) the many possible quantum states are selected against in favor of (e) a stable pointer state.

(2) It was proposed in 2003 by Wojciech Zurek and a group of collaborators including Ollivier, Poulin, Paz and Blume-Kohout. The development of the theory is due to (e) the integration of a number of Zurek's research topics pursued over the course of twenty-five years including: pointer states, einselection and decoherence.

**(3)** A study in 2010 is claimed to provide preliminary supporting evidence of quantum Darwinism with scars of a quantum dot "becoming (=) a family of mother-daughter states" indicating (eb) they could "stabilize into multiple pointer states."[1]

(4) However, the claimed evidence is also subject to (e) the circularity criticism by Kastner (see Implications below). Basically, the de facto phenomenon of decoherence that underlies the claims of Quantum Darwinism may not really arise in a unitary-only dynamics. Thus, even if there is decoherence, this does not show that macroscopic pointer states naturally emerge without (e) some form of collapse

**(5)** Along with Zurek's related theory of envariance, quantum Darwinism seeks to explain how the classical world emerges from the quantum world and proposes to answer the quantum measurement problem, the main interpretational challenge for quantum theory. The measurement problem arises because the quantum state vector, the source of all knowledge concerning quantum systems, evolves according to the Schrödinger equation into a linear superposition of different states, predicting paradoxical situations such as "Schrödinger's cat"; situations never experienced in our classical world. Quantum theory has traditionally treated this problem as being resolved by a non-unitary transformation of the state vector at the time of measurement into a definite state. It provides an extremely accurate means of predicting the value of the definite state that will be measured in the form of a probability for each possible measurement value. The physical nature of the transition from the quantum superposition of states to the definite classical state measured is not explained by the traditional theory but is usually assumed as an axiom and was at the basis of the debate between Bohr and Einstein concerning the completeness of quantum theory.

**(6)** Quantum Darwinism seeks to explain the transition of quantum systems from the vast potentiality of superposed states to the greatly reduced set of pointer states [2] as a selection process, einselection, imposed on the quantum system through its continuous interactions with the environment. All quantum interactions, including measurements, but much more typically interactions with the environment such as with the sea of photons in which all quantum systems are immersed, lead to decoherence or the manifestation of the quantum system in a particular basis dictated by the nature of the interaction in which the quantum system is involved. In the case of interactions with its environment Zurek and his collaborators have shown that a preferred basis into which a quantum system will decohere is the pointer basis underlying predictable classical states. It is in this sense that the pointer states of classical reality are selected from quantum reality and exist in the macroscopic realm in a state able to undergo further evolution. However, the 'einselection' program depends on assuming a particular division of the universal quantum state into 'system' + 'environment', with the different degrees of freedom of the environment posited as having mutually random phases. This phase randomness does not arise from within the quantum state of the universe on its own, and Kastner [3] has pointed out that this limits the explanatory power of the Quantum Darwinism program.

(7) As a quantum system's interactions with its environment results in the (eb) recording of many redundant copies of information regarding its pointer states, this information is available to numerous observers able to achieve consensual agreement concerning their information of the quantum state. This aspect of einselection, called by Zurek 'Environment as a Witness', results in

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| the potential for objective knowledge. | |
| **NOTATION** | |
| **Module One** | |
| $G_{13}$ : Category one of many possible quantum states is selected against in favor of (e) a stable pointer state.<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of emergence of the classical world from the quantum world as due to a process of Darwinian natural selection<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>many possible quantum states are selected against in favor of (e) a stable pointer state | |
| $G_{16}$ : Category one of stable pointer state<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of many possible quantum states are selected against in favor<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>pointer states, einselection and **decoherence** | |
| $G_{20}$ : Category one of **pointer states;** einselection<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of einselection **;pointer states**<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four** | |
| $G_{24}$ : Category one of einselection; **decoherence** | |

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| $G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of **decoherence** ;einselection<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five** | |
| $G_{28}$ : Category one of **decoherence;** pointer states<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of pointer states**; decoherence**<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>As a quantum system's interactions with its environment results in the (eb) recording of many redundant copies of information regarding its pointer states, this information is available to numerous observers able to achieve consensual agreement concerning their information of the quantum state. This aspect of einselection, called by Zurek 'Environment as a Witness', results in the potential for objective knowledge | |
| $G_{32}$ : Category one of quantum system's interactions with its environment<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one ofrecording of many redundant copies of information regarding its pointer states, this information is available to numerous observers able to achieve consensual agreement concerning their information of the quantum state<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br>**Pauli's equation: Wikipedia**<br>$$\left[\frac{1}{2m}(\boldsymbol{\sigma}\cdot(\mathbf{p}-q\mathbf{A}))^2+q\phi\right]\vert\psi\rangle = i\hbar\frac{\partial}{\partial t}\vert\psi\rangle$$ | |

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| $G_{36}$ : Category one of LHS of the above equation. Take term by term and apply the model after classification as is being done now.<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of RHS of the above equation. Take term by term and apply the model after classification as is being done now.<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>**Klieon Gordon Equation: Wikipedia**<br><br>$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0.$$ | |
| $G_{40}$ : Category one of LHS of the above equation. Take term by term and apply the model after classification as is being done now.<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of RHS of the above equation. Take term by term and apply the model after classification as is being done now.<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>**Schrodinger's equation: Wikipedia**<br><br>$$E\Psi(\mathbf{r}) = \left[\frac{-\hbar^2}{2\mu}\nabla^2 + V(\mathbf{r})\right]\Psi(\mathbf{r})$$ | |
| $G_{44}$ : Category one of LHS of the above equation. Take term by term and apply the model after classification as is being done now.<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |

$T_{44}$ : Category one of RHS of the above equation. Take term by term and apply the model after classification as is being done now.

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

## SECTION TWENTY EIGHT

## Pointer States, Einselection And Decoherence

### INTRODUCTION—VARIABLES USED

**Quantum Darwinism from Wikipedia, the free encyclopedia: For references please see Wikipedia**

(1) It was proposed in 2003 by Wojciech Zurek and a group of collaborators including Ollivier, Poulin, Paz and Blume-Kohout. The development of the theory is due to (e) the integration of a number of Zurek's research topics pursued over the course of twenty-five years including: pointer states, einselection and decoherence.

**(2)** A study in 2010 is claimed to provide preliminary supporting evidence of quantum Darwinism with scars of a quantum dot "becoming (=) a family of mother-daughter states" indicating (eb) they could "stabilize into multiple pointer states."[1]

(3) However, the claimed evidence is also subject to (e) the circularity criticism by Kastner (see Implications below).

(4) Basically, the de facto phenomenon of decoherence that underlies the claims of Quantum Darwinism may not really arise in (eb) a unitary-only dynamics.

(5) Thus, even if there is decoherence, (eb) this does not show that macroscopic pointer states naturally emerge without (e) some form of collapse

**(6)** Along with Zurek's related theory of (e) envariance, quantum Darwinism seeks to explain (eb) how the classical world emerges from (e) the quantum world and proposes to answer the quantum measurement problem, the main interpretational challenge for quantum theory.

**(7)** The measurement problem arises because (e) the quantum state vector, the source of all knowledge concerning (e&eb) quantum systems, evolves according to (e&eb) the Schrödinger equation into a linear superposition of different states, predicting (eb) paradoxical situations such as "Schrödinger's cat"; situations never experienced in our classical world.

**(8)** Quantum theory has traditionally treated this problem as being resolved by (e) a non-unitary transformation of the state vector at the time of measurement into (e&eb) a definite state.

**(9)** It provides (eb) an extremely accurate means of predicting (eb) the value of the definite state that will be measured in the form of (e) a probability for each possible measurement value.

**(10)** The physical nature of the transition from the quantum superposition of states to (e&eb) the definite classical state measured is not explained by the traditional theory but is usually assumed as (=) an axiom and was at the basis of the debate between Bohr and Einstein concerning the completeness of quantum theory.

**(11)** Quantum Darwinism seeks to explain the transition of quantum systems from the vast potentiality of superposed states to (e&eb) the greatly reduced set of pointer states [2] as a selection process,

einselection, imposed on the quantum system through (e&eb) its continuous interactions with (e&eb) the environment.

**(12)** All quantum interactions, including measurements, but much more typically interactions with (e&eb) the environment such as with the sea of photons in which all quantum systems are immersed, lead to (eb)decoherence or the manifestation of the quantum system in a particular basis dictated by (e&eb) the nature of the interaction in which the quantum system is involved.

**(13)** In the case of interactions with its environment Zurek and his collaborators have shown that a preferred basis into which a quantum system will decohere is (=) the pointer basis underlying (e&eb) predictable classical states.

**(14)** It is in this sense that the pointer states of classical reality are selected from (e) quantum reality and exist in the macroscopic realm in a state able to undergo (e&eb) further evolution.

**(15)** However, the 'einselection' program depends on (=) assuming a particular division of the universal quantum state into (e&eb) 'system' + 'environment', with (e&eb)the different degrees of freedom of the environment posited as having (e) mutually random phases.

**(16)** This phase randomness does not arise from (e) within the quantum state of the universe on its own, and Kastner [3] has pointed out that this limits the explanatory power of the Quantum Darwinism program.

**(17)** As a quantum system's interactions with (e&eb) its environment results in (eb) the recording of many redundant copies of information regarding its pointer states, this information is available to numerous observers able to achieve (eb) consensual agreement concerning their information of the quantum state. This aspect of einselection, called by Zurek 'Environment as a Witness', results in the potential for objective knowledge.

**NOTATION**

**Module One**

A study in 2010 is claimed to provide preliminary supporting evidence of quantum Darwinism with scars of a quantum dot "becoming (=) a family of mother-daughter states" indicating (eb) they could "stabilize into multiple pointer states."[1]

$G_{13}$ : Category one of preliminary supporting evidence of quantum Darwinism with scars of a quantum dot

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of family of mother-daughter states" indicating (eb) they could "stabilize into multiple pointer states."[1]

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

A study in 2010 is claimed to provide preliminary supporting evidence of quantum Darwinism with scars of a quantum dot "becoming a family of mother-daughter states" indicating (eb) they could "stabilize into multiple pointer states."[1]

$G_{16}$ : Category one of preliminary supporting evidence of quantum Darwinism with scars of a quantum dot

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| "becoming a family of mother-daughter states"<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of they could "stabilize into multiple pointer states."[1]<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>However, the claimed evidence is also subject to (e) the circularity criticism by Kastner (see Implications below)<br><br>Basically, the de facto phenomenon of decoherence that underlies (e&eb) the claims of Quantum Darwinism may not really arise in (eb) a unitary-only dynamics | |
| $G_{20}$ : Category one of **de facto phenomenon of decoherence;** claims of Quantum Darwinism may not really arise in (eb) a unitary-only dynamics<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of claims of Quantum Darwinism may not really arise in (eb) a unitary-only dynamics **;de facto phenomenon of decoherence**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Basically, the de facto phenomenon of decoherence that underlies the claims of Quantum Darwinism may not really arise in (eb) a unitary-only dynamics | |
| $G_{24}$ : Category one of de facto phenomenon of decoherence that underlies the claims of Quantum Darwinism may not really arise<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of unitary-only dynamics<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Thus, even if there is decoherence, (eb) this does not show that macroscopic pointer states naturally emerge | |

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| without (e) some form of collapse | |
| $G_{28}$ : Category one of decoherence persistence<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of this does not show that macroscopic pointer states naturally emerge without (e) some form of collapse<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Thus, even if there is decoherence this does not show that macroscopic pointer states naturally emerge without (e) some form of collapse | |
| $G_{32}$ : Category one of some form of collapse<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of even if there is decoherence this does not show that macroscopic pointer states naturally emerge<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Along with Zurek's related theory of envariance, quantum Darwinism seeks to explain (eb) how the classical world emerges from (e) the quantum world and proposes to answer the quantum measurement problem, the main interpretational challenge for quantum theory<br><br>All the three categories could be taken as equal. | |
| $G_{36}$ : Category one of Zurek's related theory of envariance, quantum Darwinism<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of how the classical world emerges from (e) the quantum world and proposes to answer the quantum measurement problem, the main interpretational challenge for quantum theory<br><br>$T_{37}$ : Category two of SAS | |

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| $T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Along with Zurek's related theory of envariance, quantum Darwinism seeks to explain how the classical world emerges from (e) the quantum world and proposes to answer the quantum measurement problem, the main interpretational challenge for quantum theory<br><br>All the three categories could be taken as equal. | |
| $G_{40}$ : Category one of quantum world and proposes to answer the quantum measurement problem, the main interpretational challenge for quantum theory<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one ofZurek's related theory of envariance, quantum Darwinism seeks to explain how the classical world emerges<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The measurement problem arises because (e) the quantum state vector, the source of all knowledge concerning (e&eb) quantum systems, evolves according to (e&eb) the Schrödinger equation into a linear superposition of different states, predicting (eb) paradoxical situations such as "Schrödinger's cat"; situations never experienced in our classical world | |
| $G_{44}$ : Category one ofquantum state vector, the source of all knowledge concerning (e&eb) quantum systems, evolves according to (e&eb) the Schrödinger equation into a linear superposition of different states, predicting (eb) paradoxical situations such as "Schrödinger's cat"; situations never experienced in our classical world<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of measurement problem arises<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION TWENTY NINE

## Preferred Basis Into Which A Quantum System Will Decohere Is The Pointer Basis Underlying Predictable Classical States

### INTRODUCTION—VARIABLES USED

**Source: Wikipedia**

**(1)** Quantum theory has traditionally treated this problem as being resolved by (e) a non-unitary transformation of the state vector at the time of measurement into (e&eb) a definite state.

**(2)** non-unitary transformation of the state vector at the time of measurement into (e&eb) a definite statet provides (eb) an extremely accurate means of predicting (eb) the value of the definite state that will be measured in the form of (e) a probability for each possible measurement value.

**(3)** The physical nature of the transition from the quantum superposition of states to (e&eb) the definite classical state measured is not explained by the traditional theory but is usually assumed as (=) an axiom and was at the basis of the debate between Bohr and Einstein concerning the completeness of quantum theory.

**(4)** Quantum Darwinism seeks to explain the transition of quantum systems from the vast potentiality of superposed states to (e&eb) the greatly reduced set of pointer states [2] as a selection process, einselection, imposed on the quantum system through (e&eb) its continuous interactions with (e&eb) the environment.

**(5)** All quantum interactions, including measurements, but much more typically interactions with (e&eb) the environment such as with the sea of photons in which all quantum systems are immersed, lead to (eb) decoherence or the manifestation of the quantum system in a particular basis dictated by (e&eb) the nature of the interaction in which the quantum system is involved.

**(6)** In the case of interactions with its environment Zurek and his collaborators have shown that a preferred basis into which a quantum system will decohere is (=) the pointer basis underlying (e&eb) predictable classical states.

**(7)** It is in this sense that the pointer states of classical reality are selected from (e) quantum reality and exist in the macroscopic realm in a state able to undergo (e&eb) further evolution.

**(8)** However, the 'einselection' program depends on (=) assuming a particular division of the universal quantum state into (e&eb) 'system' + 'environment', with (e&eb)the different degrees of freedom of the environment posited as having (e) mutually random phases.

**(9)** This phase randomness does not arise from (e) within the quantum state of the universe on its own, and Kastner [3] has pointed out that this limits the explanatory power of the Quantum Darwinism program.

**(10)** As a quantum system's interactions with (e&eb) its environment results in (eb) the recording of many redundant copies of information regarding its pointer states, this information is available to numerous observers able to achieve (eb) consensual agreement concerning their information of the quantum state. This aspect of einselection, called by Zurek 'Environment as a Witness', results in the potential for objective knowledge.

### NOTATION

### Module One

The measurement problem arises because the quantum state vector, the source of all knowledge concerning (e&eb) quantum systems, evolves according to (e&eb) the Schrödinger equation into a linear superposition

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| of different states, predicting (eb) paradoxical situations such as "Schrödinger's cat"; situations never experienced in our classical world | |
| $G_{13}$ : Category one of **measurement problem arises because the quantum state vector, the source of all knowledge;** quantum systems, evolves according to (e&eb) the Schrödinger equation into a linear superposition of different states, predicting (eb) paradoxical situations such as "Schrödinger's cat"; situations never experienced in our classical world<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of quantum systems, evolves according to (e&eb) the Schrödinger equation into a linear superposition of different states, predicting (eb) paradoxical situations such as "Schrödinger's cat"; situations never experienced in our classical world;**measurement problem arises because the quantum state vector, the source of all knowledge**<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>The measurement problem arises because the quantum state vector, the source of all knowledge concerning quantum systems, evolves according to (e&eb) the Schrödinger equation into a linear superposition of different states, predicting (eb) paradoxical situations such as "Schrödinger's cat"; situations never experienced in our classical world | |
| $G_{16}$ : Category one of **measurement problem arises because the quantum state vector, the source of all knowledge concerning quantum systems, evolves;** Schrödinger equation into a linear superposition of different states, predicting (eb) paradoxical situations such as "Schrödinger's cat"; situations never experienced in our classical world<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of Schrödinger equation into a linear superposition of different states, predicting (eb) paradoxical situations such as "Schrödinger's cat"; situations never experienced in our classical world **;measurement problem arises because the quantum state vector, the source of all knowledge concerning quantum systems, evolves**<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>The measurement problem arises because the quantum state vector, the source of all knowledge concerning quantum systems, evolves according to the Schrödinger equation into a linear superposition of different states, predicting (eb) paradoxical situations such as "Schrödinger's cat"; situations never experienced in our classical world | |
| $G_{20}$ : Category one of measurement problem arises because the quantum state vector, the source of all knowledge concerning quantum systems, evolves according to the Schrödinger equation into a linear superposition of different states, | |

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| $G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of paradoxical situations such as “Schrödinger's cat”; situations never experienced in our classical world<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Quantum theory has traditionally treated measurement problem as being resolved by (e) a non-unitary transformation of the state vector at the time of measurement into (e&eb) a definite state | |
| $G_{24}$ : Category one of measurement problem<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of non-unitary transformation of the state vector at the time of measurement into (e&eb) a definite state<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Quantum theory has traditionally treated measurement problem as being resolved by a non-unitary transformation of the state vector at the time of measurement into (e&eb) a definite state | |
| $G_{28}$ : Category one of **measurement problem as being resolved by a non-unitary transformation of the state vector at the time of measurement;** definite state<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of definite state; **measurement problem as being resolved by a non-unitary transformation of the state vector at the time of measurement**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>non-unitary transformation of the state vector at the time of measurement into (e&eb) a definite statet provides (eb) an extremely accurate means of predicting (eb) the value of the definite state that will be measured in the form of (e) a probability for each possible measurement value | |

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| $G_{32}$ : Category one of **non-unitary transformation of the state vector at the time of measurement;** definite statet provides (eb) an extremely accurate means of predicting (eb) the value of the definite state that will be measured in the form of (e) a probability for each possible measurement value<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of definite statet provides (eb) an extremely accurate means of predicting (eb) the value of the definite state that will be measured in the form of (e) a probability for each possible measurement value; **non-unitary transformation of the state vector at the time of measurement**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>non-unitary transformation of the state vector at the time of measurement into a definite state provides (eb) an extremely accurate means of predicting (eb) the value of the definite state that will be measured in the form of (e) a probability for each possible measurement value | |
| $G_{36}$ : Category one of non-unitary transformation of the state vector at the time of measurement into a definite state<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of extremely accurate means of predicting (eb) the value of the definite state that will be measured in the form of (e) a probability for each possible measurement value<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>non-unitary transformation of the state vector at the time of measurement into a definite state provides (eb) an extremely accurate means of predicting the value of the definite state that will be measured in the form of (e) a probability for each possible measurement value | |
| $G_{40}$ : Category one of probability for each possible measurement value<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |

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| $T_{40}$ : Category one of non-unitary transformation of the state vector at the time of measurement into a definite state provides (eb) an extremely accurate means of predicting the value of the definite state that will be measured in the form<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The physical nature of the transition from the quantum superposition of states to (e&eb) the definite classical state measured is not explained by the traditional theory but is usually assumed as (=) an axiom and was at the basis of the debate between Bohr and Einstein concerning the completeness of quantum theory | |
| $G_{44}$ : Category one of **physical nature of the transition from the quantum superposition of states;** definite classical state measured is not explained by the traditional theory but is usually assumed as (=) an axiom and was at the basis of the debate between Bohr and Einstein concerning the completeness of quantum theory<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one ofdefinite classical state measured is not explained by the traditional theory but is usually assumed as (=) an axiom and was at the basis of the debate between Bohr and Einstein concerning the completeness of quantum theory; **physical nature of the transition from the quantum superposition of states**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION THIRTY

## Vast Potentiality Of Superposed States To The Greatly Reduced Set Of Pointer States

### INTRODUCTION—VARIABLES USED

**Source: Wikipedia**

**(1)** Quantum Darwinism seeks to explain the transition of quantum systems from the vast potentiality of superposed states to (e&eb) the greatly reduced set of pointer states [2] as a selection process, einselection, imposed on the quantum system through (e&eb) its continuous interactions with (e&eb) the environment.

**(2)** All quantum interactions, including measurements, but much more typically interactions with (e&eb) the environment such as with the sea of photons in which all quantum systems are

immersed, lead to (eb) decoherence or the manifestation of the quantum system in a particular basis dictated by (e&eb) the nature of the interaction in which the quantum system is involved.

**(3)** In the case of interactions with its environment Zurek and his collaborators have shown that a preferred basis into which a quantum system will decohere is (=) the pointer basis underlying (e&eb) predictable classical states.

**(4)** It is in this sense that the pointer states of classical reality are selected from (e) quantum reality and exist in the macroscopic realm in a state able to undergo (e&eb) further evolution.

**(5)** However, the 'einselection' program depends on (=) assuming a particular division of the universal quantum state into (e&eb) 'system' + 'environment', with (e&eb)the different degrees of freedom of the environment posited as having (e) mutually random phases.

**(6)** This phase randomness does not arise from (e) within the quantum state of the universe on its own, and Kastner [3] has pointed out that this limits the explanatory power of the Quantum Darwinism program.

**(7)** As a quantum system's interactions with (e&eb) its environment results in (eb) the recording of many redundant copies of information regarding its pointer states, this information is available to numerous observers able to achieve (eb) consensual agreement concerning their information of the quantum state. This aspect of einselection, called by Zurek 'Environment as a Witness', results in the potential for objective knowledge.

**NOTATION**

**Module One**

The physical nature of the transition from the quantum superposition of states to the definite classical state measured is not explained by the traditional theory but is usually assumed as (=) an axiom and was at the basis of the debate between Bohr and Einstein concerning the completeness of quantum theory

$G_{13}$ : Category one of physical nature of the transition from the quantum superposition of states to the definite classical state measured is not explained by the traditional theory but is usually assumed

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of axiom and was at the basis of the debate between Bohr and Einstein concerning the completeness of quantum theory

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

Quantum Darwinism seeks to explain the transition of quantum systems from the vast potentiality of superposed states to (e&eb) the greatly reduced set of pointer states [2] as a selection process, einselection, imposed on the quantum system through (e&eb) its continuous interactions with (e&eb) the environment

$G_{16}$ : Category one of **Quantum Darwinism seeks to explain the transition of quantum systems from the vast potentiality of superposed states;** greatly reduced set of pointer states [2] as a selection process, einselection, imposed on the quantum system through (e&eb) its continuous interactions with (e&eb) the environment

$G_{17}$ : Category two of SAS

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| $G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of greatly reduced set of pointer states [2] as a selection process, einselection, imposed on the quantum system through (e&eb) its continuous interactions with (e&eb) the environment **;Quantum Darwinism seeks to explain the transition of quantum systems from the vast potentiality of superposed states**<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>Quantum Darwinism seeks to explain the transition of quantum systems from the vast potentiality of superposed states to the greatly reduced set of pointer states [2] as a selection process, einselection, imposed on the quantum system through (e&eb) its continuous interactions with (e&eb) the environment | |
| $G_{20}$ : Category one of**Quantum Darwinism seeks to explain the transition of quantum systems from the vast potentiality of superposed states to the greatly reduced set of pointer states [2] as a selection process, einselection, imposed on the quantum system ;** its continuous interactions with (e&eb) the environment<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of its continuous interactions with (e&eb) the environment **;Quantum Darwinism seeks to explain the transition of quantum systems from the vast potentiality of superposed states to the greatly reduced set of pointer states [2] as a selection process, einselection, imposed on the quantum system t**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Quantum Darwinism seeks to explain the transition of quantum systems from the vast potentiality of superposed states to the greatly reduced set of pointer states [2] as a selection process, einselection, imposed on the quantum system through its continuous interactions with (e&eb) the environment | |
| $G_{24}$ : Category one of **Quantum Darwinism seeks to explain the transition of quantum systems from the vast potentiality of superposed states to the greatly reduced set of pointer states [2] as a selection process, einselection, imposed on the quantum system through its continuous interactions;** environment<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of environment; **Quantum Darwinism seeks to explain the transition of quantum systems from the vast potentiality of superposed states to the greatly reduced set of pointer states [2] as a selection process, einselection, imposed on the quantum system through its continuous interactions** | |

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| $T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>All quantum interactions, including measurements, but much more typically interactions with (e&eb) the environment such as with the sea of photons in which all quantum systems are immersed, lead to (eb) decoherence or the manifestation of the quantum system in a particular basis dictated by (e&eb) the nature of the interaction in which the quantum system is involved | |
| $G_{28}$ : Category one of **All quantum interactions, including measurements, but much more typically interactions;** environment such as with the sea of photons in which all quantum systems are immersed, lead to (eb) decoherence or the manifestation of the quantum system in a particular basis dictated by (e&eb) the nature of the interaction in which the quantum system is involved<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of environment such as with the sea of photons in which all quantum systems are immersed, lead to (eb) decoherence or the manifestation of the quantum system in a particular basis dictated by (e&eb) the nature of the interaction in which the quantum system is involved;**All quantum interactions, including measurements, but much more typically interactions**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>All quantum interactions, including measurements, but much more typically interactions with the environment such as with the sea of photons in which all quantum systems are immersed, lead to (eb) decoherence or the manifestation of the quantum system in a particular basis dictated by (e&eb) the nature of the interaction in which the quantum system is involved | |
| $G_{32}$ : Category one of All quantum interactions, including measurements, but much more typically interactions with the environment such as with the sea of photons in which all quantum systems are immersed<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of decoherence or the manifestation of the quantum system in a particular basis dictated by (e&eb) the nature of the interaction in which the quantum system is involved<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |

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| **Module seven**<br><br>All quantum interactions, including measurements, but much more typically interactions with the environment such as with the sea of photons in which all quantum systems are immersed, lead to decoherence or the manifestation of the quantum system in a particular basis dictated by (e&eb) the nature of the interaction in which the quantum system is involved | |
| $G_{36}$ : Category one of **All quantum interactions, including measurements, but much more typically interactions with the environment such as with the sea of photons in which all quantum systems are immersed, lead to decoherence or the manifestation of the quantum system in a particular basis;** nature of the interaction in which the quantum system is involved<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of nature of the interaction in which the quantum system is involved; **All quantum interactions, including measurements, but much more typically interactions with the environment such as with the sea of photons in which all quantum systems are immersed, lead to decoherence or the manifestation of the quantum system in a particular basis**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Zurek and his collaborators have shown that a preferred basis in the case of interactions with its environment into which a quantum system will decohere is (=) the pointer basis underlying (e&eb) predictable classical states. | |
| $G_{40}$ : Category one of preferred basis in the case of interactions with its environment into which a quantum system will decohere<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of pointer basis underlying (e&eb) predictable classical states<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Zurek and his collaborators have shown that a preferred basis in the case of interactions with its environment into which a quantum system will decohere is the pointer basis underlying (e&eb) predictable classical states. | |

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| $G_{44}$ : Category one of **preferred basis in the case of interactions with its environment into which a quantum system will decohere is the pointer basis;** predictable classical states.<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of predictable classical states.;**preferred basis in the case of interactions with its environment into which a quantum system will decohere is the pointer basis**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION THIRTY ONE

## Universal Quantum State Into 'System' + 'Environment', With The Different Degrees Of Freedom Of The Environment Posited As Having Mutually Random Phases

### INTRODUCTION—VARIABLES USED

**Source: Wikipedia**

**(1)** It is in this sense that the pointer states of classical reality are selected from (e) quantum reality and exist in the macroscopic realm in a state able to undergo (e&eb) further evolution.

**(2)** However, the 'einselection' program depends on (=) assuming a particular division of the universal quantum state into (e&eb) 'system' + 'environment', with (e&eb)the different degrees of freedom of the environment posited as having (e) mutually random phases.

**(3)** This phase randomness does not arise from (e) within the quantum state of the universe on its own, and Kastner [3] has pointed out that this limits the explanatory power of the Quantum Darwinism program.

**(4)** As a quantum system's interactions with (e&eb) its environment results in (eb) the recording of many redundant copies of information regarding its pointer states, this information is available to numerous observers able to achieve (eb) consensual agreement concerning their information of the quantum state. This aspect of einselection, called by Zurek 'Environment as a Witness', results in the potential for objective knowledge.

### NOTATION

**Module One**

It is in this sense that the pointer states of classical reality are selected from (e) quantum reality and exist in the macroscopic realm in a state able to undergo (e&eb) further evolution

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| $G_{13}$ : Category one of **pointer states of classical reality are selected;** quantum reality and exist in the macroscopic realm in a state able to undergo (e&eb) further evolution<br><br>$G_{14}$ : Category two of SAS | |

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| $G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of quantum reality and exist in the macroscopic realm in a state able to undergo (e&eb) further evolution; **pointer states of classical reality are selected**<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>However, the 'einselection' program depends on (=) assuming a particular division of the universal quantum state into (e&eb) 'system' + 'environment', with (e&eb)the different degrees of freedom of the environment posited as having (e) mutually random phases | |
| $G_{16}$ : Category one of'einselection' program<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one ofassuming a particular division of the universal quantum state into (e&eb) 'system' + 'environment', with (e&eb)the different degrees of freedom of the environment posited as having (e) mutually random phases<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>However, the 'einselection' program depends on assuming a particular division of the universal quantum state into (e&eb) 'system' + 'environment', with (e&eb)the different degrees of freedom of the environment posited as having (e) mutually random phases | |
| $G_{20}$ : Category one of'**einselection' program depends on assuming a particular division of the universal quantum state ;** 'system' + 'environment', with (e&eb)the different degrees of freedom of the environment posited as having (e) mutually random phases<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of 'system' + 'environment', with (e&eb)the different degrees of freedom of the environment posited as having (e) mutually random phases **;'einselection' program depends on assuming a particular division of the universal quantum state**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>However, the 'einselection' program depends on assuming a particular division of the universal quantum state into 'system' + 'environment', with (e&eb)the different degrees of freedom of the environment posited | |

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| as having (e) mutually random phases | |
| $G_{24}$ : Category one of**'einselection' program depends on assuming a particular division of the universal quantum state into 'system' + 'environment';** different degrees of freedom of the environment posited as having (e) mutually random phases<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of different degrees of freedom of the environment posited as having (e) mutually random phases **;'einselection' program depends on assuming a particular division of the universal quantum state into 'system' + 'environment'**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>However, the 'einselection' program depends on assuming a particular division of the universal quantum state into 'system' + 'environment', with the different degrees of freedom of the environment posited as having (e) mutually random phases | |
| $G_{28}$ : Category one of'einselection' program depends on assuming a particular division of the universal quantum state into 'system' + 'environment', with the different degrees of freedom of the environment posited<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of mutually random phases<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>This phase randomness does not arise from (e) within the quantum state of the universe on its own, and Kastner [3] has pointed out that this limits the explanatory power of the Quantum Darwinism program | |
| $G_{32}$ : Category one of within the quantum state of the universe on its own, and Kastner [3] has pointed out that this limits the explanatory power of the Quantum Darwinism program<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of phase randomness does not arise | |

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| $T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>This phase randomness does not arise from (e) within the quantum state of the universe on its own, and Kastner [3] has pointed out that this limits the explanatory power of the Quantum Darwinism program | |
| $G_{36}$ : Category one of explanatory power of the Quantum Darwinism program<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of phase randomness does not arise from (e) within the quantum state of the universe on its own<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>As a quantum system's interactions with its environment results in (eb) the recording of many redundant copies of information regarding its pointer states, this information is available to numerous observers able to achieve (eb) consensual agreement concerning their information of the quantum state.<br><br>This aspect of einselection, called by Zurek 'Environment as a Witness', results in the potential for objective knowledge. | |
| $G_{40}$ : Category one of quantum system's interactions with its environment<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one ofrecording of many redundant copies of information regarding its pointer states, this information is available to numerous observers able to achieve (eb) consensual agreement concerning their information of the quantum state<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>As a quantum system's interactions with its environment results in the recording of many redundant copies of information regarding its pointer states, this information is available to numerous observers able to achieve (eb) consensual agreement concerning their information of the quantum state. | |

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| $G_{44}$ : Category one of quantum system's interactions with its environment results in the recording of many redundant copies of information regarding its pointer states, this information is available to numerous observers able to achieve<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of consensual agreement concerning their information of the quantum state<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION THIRTY TWO

## Lattice Spacing Imposes Any Kind Of Limitation On The Physical Processes We See In The Universe

### INTRODUCTION—VARIABLES USED

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) It's not hard to imagine that Moore's Law-type progress will allow (eb) physicists to simulate (e&eb) significantly larger regions of space.
(2) A region just a few micrometres across could encapsulate (e&eb) the entire workings of a human cell.
(3) Again, the behaviour of this human cell would be (=) indistinguishable from the real thing.
(4) It's this kind of thinking that forces (eb) physicists to consider the possibility that our entire cosmos could be (=) running on a vastly powerful computer.
(5) If so, is there any way (=) we could ever know?
(6) Today, we get an answer of sorts from (e) Silas Beane, at the University of Bonn in Germany, and a few pals. They say there is (=) a way to see evidence that we are being simulated, at least in (eb) certain scenarios.
(7) First, some background. The problem with all simulations is that the laws of physics, which appear

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| continuous, have to be superimposed onto a discrete three dimensional lattice which advances in steps of time.<br>(8) The question that Beane and co asks is whether the **lattice spacing imposes any kind of limitation on the physical processes we see in the universe.** They examine, in particular, high energy processes, which probe smaller regions of space as they get more energetic<br>**(9)** What they find is interesting. They say that the lattice spacing imposes (e&eb) a fundamental limit on the energy that particles can have. | |
| **NOTATION** | |
| **Module One**<br>It's not hard to imagine that Moore's Law-type progress will allow (eb) physicists to simulate (e&eb) significantly larger regions of space | |
| $G_{13}$ : Category one of Moore's Law-type progress<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of physicists to simulate (e&eb) significantly larger regions of space<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>It's not hard to imagine that Moore's Law-type progress will allow physicists to simulate (e&eb) significantly larger regions of space | |
| $G_{16}$ : Category one of **Moore's Law-type progress will allow physicists to simulate;** significantly larger regions of space<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of significantly larger regions of space; **Moore's Law-type progress will allow physicists to simulate**<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>A region just a few micrometres across could encapsulate (e&eb) the entire workings of a human cell | |
| $G_{20}$ : Category one of entire workings of a human cell<br>$G_{21}$ : Category two of SAS | |

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| $G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of region just a few micrometres across could encapsulate<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>Again, the behaviour of this human cell would be indistinguishable from the real thing | |
| $G_{24}$ : Category one of behaviour of this human cell<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of indistinguishable from the real thing<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>It's this kind of thinking that behaviour of this human cell would be indistinguishable from the real thing forces (eb) physicists to consider the possibility that our entire cosmos could be (=) running on a vastly powerful computer | |
| $G_{28}$ : Category one of behaviour of this human cell would be indistinguishable from the real thing<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of physicists to consider the possibility that our entire cosmos could be (=) running on a vastly powerful computer<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>It's this kind of thinking that behaviour of this human cell would be indistinguishable from the real thing forces physicists to consider the possibility that our entire cosmos could be (=) running on a vastly powerful computer | |
| $G_{32}$ : Category one of behaviour of this human cell would be indistinguishable from the real thing forces physicists to consider the possibility that our entire cosmos<br>$G_{33}$ : Category two of SAS | |

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| $G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of running on a vastly powerful computer<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>First, some background.<br><br>The problem with all simulations is that the laws of physics, which appear continuous, have to be superimposed onto a discrete three dimensional lattice which advances in steps of time | |
| $G_{36}$ : Category one of **problem with all simulations is that the laws of physics, which appear continuous, have to be superimposed;** discrete three dimensional lattice which advances in steps of time<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of discrete three dimensional lattice which advances in steps of time **;problem with all simulations is that the laws of physics, which appear continuous, have to be superimposed**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The problem with all simulations is that the laws of physics, which appear continuous, have to be superimposed onto a discrete three dimensional lattice which advances in steps of time | |
| $G_{40}$ : Category one of **problem with all simulations is that the laws of physics, which appear continuous, have to be superimposed onto a discrete three dimensional lattice which advances;** steps of time<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of steps of time **;problem with all simulations is that the laws of physics, which appear continuous, have to be superimposed onto a discrete three dimensional lattice which advances**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine** | |

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| They say that the lattice spacing imposes a fundamental limit on the energy that particles can have. | |
| $G_{44}$ : Category one of **lattice spacing;** fundamental limit on the energy that particles can have<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of fundamental limit on the energy that particles can have; **lattice spacing**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION THIRTY THREE

## Angular Distribution Of The Highest Energy Components Would Exhibit Cubic Symmetry In The Rest Frame Of The Lattice, Deviating Significantly From Isotropy

### INTRODUCTION—VARIABLES USED

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) That's because nothing can exist that is (=) smaller than the lattice itself.
(2) So if our cosmos is (=) merely a simulation, there ought to be (=) a cut off in the spectrum of high energy particles.
(3) It turns out there is (=) exactly this kind of cut off in the energy of cosmic ray particles, a limit known as the Greisen–Zatsepin–Kuzmin or GZK cut off.
(4) This cut-off has been well studied and comes about because (e) high energy particles interact with (e&eb) the cosmic microwave background and so lose (e) energy as they travel (e&eb) long distances.
(5) But Beane and co calculate that the lattice spacing imposes (e&eb) some additional features on (e&eb) the spectrum.
(6) "The most striking feature…is (=) that the angular distribution of (e) the highest energy components would exhibit (eb) cubic symmetry in the rest frame of the lattice, deviating (e&eb) significantly from isotropy," they say.
(7) In other words, the cosmic rays would travel (e&eb) preferentially along the axes of (e) the lattice, so (e) we wouldn't see them equally in all directions.
(8) That's a measurement we could do now with (e&eb) current technology.
**(9)** Finding the effect would be equivalent to (=) being able to 'see' the orientation of (e) lattice on which our universe is (=) simulated.

| NOTATION | |
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| **Module One** | |
| $G_{13}$ : Category one of nothing can exist that is smaller than the lattice itself.<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of lattice spacing imposes a fundamental limit on the energy that particles can have<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>So if our cosmos is merely a simulation, there ought to be a cut off in the spectrum of high energy particles | |
| $G_{16}$ : Category one of our cosmos is merely a simulation<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of there ought to be a cut off in the spectrum of high energy particles<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>It turns out there is (=) exactly this kind of cut off in the energy of cosmic ray particles, a limit known as the Greisen–Zatsepin–Kuzmin or GZK cut off | |
| $G_{20}$ : Category one of this kind of cut off in the energy of cosmic ray particles exist<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of Greisen–Zatsepin–Kuzmin or GZK cut off<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>This cut-off has been well studied and comes about because (e) high energy particles interact with (e&eb) the cosmic microwave background and so lose (e) energy as they travel (e&eb) long distances | |
| $G_{24}$ : Category one of high energy particles interact with (e&eb) the cosmic microwave background and so | |

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| lose (e) energy as they travel (e&eb) long distances<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of cut-off has been well studied and comes about<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>This cut-off has been well studied and comes about because high energy particles interact with (e&eb) the cosmic microwave background and so lose (e) energy as they travel (e&eb) long distances | |
| $G_{28}$ : Category one of **cut-off has been well studied and comes about because high energy particles interact;** cosmic microwave background and so lose (e) energy as they travel (e&eb) long distances<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of cosmic microwave background and so lose (e) energy as they travel (e&eb) long distances; **cut-off has been well studied and comes about because high energy particles interact**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>This cut-off has been well studied and comes about because high energy particles interact with the cosmic microwave background and so lose (e) energy as they travel (e&eb) long distances | |
| $G_{32}$ : Category one of energy as they travel (e&eb) long distances<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of cut-off has been well studied and comes about because high energy particles interact with the cosmic microwave background<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>This cut-off has been well studied and comes about because high energy particles interact with the cosmic | |

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| microwave background and so lose energy as they travel (e&eb) long distances | |
| $G_{36}$ : Category one of **cut-off has been well studied and comes about because high energy particles interact with the cosmic microwave background and so lose energy as they travel;** long distances<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of long distances; **cut-off has been well studied and comes about because high energy particles interact with the cosmic microwave background and so lose energy as they travel**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The most striking feature…is (=) that the angular distribution of (e) the highest energy components would exhibit (eb) cubic symmetry in the rest frame of the lattice, deviating (e&eb) significantly from isotropy," they say | |
| $G_{40}$ : Category one of angular distribution of the highest energy components<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of cubic symmetry in the rest frame of the lattice, deviating (e&eb) significantly from isotropy," they say<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The most striking feature…is that the angular distribution of the highest energy components would exhibit ( cubic symmetry in the rest frame of the lattice, deviating (e&eb) significantly from isotropy," they say | |
| $G_{44}$ : Category one of **angular distribution of the highest energy components would exhibit ( cubic symmetry in the rest frame of the lattice;** significantly from isotropy,<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of significantly from isotropy,;**angular distribution of the highest energy components would exhibit ( cubic symmetry in the rest frame of the lattice**<br><br>$T_{45}$ : Category two of SAS | |

$T_{46}$ : Category three of SAS

## SECTION THIRTY FOUR

## Simulate Quantum Chromodynamics On A Computer To See What Kind Of Complexity Arises

### INTRODUCTION—VARIABLES USED

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) That's cool, mind-blowing even. But the calculations by Beane and co are not without (e) some important caveats.
(2) One problem is that the computer lattice may be constructed in (e&eb) an entirely different way to the one envisaged by these guys.
(3) Another is that this effect is (=) only measurable if (e) the lattice cut off is (=) the same as the GZK cut off.
(4) This occurs when (e) the lattice spacing is about 10^-12 femtometers. If the spacing is significantly smaller than that, (eb) we'll see nothing.
(5) Nevertheless, it's surely worth looking for, if only to rule out (e) the possibility that we're part of a simulation of this particular kind but secretly in the hope that we'll find good evidence of our robotic overlords once and for all. Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation. It is your mind or self that is creating the illusion. See earlier discussions.
(6) One of modern physics' most cherished ideas is quantum chromodynamics, the theory that describes (eb) the strong nuclear force, how it binds (e&eb) quarks and gluons into (eb) protons and neutrons, how these form (eb) nuclei those themselves interact. This is the universe at its most fundamental.
**(7)** So an interesting pursuit is to **simulate quantum chromodynamics on a computer to see (eb) what kind of complexity arises.**
(8) The promise is that simulating physics on such a fundamental level is more or less equivalent to (=) simulating the universe itself.
**(9)** There are one or two challenges of course. The physics is mind-bogglingly complex and operates on a (e&eb) vanishingly small scale.

### NOTATION

**Module One**

One problem is that the computer lattice may be constructed in an entirely different way to the one envisaged by these guys

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| $G_{13}$ : Category one of **computer lattice may be constructed in an entirely different way;**one construction envisaged<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of one construction envisaged**;computer lattice may be constructed in an entirely different way**<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>Another is that this effect is only measurable if (e) the lattice cut off is (=) the same as the GZK cut off. | |
| $G_{16}$ : Category one of lattice cut off is the same as the GZK cut off<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of effect is only measurable<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>This occurs when (e) the lattice spacing is about 10^-12 femtometers.<br><br>If the spacing is significantly smaller than that, (eb) we'll see nothing | |
| $G_{20}$ : Category one of lattice spacing is about 10^-12 femtometers<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of measurability occurs<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>If the spacing is significantly smaller than that, (eb) we'll see nothing | |
| $G_{24}$ : Category one of spacing is significantly smaller than that<br><br>$G_{25}$ : Category two of SAS | |

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| $G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of we'll see nothing<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>One of modern physics' most cherished ideas is quantum chromodynamics, the theory that describes (eb) the strong nuclear force, how it binds (e&eb) quarks and gluons into (eb) protons and neutrons, how these form (eb) nuclei those themselves interact.<br><br>This is the universe at its most fundamental | |
| $G_{28}$ : Category one of quantum chromodynamics<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of strong nuclear force, how it binds (e&eb) quarks and gluons into (eb) protons and neutrons, how these form (eb) nuclei those themselves interact.<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>One of modern physics' most cherished ideas is quantum chromodynamics, the theory that describes the strong nuclear force, how it binds (e&eb) quarks and gluons into (eb) protons and neutrons, how these form (eb) nuclei those themselves interact | |
| $G_{32}$ : Category one of **quantum chromodynamics, the theory that describes the strong nuclear force, how it binds;** quarks and gluons into (eb) protons and neutrons, how these form (eb) nuclei those themselves interact<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of quarks and gluons into (eb) protons and neutrons, how these form (eb) nuclei those themselves interact **;quantum chromodynamics, the theory that describes the strong nuclear force, how it binds**<br><br>$T_{33}$ : Category two of<br><br>$T_{34}$ : Category three of | |
| **Module seven** | |

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| One of modern physics' most cherished ideas is quantum chromodynamics, the theory that describes the strong nuclear force, how it binds quarks and gluons into (eb) protons and neutrons, how these form (eb) nuclei those themselves interact | |
| $G_{36}$ : Category one of **quantum chromodynamics, the theory that describes the strong nuclear force, how it binds quarks and gluons;** protons and neutrons, how these form (eb) nuclei those themselves interact<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of protons and neutrons, how these form (eb) nuclei those themselves interact **;quantum chromodynamics, the theory that describes the strong nuclear force, how it binds quarks and gluons**<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>quantum chromodynamics, the theory that describes the strong nuclear force, how it binds quarks and gluons into protons and neutrons, how these form (eb) nuclei those themselves interact<br>Model gives the mechanism | |
| $G_{40}$ : Category one of quantum chromodynamics, the theory that describes the strong nuclear force, how it binds quarks and gluons into protons and neutrons<br>$G_{41}$ : Category two of SAS<br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of nuclei those themselves interact<br>$T_{41}$ : Category two of SAS<br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br>The promise is that simulating physics on such a fundamental level is more or less equivalent to (=) simulating the universe itself | |
| $G_{44}$ : Category one of simulating physics on such a fundamental level<br>$G_{45}$ : Category two of SAS<br>$G_{46}$ : Category three of SAS | |

$T_{44}$ : Category one of simulating the universe itself

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

## SECTION THIRTY FIVE

## Problem With All Simulations Is That The Laws Of Physics, Which Appear Continuous, Have To Be Superimposed Onto A Discrete Three Dimensional Lattice Which Advances In Steps Of Time

### INTRODUCTION—VARIABLES USED

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) So even using the world's most powerful supercomputers, physicists have only managed to simulate (e&eb) tiny corners of the cosmos just a few femtometers across. (A femtometer is 10^-15 metres.)
(2) That may not sound like much but the significant point is that the simulation is (=) essentially indistinguishable from the real thing (at least as far as we understand it).
(3) It's not hard to imagine that **Moore's Law-type progress** will allow physicists to simulate significantly larger regions of space. A region just a few micrometres across could encapsulate (e) the entire workings of a human cell.
(4) Again, the behaviour of this human cell would be (=) indistinguishable from the real thing.
(5) It's this kind of thinking that forces physicists to consider the possibility that our entire cosmos could be running on (e&eb) a vastly powerful computer. If so, is there any way we could ever know?
**(6)** First, some background. The problem with all simulations is that the laws of physics, which appear continuous, have to be superimposed onto (e&eb) a discrete three dimensional lattice which advances in steps of time.
(7) The question that Beane and co asks is whether the lattice spacing imposes (e&eb) any kind of limitation on the physical processes we see in the universe. They examine, in particular, high energy processes, which probe smaller regions of (e) space as they get more energetic
(8) What they find is interesting. They say that the lattice spacing imposes (e&eb) a fundamental limit on the energy that particles can have. That's because nothing can exist that is smaller than the lattice itself.
**(9)** So if our cosmos is merely a simulation, there ought to be a cut off in the spectrum of high energy particles.

### NOTATION

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| **Module One**<br>So even using the world's most powerful supercomputers, physicists have only managed to simulate (e&eb) tiny corners of the cosmos just a few femtometers across. (A femtometer is 10^-15 metres.) | |
| $G_{13}$ : Category one of physicists using the world's most powerful supercomputers have only managed to simulate<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of tiny corners of the cosmos just a few femtometers across. (A femtometer is 10^-15 metres.)<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>That may not sound like much but the significant point is that the simulation is (=) essentially indistinguishable from the real thing (at least as far as we understand it) | |
| $G_{16}$ : Category one of significant point is that the simulation<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of essentially indistinguishable from the real thing (at least as far as we understand it)<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>Police is chasing one hundred people mob. It is not as if that is not reality. Or let us assume it is real without getting in to the complexities of quantum mechanics. You might start interpreting it an entirely different way. You might think your friends are salvaging the situation by controlling the mob, particularly when you are in a juxtaposition and imbroglio | |
| $G_{20}$ : Category one of simulation; reality<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of reality; simulation<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four** | |

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| It's not hard to imagine that **Moore's Law-type progress** will allow physicists to simulate significantly larger regions of space. A region just a few micrometres across could encapsulate (e) the entire workings of a human cell | |
| $G_{24}$ : Category one of **Moore's Law-type progress**<br>$G_{25}$ : Category two of SAS<br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of physicists to simulate significantly larger regions of space. A region just a few micrometres across could encapsulate (e) the entire workings of a human cell<br>$T_{25}$ : Category two of SAS<br>$T_{26}$ : Category three of SAS | |
| **Module five**<br>It's not hard to imagine that Moore's Law-type progress will allow physicists to simulate significantly larger regions of space.<br>A region just a few micrometres across could encapsulate (e) the entire workings of a human cell | |
| $G_{28}$ : Category one of entire workings of a human cell<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of region just a few micrometres across could encapsulate<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>Again, the behaviour of this human cell would be (=) indistinguishable from the real thing | |
| $G_{32}$ : Category one of behaviour of this human cell<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of indistinguishable from the real thing<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |

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| **Module seven**<br><br>It's this kind of thinking that forces physicists to consider the possibility that our entire cosmos could be running on (e&eb) a vastly powerful computer.<br><br>If so, is there any way we could ever know?<br><br>That you are attached with a mind with predilections of pinching the child and rocking the cradle, there are utmost possibilities your mind itself is playing a Don John, by presuppositions and assumptions, introspections, contemplations and conjurations. | |
| $G_{36}$ : Category one of **entire cosmos could be running;** vastly powerful computer<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of vastly powerful computer **;entire cosmos could be running**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Problem with all simulations is that the laws of physics, which appear continuous, have to be superimposed onto (e&eb) a discrete three dimensional lattice which advances in steps of time. | |
| $G_{40}$ : Category one of **Problem with all simulations is that the laws of physics, which appear continuous, have to be superimposed;** discrete three dimensional lattice which advances in steps of time<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of discrete three dimensional lattice which advances in steps of time **;Problem with all simulations is that the laws of physics, which appear continuous, have to be superimposed**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>What they find is interesting.<br><br>They say that the lattice spacing imposes (e&eb) a fundamental limit on the energy that particles can have. That's because nothing can exist that is smaller than the lattice itself | |
| $G_{44}$ : Category one of **lattice spacing;** fundamental limit on the energy that particles can have | |

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| $G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of fundamental limit on the energy that particles can have **;lattice spacing**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

## SECTION NINETEEN

## Angular Distribution Of The Highest Energy Components Would Exhibit Cubic Symmetry In The Rest Frame Of The Lattice, Deviating Significantly From Isotropy

### INTRODUCTION—VARIABLES USED

**Complexified Path Integrals, Exact Saddles, and Supersymmetry Alireza Behtash, Gerald V. Dunne, Thomas Schäfer, Tin Sulejmanpasic, and Mithat Ünsal Phys. Rev. Lett. 116, 011601 – Published 6 January 2016** The Evolutionary Argument Against Reality Quanta magazine **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation and Donald Hoffman and Quanta magazine interview**

(1) It turns out there is (=) exactly this kind of cut off in (eb)the energy of cosmic ray particles, a limit known as the Greisen–Zatsepin–Kuzmin or GZK cut off.
(2) This cut-off has been well studied and comes about because (e) high energy particles interact with the cosmic microwave background and so lose energy as they travel (e&eb) long distances.
(3) But Beane and co calculate that the lattice spacing imposes (e&eb) some additional features on the spectrum. "The most striking feature…is (=) that the **angular distribution of the highest energy components would exhibit (eb) cubic symmetry in the rest frame of the lattice, deviating significantly from isotropy**," they say.
(4) In other words, the cosmic rays would travel preferentially along (e&eb) the axes of the lattice, so we wouldn't see them equally in all directions.
(5) That's a measurement we could do now with current technology. Finding the effect would be equivalent to (=) being able to 'see' the orientation of lattice on which our universe is simulated.
(6) That's cool, mind-blowing even. But the calculations by Beane and co are not without some important caveats. One problem is that the computer lattice may be constructed in (eb) an entirely different way to the one envisaged by these guys.
(7) Another is that this effect is only measurable if (e) the lattice cut off is the same as the GZK cut off. This occurs when the lattice spacing is about 10^-12 femtometers. If the spacing is significantly smaller than that, we'll see nothing.
**(8)** Nevertheless, it's surely worth looking for, if only to rule out the possibility that we're part of a simulation of this particular kind but secretly in the hope that we'll find good evidence of our robotic overlords once and for all. **Ref: arxiv.org/abs/1210.1847: Constraints on the Universe as a Numerical Simulation**

| NOTATION | |
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| **Module One**<br>They say that the lattice spacing imposes a fundamental limit on the energy that particles can have. That's because nothing can exist that is smaller than the lattice itself | |
| $G_{13}$ : Category one of **lattice spacing imposes a fundamental limit;** energy that particles can have<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of energy that particles can have **;lattice spacing imposes a fundamental limit**<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two** | |
| $G_{16}$ : Category one of nothing can exist that is smaller than the lattice itself<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of lattice spacing imposes a fundamental limit on the energy that particles can have<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>So if our cosmos is merely a simulation, there ought to be a cut off in the spectrum of high energy particles | |
| $G_{20}$ : Category one of our cosmos is merely a simulation<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of there ought to be a cut off in the spectrum of high energy particles<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>It turns out there is (=) exactly this kind of cut off in (eb)the energy of cosmic ray particles, a limit known as the Greisen–Zatsepin–Kuzmin or GZK cut off | |
| $G_{24}$ : Category one of cut off<br>$G_{25}$ : Category two of SAS | |

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| $G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of energy of cosmic ray particles, a limit known as the Greisen–Zatsepin–Kuzmin or GZK cut off<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>This cut-off has been well studied and comes about because (e) high energy particles interact with the cosmic microwave background and so lose energy as they travel (e&eb) long distances | |
| $G_{28}$ : Category one of high energy particles interact with the cosmic microwave background and so lose energy as they travel (e&eb) long distances<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of cut-off has been well studied and comes about<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>This cut-off has been well studied and comes about because high energy particles interact with the cosmic microwave background and so lose energy as they travel (e&eb) long distances | |
| $G_{32}$ : Category one of **cut-off has been well studied and comes about because high energy particles;** cosmic microwave background and so lose energy as they travel (e&eb) long distances<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of cosmic microwave background and so lose energy as they travel (e&eb) long distances; **cut-off has been well studied and comes about because high energy particles**<br><br>$T_{33}$ : Category two of<br><br>$T_{34}$ : Category three of | |
| **Module seven** | |
| $G_{36}$ : Category one of **high energy particles as they travel for long distances**<br><br>$G_{37}$ : Category two of SAS | |

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| $G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one ofloose energy<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br>That's a measurement we could do now with current technology.<br>Finding the effect would be equivalent to (=) being able to 'see' the orientation of lattice on which our universe is simulated | |
| $G_{40}$ : Category one of measurement of effect<br>$G_{41}$ : Category two of SAS<br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of being able to 'see' the orientation of lattice on which our universe is simulated<br>$T_{41}$ : Category two of SAS<br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br>Another is that this effect is only measurable if the lattice cut off is the same as the GZK cut off.<br>This effect is only measurable if the lattice cut off is the same as the GZK cut off. occurs when the lattice spacing is about 10^-12 femtometers. If the spacing is significantly smaller than that, we'll see nothing | |
| $G_{44}$ : Category one of lattice cut off is the same as the GZK cut off<br>$G_{45}$ : Category two of SAS<br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of this effect is only measurable<br>$T_{45}$ : Category two of SAS<br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$ | |

| | |
|---|---|
| $(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}, \ (b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$<br>$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$<br>$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)} \ (a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,<br>$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$<br>$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,<br>$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,<br>$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b'_{13})^{(1)} - (b''_{13})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b'_{14})^{(1)} - (b''_{14})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b'_{15})^{(1)} - (b''_{15})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a''_{13})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b''_{13})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b'_{16})^{(2)} - (b''_{16})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b'_{17})^{(2)} - (b''_{17})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b'_{18})^{(2)} - (b''_{18})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a''_{16})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b''_{16})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |

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|---|---|
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}\left((G_{27}),t\right) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}\left((G_{31}),t\right) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |

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|---|---|
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b'_{32})^{(6)} - (b''_{32})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b'_{33})^{(6)} - (b''_{33})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b'_{34})^{(6)} - (b''_{34})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a''_{32})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b'_{36})^{(7)} - (b''_{36})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b'_{37})^{(7)} - (b''_{37})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b'_{38})^{(7)} - (b''_{38})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a''_{36})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a'_{41})^{(8)} + (a''_{41})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b'_{40})^{(8)} - (b''_{40})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b'_{41})^{(8)} - (b''_{41})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b'_{42})^{(8)} - (b''_{42})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b'_{44})^{(9)} - (b''_{44})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b'_{45})^{(9)} - (b''_{45})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b'_{46})^{(9)} - (b''_{46})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a''_{44})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b''_{44})^{(9)}((G_{47}),t) =$ **First detrition factor** | |

$$\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{13} \quad (55)$$

$$\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{array}{ccc} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{14} \quad (56)$$

$$\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{15} \quad (57)$$

Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3

$\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[\begin{array}{ccc} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{array}\right] T_{13} \quad (58)$$

$$\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[\begin{array}{ccc} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{array}\right] T_{14} \quad (59)$$

$$\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)} & \boxed{-(b_{15}^{''})^{(1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15} \quad (60)$$

Where $\boxed{-(b_{13}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}^{'})^{(2)} & \boxed{+(a_{16}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{16} \quad (61)$$

$$\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}^{'})^{(2)} & \boxed{+(a_{17}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{17} \quad (62)$$

$$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}^{'})^{(2)} & \boxed{+(a_{18}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{18} \quad (63)$$

Where $\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)}$ are third augmentation coefficient for

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| category 1, 2 and 3 $+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3 $+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3 $+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)}\,-(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{16}$ | *64* |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)}\,-(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{17}$ | *65* |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)}\,-(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{18}$ | *66* |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$, $-(b_{17}^{''})^{(2)}(G_{19},t)$, $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{13}^{''})^{(1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3 $-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3 $-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3 $-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3 $-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3 $-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3 | |

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| $-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[\begin{array}{c}(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21},t)+(a_{16}^{''})^{(2,2,2)}(T_{17},t)+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)\\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)\\ +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{44}^{''})^{(9,9,9)}(T_{45},t)\end{array}\right]G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[\begin{array}{c}(a_{21}^{'})^{(3)}+(a_{21}^{''})^{(3)}(T_{21},t)+(a_{17}^{''})^{(2,2,2)}(T_{17},t)+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)\\ +(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)\\ +(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{45}^{''})^{(9,9,9)}(T_{45},t)\end{array}\right]G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[\begin{array}{c}(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21},t)+(a_{18}^{''})^{(2,2,2)}(T_{17},t)+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)\\ +(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)\\ +(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{46}^{''})^{(9,9,9)}(T_{45},t)\end{array}\right]G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[\begin{array}{c}(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23},t)-(b_{16}^{''})^{(2,2,2)}(G_{19},t)-(b_{13}^{''})^{(1,1,1,)}(G,t)\\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)\\ -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)-(b_{44}^{''})^{(9,9,9)}(G_{47},t)\end{array}\right]T_{20}$ | 70 |

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b'_{21})^{(3)} & -(b''_{21})^{(3)}(G_{23},t) & -(b''_{17})^{(2,2,2)}(G_{19},t) & -(b''_{14})^{(1,1,1,)}(G,t) \\ -(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b''_{37})^{(7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)} & -(b''_{22})^{(3)}(G_{23},t) & -(b''_{18})^{(2,2,2)}(G_{19},t) & -(b''_{15})^{(1,1,1,)}(G,t) \\ -(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b''_{38})^{(7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{22} \quad (72)$$

$-(b''_{20})^{(3)}(G_{23},t)$, $-(b''_{21})^{(3)}(G_{23},t)$, $-(b''_{22})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*

$-(b''_{16})^{(2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*

$-(b''_{13})^{(1,1,1,)}(G,t)$, $-(b''_{14})^{(1,1,1,)}(G,t)$, $-(b''_{15})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*

$-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*

$-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*

$-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*

$-(b''_{36})^{(7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7)}(G_{39},t)$ $-(b''_{38})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b''_{40})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b''_{46})^{(9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a'_{24})^{(4)} & +(a''_{24})^{(4)}(T_{25},t) & +(a''_{28})^{(5,5,)}(T_{29},t) & +(a''_{32})^{(6,6,)}(T_{33},t) \\ +(a''_{13})^{(1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{36})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a'_{25})^{(4)} & +(a''_{25})^{(4)}(T_{25},t) & +(a''_{29})^{(5,5,)}(T_{29},t) & +(a''_{33})^{(6,6)}(T_{33},t) \\ +(a''_{14})^{(1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{37})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a'_{26})^{(4)} & +(a''_{26})^{(4)}(T_{25},t) & +(a''_{30})^{(5,5,)}(T_{29},t) & +(a''_{34})^{(6,6,)}(T_{33},t) \\ +(a''_{15})^{(1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3)}(T_{21},t) \\ +(a''_{38})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a''_{24})^{(4)}(T_{25},t)$, $(a''_{25})^{(4)}(T_{25},t)$, $(a''_{26})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a_{28}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* $1,2$ *and* $3$

$+(a_{32}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* $1,2$ *and* $3$

$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$ *are fourth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$ *are fifth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$ *are sixth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$ *are seventh augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$ *are eighth augmentation coefficients for category* $1,2$ *and* $3$

$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)}\,-(b_{24}^{''})^{(4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{13}^{''})^{(1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{24} \tag{76}$$

$$\frac{dT_{25}}{dt}=(b_{25})^{(4)}T_{24}-\begin{bmatrix}(b_{25}^{'})^{(4)}\,-(b_{25}^{''})^{(4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{14}^{''})^{(1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{25} \tag{77}$$

$$\frac{dT_{26}}{dt}=(b_{26})^{(4)}T_{25}-\begin{bmatrix}(b_{26}^{'})^{(4)}\,-(b_{26}^{''})^{(4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{15}^{''})^{(1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{26} \tag{78}$$

*Where* $-(b_{24}^{''})^{(4)}(G_{27},t)$, $-(b_{25}^{''})^{(4)}(G_{27},t)$, $-(b_{26}^{''})^{(4)}(G_{27},t)$ *are first detrition coefficients for category* $1,2$ *and* $3$

$-(b_{28}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,)}(G_{31},t)$ *are second detrition coefficients for category* $1,2$ *and* $3$

$-(b_{32}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,)}(G_{35},t)$ *are third detrition coefficients for category* $1,2$ *and* $3$

$-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$ *are fourth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$ *are fifth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$

| | |
|---|---|
| *are sixth detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)}$<br>*are seventh detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)}$<br>*are eighth detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[\begin{array}{c} (a_{28}^{'})^{(5)}\boxed{+(a_{28}^{''})^{(5)}(T_{29},t)}\ \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)}\ \boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)}\ \boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)}\ \boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}\ \boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[\begin{array}{c} (a_{29}^{'})^{(5)}\boxed{+(a_{29}^{''})^{(5)}(T_{29},t)}\ \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)}\ \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)}\ \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)}\ \boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}\ \boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[\begin{array}{c} (a_{30}^{'})^{(5)}\boxed{+(a_{30}^{''})^{(5)}(T_{29},t)}\ \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)}\ \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)}\ \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)}\ \boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}\ \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}\ \boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{30}$ | *81* |
| *Where* $\boxed{+(a_{28}^{''})^{(5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5)}(T_{29},t)}$ *are first augmentation coefficients for category* 1, 2 *and* 3<br>*And* $\boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3<br>$\boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation coefficients for category 1,2,and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)}$ are sixth augmentation coefficients for category 1,2, 3<br>$\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1,2, 3<br>$\boxed{+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$ are eighth augmentation coefficients for category 1,2, 3<br>$\boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1,2, 3 | |

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b'_{28})^{(5)} & -(b''_{28})^{(5)}(G_{31},t) & -(b''_{24})^{(4,4,)}(G_{27},t) & -(b''_{32})^{(6,6,6)}(G_{35},t) \\ -(b''_{13})^{(1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b'_{29})^{(5)} & -(b''_{29})^{(5)}(G_{31},t) & -(b''_{25})^{(4,4,)}(G_{27},t) & -(b''_{33})^{(6,6,6)}(G_{35},t) \\ -(b''_{14})^{(1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b'_{30})^{(5)} & -(b''_{30})^{(5)}(G_{31},t) & -(b''_{26})^{(4,4,)}(G_{27},t) & -(b''_{34})^{(6,6,6)}(G_{35},t) \\ -(b''_{15})^{(1,1,1,1,1,)}(G,t) & -(b''_{18})^{(2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3)}(G_{23},t) & \\ -(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{30} \quad (84)$$

*where* $-(b''_{28})^{(5)}(G_{31},t)$ , $-(b''_{29})^{(5)}(G_{31},t)$ , $-(b''_{30})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{24})^{(4,4,)}(G_{27},t)$, $-(b''_{25})^{(4,4,)}(G_{27},t)$, $-(b''_{26})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3

$-(b''_{32})^{(6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b''_{13})^{(1,1,1,1,1)}(G,t)$, $-(b''_{14})^{(1,1,1,1,1)}(G,t)$ , $-(b''_{15})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3

$-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3

$-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)$, $-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)$, $-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3

$-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3

$-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3

$-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a'_{32})^{(6)} & +(a''_{32})^{(6)}(T_{33},t) & +(a''_{28})^{(5,5,5)}(T_{29},t) & +(a''_{24})^{(4,4,4,)}(T_{25},t) \\ +(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a'_{33})^{(6)} & +(a''_{33})^{(6)}(T_{33},t) & +(a''_{29})^{(5,5,5)}(T_{29},t) & +(a''_{25})^{(4,4,4,)}(T_{25},t) \\ +(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t) & \\ +(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)} G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ & +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)} T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)} T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)} T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ & -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b''_{28})^{(5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)} & +(a_{36}^{''})^{(7)}(T_{37},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)} & +(a_{37}^{''})^{(7)}(T_{37},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)} & +(a_{38}^{''})^{(7)}(T_{37},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15} \quad (93)$$

Where $(a_{36}^{''})^{(7)}(T_{37},t)$, $(a_{37}^{''})^{(7)}(T_{37},t)$, $(a_{38}^{''})^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation

coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3 $+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}(G_{39},t) & -(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

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$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}-(b_{37}^{''})^{(7)}(G_{39},t) & -(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}(G_{39},t) & -(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3 $-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3 $-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3 $-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{46}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}'')^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

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$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}')^{(8)}+(a_{40}'')^{(8)}(T_{41},t) & +(a_{16}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}'')^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13}$$

$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}')^{(8)}+(a_{41}'')^{(8)}(T_{41},t) & +(a_{17}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}'')^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}')^{(8)}+(a_{42}'')^{(8)}(T_{41},t) & +(a_{18}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}'')^{(9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$$

Where $+(a_{40}'')^{(8)}(T_{41},t)$, $+(a_{41}'')^{(8)}(T_{41},t)$, $+(a_{42}'')^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}'')^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}'')^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}'')^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}'')^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}'')^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}'')^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}'')^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & -(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & -(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & -(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3



$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{13}$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[\begin{array}{ccc} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$\frac{dT_{44}}{dt} =$

$$(b_{44})^{(9)}T_{45} - \left[\begin{array}{ccc} (b_{44}^{'})^{(9)}\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[\begin{array}{ccc} (b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[\begin{array}{ccc} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{array}\right] T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$$

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

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| **Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br>$(a_i'')^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$<br>$(b_i'')^{(1)}(G,t) \le (r_i)^{(1)} \le (b_i')^{(1)} \le (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2\to\infty}(a_i'')^{(1)}(T_{14},t) = (p_i)^{(1)}$<br>$\lim_{G\to\infty}(b_i'')^{(1)}(G,t) = (r_i)^{(1)}$<br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(1)}(T_{14}',t) - (a_i'')^{(1)}(T_{14},t)\rvert \le (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}'\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br>$\lvert(b_i'')^{(1)}(G',t) - (b_i'')^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G'\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(1)}(T_{14}',t)$ and $(a_i'')^{(1)}(T_{14},t)$ . $(T_{14}',t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i'')^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i'')^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i')^{(1)}, (b_i)^{(1)}, (b_i')^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i,j = 16,17,18$ | |

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| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17},t) \le (p_i)^{(2)} \le (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19},t) \le (r_i)^{(2)} \le (b_i')^{(2)} \le (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2\to\infty}(a_i'')^{(2)}(T_{17},t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty}(b_i'')^{(2)}((G_{19}),t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}$ are positive constants and $i = 16,17,18$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(2)}(T_{17}',t) - (a_i'')^{(2)}(T_{17},t)\rvert \le (\hat{k}_{16})^{(2)}\lvert T_{17} - T_{17}'\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i'')^{(2)}((G_{19})',t) - (b_i'')^{(2)}((G_{19}),t)\rvert < (\hat{k}_{16})^{(2)}\lvert\lvert(G_{19}) - (G_{19})'\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}',t)$ and $(a_i'')^{(2)}(T_{17},t)$ . $(T_{17}',t)$ and $(T_{17},t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}]$ . It is to be noted that $(a_i'')^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i'')^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i')^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ | 111 |

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| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(3)} (T_{21}, t) = (p_i)^{(3)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(3)} (G_{23}, t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'}, t) - (a_i^{''})^{(3)}(T_{21}, t)\rvert \leq (\hat{k}_{20})^{(3)} \lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)} t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'}, t) - (b_i^{''})^{(3)}(G_{23}, t)\rvert < (\hat{k}_{20})^{(3)} \lvert\lvert G_{23} - G_{23}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)} t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'}, t)$ and$(a_i^{''})^{(3)}(T_{21}, t)$ . $(T_{21}^{'}, t)$ And $(T_{21}, t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$<br>satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$ | 116 |

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|---|---|
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}\big((G_{27}), t\big) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(4)}\big((G_{27}), t\big) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\rvert \leq (\hat{k}_{24})^{(4)} \lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)} t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'}, t) - (b_i^{''})^{(4)}\big((G_{27}), t\big)\rvert < (\hat{k}_{24})^{(4)} \lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)} t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $\big[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\big]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |

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| $\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i^{'})^{(4)} +\ (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}\big((G_{31}), t\big) \leq\ (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(5)}\,(T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(5)}\,(G_{31}, t) =\ (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'}, t) - (b_i^{''})^{(5)}\big((G_{31}), t\big)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $\big[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\big]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br><br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\ , \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)},\ (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities | 126 |

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| $\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(6)}(T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(6)}((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)}\lvert T_{33} - T_{33}^{'}\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i^{''})^{(6)}((G_{35})^{'}, t) - (b_i^{''})^{(6)}((G_{35}), t)\rvert < (\hat{k}_{32})^{(6)}\lvert\lvert(G_{35}) - (G_{35})^{'}\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and$(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants | 130 |

$(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$
satisfy the inequalities

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$$

Where we suppose

(OOOOOO) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i, j = 36,37,38$ 131

(PPPPPP) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$:

$$(a_i^{''})^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$$

$$(b_i^{''})^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$$

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*(QQQQQQ)* $\lim_{T_2 \to \infty} (a_i^{''})^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(RRRRRR)*

$\lim_{G \to \infty} (b_i^{''})^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, \ (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)} |T_{37} - T_{37}^{'}| e^{-(\hat{M}_{36})^{(7)} t}$$

$$|(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)} ||(G_{39}) - (G_{39})^{'}|| e^{-(\hat{M}_{36})^{(7)} t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'}, t)$ and $(a_i^{''})^{(7)}(T_{37}, t)$ . $(T_{37}^{'}, t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ : 134

(SSSSSS) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants

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| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :<br><br>(TTTTTT) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}, (r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})', t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})'\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would | |

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| be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i, j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45}, t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47}, t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2 \to \infty} (a_i'')^{(9)}(T_{45}, t) = (p_i)^{(9)}$<br>$\lim_{G \to \infty} (b_i'')^{(9)}(G_{47}, t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}', t) - (a_i'')^{(9)}(T_{45}, t)\rvert \le (\hat{k}_{44})^{(9)} \lvert T_{45} - T_{45}'\rvert e^{-(\hat{M}_{44})^{(9)} t}$<br>$\lvert(b_i'')^{(9)}((G_{47})', t) - (b_i'')^{(9)}((G_{47}), t)\rvert < (\hat{k}_{44})^{(9)} \lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}', t)$ | |

and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)},(\widehat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{44})^{(9)}=1$ then the function $(a_i'')^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$ :

$(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\ ,\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}<1$$

**Definition of** $(\hat{P}_{44})^{(9)},(\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\widehat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},(\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)},(a_i')^{(9)},(b_i)^{(9)},(b_i')^{(9)},(p_i)^{(9)},\ (r_i)^{(9)}, i=44,45,46,$ satisfy the inequalities

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[(a_i)^{(9)}+(a_i')^{(9)}+\ (\hat{A}_{44})^{(9)}+\ (\hat{P}_{44})^{(9)}\ (\hat{k}_{44})^{(9)}]<1$$

$$\frac{1}{(\widehat{M}_{44})^{(9)}}[\ (b_i)^{(9)}+(b_i')^{(9)}+\ (\hat{B}_{44})^{(9)}+\ (\hat{Q}_{44})^{(9)}\ \ (\hat{k}_{44})^{(9)}]<1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0)\ ,T_i(0)$ :

$$G_i(t)\le\ \left(\hat{P}_{13}\right)^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{G_i(0)=G_i^0>0}$$

$$T_i(t)\le\ (\hat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ ,\ \boxed{T_i(0)=T_i^0>0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0)\ ,T_i(0)$

$$G_i(t)\le\ (\hat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t)\le\ (\hat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad G_i(0)=G_i^0>0$$

$$T_i(t)\le\ (\hat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}\ ,\quad T_i(0)=T_i^0>0$$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

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| **Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0)\,, T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$ | 153 B |

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| $T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[ (a_{15})^{(1)} G_{14}(s_{(13)}) - \left( (a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{15}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[ (b_{13})^{(1)} T_{14}(s_{(13)}) - \left( (b_{13}')^{(1)} - (b_{13}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{13}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[ (b_{14})^{(1)} T_{13}(s_{(13)}) - \left( (b_{14}')^{(1)} - (b_{14}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{14}(s_{(13)}) \right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[ (b_{15})^{(1)} T_{14}(s_{(13)}) - \left( (b_{15}')^{(1)} - (b_{15}'')^{(1)} (G(s_{(13)}), s_{(13)}) \right) T_{15}(s_{(13)}) \right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |

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| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br><br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| By<br><br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} G_{21}(s_{(20)}) - \left((a_{20}')^{(3)} + a_{20}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{20}(s_{(20)})\right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[(a_{21})^{(3)} G_{20}(s_{(20)}) - \left((a_{21}')^{(3)} + (a_{21}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}')^{(3)} - (b_{21}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br><br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{28})^{(5)}$, $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[ (b_{29})^{(5)} T_{28}(s_{(28)}) - \left( (b'_{29})^{(5)} - (b''_{29})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[ (b_{30})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{30})^{(5)} - (b''_{30})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{30}(s_{(28)}) \right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** | |

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| Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[ (a_{32})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{32}(s_{(32)}) \right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[ (a_{33})^{(6)} G_{32}(s_{(32)}) - \left( (a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[ (a_{34})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{34}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[ (b_{32})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{32}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[ (b_{33})^{(6)} T_{32}(s_{(32)}) - \left( (b'_{33})^{(6)} - (b''_{33})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[ (b_{34})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{34})^{(6)} - (b''_{34})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{34}(s_{(32)}) \right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |

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| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}^{'})^{(8)} + a_{40}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^{0} + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^{0} + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^{0} + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)} (G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{13}(t) \leq G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that<br>$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \leq \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}$ , $G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \leq G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that | 170 |

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| $(G_{16}(t)-G_{16}^0)e^{-(\bar{M}_{16})^{(2)}t} \le \frac{(a_{16})^{(2)}}{(\bar{M}_{16})^{(2)}}\left[((\hat{P}_{16})^{(2)}+G_{17}^0)e^{\left(-\frac{(\hat{P}_{16})^{(2)}+G_{17}^0}{G_{17}^0}\right)}+(\hat{P}_{16})^{(2)}\right]$ | |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0+(\hat{P}_{20})^{(3)}e^{(\bar{M}_{20})^{(3)}s_{(20)}}\right)\right] ds_{(20)} =$<br>$(1+(a_{20})^{(3)}t)G_{21}^0 + \frac{(a_{20})^{(3)}(\hat{P}_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left(e^{(\bar{M}_{20})^{(3)}t}-1\right)$ | 171 |
| From which it follows that<br>$(G_{20}(t)-G_{20}^0)e^{-(\bar{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\bar{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that<br>$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\bar{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$<br>$(1+(a_{24})^{(4)}t)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left(e^{(\bar{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that<br>$(G_{24}(t)-G_{24}^0)e^{-(\bar{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$<br>$(1+(a_{28})^{(5)}t)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)}t}-1\right)$ | |
| From which it follows that<br>$(G_{28}(t)-G_{28}^0)e^{-(\bar{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 5 | 175 |
| The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious | 176 |

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| that<br>$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$<br>$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$ | |
| From which it follows that<br>$(G_{32}(t) - G_{32}^0) e^{-(\hat{M}_{32})^{(6)} t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)} + G_{33}^0) e^{\left(-\frac{(\hat{P}_{32})^{(6)} + G_{33}^0}{G_{33}^0}\right)} + (\hat{P}_{32})^{(6)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 6<br>Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$ | 177 |
| (y) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)}\left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$<br>$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$ | 178 |
| From which it follows that<br>$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)} + G_{37}^0) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 7 | |
| The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)}\left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$<br>$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$ | 180 |
| From which it follows that<br>$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)} + G_{41}^0) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)}\left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$ | |

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| $\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$ | |
| From which it follows that<br>$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t}\leq\frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{P}_{44})^{(9)}+G_{45}^{0}\right)e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ and to choose<br>$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{13})^{(1)}\left\|G_{14}^{(1)}-G_{14}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$<br>$\int_0^t\{(a_{13}')^{(1)}\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{-(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}+$<br>$G_{13}^{(2)}\left\\|(a_{13}'')^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}'')^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right\\|\ e^{-(\hat{M}_{13})^{(1)}s_{(13)}}e^{(\hat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$ | |

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| Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | |
| $\left\|G^{(1)} - G^{(2)}\right\| e^{-(\widehat{M}_{13})^{(1)}t} \leq$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right) d\left(\left(G^{(1)}, T^{(1)};\ G^{(2)}, T^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)} e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(1)}$ and $(b''_i)^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(1)} - (a''_i)^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1 / (a'_{14})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2 / (a'_{15})^{(1)}$<br><br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(1)}(G(t), t)) = (b'_{14})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ : | 189 |

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| Indeed let $t_1$ be so that for $t > t_1$<br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ $and$ $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |

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| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$(a_{16}^{''})^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)} \left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)} \left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}} \left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)} (\widehat{k}_{16})^{(2)}\right) d\left(\left((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)}\right)\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)} (T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0 \quad \text{for } t > 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)} \left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$<br>In the same way , one can obtain | 201 |

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| $G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$, $G_{18}$ and $G_{16}$, $G_{17}$ respectively. | |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$. The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}$, $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}} < 1$ and to choose<br><br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$, $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{20})^{(3)}t}\}$ | 211 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :\left( \widetilde{(G_{23})}, \widetilde{(T_{23})} \right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)} \left\|G_{21}^{(1)} - G_{21}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\, ds_{(20)} +$<br><br>$\int_0^t \{(a_{20}^{'})^{(3)} \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$G_{20}^{(2)} \left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}^{''})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\} ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} t} \leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a_{20}^{'})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)} (\widehat{k}_{20})^{(3)}\right) d\left(\left((G_{23})^{(1)}, (T_{23})^{(1)};\ (G_{23})^{(2)}, (T_{23})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ *and* $(\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)} (T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)} t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating | 217 |

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| $G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}')^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}')^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$ | 225 |

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{27})},\widetilde{(T_{27})}$ : $\left(\widetilde{(G_{27})},\widetilde{(T_{27})}\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{24})^{(4)}\left|G_{25}^{(1)}-G_{25}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$$

$$\int_0^t\{(a_{24}^{'})^{(4)}\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left|G_{24}^{(1)}-G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$$

$$G_{24}^{(2)}\left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)}-(G_{27})^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}t}\le$$
$$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows

(226)

**Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}\ and\ (\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

(227)

**Remark 17:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

it results

$$G_i\ (t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]}\ge 0$$

$$T_i\ (t)\ge T_i^0e^{\left(-(b_i^{'})^{(4)}t\right)}>0\quad \text{for t} > 0$$

(228)

**Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1,\left((\widehat{M}_{24})^{(4)}\right)_2\ and\ \left((\widehat{M}_{24})^{(4)}\right)_3$ :

**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}\ and\ G_{26}$ . indeed if

(229)

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| $G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a'_{25})^{(4)} G_{25}$ and by integrating<br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a'_{25})^{(4)}$<br>In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a'_{26})^{(4)}$<br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t),t)) = (b_{25}^{'})^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i^{''})^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + \left((\widehat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\widehat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{28})^{(5)}\right] \leq (\widehat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying | |

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|---|---|
| Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}((G_{31}),(T_{31}))$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)}+$<br>$\int_0^t\{(a_{28}^{'})^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\le$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}^{'})^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}^{''})^{(5)}$ and $(b_{28}^{''})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(5)}-(a_i^{''})^{(5)}(T_{29}(s_{(28)}),s_{(28)})\}ds_{(28)}\right]}\ge 0$ | 239 |

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| $T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(5)}t)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}^{'})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1/(a_{29}^{'})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2/(a_{30}^{'})^{(5)}$<br><br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(5)}((G_{31})(t),t)) = (b_{29}^{'})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |

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| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{32})^{(6)}\right]\leq(\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}ds_{(32)}+$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a''_{32})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a'_{32})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(6)}$ and $(b_i'')^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 249 |

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| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(6)} - (a_i'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(6)} t)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}')^{(6)} G_{33}$ and by integrating<br><br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}')^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}')^{(6)}$<br><br>If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(6)}((G_{35})(t), t)) = (b_{33}')^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t > t_6$<br><br>$(b_{33})^{(6)} - (b_i'')^{(6)}\left((G_{35})(t), t\right) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty} (b_{34}'')^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}')^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have | 255 |

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| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)}+\left((\hat{P}_{36})^{(7)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{36})^{(7)}\right]\leq(\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}')^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}'')^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}'')^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}')^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}'')^{(7)}$ and $(b_{36}'')^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$. | 260 |

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| If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2 / (a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t), t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t), t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right),\ \ t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded. | 265 |

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| The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t),t\right)=(b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}<1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)}+\left((\hat{P}_{40})^{(8)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{40})^{(8)}\right]\leq(\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}ds_{(40)}+$<br>$\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\leq$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |

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| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(8)}-(a_i^{''})^{(8)}(T_{41}(s_{(40)}),s_{(40)})\}ds_{(40)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2\ and\ \left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)}G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1/(a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2/(a_{42}^{'})^{(8)}$<br><br>If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}\,((G_{43})(t),t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t),t\right) < \varepsilon_8, T_{40}\,(t) > (m)^{(8)}$ | 278 |

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| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.<br>The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}^{'})^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$ | |
| In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying 39,35,36 into itself | |
| The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{47})^{(1)}, (T_{47})^{(1)}), ((G_{47})^{(2)}, (T_{47})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \left(\widetilde{(G_{47})}, \widetilde{(T_{47})}\right) = \mathcal{A}^{(9)}\left((G_{47}), (T_{47})\right)$<br>It results<br><br>$\left\|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$<br>$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$<br>$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right) - (a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right| \; e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$<br><br>Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on 45,46,47,28 and 29 it follows | |
| $\left|(G_{47})^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t} \leq$<br>$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)}, (T_{47})^{(1)}; (G_{47})^{(2)}, (T_{47})^{(2)})\right)$ | |

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$

From 99 to 44 it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(9)}t\right)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)}G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t), t\right) < \varepsilon_9, T_{44}\,(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded. The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b''_{46})^{(9)}\left((G_{47})(t),t\right) = (b'_{46})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a'_{13})^{(1)} + (a'_{14})^{(1)} - (a''_{13})^{(1)}(T_{14}, t) + (a''_{14})^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b'_{13})^{(1)} + (b'_{14})^{(1)} - (b''_{13})^{(1)}(G,t) - (b''_{14})^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,

and $\boxed{(u_0)^{(1)} = \frac{T^0_{13}}{T^0_{14}}}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$

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| are defined | |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le \right.$<br>$\left.\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a'_{15})^{(1)})}\left[e^{(S_1)^{(1)}t} - e^{-(a'_{15})^{(1)}t}\right] + G_{15}^0 e^{-(a'_{15})^{(1)}t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b'_{15})^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b'_{15})^{(1)}t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)}t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$<br>$(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ :<br>$(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a'_{16})^{(2)} + (a'_{17})^{(2)} - (a''_{16})^{(2)}(T_{17}, t) + (a''_{17})^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b'_{16})^{(2)} + (b'_{17})^{(2)} - (b''_{16})^{(2)}((G_{19}), t) - (b''_{17})^{(2)}((G_{19}), t) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(v_1)^{(2)}, (v_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |

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| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\left(u^{(2)}\right)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}$,, $(\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\left(u^{(2)}\right)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}$ , $(m_2)^{(2)}$ , $(\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$,<br>and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)}t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le\right.$ | 312 |

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| $\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_2)^{(2)}((S_1)^{(2)}-(a_{18}')^{(2)})}[e^{(S_1)^{(2)}t}-e^{-(a_{18}')^{(2)}t}]+G_{18}^{0}e^{-(a_{18}')^{(2)}t})$ | |
| $\boxed{T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}}T_{16}^{0}e^{(R_1)^{(2)}t}\le T_{16}(t)\le\frac{1}{(\mu_2)^{(2)}}T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)}T_{16}^{0}}{(\mu_1)^{(2)}((R_1)^{(2)}-(b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t}-e^{-(b_{18}')^{(2)}t}\right]+T_{18}^{0}e^{-(b_{18}')^{(2)}t}\le T_{18}(t)\le$ $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)},(S_2)^{(2)},(R_1)^{(2)},(R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)}=(a_{16})^{(2)}(m_2)^{(2)}-(a_{16}')^{(2)}$ $(S_2)^{(2)}=(a_{18})^{(2)}-(p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)}=(b_{16})^{(2)}(\mu_2)^{(1)}-(b_{16}')^{(2)}$ $(R_2)^{(2)}=(b_{18}')^{(2)}-(r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$: $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)}\le-(a_{20}')^{(3)}+(a_{21}')^{(3)}-(a_{20}'')^{(3)}(T_{21},t)+(a_{21}'')^{(3)}(T_{21},t)\le-(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)}\le-(b_{20}')^{(3)}+(b_{21}')^{(3)}-(b_{20}'')^{(3)}(G_{23},t)-(b_{21}'')^{(3)}((G_{23}),t)\le-(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)},(v_2)^{(3)},(u_1)^{(3)},(u_2)^{(3)}$: By $(v_1)^{(3)}>0,(v_2)^{(3)}<0$ and respectively $(u_1)^{(3)}>0,(u_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_1)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_1)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ and By $(\bar{v}_1)^{(3)}>0,(\bar{v}_2)^{(3)}<0$ and respectively $(\bar{u}_1)^{(3)}>0,(\bar{u}_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_2)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$ and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_2)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ | 320 |
| **Definition of** $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$:- | 321 |

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$, ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,

and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously 322

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$$

$(p_i)^{(3)}$ is defined by equation

$$\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t} \quad (323)$$

$$\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$$

$$\left. \frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right) \quad (324)$$

$$\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}} \quad (325)$$

$$\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t} \quad (326)$$

$$\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$$

$$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t} \quad (327)$$

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- (328)

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$$

$$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$$

$$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}\big((G_{27}), t\big) - (b''_{25})^{(4)}\big((G_{27}), t\big) \le -(\tau_1)^{(4)}$$

**Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ : 329

By $(v_1)^{(4)} > 0$, $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{v}_1)^{(4)}, , (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{v}_1)^{(4)} > 0$, $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (v_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}, \ \textbf{\textit{if}}\ (v_0)^{(4)} < (v_1)^{(4)}$$

$$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}, \textbf{\textit{if}}\ (v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)},$$

and $\boxed{(v_0)^{(4)} = \frac{G^0_{24}}{G^0_{25}}}$

$$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}, \ \textbf{\textit{if}}\ (\bar{v}_4)^{(4)} < (v_0)^{(4)}$$

and analogously 331

$$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}, \ \textbf{\textit{if}}\ (u_0)^{(4)} < (u_1)^{(4)}$$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}, \textbf{\textit{if}}\ (u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)},$$

and $\boxed{(u_0)^{(4)} = \frac{T^0_{24}}{T^0_{25}}}$

$$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \textbf{\textit{if}}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)} \text{ where } (u_1)^{(4)}, (\bar{u}_1)^{(4)}$$

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| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le\right.$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}')^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}')^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)}t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(\nu_1)^{(5)}, (\nu_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, \nu^{(5)}, u^{(5)}$ :<br>By $(\nu_1)^{(5)} > 0$, $(\nu_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}(\nu^{(5)})^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)} = 0$ | 339 |

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| and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ :<br><br>By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)},(v_0)^{(5)}$ :-<br><br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)},(\mu_2)^{(5)}$ by<br><br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br><br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br><br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br><br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br><br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |

$\frac{(b_{30})^{(5)}T_{28}^{0}}{(\mu_1)^{(5)}((R_1)^{(5)}-(b_{30}^{'})^{(5)})}\left[e^{(R_1)^{(5)}t}-e^{-(b_{30}^{'})^{(5)}t}\right]+T_{30}^{0}e^{-(b_{30}^{'})^{(5)}t}\le T_{30}(t)\le$

$\frac{(a_{30})^{(5)}T_{28}^{0}}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t}-e^{-(R_2)^{(5)}t}\right]+T_{30}^{0}e^{-(R_2)^{(5)}t}$ 347

**Definition of** $(S_1)^{(5)},(S_2)^{(5)},(R_1)^{(5)},(R_2)^{(5)}$:- 348

Where $(S_1)^{(5)}=(a_{28})^{(5)}(m_2)^{(5)}-(a_{28}^{'})^{(5)}$

$(S_2)^{(5)}=(a_{30})^{(5)}-(p_{30})^{(5)}$

$(R_1)^{(5)}=(b_{28})^{(5)}(\mu_2)^{(5)}-(b_{28}^{'})^{(5)}$

$(R_2)^{(5)}=(b_{30}^{'})^{(5)}-(r_{30})^{(5)}$

**Behavior of the solutions of equation** 349

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ four constants satisfying

$-(\sigma_2)^{(6)}\le-(a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}-(a_{32}^{''})^{(6)}(T_{33},t)+(a_{33}^{''})^{(6)}(T_{33},t)\le-(\sigma_1)^{(6)}$

$-(\tau_2)^{(6)}\le-(b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(b_{32}^{''})^{(6)}\left((G_{35}),t\right)-(b_{33}^{''})^{(6)}\left((G_{35}),t\right)\le-(\tau_1)^{(6)}$

**Definition of** $(\nu_1)^{(6)},(\nu_2)^{(6)},(u_1)^{(6)},(u_2)^{(6)},\nu^{(6)},u^{(6)}$ : 350

By $(\nu_1)^{(6)}>0$ , $(\nu_2)^{(6)}<0$ and respectively $(u_1)^{(6)}>0$ , $(u_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_1)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2+(\tau_1)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$ and

**Definition of** $(\bar{\nu}_1)^{(6)},,(\bar{\nu}_2)^{(6)},(\bar{u}_1)^{(6)},(\bar{u}_2)^{(6)}$ : 351

By $(\bar{\nu}_1)^{(6)}>0$ , $(\bar{\nu}_2)^{(6)}<0$ and respectively $(\bar{u}_1)^{(6)}>0$ , $(\bar{u}_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_2)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}\left(u^{(6)}\right)^2+(\tau_2)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$

**Definition of** $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)},(\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)}$ by

$(m_2)^{(6)}=(\nu_0)^{(6)},(m_1)^{(6)}=(\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)}<(\nu_1)^{(6)}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)}<(\nu_0)^{(6)}<(\bar{\nu}_1)^{(6)}$,

and $\boxed{(\nu_0)^{(6)}=\frac{G_{32}^{0}}{G_{33}^{0}}}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)}<(\nu_0)^{(6)}$

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| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},\ \boldsymbol{if}\ (u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}, \boldsymbol{if}\ (u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)},$<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}, \boldsymbol{if}\ (\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le\right.$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b_{34}')^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b_{34}')^{(6)} t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$<br>$(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ : | |

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| $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying<br><br>$-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$<br><br>$-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}((G_{39}), t) - (b''_{37})^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ :<br><br>By $(\nu_1)^{(7)} > 0, (\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ :<br><br>By $(\bar{\nu}_1)^{(7)} > 0, (\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(\nu^{(7)})^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$<br>**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-<br><br>If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by<br><br>$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,<br>and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | 362 |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,<br>and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities | 364 |

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| $G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^{0}e^{(S_1)^{(7)}t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | |
| $\frac{1}{(m_7)^{(7)}} G_{36}^{0}e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^{0}e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^{0}e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)}G_{36}^{0}}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}^{'})^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}^{'})^{(7)}t}] + G_{38}^{0}e^{-(a_{38}^{'})^{(7)}t})$ | 366 |
| $\boxed{T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^{0}e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^{0}e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)}T_{36}^{0}}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}^{'})^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}^{'})^{(7)}t}\right] + T_{38}^{0}e^{-(b_{38}^{'})^{(7)}t} \le T_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)}T_{36}^{0}}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^{0}e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}^{'})^{(7)}$<br><br>$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$<br><br>$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}^{'})^{(7)}$<br><br>$(R_2)^{(7)} = (b_{38}^{'})^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :<br><br>$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying<br><br>$-(\sigma_2)^{(8)} \le -(a_{40}^{'})^{(8)} + (a_{41}^{'})^{(8)} - (a_{40}^{''})^{(8)}(T_{41}, t) + (a_{41}^{''})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$<br><br>$-(\tau_2)^{(8)} \le -(b_{40}^{'})^{(8)} + (b_{41}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}), t) - (b_{41}^{''})^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |
| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : | 372 |

By $(v_1)^{(8)} > 0\,, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0\,, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$
and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and

**Definition of** $(\bar{v}_1)^{(8)},, (\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :

By $(\bar{v}_1)^{(8)} > 0\,, (\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0\,, (\bar{u}_2)^{(8)} < 0$ the

roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$

and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$

**Definition of** $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :-

If we define $(m_1)^{(8)}\,, (m_2)^{(8)}\,, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by

$(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)},$ ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}\,,$ ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)},$
and $\boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (v_0)^{(8)},$ ***if*** $(\bar{v}_1)^{(8)} < (v_0)^{(8)}$

and analogously

$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},$ ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$

$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}\,,$ ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$
and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$

$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)},$ ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$

Then the solution of global equations satisfies the inequalities

$$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)}t}$$

where $(p_i)^{(8)}$ is defined by equation

$$\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$$

| | |
|---|---|
| $\left(\frac{(a_{42})^{(8)}G_{40}^{0}}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t}-e^{-(S_2)^{(8)}t}\right]+G_{42}^{0}e^{-(S_2)^{(8)}t}\leq G_{42}(t)\leq \frac{(a_{42})^{(8)}G_{40}^{0}}{(m_2)^{(8)}((S_1)^{(8)}-(a_{42}^{'})^{(8)})}[e^{(S_1)^{(8)}t}-e^{-(a_{42}^{'})^{(8)}t}]+G_{42}^{0}e^{-(a_{42}^{'})^{(8)}t}\right)$ | 377 |
| $\boxed{T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}}T_{40}^{0}e^{(R_1)^{(8)}t}\leq T_{40}(t)\leq \frac{1}{(\mu_2)^{(8)}}T_{40}^{0}e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^{0}}{(\mu_1)^{(8)}((R_1)^{(8)}-(b_{42}^{'})^{(8)})}\left[e^{(R_1)^{(8)}t}-e^{-(b_{42}^{'})^{(8)}t}\right]+T_{42}^{0}e^{-(b_{42}^{'})^{(8)}t}\leq T_{42}(t)\leq$ $\frac{(a_{42})^{(8)}T_{40}^{0}}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t}-e^{-(R_2)^{(8)}t}\right]+T_{42}^{0}e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)},(S_2)^{(8)},(R_1)^{(8)},(R_2)^{(8)}$:-<br>Where $(S_1)^{(8)}=(a_{40})^{(8)}(m_2)^{(8)}-(a_{40}^{'})^{(8)}$<br>$(S_2)^{(8)}=(a_{42})^{(8)}-(p_{42})^{(8)}$<br>$(R_1)^{(8)}=(b_{40})^{(8)}(\mu_2)^{(8)}-(b_{40}^{'})^{(8)}$<br>$(R_2)^{(8)}=(b_{42}^{'})^{(8)}-(r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)},(\sigma_2)^{(9)},(\tau_1)^{(9)},(\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)}\leq -(a_{44}^{'})^{(9)}+(a_{45}^{'})^{(9)}-(a_{44}^{''})^{(9)}(T_{45},t)+(a_{45}^{''})^{(9)}(T_{45},t)\leq -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)}\leq -(b_{44}^{'})^{(9)}+(b_{45}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47}),t)-(b_{45}^{''})^{(9)}((G_{47}),t)\leq -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(\nu_1)^{(9)},(\nu_2)^{(9)},(u_1)^{(9)},(u_2)^{(9)},\nu^{(9)},u^{(9)}$ :<br>By $(\nu_1)^{(9)}>0$ , $(\nu_2)^{(9)}<0$ and respectively $(u_1)^{(9)}>0$ , $(u_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_1)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$<br>and $(b_{45})^{(9)}(u^{(9)})^2+(\tau_1)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(9)},,(\bar{\nu}_2)^{(9)},(\bar{u}_1)^{(9)},(\bar{u}_2)^{(9)}$ :<br>By $(\bar{\nu}_1)^{(9)}>0$ , $(\bar{\nu}_2)^{(9)}<0$ and respectively $(\bar{u}_1)^{(9)}>0$ , $(\bar{u}_2)^{(9)}<0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2+(\sigma_2)^{(9)}\nu^{(9)}-(a_{44})^{(9)}=0$ | |

and $(b_{45})^{(9)}\left(u^{(9)}\right)^2+(\tau_2)^{(9)}u^{(9)}-(b_{44})^{(9)}=0$

**Definition of** $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)},(\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)},(m_2)^{(9)},(\mu_1)^{(9)},(\mu_2)^{(9)}$ by

$(m_2)^{(9)}=(\nu_0)^{(9)},(m_1)^{(9)}=(\nu_1)^{(9)},$ ***if*** $(\nu_0)^{(9)}<(\nu_1)^{(9)}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)}<(\nu_0)^{(9)}<(\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)}=\frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)}=(\nu_1)^{(9)},(m_1)^{(9)}=(\nu_0)^{(9)},$ ***if*** $(\bar{\nu}_1)^{(9)}<(\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)}=(u_0)^{(9)},(\mu_1)^{(9)}=(u_1)^{(9)},$ ***if*** $(u_0)^{(9)}<(u_1)^{(9)}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)}<(u_0)^{(9)}<(\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)}=\frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)}=(u_1)^{(9)},(\mu_1)^{(9)}=(u_0)^{(9)},$ ***if*** $(\bar{u}_1)^{(9)}<(u_0)^{(9)}$ where $(u_1)^{(9)},(\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{44}(t)\le G_{44}^0e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^0e^{((S_1)^{(9)}-(p_{44})^{(9)})t}\le G_{45}(t)\le\frac{1}{(m_2)^{(9)}}G_{44}^0e^{(S_1)^{(9)}t}$$

$$\left(\frac{(a_{46})^{(9)}G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^0e^{-(S_2)^{(9)}t}\le G_{46}(t)\le\right.$$

$$\left.\frac{(a_{46})^{(9)}G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}')^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}')^{(9)}t}]+\ G_{46}^0e^{-(a_{46}')^{(9)}t}\right)$$

$$\boxed{T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^0e^{(R_1)^{(9)}t}\le T_{44}(t)\le\frac{1}{(\mu_2)^{(9)}}T_{44}^0e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}')^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}')^{(9)}t}\right]+T_{46}^0e^{-(b_{46}')^{(9)}t}\le T_{46}(t)\le$$

$$\frac{(a_{46})^{(9)}T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^0e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain (383)

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right) - (a''_{14})^{(1)}(T_{14}, t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get (384)

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{v}_1)^{(1)}$ we find like in the previous case, (385)

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)}G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)}T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)}+(C)^{(2)}(\nu_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_0)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)}-(\nu_0)^{(2)}}{(\nu_0)^{(2)}-(\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)}-(\nu_0)^{(2)}}{(\nu_0)^{(2)}-(\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)}+(C)^{(2)}(\nu_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$$

If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain 394

$$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(2)}(t)$ :- 395

$$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}, \quad \boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$$

In a completely analogous way, we obtain 396

**Definition of** $u^{(2)}(t)$ :-

$$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}, \quad \boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$$

| | |
|---|---|
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$ if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}')^{(3)} - (a_{21}')^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| From which one obtains<br><br>For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$<br><br>$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)} + (C)^{(3)}(\nu_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)} - (\nu_0)^{(3)}\right)t\right]}}$ , $\boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\nu_2)^{(3)}}}$<br><br>it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$ | 400 |
| In the same manner , we get<br><br>$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)} + (\bar{C})^{(3)}(\bar{\nu}_2)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)} e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)} - (\bar{\nu}_2)^{(3)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)} - (\nu_0)^{(3)}}{(\nu_0)^{(3)} - (\bar{\nu}_2)^{(3)}}}$<br><br>**Definition of** $(\bar{\nu}_1)^{(3)}$ :-<br><br>From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$ | 401 |
| If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, | 402 |

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain

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$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right) - (a_{25}^{''})^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\quad \boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

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$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} =$

$(\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

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In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\quad \boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get 413

$$\nu^{(6)}(t) \le \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \le \nu^{(6)}(t) \le (\bar{\nu}_1)^{(6)}$

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If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

$$(\nu_1)^{(6)} \le \frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_2)^{(6)}\right)t\right]}} \le \nu^{(6)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \le (\bar{\nu}_1)^{(6)}$$

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If $0 < (\nu_1)^{(6)} \le (\bar{\nu}_1)^{(6)} \le \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(\nu_1)^{(6)} \le \nu^{(6)}(t) \le \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \le (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

$$(m_2)^{(6)} \le \nu^{(6)}(t) \le (m_1)^{(6)}, \quad \boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)} \le u^{(6)}(t) \le (\mu_1)^{(6)}, \quad \boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)}G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right) - (a_{37}^{''})^{(7)}(T_{37},t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

$$\nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \leq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}} \leq \nu^{(7)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\bar{\nu}_1)^{(7)}$$

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If $0 < (\nu_1)^{(7)} \leq (\bar{\nu}_1)^{(7)} \leq \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain

$$(\nu_1)^{(7)} \leq \nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \leq \nu^{(7)}(t) \leq (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

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In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :- 420

$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)} G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain 421

$$\frac{d\nu^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right) - (a_{41}^{''})^{(8)}(T_{41}, t)\nu^{(8)} - (a_{41})^{(8)}\nu^{(8)}$$

**Definition of** $\nu^{(8)}$ :- $\boxed{\nu^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right) \leq \frac{d\nu^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)}-(\nu_0)^{(8)}}{(\nu_0)^{(8)}-(\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \le \nu^{(8)}(t) \le (\bar{\nu}_8)^{(8)}$

If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case, 423

$$(\nu_1)^{(8)} \le \frac{(\nu_1)^{(8)}+(C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}}{1+(C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)}-(\nu_2)^{(8)}\right)t\right]}} \le \nu^{(8)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\bar{\nu}_1)^{(8)}$$

If $0 < (\nu_1)^{(8)} \le (\bar{\nu}_1)^{(8)} \le \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$, we obtain 424

$$(\nu_1)^{(8)} \le \nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)}+(\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}}{1+(\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)}-(\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have
**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \le \nu^{(8)}(t) \le (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain
**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \le u^{(8)}(t) \le (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)}G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)}T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$. 424

**Proof** : From 99,20,44,22,23,44 we obtain 424 A

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right) - (a''_{45})^{(9)}(T_{45},t)v^{(9)} - (a_{45})^{(9)}v^{(9)}$$

**Definition of** $v^{(9)}$ :- $\boxed{v^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)}\right) \le \frac{dv^{(9)}}{dt} \le -\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(9)}, (v_0)^{(9)}$ :-

For $0 < \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (v_1)^{(9)} < (\bar{v}_1)^{(9)}$

$$v^{(9)}(t) \ge \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(v_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (v_2)^{(9)}}}$$

it follows $(v_0)^{(9)} \le v^{(9)}(t) \le (v_9)^{(9)}$

In the same manner , we get

$$v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{v}_1)^{(9)} - (v_0)^{(9)}}{(v_0)^{(9)} - (\bar{v}_2)^{(9)}}}$$

From which we deduce $(v_0)^{(9)} \le v^{(9)}(t) \le (\bar{v}_1)^{(9)}$

If $0 < (v_1)^{(9)} < (v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}} < (\bar{v}_1)^{(9)}$ we find like in the previous case,

$$(v_1)^{(9)} \le \frac{(v_1)^{(9)} + (C)^{(9)}(v_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((v_1)^{(9)} - (v_2)^{(9)}\right)t\right]}} \le v^{(9)}(t) \le$$

$$\frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (\bar{v}_1)^{(9)}$$

If $0 < (v_1)^{(9)} \le (\bar{v}_1)^{(9)} \le \boxed{(v_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$ , we obtain

$$(v_1)^{(9)} \le v^{(9)}(t) \le \frac{(\bar{v}_1)^{(9)} + (\bar{C})^{(9)}(\bar{v}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{v}_1)^{(9)} - (\bar{v}_2)^{(9)}\right)t\right]}} \le (v_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(9)}(t)$ :-

$$(m_2)^{(9)} \le v^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{v^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}$, $\boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ 430

$with\ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations 431

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$

$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,

$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$

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| $with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | |
| We can prove the following<br><br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br><br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br><br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br><br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br><br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br><br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br><br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br><br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br><br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br><br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br><br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br><br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$ | 435 |

| | |
|---|---|
| $with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |

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| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27})\big)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\big((G_{27})\big)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\big((G_{27})\big)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |

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| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |

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| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(y) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) +$ | 489 |

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| $(a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that | 496 |

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| there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{32}=\frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}=\frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{36}=\frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}=\frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |
| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*})=0$. With this value , we obtain from the three first equations | 501 |

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| $G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G) + (b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)]+(b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*}) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19}) + (b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})]+(b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})]+(b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br>$[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ | 506 |

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| is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution | 511 |

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| $G_{14}^{*}$ given by $\varphi(G^{*})=0$ , $T_{14}^{*}$ given by $f(T_{14}^{*})=0$ and<br>$G_{13}^{*}=\frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*}=\frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br>$T_{13}^{*}=\frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*}=\frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)}-(b_{15}^{''})^{(1)}(G^{*})]}$<br>Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*})=0$ , $T_{17}^{*}$ given by $f(T_{17}^{*})=0$ and | 512 |
| $G_{16}^{*}=\frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*}=\frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*}=\frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*}=\frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*})=0$ , $T_{21}^{*}$ given by $f(T_{21}^{*})=0$ and<br>$G_{20}^{*}=\frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*}=\frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br>$T_{20}^{*}=\frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*}=\frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^{*}$ given by $\varphi(G_{27})=0$ , $T_{25}^{*}$ given by $f(T_{25}^{*})=0$ and<br>$G_{24}^{*}=\frac{(a_{24})^{(4)}G_{25}^{*}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*}=\frac{(a_{26})^{(4)}G_{25}^{*}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*}=\frac{(b_{24})^{(4)}T_{25}^{*}}{[(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*}=\frac{(b_{26})^{(4)}T_{25}^{*}}{[(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}((G_{27})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*})=0$ , $T_{29}^{*}$ given by $f(T_{29}^{*})=0$ and<br>$G_{28}^{*}=\frac{(a_{28})^{(5)}G_{29}^{*}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*}=\frac{(a_{30})^{(5)}G_{29}^{*}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |

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| $T_{28}^{*} = \frac{(b_{28})^{(5)}T_{29}^{*}}{[(b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)}T_{29}^{*}}{[(b_{30}^{'})^{(5)}-(b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br>$G_{32}^{*} = \frac{(a_{32})^{(6)}G_{33}^{*}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)}G_{33}^{*}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)}T_{33}^{*}}{[(b_{32}^{'})^{(6)}-(b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)}T_{33}^{*}}{[(b_{34}^{'})^{(6)}-(b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br>$G_{36}^{*} = \frac{(a_{36})^{(7)}G_{37}^{*}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)}G_{37}^{*}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br>$T_{36}^{*} = \frac{(b_{36})^{(7)}T_{37}^{*}}{[(b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)}T_{37}^{*}}{[(b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br>$G_{40}^{*} = \frac{(a_{40})^{(8)}G_{41}^{*}}{[(a_{40}^{'})^{(8)}+(a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)}G_{41}^{*}}{[(a_{42}^{'})^{(8)}+(a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br>$T_{40}^{*} = \frac{(b_{40})^{(8)}T_{41}^{*}}{[(b_{40}^{'})^{(8)}-(b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)}T_{41}^{*}}{[(b_{42}^{'})^{(8)}-(b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)}G_{45}^{*}}{[(a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)}G_{45}^{*}}{[(a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)}T_{45}^{*}}{[(b_{44}^{'})^{(9)}-(b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)}T_{45}^{*}}{[(b_{46}^{'})^{(9)}-(b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | 523 A |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^*) = (q_{14})^{(1)} \quad , \frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^*) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)}G_{13}^*\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)}G_{14}^*\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)}G_{15}^*\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)}T_{13}^*\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)}T_{14}^*\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)}T_{15}^*\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)} \quad , \frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |

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| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ $and$ $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)}$ , $\frac{\partial(b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_{j}\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_{j}\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_{j}\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 548 |

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| $\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^{*}) = (q_{25})^{(4)}$ , $\frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^{*}) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^{*}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^{*}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^{*}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^{*}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^{*}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^{*}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^{*}) = (q_{29})^{(5)}$ , $\frac{\partial (b_i^{''})^{(5)}}{\partial G_j}((G_{31})^{*}) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^{*}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^{*}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^{*}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^{*}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^{*}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^{*}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 563 |

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| **Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ $and$ $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^*\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^*\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^*\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^*\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ $and$ $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}((G_{39})^{**}) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |

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|---|---|
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{41})^{(8)}}{\partial T_{41}}(T^*_{41}) = (q_{41})^{(8)}$ , $\frac{\partial(b_i^{''})^{(8)}}{\partial G_j}((G_{43})^*) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a'_{40})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G^*_{40}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a'_{41})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G^*_{41}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a'_{42})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G^*_{42}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |

| | |
|---|---|
| | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \ , \ \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}(\ (G_{47})^*\ ) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b_{44}^{'})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T_{44}^*\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b_{45}^{'})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T_{45}^*\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b_{46}^{'})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T_{46}^*\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)$<br>$\left[\left(\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(q_{13})^{(1)}G_{13}^*\right)\right]$<br>$\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T_{14}^* + (b_{14})^{(1)}s_{(13),(14)}T_{14}^*\right)$<br>$+\left(\left((\lambda)^{(1)} + (a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^* + (a_{13})^{(1)}(q_{14})^{(1)}G_{14}^*\right)$<br>$\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^* + (b_{14})^{(1)}s_{(13),(13)}T_{13}^*\right)$<br>$\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$<br>$\left(\left((\lambda)^{(1)}\right)^2 + \left((b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$<br>$+\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$<br>$+\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^*\right)$<br>$\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^* + (b_{14})^{(1)}s_{(13),(15)}T_{13}^*\right)\} = 0$ | |

$$+$$

$$((\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)})\{((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)})$$

$$\left[\left(((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)})(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)\right]$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(17)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^{*}_{17}\right)$$

$$+\left(((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)})(q_{16})^{(2)}\mathrm{G}^{*}_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(16)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^{*}_{16}\right)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left(\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)$$

$$\left(((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)})s_{(17),(18)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(18)}\mathrm{T}^{*}_{16}\right)\} = 0$$

$$+$$

$$((\lambda)^{(3)} + (b'_{22})^{(3)} - (r_{22})^{(3)})\{((\lambda)^{(3)} + (a'_{22})^{(3)} + (p_{22})^{(3)})$$

$$\left[\left(((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)})(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)\right]$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(21)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(21)}T^{*}_{21}\right)$$

$$+\left(((\lambda)^{(3)} + (a'_{21})^{(3)} + (p_{21})^{(3)})(q_{20})^{(3)}G^{*}_{20} + (a_{20})^{(3)}(q_{21})^{(1)}G^{*}_{21}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(20)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(20)}T^{*}_{20}\right)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((b'_{20})^{(3)} + (b'_{21})^{(3)} - (r_{20})^{(3)} + (r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left(\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)$$

$$\left(((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)})s_{(21),(22)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(22)}T^{*}_{20}\right)\} = 0$$

$$+$$

$$\left((\lambda)^{(4)}+(b'_{26})^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a'_{26})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a'_{24})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G^{*}_{25}+(a_{25})^{(4)}(q_{24})^{(4)}G^{*}_{24}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(25)}T^{*}_{25}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a'_{25})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G^{*}_{24}+(a_{24})^{(4)}(q_{25})^{(4)}G^{*}_{25}\right)$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(24)}T^{*}_{24}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((a'_{24})^{(4)}+(a'_{25})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((b'_{24})^{(4)}+(b'_{25})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^{2}+\left((a'_{24})^{(4)}+(a'_{25})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a'_{24})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G^{*}_{25}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G^{*}_{24}\right)$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(26)}T^{*}_{24}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(5)}+(b'_{30})^{(5)}-(r_{30})^{(5)}\right)\{\left((\lambda)^{(5)}+(a'_{30})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a'_{28})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G^{*}_{29}+(a_{29})^{(5)}(q_{28})^{(5)}G^{*}_{28}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(29)}T^{*}_{29}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a'_{29})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G^{*}_{28}+(a_{28})^{(5)}(q_{29})^{(5)}G^{*}_{29}\right)$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(28)}T^{*}_{28}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((a'_{28})^{(5)}+(a'_{29})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((b'_{28})^{(5)}+(b'_{29})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^{2}+\left((a'_{28})^{(5)}+(a'_{29})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a'_{28})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G^{*}_{29}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G^{*}_{28}\right)$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(30)}T^{*}_{28}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\right)\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\right)\}=0$$

$$+$$

$$\left(\lambda\right)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\right)\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)\right]$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(41)}T^{*}_{41}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^{*}_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^{*}_{41}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(40)}T^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$+\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(42)}T^{*}_{40}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\right)\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)\right]$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(45)}T^{*}_{45}\right)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^{*}_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^{*}_{45}\right)$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(44)}T^{*}_{44}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^{2}+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$+\left(\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)$$

$\left(\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\right)\} = 0$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.**

# Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured populations can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i \in R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i\ X_i(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator-prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor

bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k \tag{25}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 25. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,...x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone $\{ x_i \epsilon C_i: \prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i(x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \in I, x \in K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow *x.t restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\,\delta\,(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( $\Gamma(A)$), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *($\Gamma(A)$).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C\backslash C^{+}) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^{n}_{i}$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^{+}$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^{+}$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x) = 0$. Alternatively, if $\lambda_i(x) < 0$,then population *i* is doomed to extinction. **Proposition 1.** $\lambda_i(x)$ *satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v > 0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int_0^t (A_i(x.s)u_i(x.s), u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \epsilon R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}$ $F_i(x)$, we can write $v=au_i(x)+w$ with $a \epsilon R$ and $w \epsilon F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|\ B_i(t,x)u_i(x)| + |B_i(t,x)w| \leq |\ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)|\ w|)$. Hence, $\limsup_{t\to\infty} \frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\limsup_{t\to\infty} \frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1 B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |\ B_i(t,x)u_i(x)|(a- \alpha+ exp\ (-\beta t)|\ w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In } b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x)$ =0 implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

=

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \le 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \le \gamma$ for all $t \ge 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \operatorname{In} |B_i(t,x)x_i| = \operatorname{limsup}_{t\to\infty} \frac{1}{t} \operatorname{In} |\, (x.t)_i\, |$$

$$\le \operatorname{limsup}_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0. If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\ge 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\operatorname{limsup}_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \ge 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure*. $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0$ *for all* $i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h_i(x.s)ds$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu(\Pi_{i\in I} C_i^+)$=1. Let K $\subset \Pi_{i\in I} C_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} C_i^+$such that $x.t_n \in K$ for some $t_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$ , $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x) \rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1 \le j \le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \le \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \le \in$, and $|h(x.t) - h(x^n.t)| \le \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\le \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |\mathrm{h}(x.t) - h(x^n.t)|d\mu_n(x)| + |\int_c h(x^n.t)|d\mu_n(x) - \int_c \mathrm{h}(x)d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x)d\mu_n(x) - \int_c h(x)d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $\mathrm{h}_i^n(x) = \langle \mathrm{A}_i^n(x)\mathrm{u}_i^n(x), \mathrm{u}_i^n(x)\rangle$ where $\mathrm{u}_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $\mathrm{A}_i^n(x) \to \mathrm{A}_i(x)$ uniformly for $x \in N_1((\mathrm{A}))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $\mathrm{u}_i^n(x) \to \mathrm{u}_i(x)$ converges uniformly for $x \in N_1(\Lambda(\mathrm{A}))$. Given $\in > 0$, choose $n$ sufficiently large so that $|\mathrm{h}_i^n(x) - \mathrm{h}_i(x)| \leq \in$ for all $x \in N_1(\Lambda(\mathrm{A}))$ and $|\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$\lambda_i(\mu) \leq \int_c |\mathrm{h}_i(x) - \mathrm{h}_i^n(x)|d\mu(x) + |\int_c \mathrm{h}_i^n(x)\, d\mu(x) - \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)|$

$+ \int_c \mathrm{h}_i^n(x)\, d\mu_n(x)$

$\leq \quad 2\in$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in \mathrm{W}^s(\mathrm{K}) \cap \mathrm{C}^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\, \mathrm{ds}$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument

implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i \, (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

(6) $\min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$ $\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that (□$i \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceKc\,.\partial$□,$\{1, \ldots . k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V $c$ N$\delta$ (*Γ*(A)) to be a compact neighborhood of Γ(A) such that V.t $c$ int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))$c$ *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \begin{cases} \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} & if\ i \epsilon I \\ if\ i \notin I \end{cases}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})c\ V\ c\ N\delta\ (\ (A))). We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$□ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$c$ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set $K$ if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$?then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$?and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume x* $\mapsto$ *A(x) is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists j* $\epsilon$ { 1,….,k} *such that $M_i$ ----- { $x \in C : x_l \gg 0$ for $l \neq j$} Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_iM_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----{1,…,k} such that $\mu(---_i \in_I C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures μ with support in ∂C, $\lambda_i(\mu)>0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $$\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j \qquad i = 1,..,k \qquad j=1,\ldots,m$$

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge

to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j$ ($x^j_i$) = 0 i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product ($E_1^1$ ,…,$E_1^m$), and the matrix $A^2$ ($E_1$) is close to the diagonal matrix diag($f^i_2(E_1{}^j)$)$^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f_2^j (E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j (E^j_1)>0$ and $\max_j f_2^j (E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial$C is given by two equilibria {0,$E_1$}, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j (E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$

$\frac{dx_2^j}{dt} = x_2^j(1 - \alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1 - \beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > 1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria $E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$ and $\lambda_3(E_1)$ are close to $\max_j 1 - a^j$ and $\max_j 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1 - a^j$ and $\max_j 1 - \beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1 - a^j) > -\max_j(1 - \beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $r$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \ \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i\left(r_i - \sum_s a_{is}x_s\right), \qquad i = 1,\dots,k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$$

It has been shown [12,ch.16.1,16.2] That for *k* =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \ \ with\ \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1 \to E_2 \to E_3 \to E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing

holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & O \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,($K$,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance $N$ and exhibits logistic dynamics f($N$)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left[ \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1 - y) - b(N)\ y \right]$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*′ Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\curlywedge_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\curlywedge_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d\ and$

$$\curlywedge_3(\mu) = \beta \int \frac{x_2}{\epsilon+x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords

Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation

of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE.

**Introduction**

Let $P_1$ *(t)* and $P_2$ *(t)* denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial

alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.73) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.74) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.75) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_{1,}$ $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_{i,}$ $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic ($as$ $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 25. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2)) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, \text{R}^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1, 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) \quad P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^\infty f_i(a + s, P(t + S)) ds \right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[ - \int_o^x f_1(a + s, p(t + s)) ds \right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0) \exp\left[ - \int_o^t f_1\left(a - t + s, p_1(s)\right) ds \right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$$

$$(2.5)\ L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(\text{a+s}, p_i(s))ds]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da+\int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$$t> 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \epsilon[-\tau, 0], i = 1,2,$

(2.8) $\beta_i(\text{t}) = \int_0^t \beta_i(\text{a}) b_i(\text{t-a}, P_i(t)) M_i.(a, t, P_i(t))da$

$$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$$

$$i = 1, 2, \text{t} > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi(0) d_i^* t$$

$$\le [\beta_i^* \phi(0)\, exp(\beta_i^* t) \phi(0) d_i^*]t \quad for\ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T],R^+), \quad B_i \in C([0,T],R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau,T] = \{f \in C([-\tau,T],R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau,T]$. Then for any

$(x, y) \in C^+[-\tau,T] \times C^+[-\tau,T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau,T], \quad C_T^2 = C^+[-\tau,T] \times C^+[-\tau,T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau,T]$ *and* $t \in [0,T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$B: C_T^2 \to C_T^2$ *and* $\Pi C_T^2 \to C_T^2$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = P_t \left\{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

$$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \right) t^*$$ (2.13)

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Pi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Pi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Pi(P_t^{(1)}) - \Pi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Pi(P^{(1)}(t)) - \Pi(P^{(2)}(t)| \le \int_0^t B_1^{(1)}(a) \left\{ M_1\left(a, t, P_t^{(1)}\right) - M_1\left(a, t, P_t^{(2)}\right) \right\} da$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \le |x-y| \ for\ x,y \le 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\le \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\le d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} \text{(using } A_1\text{)}$$

*Now,using (2.10),*

$$\left| J_1 \right| \le \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left( \int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_* ds \right) da$$

$$\le d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \le \int_o^\infty \varphi(a,o) \left[ \int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds \right] da$$

$$\le d_1 \Phi_1(0) \| P_t^{(1)} - P_t^{(2)} \|_t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$=\int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\mathrm{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right]\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left| b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right) \right| \leq B_1 \, /\!/ P_t^{(1)} - P_t^{(2)} /\!/ \text{ for } a \, \varepsilon [o, t^*]$$

And hence we have

$$|R_3(a)+R_4(a)| \leq \beta_1 \Phi_1(0) || P_t^{(1)} - P_t^{(2)} ||_t^* + \Phi_1(0) \beta_1 d_1 || P_t^{(1)} - P_t^{(2)} ||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) \, e^{\delta 1^a} \beta_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) \, e^{\delta 1^a} /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| B_1^{*d}\sigma$$

$$+ \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \left\{ B_1 + B_2 d_1 a + B_1^* \left(B_2 a + d_1 a^2/2\right) e^{\delta 1^a} \right\}$$

$$\leq \beta_1^* \int_o^a \left| B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) \right| do + \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} g_1^{(a)}$$

Where $g_1(a) \rightarrow \beta_1$ as $a \rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{g_1(a) + 0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a)) - b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a)) do \right|$$

$$\left| B_1^* \left(r^* + /\!/ \Phi_t /\!/_{t*}\right) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} 0(a) as\ a \rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi (P_t^{(1)} - \pi \left(P_t^{(2)}\right) /\!/ \leq (r^* + /\!/ \Phi_t /\!/_{t*}) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t*} o(t^*)$$

As $t^* \rightarrow o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp P_2^*(a)}{da} + f_i(a, P_1^*, P_2^*) P_i^*(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^* = \int_o^\infty P_i^*(a) da, i = 1,\ 2,$$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i=1.\ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s,P_1^*,P_2^*)ds\right], i=1,\quad 2, a\geq 0,$

Which together with (3.3) lead to

$$1=\int_o^\infty b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^\infty b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1,P_2) = \int_o^\infty b_i(a,P_1,P_2)\exp\left[-\int_o^\infty f_i(s,P_1,P_2)ds\right]da,$

$i=1,\ 2,\quad P_1\geq 0,\quad P_2\geq 0,$

**And suppose the following hold:**

(3.7) $A_4.\quad F_{i.}\ (0,\ 0)>1,\ i=1,2$:

$A_5.$ *there exists a positive constant c for which*

$$F_1(C,0)< F_{2.}(C,0),$$

(3.8) $$F_1(0.C)< F_{2.}(0,C),$$

Then there exists a unique pair $(P_1^*,P_2^*)$ of real numbers $P_1^*>o,$

$P_2^*>0$ such that

(3,9) $F_1(P_1^*,P_2^*)=1=F_2(P_1^*,P_2^*)$

And the unique nomegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a)=[P_2^*\exp[-f_i(s,P_1^*,P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s,P_1^*,P_2^*)ds]da),$

$i=1,2,\quad a\geq 0,$

Proof Consider the elements of the surfaces defined by

Z= $F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0,$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

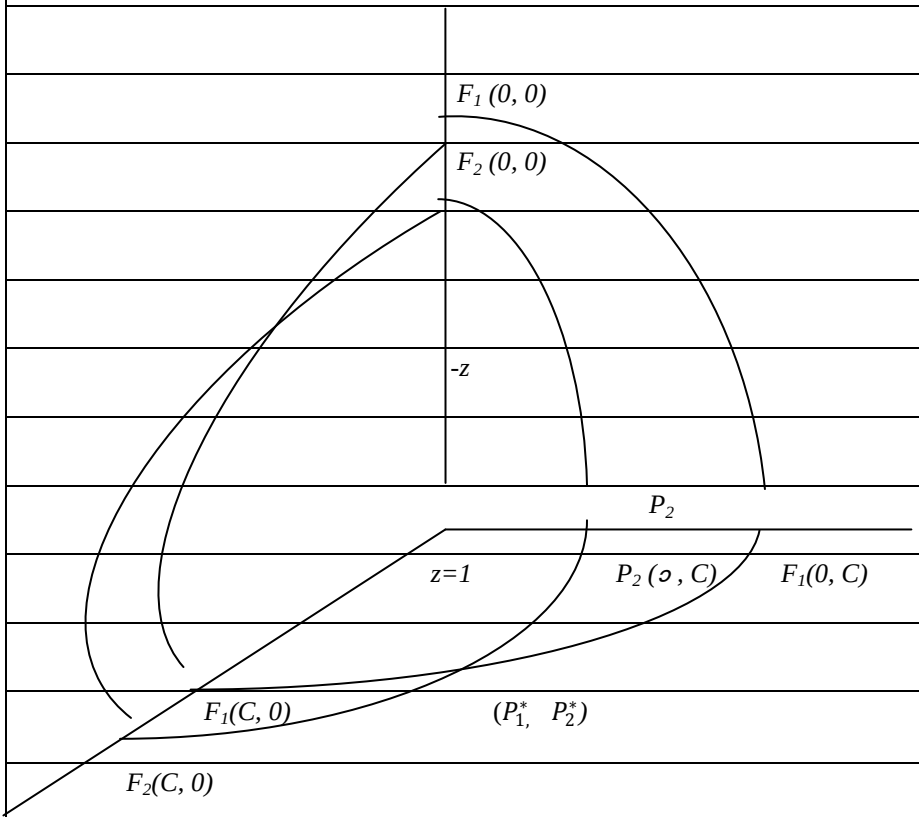
$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0,$ These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, F_1(P_1, P_2 \geq 0,$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11) $$\begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12) $$\begin{cases} Du_i(a,t) = -f_i\left(a, p_1^*, p_2^*\right) u_i(a,t) - p_i^*(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) u_i(a,t)da + \int_0^\infty p_i^*(a)\left\{\sum_{\square=1}^{2} \frac{\partial b_i}{\partial p_{\square}^*} p_{i\square}(t)\right\}da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14) $$\begin{cases} u_i(a,t) = u\square(a)\exp[\lambda t], \quad \square(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p_i^* \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i^* = \int_0^\infty \square(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d\square}{da} + \lambda\square = -f_i\left(a, p_1^*, p_2^*\right)\square(a) - p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_{i\square}^*\right\},$$

(3.15)

$$\square(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\square(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*} p_{i\square}^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^{*} = p_{1}^{*}, \quad p_{12}^{*} = p_{2}^{*} \exp\{-\lambda\tau\},$$

$$p_{21}^{*} = p_{1}^{*}, \exp\{-\lambda\tau\}, \ p_{22}^{*} = p_{2}^{*}.$$

If we integrate the first of (3.15) and use the fact that $\Omega_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\Omega_1(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\Omega_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\Omega_i(a) = \{\Omega_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\Omega_i\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\Omega_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\Omega_i\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\Omega_i\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\ p_i^*\Omega_i\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)}\int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)e^{-\lambda a}da, \quad i = 1,\ 2,$$

Now supplying $\Omega_1(a)$ and $\Omega_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^*\exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^*\exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^* \exp(-\lambda\tau) + (\lambda - A_{22})\, p_2^* = g_{21}(\lambda)\, p_1^* \exp(-\lambda\tau) + g_{22}(\lambda)\, p_2^*$$

Where

(3.22) $$A_i = \int_0^\infty p_i^*(a)\left[\frac{\partial b_i}{\partial p_i^*} - \frac{\partial f_i}{\partial p_i^*}\right]da,$$

(3.23) $$g_i(\lambda) = \left[p_i^*(0)c_i\right]^{-1}\left\{\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\, da\right\}$$
$$\left\{\int_0^\infty \left[p_i^*(a)\frac{\partial b_i}{\partial p_i^*} - b_i(a, p_1^*, p_2^*)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_i^*}\, e^{\lambda s}\, ds\right)\right]da\right\}$$

$$-\int_0^\infty \sigma_i(a)p_i^*(a)\, e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_i^*}\, e^{\lambda s}\, ds\right)da, \quad i = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*), \quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12})e^{-\lambda\tau} \\ -(A_{21} + g_{21})\, e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$$S_2(\lambda, \tau) = \left[A_{21}\, g_{12}(\lambda) + A_{12}\, g_{21}(\lambda) + g_{12}(\lambda)\, g_{21}(\lambda)\right]\exp[-2\lambda\tau]$$

$$-A_{11}\, g_{22}(\lambda) - A_{22}\, g_{11}(\lambda) - g_{11}(\lambda)\, g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6$. $0 < \beta_i^* < d_i^*$ $(i = 1, 2.)$ $(see(A_1) and (A_2))$;

$A_7$. $\{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \text{[illegible]}(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$

$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)$ ,

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \left|\right.$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2$ ,

$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \,| - |\, A_{11} + A_{22} + \| S_1(0) \| \,|^2 \right\}/2,$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than utility; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational

ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the

Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have

obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is:

individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

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